RATIONAL CURVES AND AMPLENESS PROPERTIES OF THE TANGENT BUNDLE OF ALGEBRAIC VARIETIES

FRÉDÉRIC CAMPANA AND THOMAS PETERNELL

Abstract. The purpose of this paper is to translate positivity properties of the tangent bundle (and the anti-canonical bundle) of an algebraic manifold into existence and movability properties of rational curves and to investigate the impact on the global geometry of the manifold X. Among the results we prove are these:

If X is a projective manifold, and $\mathcal{E} \subset T_X$ is an ample locally free sheaf with $n-2 \geq rk\mathcal{E} \geq n$, then $X \simeq \mathbb{P}_n$.

Let X be a projective manifold. If X is rationally connected, then there exists a free T_X-ample family of (rational) curves. If X admits a free T_X-ample family of curves, then X is rationally generated.

Introduction

The purpose of this paper is, vaguely speaking, to translate positivity properties of the tangent bundle (and the anti-canonical bundle) of an algebraic manifold into existence and movability properties of rational curves and to investigate the impact on the global geometry of the manifold X. This study falls into two parts:

(1) a biregular one in which ampleness of some subsheaf of the tangent bundle T_X is assumed; one then expects a classification,
(2) a birational one, where the ampleness of T_X is assumed only generically; one then expects X to be rationally connected.

In 1979 in his famous solution of the Hartshorne-Frankel conjecture, S. Mori proved that a projective manifold X whose tangent bundle T_X is ample, must be the projective space $\mathbb{P}_n$. Previously, Mori and Sumihiro showed that $\mathbb{P}_n$ is the only manifold admitting a vector field vanishing along an ample divisor (and this is an Euler vector field if $n \geq 2$). This was generalised by J. Wahl to the extent that if $T_X \otimes L^{-1}$ has a non-zero section with L an ample line bundle (not necessarily effective), then $X \simeq \mathbb{P}_n$.

Although these two theorems look at first quite different in nature, they might be only special cases of a more general theorem.

Parts of this paper were written up during a stay of one of the authors at MSRI. He would like to thank the institute for support and the excellent working conditions. Research at MSRI is supported on part by NSF grant DMS-9022140.

Question: Let X be a projective manifold, $\mathcal{E}$ an ample locally free sheaf, $\mathcal{E} \subset T_X$. Is $X \simeq \mathbb{P}_n$?

We shall prove

Theorem (1.1) *The question has a positive answer if $rk\mathcal{E}$ is $n-2$, $n-1$ or n.*

A manifold is said to be rationally connected if two general points can be joined by a chain of rational curves. Examples are Fano manifolds as shown by Campana and Kollár-Miyaoka-Mori. We ask whether it is possible to characterize rational connectedness by a weak positivity property of the tangent bundle. Philosophically speaking we require that T_X should be ample on sufficiently many curves; more precisely T_X should be ample on the generic curve of a *free* family (2.5).
We shall prove in this spirit (for definitions see (2.1))

Theorem (2.7) *Let X be a projective manifold.*

1. *If X is rationally connected, then there exists a free T_X-ample family of (rational) curves.*
2. *If X admits a free T_X-ample family of curves, then X is rationally generated.*

(1) is actually an easy consequence of [KoMiMo92]. A projective variety X is rationally generated if every variety dominated by X is uniruled, see sect.2. This notion was introduced in [Ca95]); we expect that rationally generated varieties are actually rationally connected; this holds in dimension 3 by [KoMiMo92].
Important special cases of free family are given by complete intersections of hyperplanes. To make this precise we recall the following

Definition (Miyaoka) Let X be a $n-$dimensional projective manifold, $H_1, ..., H_{n-1}$ ample divisors on X. Let $\mathcal{E}$ be a vector bundle on X. Then $\mathcal{E}$ is said to be generically $(H_1, ..., H_{n-1})$ -ample (nef) if for $m_i >> 0$ and for the general curve C cut out by $m_1\ H_1, ..., m_{n-1}H_{m-1}$, the restriction $\mathcal{E}|_C$ is ample (nef).

Miyaoka [Mi 87] has shown that X is not uniruled iff Ω^1_X is generically $(H_1, ..., H_{n-1})$-<u>nef</u> for some H_i.
We obtain as a special case of (2.5):

Corollary *If T_X is generically $(H_1, ..., H_{n-1})$-ample, then X is rationally generated.*

The converse of the corollary can be expected but should be hard (at least with the H_i big instead of ample). It is much easier to look only at positivity of large families of curves, not necessarily cut out by divisors. Also a relation to stability is pointed out.

In practice the positivity of T_X is difficult to handle. Much easier is the anticanonical bundle $-K_X = det\ T_X$. Therefore we shall investigate positivity properties of $-K_X$ in sect. 3 for varieties of negative Kodaira dimension.

0. Notations around rational curves.

Herewe collect for the convenience of the reader various concepts dealing with rational curves. General references are [Ca 92], [KoMiMo 92].

(0.1) A variety X is called rationally connected if and only if two general points on X can be joined by a chain of rational curves.

(0.2) Let (C_t) be a covering family of rational curves on X. Introduce the following equivalence relation: $x \sim y \Longleftrightarrow x, y$ can be joined by a chain of $C_t's$. Then there is a variety Y and a rational map $f : X \rightharpoonup Y$ such that for general x, y we have $f(x) = f(y) \Longleftrightarrow x \sim y$. The map $f : X \rightharpoonup Y$ is called the rational quotient of X w.r.t. (C_t) (see [Ca94] for a proof, in a more general situation).

(0.3) One can do the construction in (0.2) for all (C_t) at the same time and obtains the notion of a "rational quotient" of X.
More precisely define $x \approx y \Longleftrightarrow x, y$ can be joined by a chain of rational curves.
Then there exists a variety Z and a rational map $g : X \rightharpoonup Z$ such that for very general x, y (i.e. outside a countable union of subvarieties of X) we have $g(x) = g(y) \Longleftrightarrow x \approx y$.
An important property of the rational maps $f : \rightharpoonup Y$ and $g : \rightharpoonup Z$ is the almost holomorphicity, i.e. they are holomorphic and *proper* on a suitable Zariski dense open subset of X.

(0.4) Let X be a variety. Define inductively: $X_o = X$, $X_i =$ rational quotient of X_{i-1}.
Then X is said to be rationally generated if there exists m such that $dim\, X_m = 0$. It is conjectured (but known only up to dimension 3, [KoMiMo92]) that X is rationally connected iff X is rationally generated.
Rational connectedness and rational generatedness are equivalent provided the following holds: for every holomorphic surjective map $f : X\mathbb{P}_1$ with X a projective manifold and f having rationally connected fibers, there exists a rational curve $C \subset X$ with $f(C) = \mathbb{P}_1$. See [Ca95].

1. Ample subsheaves of tangent bundles

In this section we shall prove

1.1 Theorem *Let X be a projective manifold of dimension n. Let $\mathcal{E} \subset \mathcal{T}_X$ be a locally free sheaf of rank r. If $\mathcal{E}$ is ample and $rk\ \mathcal{E} \geq n-2$, then $X \simeq \mathbb{P}_n$ and $\mathcal{E} = \mathcal{T}_X$ or $\mathcal{O}(1)^{\oplus r}$.*

Actually (1.1) should hold without any assumption on the rank. If $\mathcal{E} = \mathcal{T}_X$, then (1.1) is nothing than Mori's theorem [Mo 79]. Also the other extremal case is known: if $rk\ \mathcal{E} = 1$, then $X \simeq \mathbb{P}_n$ by J. Wahl [Wa 83].

The proof of (1.1) proceeds by induction on $dim\ X$. In the induction step we will need the following

1.2 Lemma *Let X be a n-dimensional projective manifold, $\varphi\colon X \to Y$ a $\mathbb{P}_k$-bundle $(k < n)$ of the form $X = \mathbb{P}(V)$ with a vector bundle V of rank $k+1$. Then the relative tangent sheaf $\mathcal{T}_{X|Y}$ does not contain an ample locally free subsheaf.*

Remark Lemma 1.2 should only be a special case of a much curve general fact: if $\varphi\colon X \to Y$ is a fiber space (sufficiently smooth?), then $\mathcal{T}_{X|Y} = (\Omega^1_{X|Y})^*$ does not contain an ample subsheaf unless Y is a point and X projective space.

Proof. Suppose $\mathcal{E} \subset \mathcal{T}_{X|Y}$ is an ample locally free subsheaf. Let F be a fiber of φ. Then $\mathcal{E}|_F \subset \mathcal{T}_{X|Y}|F = \mathcal{T}_F$. Hence either $\mathcal{E} = \mathcal{T}_{X|Y}$ or $\mathcal{E} = \mathcal{O}(1)^{\oplus r}_{|F}$ for all F. In the first case, apply the Euler sequence

(E) $\qquad 0 \to \mathcal{O} \to \varphi^*(V^*) \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \mathcal{T}_{X|Y} \to 0$

to derive the ampleness of $\varphi^*(V^*) \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$ ((E) does not split !) which is absurd (or argue as in the second case). So let $\mathcal{E}|_F = \mathcal{O}(1)|^{\oplus r}_F$ for all F. Then

(*) $\qquad \mathcal{E} \simeq \varphi^*(\mathcal{E}') \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$

with a bundle $\mathcal{E}'$ of rank r on Y. Consider again (E). We claim that the inclusion $\mathcal{E} \hookrightarrow \mathcal{T}_{X|Y}$ lifts to an inclusion $\mathcal{E} \hookrightarrow \varphi^*(V^*) \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$.
=46or this we need to show that the canonical map

$$H^o(\mathcal{E}^* \otimes \varphi^*(V^*) \otimes \mathcal{O}_{\mathbb{P}(V)}(1)) \to H^o(\mathcal{E}^* \otimes \mathcal{T}_{X|Y})$$

is onto, hence that $H^1(\mathcal{E}^*) = 0$.
By Leray's spectral sequence,

$$H^1(X, \mathcal{E}^*) \simeq H^1(Y, \mathcal{E}'^* \otimes \varphi_*(\mathcal{O}_{\mathbb{P}(V)}(-1)) = 0.$$

So we have $\mathcal{E} = \varphi^*(\mathcal{E}') \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \subset \varphi^*(V^*) \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$, therefore $\mathcal{E}' \subset V^*$. Consider the dual map

$$\alpha\colon V \to \mathcal{E}'^*,$$

which is generically onto. Let $\mathcal{S} = Im\ \alpha$. Then
$i : \mathbb{P}(\mathcal{S}) \hookrightarrow \mathbb{P}(V)$ and

$$i^*(\mathcal{E}) = i^*(\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \varphi^*(\mathcal{E}')) = \mathcal{O}_{\mathbb{P}(\mathcal{S})}(1) \otimes \varphi'^*(\mathcal{E}'),$$

where $\varphi' \colon \mathbb{P}(\mathcal{S}) \to Y$ is the projection. By construction, $\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1) \otimes \varphi'^*(\mathcal{E})$ is ample, hence, taking det,

$$\mathcal{O}_{\mathbb{P}(\mathcal{S})}(r) \otimes \varphi'^*(det\mathcal{E}') = \mathcal{O}_{\mathbb{P}(\mathcal{S}\otimes \frac{det\,\mathcal{E}'}{r})}(r)$$

is ample. Hence $\mathcal{S} \otimes \frac{det\,\mathcal{E}'}{r}$ is ample. Now take a general curve $C \subset Y$. Then $\mathcal{E}'^* \otimes \frac{det\mathcal{E}'}{r}|C$ is ample. But $c_1(\mathcal{E}'^* \otimes \frac{det\mathcal{E}'}{r}) = 0$, contradiction.

Now we come to the proof of (1.1). We treat the cases $r = n, n-1$ and $n-2$ separately.

(1.3) Case $r = n$.
The inclusion $\mathcal{E} \subset \mathcal{T}_X$ yields $det\,\mathcal{E} \subset -K_X$. If $det\,\mathcal{E} = -K_X$, then $\mathcal{E} = \mathcal{T}_X$ and the claim is Mori's theorem. So assume $det\,\mathcal{E} \neq -K_X$, hence $K_X + det\,\mathcal{E} \neq \mathcal{O}_X$, but $K_X + det\,\mathcal{E} \subset \mathcal{O}_X$.
So $K_X + det\,\mathcal{E}$ is not nef. Now vector bundle adjunction theory [Fu 90, YZ 90] gives $X \simeq \mathbb{P}_n$ and $\mathcal{E} = \mathcal{O}(1)^n$.

(1.4) Case $r = n-1$.
Since all arguments in this case appear also in the more difficult case $r = n-2$, we just give a very rough sketch. $\mathcal{E} \subset \mathcal{T}_X$ yields $det\,\mathcal{E} \subset \wedge^{n-1}\mathcal{T}_X = \Omega^1_X \otimes -K_X$, hence $L = K_X + det\,\mathcal{E} \subset \Omega^1_X$.
If L is not nef, then all such pairs $(X, \mathcal{E})$ can be classified ([Fu 90, YZ 90]) and examined to prove our claim. So let L be nef. Hence

$$H^0(X, \Omega^1_X \otimes L^*) \neq 0$$

for some nef line bundle L.

Now $H^o(\Omega^1_X \otimes L^*) \neq 0$ implies that X is not rationally connected. However it is uniruled by [Mi 87], so one can consider the rational quotient $f \colon X \rightharpoonup Z$ w.r.t. some covering family of rational curves. Then the general compact fiber of f is a projective space which leads easily to a contradiction to the nefness of L.

(1.5) Case $r = n-2$.
Taking det of $\mathcal{E} \hookrightarrow \mathcal{T}_X$ gives $det\,\mathcal{E} \hookrightarrow \wedge^{n-2}\mathcal{T}_X = \Omega^2_X \otimes -K_X$ hence

$$L := K_X + det\,\mathcal{E} \subset \Omega^2_X.$$

(1.5.1) We first assume that L is not nef.
Then by [AM 95], $(X, \mathcal{E})$ is one of the following.

(a) There is a blow-up $\varphi \colon X \to W$ of a finite set $B \subset W$ of smooth points and an ample bundle $\mathcal{E}'$ on W with $\mathcal{E} = \varphi^*(\mathcal{E}') \otimes \mathcal{O}_X(-A)$, $\qquad A = \varphi^{-1}(B)$

(b) $X = \mathbb{P}_n, \mathcal{E}$ splits
(c) $X = Q_n$ and $\mathcal{E}$ splits or is a twist of the spinor bundle
(d) $X = \mathbb{P}_2 \times \mathbb{P}_2,\ \mathcal{E} = \mathcal{O}(1,1) \oplus \mathcal{O}(1,1)$
(e) X is a del Pezzo manifold with $b_2 = 1, \mathcal{E} = \mathcal{O}(1) \oplus n-2$
(f) X is a $\mathbb{P}_{n-1}$ - bundle or a quadric bundle over a curve Y
(g) X is a $\mathbb{P}_{n-2}$-bundle over a surface.

We comment on those cases separately.

(a) We have $\mathcal{E} = \varphi^*(\mathcal{E}') \otimes \mathcal{O}_X(-A) \subset \mathcal{T}_X \subset \varphi^*(\mathcal{T}_W)$, so that $\mathcal{E}' \otimes I_B \subset \varphi_*(\mathcal{T}_X) \subset \mathcal{T}_W$, hence
$$\mathcal{E}' \subset \mathcal{T}_W.$$
By induction on $b_2(X)$ we may assume $W = \mathbb{P}_n$ and $\mathcal{E}' = \mathcal{O}(1)^{\otimes(n-2)}$. But then $\mathcal{E} = \varphi^*(\mathcal{O}(1))^{n-2} \otimes \mathcal{O}_X(-A)$ is clearly not ample, at most nef.
(b) Obviously $\mathcal{E} = \mathcal{O}(1)^{\oplus(n-2)}$.
(c) If $\mathcal{E}$ splits, then T_{Q_n} would have a section vanishing on ample divisor which is not possible. So $\mathcal{E}$ is the twist $F(2)$ of a spinor bundle, see e.g. [PSW 90]. Therefore $\mathcal{E}$ has a section vanishing on a hyperplane section, so does T_{Q_n} which would imply $H^o(T_{Q_n}(-1)) \neq 0$, contradiction.
(d) Again $\mathbb{P}_2 \times \mathbb{P}_2$ has no vector field vanishing on an ample divisor.
(e) Here $K_X + det\ \mathcal{E} = \mathcal{O}_X(-1)$ and $\mathcal{E} = \mathcal{O}_X(1)^{\oplus(n-2)}$. Now either apply [Wa 83], [MS 78] or apply the Fujita classification of del Pezzo manifolds or argue directly as follows. We have $H^o(X, \mathcal{T}_X(-1)) \neq 0$. Choose $H \in |\mathcal{O}_X(1)|$ smooth and consider the sequence
$$0 \to \mathcal{T}_H \to \mathcal{T}_X|H \to N_{H|X} = \mathcal{O}(1) \to 0.$$
If the induced map $\mathcal{E}|_H \to N_H$ is non-zero, we obtain a map $\mathcal{O}(1)^{n-3} \hookrightarrow \mathcal{T}_H$, otherwise $\mathcal{O}(1)^{n-2} \subset \mathcal{T}_H$. In both cases we get $H \simeq \mathbb{P}_{n-1}$ by induction, hence $X \simeq \mathbb{P}_n$.
(f) Let F be the general fiber of $\varphi\colon X \to Y$. Then $\mathcal{E}|_F \hookrightarrow \mathcal{T}_X|F$ and since the composition
$$\mathcal{E}|_F \to \mathcal{N}_{F|X} = \mathcal{O}_F^a$$
is zero, we obtain $\mathcal{E}|_F \hookrightarrow \mathcal{T}_F$. Inductively, $F \simeq \mathbb{P}_{n-1}$, hence f cannot be a quadric bundle therefore must be a $\mathbb{P}_{n-1}$-bundle; moreover $\mathcal{E} \subset \mathcal{T}_{X|Y}$. But this contradicts (1.2).
(g) Finally φ is a $\mathbb{P}_{n-2}$ bundle. As in (f), $\mathcal{E} \subset \mathcal{T}_{X|Y}$, contradicting (1.2).

This finishes the case that L is not nef.

(1.5.2) $L = K_X + det\,\mathcal{E}$ is nef.
So (*) $\qquad H^o(X, \Omega^2_X \otimes L^*) \neq 0$

with L nef. This implies that X is not rationally connected: otherwise we find by [KoMiMo 92] a family of rational curves $C_t \subset X$ with $T_X|C_t$ ample for t general contradicting (*).
Hence we have a rational quotient $f\colon X \rightharpoonup Z$, [Ca92] and $dim\, Z > 0$ since X is not rationally connected. Moreover $dim\, Z < dim\, X$, since X is uniruled by [Mi 87]. Since f is almost holomorphic, i.e. proper on an open set, it has some compact fibers. The general fiber F therefore must be $\mathbb{P}_k$ (since $\mathcal{E}|_F \hookrightarrow \mathcal{T}_F$) with $k \geq n-2$. Therefore L_F is never nef, contradiction. $\square$

(1.6) We already mentioned Wahl's theorem several times: if X is a projective manifold, L an ample line bundle, $H^o(X, T_X \otimes L^*) \neq 0$, then $X \simeq \mathbb{P}_n$. Previously Mori and Sumihiro [MS 78] proved this under the additional assumption that L is effective.
It should actually be possible to deduce Wahl's theorem from Mori-Sumihiro by covering tricks in the following way. Take $d \geq 2$ with $H^o(X, L^d) \neq 0$, and $H^o(X, T_X \otimes L^*) \neq O$. Suppose $s \in H^o(X, L^d)$, $s \neq 0$ with $\{s = 0\}$ smooth. Let $f\colon Y \to X$ be the associated cyclic cover so that $H^o(Y, f^*(L)) \neq 0$. Then there is an exact sequence

$$0 \to T_Y \to f^*T_X \to N_R \to 0$$

where N_R is supported on the ramification divisor $R \subset Y$. The map $f^*(L) \hookrightarrow f^*(T_X)$ yields a map $\gamma\colon f^*(L) \to N_R$. Now it can be shown that $\gamma = 0$, hence we obtain $f^*(L) \subset T_Y$. By [MS 78], $Y \simeq \mathbb{P}_n$, moreover $f^*(L) = \mathcal{O}(1)$, so that $deg\, f = 1$ and $X \simeq \mathbb{P}_n$.
Of course the zero set $\{s = 0\}$ could be non-smooth. If $\{s = 0\}$ is at least reduced, then the same arguments basically work, only that Y is merely normal. So one should first prove [MS 78] in case Y is normal. The case where $\{s = 0\}$ contains a non-reduced component seems more complicated, one should proceed taking roots.

2. The tangent bundle of rationally connected varieties

2.1 Definition Let X be a projective manifold. A covering family $(C_t)_{t\in T}$ (T compact) of curves is free if
(1) T is irreducible
(2) C_t is irreducible for $t \in T$ general
(3) for all $A \subset X$ at least 2-codimensional there exists $t \in T$ with $C_t \cap A = \emptyset$.

We shall say that a family (C_t) is T_X-ample, if $T_X|C_t$ is ample for general t.

2.2 Theorem *Let X be a projective manifold, (C_t) a free T_X-ample family. Then X is rationally generated. Conversely, a rationally connected manifold has a free T_X-ample family of (rational) curves.*

Proof : From the definition of a free family it is clear that X is uniruled, applying [MM 86]. Then we consider "the" rational reduction $f\colon X \rightharpoonup Y$ and have $dim Y < dim X$, Y smooth. By construction, Y is not uniruled.
We need to show that $dim Y = 0$. So assume $dim Y > 0$. By condition (2) of (2.1), f is holomorphic near C_t for general t. Since $T_X|C_t$ is ample, we have $dim\, f(C_t) > 0$. Now consider the generically surjective map (t general)

$$\alpha\colon T_X|C_t \to f^*T_Y|C_t.$$

Then $Im\ \alpha$ is an ample sheaf, hence $f^*T_Y|C_t$ is ample and so does $T_Y|f(C_t)$. The family $C'_t = f(C_t)$ still covers Y with $T_Y|C'_t$ ample for general t, hence $-K_Y.C'_t > 0$ and Y is uniruled by [MM 86], contradiction. The other direction follows from [KoMiMo92] using (2.5(2)). $\square$

Next we recall a notion due to Miyaoka.

2.3 Definition Let X be a projective manifold, $n = dim\, X, H_1, \cdots, H_{n-1}$ ample divisors X. A vector bundle E on X is generically $(H_1, \cdots, H_{n-1})$-ample (nef) iff for $m_1 >> 0, \cdots, m_{n-1} >> 0$ and for general C cut out by $m_1\, H_1, \cdots, m_{n-1}H_{n-1}$ the restriction $E|C$ is ample (nef).

(2.4) The same definition makes sense for $H_1, \cdots, H_{n-1}$ big, as long as the general element cut out by $m_1\, H_1, \cdots, m_{n-1}H_{n-1}$ is really a curve, i.e. if $dim \bigcap\limits_{i=1}^{n-1} Bs(m_iH_i) \leq 1$.

2.5 Obvious Example (1) If $H_1, \cdots, H_{n-1}$ are very ample, then the general C_t cut out by $H_1, \cdots, H_{n-1}$ deforms to build free families.
(2) If $C \subset X$ is a rational curve with $T_X|C$ ample, then C moves in a free T_X-ample family, provided X is smooth near C.

(2.6) Miyaoka has shown in [Mi 87] that X is not uniruled iff Ω^1 is generically $(H_1, \cdots, H_{n-1})$-nef for all ample $H_1, \cdots, H_{n-1}$. We are now interested in the "dual" case.

2.7 Theorem *If T_X is generically $(H_1, \cdots, H_{n-1})$-ample, then X is rationally generated.*

This follows from the more general theorem 2.2.

2.8 Corollary *Let X be a smooth projective 3-fold, (C_t) a free family of curves such that $T_X|C_t$ is ample for general t. Then X is rationally connected.*

Proof. Rationally generated 3-fold are rationally connected [KoMiMo 92].

2.9 Remark One might expect that (2.2) remains true if one assumes only the ampleness of $T_X|C_t, t$ general, where (C_t) is a family with the following property: for $x, y \in X$ general, there is some t such $x, y \in C_t$ (or joined by a chain of $C_t's$). However the proof of (2.2) does not give this claim. Probably one should construct from such a family a new, free family of curves.

2.10 Problem Let X be rationally connected. Do there exist $H_1, \cdots, H_{n-1}$ ample (or big), such that T_X is generically $(H_1, \cdots, H_{n-1})$ - ample? Is there at least a free family (C_t) such $T_X|C_t$ is ample, t general?

(2.11) We discuss (2.10) in several special cases.

(1) If $n = dim\, X = 2$, then (2.10) "clearly" holds.

(2) Assume T_X to be $(H_1, \cdots, H_{n-1})$ - semi-stable.

Then $T_X|C$ is semi-stable for C cut out by $m_1\, H_1, \cdots, m_{n-1}\ H_{n-1}$ for $m_i >> 0$ (Mehta-Ramanathan). By [Mi 87a], this is equivalent to saying that $(T_X \otimes \frac{K_X}{n})|C$ is nef (i.e. $S^n T_X \otimes K_X|C$ is nef) so if we know additionally that $-K_X \cdot H_1 \cdot ... \cdot H_{n-1} > 0$, (2.10) holds.

So (2.10) has a positive answer if we can find $H_1, \cdots, H_{n-1}$ ample such that

(a) T_X is $(H_1, \cdots, H_{n-1})$ semi-stable.
(b) $H_1, \cdots, H_{n-1}$ is in the cone generated by classes of covering families of rational curves.
(since (b) implies $-K_X.H_1. \cdots .H_{n-1} > 0$!)

In sect. 3 we shall discuss the relation of various cones of curves.

(3) If X is Fano, then (b) is always fulfilled.
It is expected that for $b_2 = 1, T_X$ is always stable; a lot of cases being checked in [PW 94], so that (2.10) should hold for Fano manifolds with $b_2 = 1$ (and probably also for higher b_2).

2.12 Remarks (1) If X is rationally generated, does X carry a free, T_X-ample family of curves? This could also be considered as a converse of the statement of (2.2).
(2) Is there a direct way to prove that a projective manifold carrying a free T_X-ample family of rational curves is rationally connected?
(3) From (2) and (3) it would follow that rationally generated manifolds are rationally connected. While (2) seems accessible, (3) is probably hard.

3. Various cones and the canonical bundle

3.1 Notation K_{nef} denotes the (closed) cone of nef divisors, K_{eff}, that one effective divisors (always modulo numerical equivalence, of course). We let N_{nef} be the dual cone of K_{eff} so that $C \in N_1(X)$ is in N_{nef} iff $D.C \geq 0$ for all $D \in K_{eff}$ or all effective divisors D) and call an element in N_{nef} a nef curve. The dual cone of K_{nef} is of course $\overline{NE}(X)$, the closed cone of curves. Furthermore we let

a) N_{rat} be the closed cone generated by classes $[C_t]$, where (C_t) is a covering family of rational curves, i.e. C_t is a possibly singular irreducible rational curve for general t. Without requiring C_t to be irreducible, but connected, with all components of C_t being rational, we call the resulting cone $\widetilde{N}_{rat}$.
b) N_{cov} be the analogous cone, omitting "rational"
c) N_{ci} be the closed cone generated by curves $H_1, \cdots, H_{n-1}; H_i$ very ample ("ci" stands for complete intersection)
d) N_{sc} be the closed cone generated by curves H^{n-1}; H very ample ("sc" stands for special complete intersection).

We have the obvious inclusions $N_{rat} \subset \tilde{N}_{rat} \subset N_{cov}$ and $N_{sc} \subset N_{ci} \subset N_{cov}$.

3.2 Problems (cp. [DPS 95].

(1) Is $N_{nef} = N_{cov}$? (Clearly $N_{cov} \subset N_{nef}$).
(2) Assume X Fano. Is $N_{rat} = N_{nef}$ or at least $N_{rat} = N_{cov}$?
(3) Assume X rationally connected or Fano. Is $N_{rat} \cap N_{ci} \neq \{0\}$?
(4) Assume X rationally connected or Fano. Is $N_{rat} \neq \widetilde{N}_{rat}$?

3.3 Remarks

(1) This problem was posed in [DPS 95]. It "essentially" (up to a limit process) means that given an nef curve $C \subset X$, i.e. $D.C \geq 0$ for all effective D, then there should exist a covering family (C'_t) such that $C'_t \equiv \alpha\, C_t$ for some $\alpha \in \mathbb{Q}_t$. In particular, if every effective divisor is nef, e.g. T_X is nef, then every curve C should be "numerically movable" up to a multiple. For surfaces this is clear. This would give new evidence for the conjecture that Fano manifolds with T_X nef are homogeneous ([CP 91, DPS 94]).
(2) Assume X Fano. The intention is to investigate "how much of the geometry is dictated by the moving rational curves." The cone theorem says that $\overline{NE}(X)$ is already generated by the classes of rational curves. So it seems very natural to suspect that the cone of "covering curves" is generated by the cone of "covering rational curves". As an application, a divisor D would be effective iff $D.C \geq 0$ for all rational curves C with $T_X|C$ nef. If T_X itself is nef, this follows from [CP 91] . Note that for any X the equality $N_{rat} = N_{nef}$ implies in particular that

$-K_X \in K_{eff}$. So (2) cannot hold for every rationally connected manifold. But it could hold for every rationally connected manifold with $-K_X \in K_{eff}$.

(3) This is motivated by stability, see (2.10), (2.11).

3.4 Proposition *Let X be a del Pezzo surface. Then $N_{rat} = N_{nef}$.*

Proof. Since $dim\, X = 2$, we have $N_{amp} = K_{amp}$. Now K_{amp} is generated by $\varphi^* H'$, where $\varphi\colon X \to X'$ is the contraction of a codimension 1-face R in $\overline{NE}(X)$ and H' is ample on X'. Moreover, X being del Pezzo, $X' = \mathbb{P}_1$ or $X' = \mathbb{P}_2$.
If $X' = \mathbb{P}_1$, then R is given by φ^{-1} (point p), if $X' = \mathbb{P}_2$, then R is given by φ^{-1} (general line ℓ). Both $\varphi^{-1}(p)$ and $\varphi^{-1}(\ell)$ are nef curves, finishing the proof.

Concerning 3.2.(3) we prove:

3.5 Theorem *Let X be a Fano manifold.*
Then $N_{sc} \subset \widetilde{N}_{rat}$.

Proof. By the cone theorem (as in (3.4)) it is sufficient to show the following

Claim (*) Let $\varphi\colon X \to Y$ be the contraction ofa codimension 1 face R and H ample on Y. Then $\varphi^*(H^{n-1}) \in \widetilde{N}_{rat}$.
a) If $dim\, Y < dim\, X$, we proceed as follows. First note if $dim\, Y \leq n-2$, then $H^{n-1} = 0$, so the claim is obvious. So let $dim\, Y = n-1$. Then H^{n-1} can be thought of as a general point of Y (up to a positive rational multiple of H^{n-1}), so that $\varphi^*(H^{n-1}) \equiv$ general fiber of φ, which is a smooth moving $\mathbb{P}_1$, hence in N_{rat}, proving (*) in case $dim\, Y < dim\, X$.
b) φ is birational.
Note that $\rho(Y) = 1$ and that Y is rationally connected (but may be singular).
=46irst let us prove the following assertion

(**) Let $C \subset Y$ be an irreducible curve not contained in the degeneracy set in Y of $\varphi\colon X \to Y$.
Then there is a - up to numerical equivalence - unique effective curve $\widetilde{C} \subset X$ such that
(1) $\varphi_*(\widetilde{C}) = C$
(2) $\widetilde{C}.E_i = 0$ for all exceptional components of φ.

In fact, let $\widehat{C} \subset X$ be the strict transform of C in X. We make the following ansatz

$$\widetilde{C} = \widehat{C} + \sum \lambda_i C_i,$$

where C_i are the extremal rational curves contracted by φ. We choose the λ_i in such a way that $\widetilde{C}.E_j = 0$ for all divisors E_j contracted by φ. Then the λ_i are uniquely determined. What remains to be shown is $\lambda_i \geq 0$. Take a basis of the φ-ample divisors

D, with $\varphi^*(C).D = 0$, say $D_1, \cdots, D_{\rho-1}$. Every D_j is of the form $D_j = -\sum \kappa_\mu E_\mu$ with $\kappa_\mu \geq 0$ by Lemma 3.7. Hence we have

$$D_j. \sum \lambda_i C_i = D_j.\widetilde{C} - D_j.\widehat{C} - D_j.\widehat{C} \sum \kappa_k E_k.\widehat{C} \geq 0.$$

Therefore $\sum \lambda_i C_i \in \overline{NE}(X|Y)$, i.e. all $\lambda_i \geq 0$. So (**) holds.

Now the verification of (*) in the birational case is easy; choose a covering family (C_t) of rational curves in Y. Then, applying (**), we have for t general an effective curve $\widetilde{C}_t \subset X$ with $\varphi_*(\widetilde{C}_t) = C_t$ and $\widetilde{C}_t.E_i = 0$ for all $\varphi-$exceptional divisors $E_i \subset X$. We can choose $\widetilde{C}_t$ rational because φ is the contraction of an extremal face. Now let H be the ample generator of $Pic(Y) \simeq \mathbb{Z}$. Take $\mu \in \mathbb{N}$ such that μH is very ample so that we find a smooth curve $\Gamma \equiv (\mu H)^{n-1}$. Then $\Gamma \equiv m C_t$ for some $m \in \mathbb{Q}_+$ since $\rho(Y) = 1$. It follows that $\widetilde{\Gamma} \equiv m\widetilde{C}_t$, where $\widetilde{\Gamma}$ is again constructed by (**). On the other hand, $\widetilde{\Gamma} \equiv \varphi^*(H)^{n-1}$. Hence the $(\widetilde{C}_t)$ yield (taking closure) a covering family of rational curves, numerically of the form $\varphi^*(H^{n-1})$, so that $\varphi^*(H^{n-1}) \in \widetilde{N}_{rat}(X)$.
There are two problems left: first, our $\widetilde{C}$ might not be connected and second, the λ_i might only be rational and anyway no multiplicities are allowed for the curves in N_{sc}. To get around with these difficulties we first choose C to be a general member of a large family of rational curves in Y, say a free family. Then choose a positive integer k such that $k\lambda_i$ are integers for all i. Now substitute kC by general deformations C_j of C. The λ_j will then be the same for C and C_j. Since all fibers of extremal contractions are uniruled by [Ka92], we can substitute the $\lambda_i \widetilde{C}_i$ by rational curves $C_{i,1}, \ldots, C_{i,k\lambda_i}$ homologous to multiples of $\widetilde{C}_i$ such that the new $\widetilde{C} = \bigcup \hat{C}_j \cup \bigcup C_{i,j}$ is connected.

3.6 Remark. In order to prove $N_{sc} \subset N_{rat}$ in (3.5) we would need the following. Let Y be a singular Fano variety (terminal singularities), $\rho(Y) = 1, A \subset Y$ an algebraic subvariety of codimension ≥ 2, then there exists a covering family (C_t) of rational curves, C_t irreducible for t general, such that $A \cap C_t = \emptyset$ for t general.

3.7 Lemma *Let X be a projective manifold, $f\colon X \to Y$ be a birational morphism to a normal projective variety Y. Let $E_1, \ldots, E_k$ be the codimension 1 components of the exceptional set of f. Let D be f-ample. Then*

$$D = -\sum \kappa_\mu \, E_\mu + f^*(L)$$

with $\kappa_\mu \geq 0$ for some line bundle L on Y.

Proof. Of course an equation as above always exists; we only need to show $\kappa_\mu \geq 0$. Taking hyperplane sections we reduce to the case $dim \ X = 2$. Let $B = \{\mu | \kappa_\mu \leq 0\}$ and A be its complement; assume $B \neq \emptyset$. Since D is f-ample, we have

$$\sum_{\mu} \kappa_\mu E_\mu \cdot \sum_{\nu \in B} (-\kappa_\nu) E_\nu < 0.$$

Hence

$$\sum_{\substack{\mu \in A \\ \nu \in B}} \kappa_\mu (-\kappa_\nu) E_\mu \cdot E_\nu < \sum_{\mu,\nu \in B} \kappa_\mu \kappa_\nu E_\mu \cdot E_\nu.$$

Now the right hand side of the last equation is negative, the intersection matrix $(E_\mu \cdot E_\nu)$ being negative definite, while the left hand side is obviously non-negative, contradiction. □

We now relate the canonical bundle to various cones. The question we have mind is the following: how negative is the canonical bundle K_X of a manifold X with $\kappa(X) = -\infty$? In dimension > 3 it is even unknown whether K_X is not nef, so we will mainly restrict ourselves to $dim\, X = 3$.
We introduce the following notation:

$$\overline{NE}_-(X) = \{C \in \overline{NE}(X) | K_X \cdot C < 0\}.$$

Then vaguely speaking, we ask for the size of $\overline{NE}_-(X)$. E.g.:
3.8 Problem Does $\kappa(X) = -\infty$ imply that $\overline{NE}_-(X) \cap N_{clt} \neq \{0\}$?
In other words: are there $H_1, \cdots, H_{n-1}$ ample such that $K_X \cdot H_1. \cdots .H_{n-1} < 0$?

A weaker problem would ask for H_i only big. Note that this problem is related to the stability considerations in § 2 . We shall now consider the case $dim X = 3$.

3.9 Proposition *Let X be a normal ($\mathbb{Q}$-factorial) threefold with at most terminal singularities, $\varphi\colon X \to Y$ the contraction of an extremal ray with $dim\, Y \leq 2$. Then there exists an ample divisor H on X with $K_X.H^2 < 0$.*

Proof. If $\rho(X) = 1$, the claim is obvious, so let $\rho(X) \geq 2$, i.e. dim $Y > 0$. Fix $H_o \in Pic(X)$ ample. Choose any ample divisor L on Y and let

$$H = H_o + \varphi^*(mL),$$

$m >> 0$ to be specified in a moment. H is ample and

$$K_X \cdot H^2 = K_X \cdot H_o^2 + 2K_X \cdot H_o \cdot \varphi^*(mL) + K_X \cdot \varphi^*(mL)^2.$$

If $dim\, Y = 2$, then $K_X \cdot \varphi^*(L)^2 < 0$ since $-K_X$ is φ- ample, hence $K_X \cdot H^2 < 0$ for $m >> 0$.
If $dim\, Y = 1$, then $L^2 = 0$, but now $K_X \cdot H_o \cdot \varphi^*(mL) < 0$, hence again the claim for $m >> 0$.

Note that we only used that $-K_X$ is ample on the general fiber instead of φ being an extremal contraction.

3.10 Proposition *Let X, Y be normal projective ($\mathbb{Q}$-factorial) terminal 3-folds, $\varphi\colon X \to Y$ a divisorial extremal contraction. Assume there exists H' ample on Y with $K_Y \cdot H'^2 < 0$. Then there is H ample on X with $K_X \cdot H^2 < 0$.*

Proof. Choose H_o ample on X and let $H = H_o + \varphi^*(mH)$. Then H is ample and $K_X \cdot H_o^2 + 2K_X \cdot H_o \cdot \varphi^*(mH) + K_X \cdot \varphi^*(mH)^2$. Since $K_X \cdot \varphi^*(H)^2 = K_Y \cdot H^2 < 0$, we have $K_X \cdot H^2 < 0$ for $m >> 0$.

3.11 Proposition *Let X be as in (3.10). Let*

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ \varphi \searrow & & \swarrow \varphi^+ \\ & Y & \end{array} \qquad \textit{be a flip}$$

so $-K_X$ is φ-ample and K_X+ is φ^+-ample.
Let H^+ be effective on X^+ whith $H^{+2} \in NE(X)$. Let H be its strict transform on X. If $K_X \cdot H^2 < 0$, then $K_{X^+} \cdot H^{+2} < 0$.
Proof. Write

$H^{+2} = \sum \lambda_i C_i^+ + \sum \mu_j \cdot A_j^+, \qquad \lambda_j > 0, \quad \mu_j \geq 0,$

where the A_j^+ are exactly the exceptional curves for φ^+ .

Analogously: $H^2 = \sum \tau_i C_i + \sum \nu_j A_j$.
(it is clear that $H^2 \in NE(X)$!). We arrange things, such that C_i is the strict tansform of C_i^+. Since $K_{X^+}.A_j^+ > 0$ and $K_X.A_j < 0$ it suffices to show
(*) $K_X.C_i \geq K_{X^+}.C_i^+$.
Let $C = C_i,\ C^+ = C_i^+$ for simplicity. Let

$$\begin{array}{ccc} & \hat{X} & \\ \pi \swarrow & & \searrow \sigma \\ X & \dashrightarrow & X^+ \end{array}$$

be a common desingularisation. By [KMM 87, 5-1-11] we have

$$K_{\widehat{X}} = \sigma^*(K_{X^+}) + \sum a_i^+ E_i$$

$$K_{\widehat{X}} = \pi^*(K_X) + \sum\ a_i\ E_i$$

and $a_i \leq a_i^+$.
Therefore (*) follows by intersecting with $\widehat{C}$, the strict transform of C_i resp. C^+ in $\widehat{X}$.

3.12 Remarks Of course we would like to show that given a 3-fold X with $\kappa(X) = -\infty$, then there exists H ample (or at least big) such that $K_X.H^2 < 0$. The strategy would be to apply the Minimal Model Program (MMP) to finally abtain a contraction $X' \to Y$ with $dimY \leq 2$, then apply (3.9) and go back successively. However the flips cause trouble because the inequalities work in the wrong direction. In fact, in the situation of (3.11) we would need $H^{+2} \in NE(X)$ and $(H^+)^2 \cap$ (the exceptional locus of the flip) $= \emptyset$. If say H^+ is big and globally generated, this could be achieved. However H would in general no longer be globally generated, so we cannot proceed.
In special situations however, the problem has a positive solution:

a) there exists $\varphi\colon X \to Y$, $dim\,Y \leq 2, -K_X$ is φ-ample or at least $-K_X$ is ample on the general fiber,
b) the Albanese dimension of X is 2 (as a consequence of (a)),
c) $|-mK_X| \neq \emptyset$ for some $m \in \mathbb{N}$ (obvious)
d) there is a divisorial modification $\varphi\colon X \to X'$ and X' is of type a) (e.g. no flips occur in MMP).

It is clear from Prop. 3.9 that in general we have at least:

3.13 Proposition *Let X be a normal projective 3-fold, $\mathbb{Q}$-factorial, terminal with $\kappa(X) = -\infty$. Then there exists a covering family (C_t) of curves, C_t irreducible with <u>ample</u> normal bundle for t general, such that $K_X.C_t < 0$.*

Uniruledness implies only the existence of a family (C_t) (however rational) whose normal bundles are nef (t general).

References

[AM 95] Andreatta, M.; Mella, M.: Contractions on a manifold polarized by an ample vector bundle. Preprint 1995

[Ca 92] Campana, F.: Connexité rationelle des variétés de Fano. Ann. Sci. Ec. Norm. Sup. 25, 539-545 (1992)

[Ca94] Campana, F.: Remarques sur le revetement universel des variétés Kählériénnes compactes. Bull.SMF 122, 255-284 (1994)

[Ca95] Campana, F.: Fundamental group and positivity of cotangent bundles of compact Kähler manifolds. J. Alg. Geom. 4, 487-502 (1995)

[CP 91] Campana, F.; Peternell, Th.: Projective manifolds whose tangent bundles are numerically effective. Math. Ann. 289, 169-187 (1991)

[DPS 94] Demailly, J.P.; Peternell, Th.; Schneider, M.: Compact complex manifolds, with numerically effective tangent bundles. J. Alg. Geometry 3, 295-345 (1994)

[DPS 95] Demailly, J.P.; Peternell, Th.; Schneider, M.: Holomorphic line bundles with partially vanishing cohomology. Proc. Tel Aviv 1993, volume in honour of F. Hirzebruch (1995)

[Fu 90] Fujita, T.: On adjoint bundles of ample vector bundles. Proc. Alg. Geom. Conf. Bayreuth 1990; Lecture Notes in Math. 1507, 105-112, Springer 1992

[Ka 92] Kawamata, Y.: Moderate degenerations of algebraic surfaces. Proc. Alg. Var. Bayreuth 1990. Lecture Notes in Math. 1507, 113–132. Springer 1992

[KMM 87] Kawamata, Y.; Matsuda, K.; Matsuki, K.: Introduction to the minimal model program. Adv. Stud. Pure Math. 10, 283-360 (1987)

[KoMiMo 92] Kollár, J.; Miyaoka, Y.; Mori, S.: Rationally connected varieties. J. Alg. Geom. 1, 429-448 (1992)

[Mi 87] Miyaoka, Y.: Deformations of a morphism along a foliation and applications. Proc. Symp. Pure Math. 46, 245-268 (1987)

[Mi 87a] Miyaoka, Y.: The Chern classes and Kodaira dimension of a minimal variety. Adv. Stud. Pure Math. 10, 449-476 (1987)

[MM 86] Miyaoka, A.; Mori, S.: A numerical criterion of uniruledness. Ann. Math. 124, 65-69 (1986)

[Mo 79] Mori, S.; Projective manifolds with ample tangent bundles. Ann. Math. 110, 593-606 (1979)

[MS 78] Mori, S.; Sumihiro, Y.: On Hartshorne's conjecture. J. Math. Kyoto Univ. 18, 523-533 (1978)

[PW 94] Peternell, Th.; Wisniewski, J.: On stability of tangent bundles of Fano manifolds with $b_2 = 1$. J. Alg. Geom. 4, 363-384 (1994)

[Wa 83] Wahl, J.: A cohomological characterisation of $\mathbb{P}_n$. Inv. math. 72, 315-322 (1983)

[YZ 90] Ye, Y.; Zhang, Q.: On ample vector bundles whose adjoint bundles are not numerically effective. Duke Math. J. 60, 671-688 (1990)

Frédéric Campana, Université de Nancy I, Département de Mathématiques, B.P. 239, 54506 Vandoeuvre les Nancy Cedex, France
E-mail address: `frederic.campana@iecn.u-nancy.fr`

Thomas Peternell, Mathematisches Institut, Universität Bayreuth, D - 95440 Bayreuth, Germany
E-mail address: `peternel@btm8x1.mat.uni-bayreuth.de`