TOPOLOGICAL ARRANGEMENT OF CURVES OF DEGREE 6 ON CUBIC SURFACES IN $\mathbb{R}P^3$

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Abstract. A quadric in $\mathbb{R}P^3$ cuts a curve of degree 6 on a cubic surface in $\mathbb{R}P^3$. The papers classifies the nonsingular curves cut in this way on non-singular cubic surfaces up to homeomorphism.

Two issues new in the study related to the first part of the 16th Hilbert problem appear in this classification. One is the distribution of the components of the curve between the components of the non-connected cubic surface which turns out to depend on the patterns of arrangements (see Theorem 1). The other is presence of positive genus among the components of the complement and genus-related restrictions (see Theorems 3 and 4).

1. Introduction

Let $\mathbb{R}B$ be a real algebraic cubic surface in $\mathbb{R}P^3$ and let $\mathbb{R}Q$ be a real algebraic quadric surface in $\mathbb{R}P^3$. This just means that for some homogeneous polynomials f, $\deg(f) = 3$, and g, $\deg(g) = 2$, surfaces $\mathbb{R}B$ and $\mathbb{R}Q$ are given by the following equations

$$\mathbb{R}B = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{R}P^3 \mid f(x_0, x_1, x_2, x_3) = 0\}$$
$$\mathbb{R}Q = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{R}P^3 \mid g(x_0, x_1, x_2, x_3) = 0\}.$$

Let $\mathbb{C}B$ and $\mathbb{C}Q$ be the complexifications of surfaces $\mathbb{R}B$ and $\mathbb{R}Q$

$$\mathbb{C}B = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{C}P^3 \mid f(x_0, x_1, x_2, x_3) = 0\}$$
$$\mathbb{C}Q = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{C}P^3 \mid g(x_0, x_1, x_2, x_3) = 0\}.$$

We suppose that $\mathbb{C}B$ and $\mathbb{C}Q$ are nonsingular and intersect transversely along a curve $\mathbb{C}A$. The intersection $\mathbb{C}A = \mathbb{C}B \cap \mathbb{C}Q$ is a smooth curve in $\mathbb{C}P^3$, its real part $\mathbb{R}A$ is $\mathbb{R}B \cap \mathbb{R}Q$.

Consider now the diffeomorphism type of $\mathbb{C}B$, $\mathbb{C}Q$, $\mathbb{C}A$. It is well-known that $\mathbb{C}Q$ is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}B$ is diffeomorphic to $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P}^2$ and we can construct $\mathbb{C}B$ from $\mathbb{C}P^2$ by blowing up 6 points not sitting on the same conic. The

Research at MSRI is supported in part by NSF grant DMS-9022140

curve $\mathbb{C}A$ is the proper transform of a plane curve of degree 6 with 6 ordinary double points in the blowup points so the genus of $\mathbb{C}A$ is 4.

Consider now the diffeomorphism type of $\mathbb{R}B$, $\mathbb{R}Q$, $\mathbb{R}A$. It is well-known that a quadric $\mathbb{R}Q$ is a hyperboloid (i.e. diffeomorphic to $\mathbb{R}P^1 \times \mathbb{R}P^1$), an ellipsoid (i.e. diffeomorphic to S^2) or empty. It is also well-known that a cubic $\mathbb{C}B$ is diffeomorphic to $\#(2n+1)\mathbb{R}P^2 \approx \mathbb{R}P^2 \# nT^2$, where $n \leq 3$ or $\mathbb{R}P^2 \sqcup S^2$. Since $\mathbb{R}A$ is smooth, every component of $\mathbb{R}A$ is diffeomorphic to S^1. Note that the embedding of this S^1 into $\mathbb{R}B$ is two-sided, the sides of $\mathbb{R}A$ are defined by the sign of polynomial g (since $\deg g$ is even).

Since the genus of $\mathbb{C}A$ is 4, the Harnack inequality implies that the number of components of $\mathbb{R}A$ is at most 5 ($\mathbb{C}A/\operatorname{conj}$ is a connected surface and, therefore, the number of components of its boundary is at most $2-\chi(\mathbb{C}A/\operatorname{conj}) = 2-\frac{1}{2}\chi(\mathbb{C}A) = 5$).

The topological classification of pairs $(\mathbb{R}Q, \mathbb{R}A)$ was done by Gudkov [3] in the case where $\mathbb{R}Q$ is a hyperboloid and by Gudkov and Shustin [6] in the case where $\mathbb{R}Q$ is an ellipsoid (see [10] for the classification of flexible curves).

This paper gives the topological classification of pairs $(\mathbb{R}B, \mathbb{R}A)$. Two new phenomena appear in this classification — the arrangement of $\mathbb{R}A$ with respect to the components of $\mathbb{R}B$ in the case when $\mathbb{R}B$ is not connected and the arrangement of the components of $\mathbb{R}A$ with respect to the handles of $\mathbb{R}B$ in the case when $\mathbb{R}B$ is diffeomorphic to a projective plane with handles. These possibilities for arrangement turn out to be controlled by the degree of the curve.

The topological type of a pair $(\mathbb{R}B, \mathbb{R}A)$ is also called the type of *topological arrangement* of $\mathbb{R}A$ in $\mathbb{R}B$ or the *real scheme* of $\mathbb{R}A$ in $\mathbb{R}B$.

Acknowledgment. A part of the work was done during my stay at the Institute for Advanced Study (supported by the NSF grant DMS 9304580) and at the Max-Planck-Institut.

2. Curves on cubic surfaces of positive Euler characteristic

There are two diffeomorphism types of cubic surfaces of positive Euler characteristic in $\mathbb{R}P^3$:

$$\mathbb{R}P^2 \sqcup S^2 \quad \text{and} \quad \mathbb{R}P^2.$$

We consider them separately since if $\mathbb{R}B$ is one of these surfaces than any component of $\mathbb{R}A$ bounds a disk in $\mathbb{R}B$ and even the notations for the topological type of $(\mathbb{R}B, \mathbb{R}A)$ are simpler in this case.

2.1. Notations. Recall the system of codes for topological arrangements of a collection of two-sided circles (or *ovals*) in $\mathbb{R}P^2$ introduced by Viro [15]. Every oval separates $\mathbb{R}P^2$ into a disk (caller the *interior* of the oval) and a Möbius band. Two ovals are called *disjoint* if their interiors are disjont. The system of codes is inductive, n disjoint ovals are encoded by $< n >$ and if we add a new oval containing a collection $< A >$ in its interior then the code of the new collection is $< 1 < A > >$.

To make the same system suitable for encoding of topological arrangements of embedded curves in S^2 it suffices to fix a point in S^2 in the complement of the curve — then the interior of an oval is defined as the component of the complement of the oval not containing the fixed point. Such a code determines uniquely a topological type of curve in S^2, but because of ambiguity in fixing a point there is more than one code corresponding to the same topological type. We choose codes minimizing the number of ovals in the interior of other ovals. To encode types of topological arrangement of $\mathbb{R}A$ into a cubic surface diffeomorphic to $\mathbb{R}P^2 \sqcup S^2$ we combine systems for $\mathbb{R}P^2$ and S^2.

Parameters α and β in the theorems are nonnegative integer numbers.

2.2. Classification.

Theorem 1. *Topological arrangement of a nonsingular algebraic curve of degree 6 on a nonsingular cubic surface diffeomorphic to $\mathbb{R}P^2 \sqcup S^2$ is of one of the following types:*

a: $<\alpha \sqcup 1<1>>_{\mathbb{R}P^2} \sqcup <\emptyset>_{S^2}$, $\alpha \leq 3$,
b: $<1<\alpha>>_{\mathbb{R}P^2} \sqcup <\emptyset>_{S^2}$, $2 \leq \alpha \leq 4$,
c: $<\alpha>_{\mathbb{R}P^2} \sqcup <\beta>_{S^2}$, $\alpha + \beta \leq 5$,
d: $<1<1<1>>>_{\mathbb{R}P^2} \sqcup <\emptyset>_{S^2}$,
e: $<1<1>>_{\mathbb{R}P^2} \sqcup <1>_{S^2}$,
f: $<\emptyset>_{\mathbb{R}P^2} \sqcup <1 \sqcup 1<1>>_{S^2}$.

Each of the 31 types listed above is realizable by a nonsingular algebraic curve of degree 6.

Theorem 2. *Topological arrangement of a nonsingular algebraic curve of degree 6 on a nonsingular cubic surface diffeomorphic to $\mathbb{R}P^2$ is of one of the following types:*

a: $<\alpha \sqcup 1<\beta>>$, $\alpha + \beta \leq 4$,
b: $<1<1<1>>>$,
c: $<\emptyset>$.

Each of the 17 types listed above is realizable by a nonsingular algebraic curve of degree 6.

3. Curves on cubic surfaces of negative Euler characteristic

There are three diffeomorphism types of cubic surfaces of negative Euler characteristic in $\mathbb{R}P^3$:

$$\mathbb{R}P^2\#3T^2, \quad \mathbb{R}P^2\#2T^2 \quad \text{and} \quad \mathbb{R}P^2\#T^2.$$

3.1. Notations. We define

$$B_+ = \{x \in \mathbb{R}B \mid g(x) \geq 0\},\ B_- = \{x \in \mathbb{R}B \mid g(x) \leq 0\}.$$

Then $\partial B_+ = \partial B_- = \mathbb{R}A$. Note that at least one of B_+, B_- is non-orientable since

$$\chi(\mathbb{R}B) \equiv 1 \pmod 2.$$

Changing the sign of g we may assume that

$$\chi(B_+) \equiv b_0(\mathbb{R}A) \pmod 2$$
$$\chi(B_-) \equiv 1 + b_0(\mathbb{R}A) \pmod 2,$$

where b_0 stands for the number of components (the 0-dimensional Betti number). Then B_- is always nonorientable.

Note that the topological type of $(\mathbb{R}B, \mathbb{R}A)$ is determined by the topological types of B_+ and B_- if the number of non-disk components is not more than one for both B_+ and B_- (since then there is only one way to glue B_+ and B_- along the boundary to obtain a connected $\mathbb{R}B$). In this case we code the topology of $\mathbb{R}A$ on $\mathbb{R}B$ with

$$< j \sqcup F, n \sqcup G >$$

where j is the number of disks in B_+, n is the number of disks in B_-, F is the topological type of the only non-disk component of B_+ (if any) and G is the only non-disk component of B_-. We denote the topological type of the connected orientable surface of genus g with gT^2 and that of the non-orientable one with $g\mathbb{R}P^2$. Notation F_k stands for the result of puncturing of a closed surface F k times.

Lemma 1 implies that if B_+ or B_- has more than one non-disk components then the topological type of $(\mathbb{R}B, \mathbb{R}A)$ is still determined by the topological types of B_+, B_- and $\mathbb{R}B$. If $\mathbb{R}B \approx \mathbb{R}P^2 \# nT^2$, $n = 1, 2, 3$ then the code of $(\mathbb{R}B, \mathbb{R}A)$ is

$$< 1 \sqcup nT^2_2, S^2_2 \sqcup \mathbb{R}P^2_1 >$$

if the core of the annulus is null-homologous and

$$< (n-1)T^2_3, S^2_2 \sqcup \mathbb{R}P^2_1 >$$

if not.

3.2. Classification.

Theorem 3. *Topological arrangement of a nonsingular algebraic curve of degree 6 on a nonsingular cubic surface diffeomorphic to* $\mathbb{R}P^2 \# 3T^2$ *is of one of the following types:*

1. $< \alpha \sqcup S^2_{\beta+\gamma}, \beta \sqcup (9-2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$
$\alpha = 0$	$\gamma = 1, 2, 3, 4$	$\gamma = 1, 2, 3, 4$	$\gamma = 1, 2, 3$	$\gamma = 2$
$\alpha = 1$	$\gamma = 1, 2, 3, 4$	$\gamma = 1, 2, 3$	$\gamma = 2$	
$\alpha = 2$	$\gamma = 1, 2, 3$	$\gamma = 1, 2$		
$\alpha = 3$	$\gamma = 1, 2$	$\gamma = 1$		

2. $< \alpha \sqcup T^2_{\beta+\gamma}, \beta \sqcup (7 - 2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$
$\alpha = 0$	$\gamma = 1, 2, 3$	$\gamma = 1, 2, 3$	$\gamma = 1, 2, 3$	$\gamma = 1, 2$	$\gamma = 1$
$\alpha = 1$	$\gamma = 1, 2, 3$	$\gamma = 1, 2, 3$	$\gamma = 1, 2$		
$\alpha = 2$	$\gamma = 1, 2, 3$	$\gamma = 1, 2$	$\gamma = 1$		
$\alpha = 3$	$\gamma = 1, 2$				
$\alpha = 4$	$\gamma = 1$				

3. $< \alpha \sqcup 2T^2_{\beta+\gamma}, \beta \sqcup (5 - 2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$
$\alpha = 0$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1, 2$
$\alpha = 1$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1$
$\alpha = 2$	$\gamma = 1, 2$	$\gamma = 1, 2$		
$\alpha = 3$	$\gamma = 1, 2$	$\gamma = 1$		

4. $< \alpha \sqcup 3T^2_{\beta+1}, \beta \sqcup \mathbb{R}P^2_{\alpha+1} >$,

$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$
$\alpha = 0, 1, 2, 3$	$\alpha = 0$	$\alpha = 0$	$\alpha = 0$	$\alpha = 0$

5. $< \alpha \sqcup 2\mathbb{R}P^2_{\beta+\gamma}, \beta \sqcup (7 - 2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$
$\alpha = 0$	$\gamma = 1, 2, 3$	$\gamma = 1, 2, 3$	$\gamma = 1, 2, 3$	$\gamma = 1, 2$	$\gamma = 1$
$\alpha = 1$	$\gamma = 1, 2, 3$	$\gamma = 1, 2, 3$	$\gamma = 1, 2$	$\gamma = 1$	
$\alpha = 2$	$\gamma = 1, 2, 3$	$\gamma = 1, 2$			
$\alpha = 3$	$\gamma = 1, 2$	$\gamma = 1$			
$\alpha = 4$	$\gamma = 1$				

6. $< \alpha \sqcup 4\mathbb{R}P^2_{\beta+\gamma}, \beta \sqcup (5 - 2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$
$\alpha = 0$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1$
$\alpha = 1$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1$	
$\alpha = 2$	$\gamma = 1, 2$	$\gamma = 1, 2$	$\gamma = 1$		
$\alpha = 3$	$\gamma = 1, 2$	$\gamma = 1$			
$\alpha = 4$	$\gamma = 1$				

7. $<\alpha\sqcup 6\mathbb{R}P^2_{\beta+1}, \beta\sqcup \mathbb{R}P^2_{\alpha+1}>$,

$\beta=0$	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$
$\alpha=0,1,2,3,4$	$\alpha=0,1$	$\alpha=0,1$	$\alpha=0,1$	$\alpha=0$

8. $<1\sqcup 3T^2_2, S^2_2\sqcup \mathbb{R}P^2_1>$, $<2T^2_3, S^2_2\sqcup \mathbb{R}P^2_1>$
9. $<\emptyset, 7\mathbb{R}P^2>$.
Each of the 157 types listed above is realizable by a nonsingular algebraic curve of degree 6.

Theorem 4. *Topological arrangement of a nonsingular algebraic curve of degree 6 on a nonsingular cubic surface diffeomorphic to $\mathbb{R}P^2\#2T^2$ is of one of the following types:*
1. $<\alpha\sqcup S^2_{\beta+\gamma}, \beta\sqcup (7-2\gamma)\mathbb{R}P^2_{\alpha+\gamma}>$, $\alpha+\beta+\gamma\leq 5,\ \gamma=1,2,3,$
2. $<\alpha\sqcup T^2_{\beta+\gamma}, \beta\sqcup (5-2\gamma)\mathbb{R}P^2_{\alpha+\gamma}>$, $\alpha+\beta+\gamma\leq 5,\ \gamma=1,2,$
3. $<\alpha\sqcup 2T^2_{\beta+1}, \beta\sqcup \mathbb{R}P^2_{\alpha+1}>$, $\alpha+\beta\leq 4,\ \alpha\neq 4,$
4. $<\alpha\sqcup 2\mathbb{R}P^2_{\beta+\gamma}, \beta\sqcup (5-2\gamma)\mathbb{R}P^2_{\alpha+\gamma}>$, $\alpha+\beta+\gamma\leq 5,\ \gamma=1,2,$
5. $<\alpha\sqcup 4\mathbb{R}P^2_{\beta+1}, \beta\sqcup \mathbb{R}P^2_{\alpha+1}>$, $\alpha+\beta\leq 4,$
6. $<1\sqcup 2T^2_2, S^2_2\sqcup \mathbb{R}P^2_1>$, $<T^2_3, S^2_2\sqcup \mathbb{R}P^2_1>$
7. $<\emptyset, 5\mathbb{R}P^2>$.
Each of the 113 types listed above is realizable by a nonsingular algebraic curve of degree 6.

Theorem 5. *Topological arrangement of a nonsingular algebraic curve of degree 6 on a nonsingular cubic surface diffeomorphic to $\mathbb{R}P^2\#T^2$ is of one of the following types:*
1. $<\alpha\sqcup S^2_{\beta+\gamma}, \beta\sqcup (5-2\gamma)\mathbb{R}P^2_{\alpha+\gamma}>$, $\alpha+\beta+\gamma\leq 5,\ \gamma=1,2,$
2. $<\alpha\sqcup T^2_{\beta+1}, \beta\sqcup \mathbb{R}P^2_{\alpha+1}>$, $\alpha+\beta\leq 4,$
3. $<\alpha\sqcup 2\mathbb{R}P^2_{\beta+1}, \beta\sqcup \mathbb{R}P^2_{\alpha+1}>$, $\alpha+\beta\leq 4,$
4. $<1\sqcup T^2_2, S^2_2\sqcup \mathbb{R}P^2_1>$, $<S^2_3, S^2_2\sqcup \mathbb{R}P^2_1>$
5. $<\emptyset, 3\mathbb{R}P^2>$.
Each of the 58 types listed above is realizable by a nonsingular algebraic curve of degree 6.

4. Auxiliary lemma

Lemma 1 (cf. [1]). *B_+ contains not more than one component of non-positive Euler characteristic. If B_- containes more then one component of non-positive Euler characteristic then $B_-\approx S^2_2\sqcup \mathbb{R}P^2_1$ and the cores of S^2_2 and $\mathbb{R}P^2_1$ of B_- form a cycle dual to $w_1(\mathbb{R}B)$.*

Remark. Of course, if $\mathbb{R}A$ is empty and $\mathbb{R}B$ is not $S^2 \sqcup \mathbb{R}P^2$ then all components of positive Euler characteristic are disks.

Proof. Consider the double covering $\mathbb{C}X$ of $\mathbb{C}B$ branched along $\mathbb{C}A$. The complex conjugation of $\mathbb{C}B$ can be lifted in two different ways to an involution on $\mathbb{C}Y$ (which differ by the covering automorphism). The fixed point set Y_+ and Y_- of these two liftings c_+ and c_- are double branched coverings of B_+ and B_-. Note that Y_+ and Y_- are orientable (even though B_- is never orientable and B_+ may be non-orientable) since $w_1(\mathbb{R}B)$ is induced by the plane section in $\mathbb{R}P^3$.

The homology vector space $H_2(\mathbb{C}Y;\mathbb{R})$ splits into the orthogonal direct sum of the subspaces $H_2^+(\mathbb{C}Y;\mathbb{R})$ and $H_2^-(\mathbb{C}Y;\mathbb{R})$ invariant under actions of both c_+ and c_-, where the intersection form of $\mathbb{C}Y$ is positive definite on $H_2^+(\mathbb{C}Y;\mathbb{R})$ and negative definite on $H_2^-(\mathbb{C}Y;\mathbb{R})$. An easy computation shows that

$$\dim H_2^+(\mathbb{C}Y;\mathbb{R}) = 3$$

($\mathbb{C}Y$ is a K3-surface). Furthermore, $H_2^+(\mathbb{C}Y;\mathbb{R})$ splits under the action of c_+ and c_- into (-1)-eigenspaces $E_{c_+}^{-1}$ and $E_{c_-}^{-1}$ and $(+1)$-eigenspaces $E_{c_+}^{+1}$ and $E_{c_-}^{+1}$. Since $c_+ \circ c_-$ is the covering authomorphism

$$\dim(E_{c_+}^{+1} \cap E_{c_-}^{-1}) = 1$$
$$\dim(E_{c_+}^{-1} \cap E_{c_-}^{+1}) = 1.$$

Note that c_- reverses orientation of Y_+. Therefore if the linear combination of the homology classes realized by components of Y_+ is in $H_2^+(\mathbb{C}Y;\mathbb{R})$ then it is in $(E_{c_+}^{+1} \cap E_{c_-}^{-1})$. Similarily, if the linear combination of the homology classes realized by components of Y_- is in $H_2^+(\mathbb{C}Y;\mathbb{R})$ then it is in $(E_{c_+}^{-1} \cap E_{c_-}^{+1})$.

Note that the self-intersection number of a component of Y_+ (Y_-) in $\mathbb{C}Y$ is equal to (-1) times its Euler characteristic since the multiplication by i provides an orientation-reversing isomorphism between the tangent and the normal bundles of Y_+ (Y_-) in $\mathbb{C}Y$. Therefore, the number of the components of Y_+ (Y_-) with non-positive Euler characteristic is not greater then $\dim(E_{c_+}^{+1} \cap E_{c_-}^{-1}) = 1$ ($\dim(E_{c_+}^{-1} \cap E_{c_-}^{+1}) = 1$) plus the number of linear relations on the $\mathbb{R}$-homology classes of these components.

Since $H_1(\mathbb{C}Y) = 0$ the universal coefficients theorem implies that the number of linear relations on the $\mathbb{R}$-homology classes of the components of Y_+ (Y_-) with non-positive Euler characteristic is not more than the number of corresponding $\mathbb{Z}_2$-relations. The Smith exact sequence (see e.g. appendix in [16]) assures that the only possible non-trivial linear relation on the Z_2-homology classes of Y_+ (Y_-) is the sum of all of them. Therefore the maximal number of the $\mathbb{R}$-relations is 1 and it can be reached only if all the components of Y_+ (Y_-) are of non-positive Euler characteristic. Furthermore, if at least one of them is of positive Euler characteristic then they are linearily independent over $\mathbb{R}$ since the only possible $\mathbb{R}$-relation (the sum of them taken with non-zero coefficients) intersects that component non-trivially.

Therefore, if there is more than one component of B_+ (B_-) of non-positive Euler characteristic then B_+ (B_-) consists of two components of zero Euler characteristic. In addition Y_+ (Y_-) is $\mathbb{Z}_2$-homologous to zero in $\mathbb{C}Y$. The latter implies that $\mathbb{R}A$ is $\mathbb{Z}_2$-homologous to zero in $\mathbb{C}A$. Indeed, if there exists a loop in $\mathbb{C}A$ which intersects $\mathbb{R}A$ once then a disk bounded by this loop in $\mathbb{C}B$ (recall that $\pi_1(\mathbb{C}B) = 0$) is a closed surface which intersects Y_+ (Y_-) in odd number of points. This implies in turn that the number of components of $\mathbb{R}A$ is of the opposite pairity with the genus of $\mathbb{C}A$ since $\mathbb{R}A$ separates $\mathbb{C}A$ into two diffeomorphic complex conjugate orientable surfaces.

There are two types of connected surfaces of zero Euler characteristic with boundary — annuli and Möbius bands. The above discussion implies that the only possibility for the surface B_+ (B_-) to have more than one component of non-positive Euler characteristic is to consist of an annulus and a Möbius band. By our convention (see 3.1) this surface is not B_+.

If the Euler characteristic of $\mathbb{R}B$ is positive ($\mathbb{R}B \approx S^2 \sqcup \mathbb{R}P^2$ or $\mathbb{R}B \approx \mathbb{R}P^2$) then the proof of the Lemma is finished since $w_1(\mathbb{R}B)$ is the only non-trivial element in $H_1(\mathbb{R}B; \mathbb{Z}_2)$. To finish the proof for $\mathbb{R}B$ of negative Euler characteristic we consider the birational equivalence $\beta : \mathbb{C}B \to \mathbb{C}P^2$ which is the blowup in 6 points. The image of $\mathbb{R}A$ under β is a curve $\mathbb{R}C$ of degree 6 in $\mathbb{R}P^2$. which has singularities of multiplicity 2 at the points of blowup. We have to prove that the image of the union of the cores of the annulus and the Möbius band is dual to $w_1(RP^2)$ and contains all points of blowup.

We deduce it from the Bezout theorem. The cores are disjoint so if the image of the cores is not dual to $w_1(\mathbb{R}P^2)$ then the image of the each core bounds a disk in $\mathbb{R}P^2$. Pick a point in each of these disks which does not belong to the image of B_- and draw a line through these points. This line intersects $\mathbb{R}C$ at least in 8 points which contradicts to the Bezout theorem. Therefore, the image of exactly one of the cores bounds a disk. If there is a blowup point on $\mathbb{R}P^2$ which does not belong to the image of either of the cores then we draw a line through that blowup point and a point on the disk which does not belong to B_-. This line intersects $\mathbb{R}C$ at least in 6 points other than the blowup point (which is of multiplicity 2). □

5. Proof of Theorem 1

The proof of Theorem 1 (as well as the proofs of Theorems 2, 3, 4 and 5) consists of two parts — the part "Restrisctions" where we prove that the topological arrangements not listed in the theorem are not realizable and the part "Constructions" where we construct the curves with the listed topological arrangements.

5.1. Restrictions. Recall (see section 1) that by the Harnack inequality the maximal number of components of $\mathbb{R}A$ is 5. Lemma 1 implies that unless $(\mathbb{R}B, \mathbb{R}A)$ is of type d of Theorem 1 there is no more than one component of $\mathbb{R}A$ which does not bound a disk in $\mathbb{R}B$ disjoint from the other components of $\mathbb{R}A$. If this component belongs to the sphere component of $\mathbb{R}B$ then Lemma 1 implies that $(\mathbb{R}B, \mathbb{R}A)$ is of type $<\emptyset>_{\mathbb{R}P^2} \sqcup <1 \sqcup 1<1>>_{S^2}$. If this component belongs to the plane component of $\mathbb{R}B$ then Lemma 1 implies that there is no more than one component of $\mathbb{R}A$ on the sphere component of $\mathbb{R}B$.

Thus we have only to prove that the following arrangements do not appear

$$<\alpha \sqcup 1<\beta>>_{\mathbb{R}P^2} \sqcup <\emptyset>_{S^2}, \text{ if } \alpha > 0,\ \beta > 1,$$
$$<\alpha \sqcup 1<\beta>>_{\mathbb{R}P^2} \sqcup <1>_{S^2}, \text{ if } \beta > 0, \text{ unless } \alpha = 0,\ \beta = 1.$$

Lemma 5.1. *$\mathbb{R}B$ is of type Irel, i.e. $\mathbb{R}B$ is dual to the second Stiefel-Whitney class $w_2(\mathbb{C}B)$ of $\mathbb{C}B$.*

Proof. The sphere component of $\mathbb{R}B$ bounds a ball in $\mathbb{R}P^3$. Let x be a point inside of that ball. Then any line in $\mathbb{R}P^3$ passing through x intersects the sphere component of $\mathbb{R}B$ in 2 points. But it also intersects the plane component of $\mathbb{R}B$ because of homology reasons. Therefore, $\mathbb{R}B$ is the full inverse image of $\mathbb{R}P^2$ under the projection $\mathbb{C}B \to \mathbb{C}P^2$ from x. The lemma now follows since $\mathbb{R}P^2$ is dual to $w_2(\mathbb{C}P^2)$. □

This lemma allows us to apply the results of [11]. By Theorem 1 and Addendum 1 of [11] $\mathbb{R}A$ bounds such a surface B_1 in $\mathbb{R}B$ that

$$\chi(B_1) \equiv 5 \pmod 8, \text{ if the number of components of } \mathbb{R}A \text{ is } 5,$$
$$\chi(B_1) \equiv 4 \text{ or } 6 \pmod 8, \text{ if the number of components of } \mathbb{R}A \text{ is } 4.$$

Therefore if $<\alpha \sqcup 1<\beta>>_{\mathbb{R}P^2} \sqcup <1>_{S^2}$ is realisable then

$$2 + \alpha - \beta \equiv 5 \pmod 8, \text{ if } \alpha + \beta = 3,$$
$$2 + \alpha - \beta \equiv 4 \text{ or } 6 \pmod 8, \text{ if } \alpha + \beta = 2.$$

Neither of these congruences is possible if $\beta > 0$.

To apply the congruence for $\chi(B_1)$ to $<\alpha \sqcup 1<\beta>>_{\mathbb{R}P^2} \sqcup <\emptyset>_{S^2}$ we have to consider two possibilities for B_1 — the sphere component of $\mathbb{R}B$ is either contained in B_1 or not. We get

$$1 + \alpha - \beta \equiv 5 \pmod 8 \text{ or } 3 + \alpha - \beta \equiv 5 \pmod 8, \text{ if } \alpha + \beta = 4,$$
$$1 + \alpha - \beta \equiv 4 \text{ or } 6 \text{ or } 3 + \alpha - \beta \equiv 4 \text{ or } 6, \text{ if } \alpha + \beta = 3.$$

This is not possible if $\alpha > 0$ and $\beta > 1$.

5.2. Constructions. To construct $<3 \sqcup 1<1>>_{\mathbb{R}P^2} \sqcup <\emptyset>_{S^2}$ we perturb a union of two plane sections of $\mathbb{R}B$.

Lemma 5.2. *There exists a plane section $\mathbb{R}H$ of $\mathbb{R}B$ which consists of two components which both belong to the plane component of $\mathbb{R}B$.*

FIGURE 1. Construction of $\mathbb{R}H$.

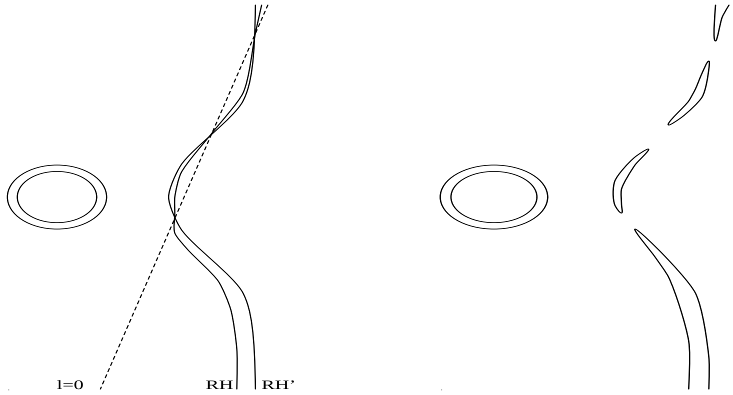

FIGURE 2. $<3 \sqcup 1 < 1 >>_{\mathbb{R}P^2} \sqcup < \emptyset >_{S^2}$.

Proof. Consider a generic plane P in $\mathbb{R}P^3$ tangent to the plane component of $\mathbb{R}B$. The intersection $P \cap \mathbb{R}B$ is a singular curve with an ordinary double point. Varying P a little we can make $\mathbb{R}H = P \cap \mathbb{R}B$ non-singular and non-connected (see Fig. 1). □

Let $l = 0$ be the equation of the plane in $\mathbb{R}P^3$ intersecting $\mathbb{R}H$ in three distinct points. Let $\mathbb{R}H' \subset \mathbb{R}B$ be the curve defined by equation $h + \epsilon l = 0$, where h is the equation of $\mathbb{R}H$ and $\epsilon > 0$ is small. Perturbing $\mathbb{R}H \cup \mathbb{R}H'$ we get the topological arrangement $< 3 \sqcup 1 < 1 >>_{\mathbb{R}P^2} \sqcup < \emptyset >_{S^2}$ (see Fig. 2). The same technique produces $< 2 \sqcup 1 < 1 >>_{\mathbb{R}P^2} \sqcup < \emptyset >_{S^2}$ and $< 1 \sqcup 1 < 1 >>_{\mathbb{R}P^2} \sqcup < \emptyset >_{S^2}$ if $l = 0$ intersects $\mathbb{R}H$ in 2 points (one is a tangent point) and in 1 point respectively. This constructs the type a of Theorem 1 with the exception of $< 1 < 1 >>_{\mathbb{R}P^2} \sqcup < \emptyset >_{S^2}$.

We construct $< 1 < 1 >>_{\mathbb{R}P^2} \sqcup < \emptyset >_{S^2}$, $< 1 < 1 < 1 >>>_{\mathbb{R}P^2} \sqcup < \emptyset >_{S^2}$ (type d), $< 1 < 1 >>_{\mathbb{R}P^2} \sqcup < 1 >_{S^2}$ (type e) and $< \emptyset >_{\mathbb{R}P^2} \sqcup < 1 \sqcup 1 < 1 >>_{S^2}$ (type f) on the cubic surface $\mathbb{R}B \subset \mathbb{R}P^3$ given in the affine coordinates (x, y, z) in $\mathbb{R}^3 \subset \mathbb{R}P^3$ by equation

$$x^2 + y^2 = z(z^2 - 1).$$

The required topological arrangements are cut on $\mathbb{R}B$ by equations $(z - C_1)(z - C_2)(z - C_3) = 0$ where we vary constants C_1, C_2 and C_3 (see Fig.3).

To construct types b and c we perturb the union of a plane $P \subset \mathbb{R}P^3$ and an ellipsoid $\mathbb{R}Q \subset \mathbb{R}P^3$ such that $\mathbb{R}H = P \cap \mathbb{R}Q$ is a non-singular non-empty conic. If $p = 0$ is the equation of the plane P and $q = 0$ is the equation of the ellipsoid $\mathbb{R}Q$

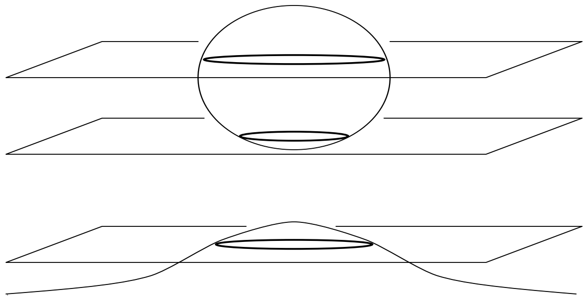

FIGURE 3. Curves on $x^2 + y^2 = z(z^2 - 1)$.

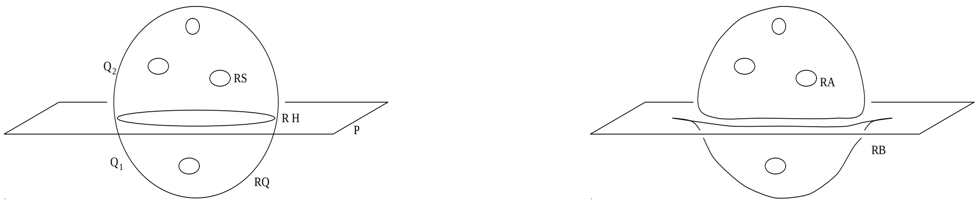

FIGURE 4. Construction of the types b and c of Theorem 1.

then $\mathbb{R}B$ is given by equation

$$pq + \epsilon s = 0,$$

where s is an auxiliary cubic polynomial which cuts a non-singular curve $\mathbb{R}S \subset \mathbb{R}Q$ not intersecting $P \cap \mathbb{R}Q$ and $\epsilon \neq 0$ is small.

Let $\mathbb{R}A = \mathbb{R}B \cap \mathbb{R}Q$. The topological type of $(\mathbb{R}B, \mathbb{R}A)$ depends only on the mutual position of $\mathbb{R}H$ and $\mathbb{R}S$ in $\mathbb{R}Q$ and the sign of ϵ (see Fig. 4). The curve $\mathbb{R}H$ splits $\mathbb{R}Q$ into 2 disks Q_1 and Q_2. For the arrangements of type c we need to construct for any α, β, $\alpha + \beta \leq 5$, a curve $\mathbb{R}S$ such that $\mathbb{R}S \cap Q_1$ bounds α disjoint disks in Q_1 and $\mathbb{R}S \cap Q_2$ bounds β disjoint disks in Q_2. For the arrangements of type 1.**b** we need to construct for any $2 \leq \alpha \leq 4$ a curve $\mathbb{R}S$ such that $\mathbb{R}S \cap Q_1$ has a component which bounds a disk in Q_1 which contains all other components of $\mathbb{R}S$ (see Fig. 5).

To construct $\mathbb{R}S \subset \mathbb{R}Q$ consisting of 5 components we petrurb the union of 3 plane sections of $\mathbb{R}Q$ which intersect as shown on Fig. 6. To construct $\mathbb{R}S \subset \mathbb{R}Q$ consisting of 4, 3, 2, 1 or 0 components we just make some of the plane sections (or their intersections) on Fig. 6 imaginary. It is easy to find all the needed positions of

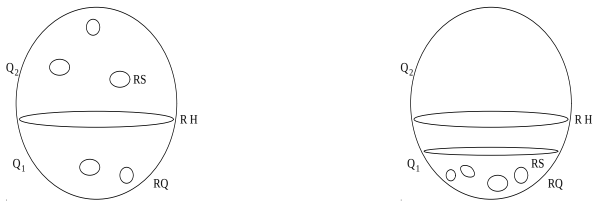

FIGURE 5. Auxiliary curves for the types c and b of Theorem 1.

FIGURE 6. Construction of $\mathbb{R}S$.

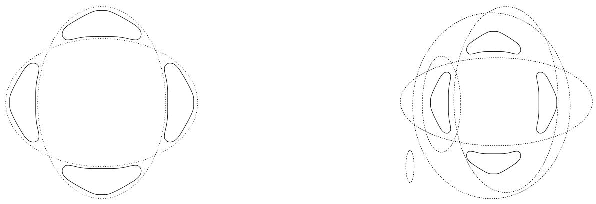

FIGURE 7. Different arrangements of a quartic and a conic.

the plane section $\mathbb{R}H$ directly from Fig. 6.

6. Proof of Theorem 2

All the needed restrictions for Theorem 2 are provided by Lemma 1. To construct the curves we obtain the cubic surface $\mathbb{R}B$ as the real part of the blowup $\beta : \mathbb{C}B \to \mathbb{C}P^2$ in 6 imaginary points of $\mathbb{C}P^2$ (these 6 points come in 3 complex conjugate pairs and they do not belong to the same conic). Thus to construct a curve $\mathbb{R}A \subset \mathbb{R}B$ of degree 6 it suffices to construct a plane curve $\mathbb{R}C \subset \mathbb{R}P^2$ such that the singular points of its complexifications $\mathbb{C}C$ are 6 imaginary ordinary double points which do not belong to the same conic. It is easy to construct a degree 6 curve decomposing to the union of a quartic and a conic transverse to each other and with no real intersection points for every topological arrangement listed in Theorem 2 (see Fig. 7). This curve has 8 imaginary ordinary double points which belong to the same conic. To get $\mathbb{R}C$ we perturb this curve so that 2 double points disappear and the other 6 survive and move to a general position (cf. [5]).

7. Proof of Theorem 3

7.1. Restrictions. Lemma 1 implies that if B_- contains more than one component of non-positive Euler characteristic then $(\mathbb{R}B, \mathbb{R}A)$ is of type 8 of Theorem 3. Otherwise the type of $(\mathbb{R}B, \mathbb{R}A)$ is

$$< \alpha \sqcup S^2_{\beta+\gamma}, \beta \sqcup k\mathbb{R}P^2_{\alpha+\gamma} >,$$
$$< \alpha \sqcup jT^2_{\beta+\gamma}, \beta \sqcup k\mathbb{R}P^2_{\alpha+\gamma} >,$$
$$\text{or } < \alpha \sqcup 2j\mathbb{R}P^2_{\beta+\gamma}, \beta \sqcup k\mathbb{R}P^2_{alpha+\gamma} >,$$
$$\text{where } \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 1, \alpha+\beta+\gamma \leq 5.$$

The number k is determined by the equality $\chi(B_+) + \chi(B_-) = \chi(\mathbb{R}B) = -5$. We get $k = 9 - 2j - 2\gamma$ (we assume that $j = 0$ in the first case). Since $k \geq 0$ we obtain

$$\gamma < 4 - j.$$

Therefore to get all the restrictions for Theorem 3 we have to prove that the following 22 types are not realisable.

- $< \alpha \sqcup S^2_{\beta+\gamma}, \beta \sqcup (9-2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$, where $(\alpha, \beta, \gamma) = (0,3,1)$, $(0,4,1)$, $(1,2,1)$, $(1,3,1)$, $(2,2,1)$ or $(4,0,1)$. These types are not realisable because of the Rokhlin ([13]) and Kharlamov-Gudkov-Krakhnov ([8], [4]) congruences

  $$\chi(B_+) = \alpha - \beta + 1 \equiv 3 \pmod 8, \text{ if } \alpha + \beta = 4 \text{ and } \gamma = 1,$$
  $$\chi(B_+) = \alpha - \beta + 1 \equiv 2 \text{ or } 4 \pmod 8, \text{ if } \alpha + \beta = 3 \text{ and } \gamma = 1.$$

  We apply the congruences in the form of Theorem 7.1 of [11] which is more convenient for our application (note that it is the condition that $\gamma = 1$ which implies the hypothesis $e = 0$ of the theorem).
- $< \alpha \sqcup T^2_{\beta+\gamma}, \beta \sqcup (7 - 2\gamma\mathbb{R}P^2_{\alpha+\gamma} >$, where $(\alpha, \beta, \gamma) = (1,3,1)$ or $(3,1,1)$. These types are not realizable since Theorem 7.4.d of [11] implies that

  $$\alpha - \beta - 1 \equiv 3 \pmod 4, \text{ if } \alpha + \beta = 4 \text{ and } \gamma = 1.$$

- $< \alpha \sqcup 2T^2_{\beta+\gamma}, \beta \sqcup (5 - 2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$, where $(\alpha, \beta, \gamma) = (0,4,1)$, $(2,2,1)$ or $(4,0,1)$. These types are not realizable since Theorem 7.4.d of [11] implies that

  $$\alpha - \beta - 3 \equiv 3 \pmod 4, \text{ if } \alpha + \beta = 4 \text{ and } \gamma = 1.$$

- $< \alpha \sqcup 3T^2_{\beta+1}, \beta \sqcup \mathbb{R}P^2_{\alpha+1} >$, where $(\alpha, \beta) = (4,0)$, $(1,1)$, $(2,1)$, $(3,1)$, $(1,2)$, $(2,2)$ or $(1,3)$. These types are not realizable since Theorem 1 of [11] implies

  $$\chi(B_-) \equiv 3 + B \pmod 8,$$

  where B is the Brown invariant of the Guillou-Marin form of $\mathbb{C}A/\operatorname{conj} \cup B_-$. But $B \equiv 1 \pmod 8$ if $\mathbb{R}A$ is an M-curve and $B \equiv 0$ or $2 \pmod 8$ if $\mathbb{R}A$ is an (M-1)-curve (cf. Addenda 1.a, 1.b of [11]) since the only class in $H_1(B_-; \mathbb{Z}_2)$ which does not vanish in $H_1(\mathbb{R}B; \mathbb{Z}_2)$ is the class dual to $w_1(\mathbb{R}B)$ and the value of the Guillou-Marin form of $\mathbb{C}A/\operatorname{conj} \cup B_-$ on such a class is 1. Therefore

  $$\beta - \alpha \equiv 4 \pmod 8, \text{ if } \alpha + \beta = 4 \text{ and } \gamma = 1,$$
  $$\beta - \alpha \equiv 3 \text{ or } 5 \pmod 8, \text{ if } \alpha + \beta = 3 \text{ and } \gamma = 1.$$

  This rules out the pairs $(\alpha, \beta) = (2,1), (3,1), (1,2), (2,2), (1,3)$. In a similar way Addendum 1.c of [11] implies that $(\alpha, \beta) = (1,1)$ is not realizable by a curve of type II (i.e. if $\mathbb{C}A/\operatorname{conj}$ is non-orientable) since $\beta - \alpha \equiv 0 \pmod 8$ in this case.

To see that $(\alpha,\beta)=(4,0)$ is not realizable we apply a version of the Rokhlin complex orientation formula [14]. Note that B_+ in this case is a disjoint union of 4 disks and a punctured sphere with 3 handles and therefore $F = \mathbb{C}A/\operatorname{conj} \cup B_+ \subset \mathbb{C}B/\operatorname{conj} \approx S^4$ (see [9]) is orientable. Therefore $F.F=0$ but on the other hand $F.F=\frac{1}{2}\mathbb{C}B.\mathbb{C}B - 2\chi(B_+) = 6+2=8$.

A similar argument shows that $(\alpha,\beta)=(1,1)$ is not realizable by a curve of type I (i.e. if $\mathbb{C}A/\operatorname{conj}$ is orientable). In this case B_+ is a disjoint union of a disk and a sphere with 3 handles punctured two times and B_- is a disjoint union of a disk and a projective plane punctured two times. Denote the disk component of B_- by D_- the disk component of B_+ by D_+ and the union of D and the non-disk component of B_+ by E. Note that $G = \mathbb{C}A/\operatorname{conj} \cup D_+ \cup E \cup D_-$ is a $\mathbb{Z}$-cycle in $\mathbb{C}B/\operatorname{conj} \approx S^4$ and thus $G.G=0$. If the orientations of $\mathbb{R}A = \partial\mathbb{C}A/\operatorname{conj} = \partial B_+$ induced from $\mathbb{C}A/\operatorname{conj}$ and from B_+ agree then $G.G = \frac{1}{2}\mathbb{C}B.\mathbb{C}B - 2(\chi(D_+)+\chi(E)+\chi(D_-))+4 = 16 \neq 0$. If the orientations do not agree then $G.G = \frac{1}{2}\mathbb{C}B.\mathbb{C}B - 2(\chi(D_+)+\chi(E)+\chi(D_-))-4 = 8 \neq 0$.

- $< 2\sqcup 2\mathbb{R}P^2_3, 2\sqcup 5\mathbb{R}P^2_3 >$. This type is not realisable because of Theorem 7.4.c of [11], since $\chi(B_+)=-1$ and B_+ is not orientable.
- $< \alpha \sqcup 6\mathbb{R}P^2_{\beta+1}, \beta\sqcup \mathbb{R}P^2_{\alpha+1} >$, where $(\alpha,\beta)=(2,1)$, $(3,1)$ or $(2,2)$. The pair $(2,1)$ is ruled out by Theorem 7.4.c of [11] applied to B_- since $\chi(B_-) = \beta-\alpha = -1$ and B_- is not orientable. The pairs $(3,1)$ and $(2,2)$ are ruled out by Theorem 7.4.b of [11].

7.2. Constructions. Similarily to the proof of Theorem 2 we construct the curves for Theorem 3 perturbing the union of a quartic and a conic. This time we have to use a quartic and a conic which intersect in at least 6 real points since we need 6 points in $\mathbb{R}P^2$ to blow up to get the cubic surface diffeomorphic to $7\mathbb{R}P^2$. We use Polotovskii's catalog [12] which lists arrangements of a conic and a quartic which intersect in 8 real points. Then we smooth 2 out of the 8 points so that the other 6 move to a general position with respect to each other (cf. [5]). One curve in Polotovskii's catalog may produce more than one type for Theorem 3 because of the ambiguity in the choice of the 2 points to smooth and in (real) smoothing of these points.

Example 7.1. The curve encoded by (12387456)[0] in Polotovskii's catalog [12] produces $< 3\sqcup S^2_2, 5\mathbb{R}P^2_5 >$, $< 4\sqcup T^2_1, 5\mathbb{R}P^2_5 >$, $< 3\sqcup T^2_2, 3\mathbb{R}P^2_5 >$, $< 3\sqcup 2\mathbb{R}P^2_2, 1\sqcup 5\mathbb{R}P^2_4 >$ and $< 3\sqcup 2\mathbb{R}P^2_2, 3\mathbb{R}P^2_5 >$ out of M-curves (see Fig. 8) and $< 3\sqcup T^2_1, 5\mathbb{R}P^2_4 >$ and $< 3\sqcup 2\mathbb{R}P^2_1, 5\mathbb{R}P^2_4 >$ out of $(M-1)$-curves.

The catalog [12] lists only arrangements of M-decomposing curves thus the quartic in the union consists of 4 ovals. But if an oval of the quartic is disjoint from the conic then we can remove it so that the resulting arrangement is still realizable by a decomposing curve of degree 6. This means that if a triple (α,β,γ) is obtained from a decomposing curve then (α',β',γ) with $\alpha'<\alpha$ and $\beta'<\beta$ can be also obtained in this way. In the following tables we indicate the code of a union of a conic and a

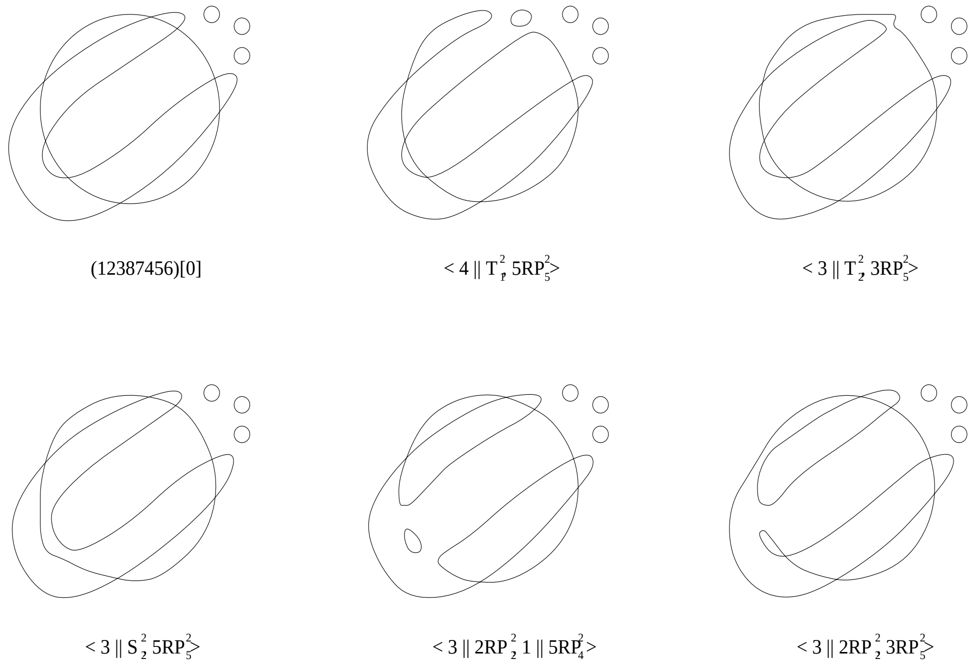

FIGURE 8. Curves produced by (12387456)[0]

quartic in [12] for arrangements in Theorem 3 with maximal (α, β).

1. $< \alpha \sqcup S^2_{\beta+\gamma}, \beta \sqcup (9-2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$
$\alpha = 0$		$\gamma = 4$ (12)(34)(56)(78)	$\gamma = 3$ (1867)(3452)[2] $\gamma = 1$ $*$	$\gamma = 2$ (18276543)[3]
$\alpha = 1$	$\gamma = 4$ (12)(34)(56)(78)	$\gamma = 3$ (1678)(2345)[1]	$\gamma = 2$ (18723456)[2]	
$\alpha = 2$	$\gamma = 3$ (145678)(23)[0]	$\gamma = 2$ (12345678)[1]		
$\alpha = 3$	$\gamma = 2$ (12387456)[0]	$\gamma = 1$ (12345678)[1]		

2. $< \alpha \sqcup T^2_{\beta+\gamma}, \beta \sqcup (7-2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta=0$	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$
$\alpha=0$			$\gamma=3$ $(1876)(2345)[2]$	$\gamma=2$ $(18743256)[3]$	$\gamma=1$ $(18276543)[3]$
$\alpha=1$		$\gamma=3$ $(187654)(23)[1]$	$\gamma=2$ $(18723456)[2]$		
$\alpha=2$	$\gamma=3$ $(1867)(3452)[2]$	$\gamma=2$ $(18765234)[1]$	$\gamma=1$ $(18723456)[2]$		
$\alpha=3$	$\gamma=2$ $(12387456)[0]$				
$\alpha=4$	$\gamma=1$ $(12387456)[0]$				

3. $< \alpha \sqcup 2T^2_{\beta+\gamma}, \beta \sqcup (5-2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta=0$	$\beta=1$	$\beta=2$	$\beta=3$
$\alpha=0$				$\gamma=2$ $(18743256)[3]$
$\alpha=1$			$\gamma=2$ $(18765432)[2]$	$\gamma=1$ $(18234765)[2]$
$\alpha=2$		$\gamma=2$ $(18765234)[1]$		
$\alpha=3$	$\gamma=2$ $(18276543)[3]$	$\gamma=1$ $(18765234)[1]$		

4. $< \alpha \sqcup 3T^2_{\beta+1}, \beta \sqcup \mathbb{R}P^2_{\alpha+1} >$,

$\beta=0$	$\beta=4$
$\alpha=3$ $(18276543)[3]$	$\alpha=0$ $(18234567)[3]$

5. $< \alpha \sqcup 2\mathbb{R}P^2_{\beta+\gamma}, \beta \sqcup (7-2\gamma)\mathbb{R}P^2_{\alpha+\gamma} >$,

	$\beta=0$	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$
$\alpha=0$			$\gamma=3$ $(1867)(3452)[2]$	$\gamma=2$ $(18276543)[3]$	$\gamma=1$ $*$
$\alpha=1$		$\gamma=3$ $(1845)(23)(67)[0]$	$\gamma=2$ $(18723456)[2]$	$\gamma=1$ $(18723456)[2]$	
$\alpha=2$	$\gamma=3$ $(1845)(3672)[0]$	$\gamma=2$ $(18432765)[1]$			
$\alpha=3$	$\gamma=2$ $(12387456)[0]$	$\gamma=1$ $(12387456)[0]$			
$\alpha=4$	$\gamma=1$ $(16254378)[0]$				

6. $<\alpha \sqcup 4\mathbb{R}P^2_{\beta+\gamma}, \beta \sqcup (5-2\gamma)\mathbb{R}P^2_{\alpha+\gamma}>$,

	$\beta=0$	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$
$\alpha=0$				$\gamma=2$ (18437625)[3]	$\gamma=1$ (18437625)[3]
$\alpha=1$			$\gamma=2$ (18432765)[1]	$\gamma=1$ (18437625)[3]	
$\alpha=2$		$\gamma=2$ (18765234)[1]	$\gamma=1$ (18765234)[1]		
$\alpha=3$	$\gamma=2$ (18276345)[0]	$\gamma=1$ (18276345)[0]			
$\alpha=4$	$\gamma=1$ (18276345)[0]				

7. $<\alpha \sqcup 6\mathbb{R}P^2_{\beta+1}, \beta \sqcup \mathbb{R}P^2_{\alpha+1}>$,

$\beta=0$	$\beta=3$	$\beta=4$
$\alpha=4$ *	$\alpha=1$ (18765432)[2]	$\alpha=0$ (18234567)[3]

To construct the arrangements marked by * in the table we start from a plane curve of degree 6 and collapse 6 of its empty ovals to points using [7]. To construct $<2\mathbb{R}P^2_5, 4\sqcup 5\mathbb{R}P^2_1>$ and $<4\sqcup 6\mathbb{R}P^2_1, \mathbb{R}P^2_5>$ we use the Gudkov curve $<5\sqcup 1<5>>$ constructed in [2]. To get $<2\mathbb{R}P^2_5, 4\sqcup 5\mathbb{R}P^2_1>$ we collapse the 5 inner ovals and one of the outer ovals; to construct $<4\sqcup 6\mathbb{R}P^2_1, \mathbb{R}P^2_5>$ we collapse the 5 outer ovals and one of the inner ovals (see Fig. 9). To construct $<S^2_3, 2\sqcup 7\mathbb{R}P^2_1>$ we collapse the 6 outer ovals of $<6\sqcup 1<2>>$.

8. $<1\sqcup 3T^2_2, S^2_2 \sqcup \mathbb{R}P^2_1>$, $<2T^2_3, S^2_2\sqcup \mathbb{R}P^2_1>$

To construct the plane curves of degree 6 with 6 double points for these arrangements we start from the union of a two-component cubic in $\mathbb{R}P^2$ and its "parallel" copy which intersect in 9 disjoint point and smooth 3 out of the 9 nodes of the union (see Fig. 10).

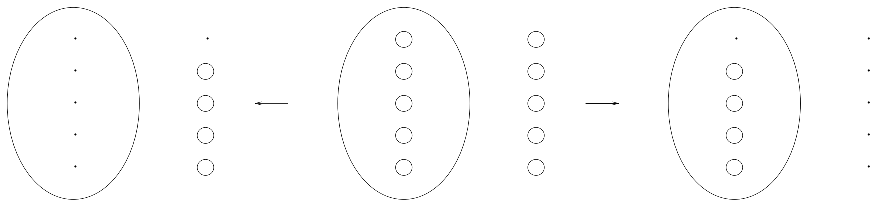

FIGURE 9. $<2\sqcup \mathbb{R}P^2_5, 4\sqcup 5\mathbb{R}P^2_1>$ and $<4\sqcup 6\mathbb{R}P^2_1, \mathbb{R}P^2_5>$.

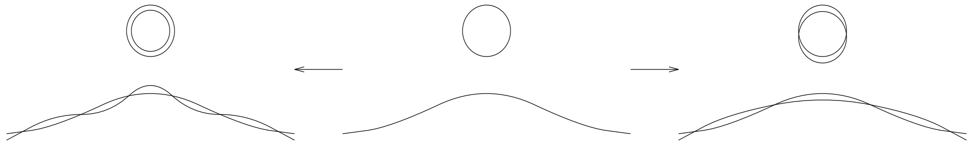

FIGURE 10. $< 1 \sqcup 3T^2_2, S^2_2 \sqcup \mathbb{R}P^2_1 >$ and $< 2T^2_3, S^2_2 \sqcup \mathbb{R}P^2_1 >$.

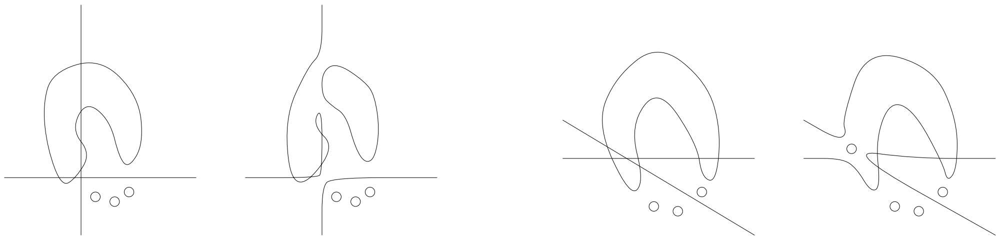

FIGURE 11. $< 1 \sqcup S^2_4, 3 \sqcup 5\mathbb{R}P^2_2 >$ and $< 2 \sqcup 2T^2_3, 2 \sqcup \mathbb{R}P^2_3 >$.

8. Proof of Theorem 4 and Theorem 5

8.1. Restrictions. Lemma 1 implies that unless $(\mathbb{R}B, \mathbb{R}A)$ is of type 6 of Theorem 4 or of type 4 of Theorem 5 it is of type

$$
\begin{gathered}
< \alpha \sqcup S^2_{\beta+\gamma}, \beta \sqcup k\mathbb{R}P^2_{\alpha+\gamma} >, \\
< \alpha \sqcup jT^2_{\beta+\gamma}, \beta \sqcup k\mathbb{R}P^2_{\alpha+\gamma} >, \\
\text{or } < \alpha \sqcup 2j\mathbb{R}P^2_{\beta+\gamma}, \beta \sqcup k\mathbb{R}P^2_{\alpha+\gamma} >, \\
\text{where } \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 1, \alpha+\beta+\gamma \leq 5.
\end{gathered}
$$

In the case of Theorem 4 $k = 7 - 2j - 2\gamma \geq 0$ and in the case of Theorem 5 $k = 5 - 2j - 2\gamma \geq 0$ since $\chi(B_+) + \chi(B_-) = \chi(\mathbb{R}B)$. To finish the restrictions for Theorem 4 and Theorem 5 we have to prove that $< 4 \sqcup 2T^2_1, \mathbb{R}P^2_5 >$ is not realizable (for Theorem 4). Note that $F = \mathbb{C}A/\operatorname{conj} \cup B_+ \subset \mathbb{C}B/\operatorname{conj} \approx \overline{\mathbb{C}P}^2$ (see [9]) is orientable and therefore $F.F \leq 0$ ($\overline{\mathbb{C}P}^2$ is negative definite). On the other hand $F.F = \frac{1}{2}\mathbb{C}B.\mathbb{C}B - 2\chi(B_+) = 6 - 2 = 4$.

8.2. Constructions. If $< j \sqcup F\#T^2, n \sqcup G >$ or $< j \sqcup F, n \sqcup G\#T^2 >$ is constructed in 7.2 out of Polotovskii's catalog then we can construct $< j \sqcup F, n \sqcup G >$ out of a similar pair of a conic and a quartic which intersect in 6 real points. Considering similar pairs of a conic and a quartic which intersect in 4 real points produces the arrangements for Theorem 5. A similar modification works for $< 1 \sqcup 2T^2_2, S^2_2 \sqcup \mathbb{R}P^2_1 >$ and $< T^2_3, S^2_2 \sqcup \mathbb{R}P^2_1 >$. This produces all the arrangements for Theorem 4 except for $< 1 \sqcup S^2_4, 3 \sqcup 5\mathbb{R}P^2_2 >$ and $< 2 \sqcup 2T^2_3, 2 \sqcup \mathbb{R}P^2_3 >$. To construct them we perturb a union of a quartic and two lines as shown on Fig. 11.

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