RANDOM VECTORS IN THE ISOTROPIC POSITION

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Abstract. Let y be a random vector in $\mathbb{R}^n$, satisfying

$$\mathbb{E}\, y \otimes y = id.$$

Let M be a natural number and let $y_1, \dots, y_M$ be independent copies of y. We prove that for some absolute constant C

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \le C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left(\mathbb{E}\, \|y\|^{\log M} \right)^{1/\log M},$$

provided that the last expression is smaller than 1.

We apply this estimate to obtain a new proof of a result of Bourgain concerning the number of random points needed to bring a convex body into a nearly isotropic position.

1. Introduction

The problem we consider has arisen from a question studied by R. Kannan, L. Lovász and M. Simonovits [K-L-S]. To construct a fast algorithm for calculating the volume of a convex body, they needed to bring it into some symmetric position. More precisely, let K be a convex body in $\mathbb{R}^n$. We shall say that it is in the isotropic position if for any $x \in \mathbb{R}^n$

$$\frac{1}{\mathrm{vol}\ (K)} \int_K \langle x, y \rangle^2 \, dy = \|x\|^2$$

By $\|\cdot\|$ we denote the standard Euclidean norm.

The notion of isotropic position was extensively studied by V. Milman and A. Pajor [M-P]. Note that our definition is consistent with [K-L-S]. The normalization in [M-P] is slightly different.

If the information about the body K is uncomplete it is impossible to bring it exactly to the isotropic position. So, the definition of the isotropic position has to be modified to allow a small error. We shall say that the body K is in ε-isotropic position if for any $x \in \mathbb{R}^n$

$$(1-\varepsilon) \cdot \|x\|^2 \le \frac{1}{\mathrm{vol}\ (K)} \int_K \langle x, y \rangle^2 \, dy \le (1+\varepsilon) \cdot \|x\|^2 .$$

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Let $\varepsilon > 0$ be given. Consider M random points $y_1, \dots, y_M$ independently uniformly distributed in K and put

$$T = \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i.$$

If M is sufficiently large, than with high probability

$$\left\| T - \frac{1}{\mathrm{vol}(K)} \int_K y \otimes y \right\|$$

will be small, so the body $T^{-1/2}K$ will be in ε-isotropic position. R. Kannan, L. Lovász and M. Simonovits ([K-L-S]) proved that it is enough to take

$$M = c\frac{n^2}{\varepsilon}$$

for some absolute constant c. This estimate was significantly improved by J. Bourgain [B]. Using rather delicate geometric considerations he has shown that one can take

$$M = C(\varepsilon) n \log^3 n.$$

Since the situation is invariant under a linear transformation, we may assume that the body K is in the isotropic position. Then the result of Bourgain may be reformulated as follows:

Theorem 0. [B] *Let K be a convex body in $\mathbb{R}^n$ in the isotropic position. Fix $\varepsilon > 0$ and choose independently M random points $x_1, \dots, x_M \in K$,*

$$M \geq C(\varepsilon) n \log^3 n.$$

Then with probability at least $1 - \varepsilon$ for any $x \in \mathbb{R}^n$ one has

$$(1 - \varepsilon) \|x\|^2 \leq \frac{1}{M} \sum_{i=1}^{M} \langle x, y \rangle^2 \leq (1 + \varepsilon) \|x\|^2 .$$

We shall show that this theorem follows from a general result about random vectors in $\mathbb{R}^n$. Let y be a random vector. Denote by $\mathbb{E}\, X$ the expectation of a random variable X. We say that y is in the isotropic position if

$$\mathbb{E}\, y \otimes y = id. \tag{1.1}$$

If y is uniformly distributed in a convex body K, then this is equivalent to the fact that K is in the isotropic position.

We prove the following

Theorem 1. *Let $y \in \mathbb{R}^n$ be a random vector in the isotropic position. Let M be a natural number and let $y_1, \dots, y_M$ be independent copies of y. Then*

$$\mathbb{E} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \le C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot \left(\mathbb{E} \, \|y\|^{\log M} \right)^{1/\log M}, \tag{1.2}$$

provided that the last expression is smaller than 1.

Here and later C, c, etc. denote absolute constants whose values may vary from line to line.

Remark. Taking the trace of (1.1) we obtain that $\mathbb{E} \, \|y\|^2 = n$, so to make the right hand side of (1.2) smaller than 1, we have to assume that $M \ge cn \log n$.

Using Theorem 1 we prove a better estimate of the length of approximate John's decompositions [R1] and thus improve the results about approximating a convex body by another one having a small number of contact points, obtained in [R2]. Estimating the moment of the norm of random vector in a convex body, we obtain a different proof of Theorem 0 which gives also a better estimate.

2. Main results.

The proof of Theorem 1 consists of two steps. First we introduce a Bernoulli random process and estimate the expectation of the norm in (1.2) by the expectation of its supremum. Then we construct a majorizing measure to obtain a bound for the latest.

The first step is relatively standard. Let $\varepsilon_1, \dots, \varepsilon_M$ be independent Bernoulli variables taking values $1, -1$ with probability $1/2$ and let $y_1, \dots, y_M, \quad \bar{y}_1, \dots, \bar{y}_M$ be independent copies of y. Denote $\mathbb{E}_y$, $\mathbb{E}_\varepsilon$ the expectation according to y and ε respectively. Since $y_i \otimes y_i - \bar{y}_i \otimes \bar{y}_i$ is a symmetric random variable, we have

$$\mathbb{E}_y \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - id \right\| \le \mathbb{E}_y \, \mathbb{E}_{\bar{y}} \left\| \frac{1}{M} \sum_{i=1}^{M} y_i \otimes y_i - \frac{1}{M} \sum_{i=1}^{M} \bar{y}_i \otimes \bar{y}_i \right\| =$$

$$\mathbb{E}_\varepsilon \, \mathbb{E}_y \, \mathbb{E}_{\bar{y}} \left\| \frac{1}{M} \sum_{i=1}^{M} \varepsilon_i (y_i \otimes y_i - \bar{y}_i \otimes \bar{y}_i) \right\| \le 2 \, \mathbb{E}_y \, \mathbb{E}_\varepsilon \left\| \frac{1}{M} \sum_{i=1}^{M} \varepsilon_i y_i \otimes y_i \right\|.$$

To estimate the last expectation, we need the following Lemma, which generalizes Lemma 1 [R3].

Lemma. *Let $y_1, \dots, y_M$ be vectors in $\mathbb{R}^n$ and let $\varepsilon_1, \dots, \varepsilon_M$ be independent Bernoulli variables taking values $1, -1$ with probability $1/2$. Then*

$$\mathbb{E} \left\| \sum_{i=1}^{M} \varepsilon_i y_i \otimes y_i \right\| \le C \sqrt{\log M} \cdot \max_{i=1,\dots,M} \|y_i\| \cdot \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\|^{1/2}.$$

We postpone the proof of the Lemma to the next section.

Applying the Lemma, we get

$$\mathbb{E}\left\|\frac{1}{M}\sum_{i=1}^{M} y_i\otimes y_i - id\right\| \le$$
$$C\cdot\frac{\sqrt{\log M}}{M}\cdot\left(\mathbb{E}\max_{i=1,\dots,M}\|y_i\|^2\right)^{1/2}\cdot\left(\mathbb{E}\left\|\sum_{i=1}^{M} y_i\otimes y_i\right\|\right)^{1/2}. \tag{2.1}$$

We have

$$\left(\mathbb{E}\max_{i=1,\dots,M}\|y_i\|^2\right)^{1/2} \le \left(\mathbb{E}\left(\sum_{i=1}^{M}\|y_i\|^{\log M}\right)^{2/\log M}\right)^{1/2} \le$$
$$M^{1/\log M}\cdot\left(\mathbb{E}\,\|y\|^{\log M}\right)^{1/\log M}.$$

Thus, denoting

$$D = \mathbb{E}\left\|\frac{1}{M}\sum_{i=1}^{M} y_i\otimes y_i - id\right\|,$$

we obtain by (2.1)

$$D \le C\cdot\frac{\sqrt{\log M}}{\sqrt{M}}\cdot\left(\mathbb{E}\,\|y\|^{\log M}\right)^{1/\log M}\cdot(D+1)^{1/2}.$$

If

$$C\cdot\frac{\sqrt{\log M}}{\sqrt{M}}\cdot\left(\mathbb{E}\,\|y\|^{\log M}\right)^{1/\log M} \le 1,$$

we get

$$D \le 2C\cdot\frac{\sqrt{\log M}}{\sqrt{M}}\cdot\left(\mathbb{E}\,\|y\|^{\log M}\right)^{1/\log M},$$

which completes the proof of Theorem 1.

We turn now to the applications of Theorem 1. Applying Theorem 1 to the question of Kannan, Lovász and Simonovits, we obtain the following

Corollary 2.1. *Let $\varepsilon > 0$ and let K be an n-dimensional convex body in the isotropic position. Let*

$$M \ge C\cdot\frac{n}{\varepsilon^2}\cdot\log^2\frac{n}{\varepsilon^2}$$

and let $y_1,\dots,y_M$ be independent random vectors uniformly distributed in K. Then

$$\mathbb{E}\left\|\frac{1}{M}\sum_{i=1}^{M} y_i\otimes y_i - id\right\| \le \varepsilon.$$

Proof. It follows from a result of S. Alesker [A], that

$$\mathbb{E}\exp\left(\frac{\|y\|^2}{c\cdot n}\right) \le 2$$

for some absolute constant c. Then

$$\mathbb{E}\ \|y\|^{\log M} \le \left(\mathbb{E}\, e^{\frac{\|y\|^2}{c\cdot n}}\right)^{1/2} \cdot \left(\mathbb{E}\ \left(\|y\|^{2\log M}\cdot e^{-\frac{\|y\|^2}{c\cdot n}}\right)\right)^{1/2} \le$$
$$\sqrt{2}\cdot\left(\max_{t\ge 0} t^{\log M}\cdot e^{-\frac{t}{c\cdot n}}\right)^{1/2} \le (C\cdot n\cdot \log M)^{\frac{\log M}{2}}.$$

Corollary 2.1 follows from this estimate and Theorem 1. $\square$

By a Lemma of Borell [M-S, Appendix III], most of the volume of a convex body in the isotropic position is concentrated within the Euclidean ball of radius $c\sqrt{n}$. So, it might be of interest to consider a random vector uniformly distributed in the intersection of a convex body and such a ball. In this case the previous estimate may be improved as follows.

Corollary 2.2. *Let $\varepsilon, R > 0$ and let K be an n-dimensional convex body in the isotropic position. Suppose that $R \ge c\sqrt{\log 1/\varepsilon}$ and let*

$$M \ge C_0 \cdot \frac{R^2\cdot n}{\varepsilon^2}\cdot \log\frac{R^2\cdot n}{\varepsilon^2} \tag{2.2}$$

and let $y_1,\dots,y_M$ be independent random vectors uniformly distributed in $K\cap R\sqrt{n})\cdot B_2^n$. Then

$$\mathbb{E}\left\|\frac{1}{M}\sum_{i=1}^M y_i\otimes y_i - id\right\| \le \varepsilon.$$

Proof. Denote $a = R\cdot\sqrt{n}$ and let z be a random vector uniformly distributed in $K\cap aB_2^n$. Then for $x\in B_2^n$

$$\mathbb{E}\,\langle z,x\rangle^2 = \frac{\mathrm{vol}\ (K)}{\mathrm{vol}\ (K\cap aB_2^n)}\cdot\Big(\frac{1}{\mathrm{vol}\ (K)}\int_K \langle y,x\rangle^2\,dy\ -$$
$$\frac{1}{\mathrm{vol}\ (K)}\int_K \langle y,x\rangle^2\cdot \mathbf{1}_{\{u\,\big|\,\|u\|\ge a\}}(y)\,dy\Big).$$

By a result of S. Alesker [A] and Khinchine type inequality [M-P], we have

$$\frac{\mathrm{vol}\ (K)}{\mathrm{vol}\ (K\cap aB_2^n)} \le 1 + e^{-ca^2/n} \le 1+\frac{\varepsilon}{4}$$

and

$$\frac{1}{\mathrm{vol}\ (K)}\int_K \langle y,x\rangle^2\cdot \mathbf{1}_{\{u\,\big|\,\|u\|\ge a\}}(y)\,dy \le$$
$$\left(\frac{1}{\mathrm{vol}\ (K)}\int_K \langle y,x\rangle^4\,dy\right)^{1/2}\cdot\left(\frac{1}{\mathrm{vol}\ (K)}\int_K \mathbf{1}_{\{u\,\big|\,\|u\|\ge a\}}(y)\,dy\right)^{1/2} \le$$
$$Ce^{-ca^2/2n} \le \frac{\varepsilon}{4}.$$

Thus for any $x \in B_2^n$

$$|\mathbb{E}\,\langle z, x\rangle^2 - 1| \le \frac{\varepsilon}{2}.$$

Define a random vector

$$y = (\mathbb{E}\, z \otimes z)^{-1/2} z.$$

Then y is in the isotropic position and

$$\left(\mathbb{E}\ \|y\|^{\log M}\right)^{1/\log M} \le \left\|(\mathbb{E}\, z \otimes z)^{-1/2}\right\| \cdot \left(\mathbb{E}\ \|z\|^{\log M}\right)^{1/\log M} \le 2a,$$

so

$$\mathbb{E}\ \left\|\frac{1}{M}\sum_{i=1}^M y_i \otimes y_i - id\right\| \le C \cdot \frac{\sqrt{\log M}}{\sqrt{M}} \cdot 2a \le \frac{\varepsilon}{2}$$

provided the constant C_0 in (2.2) is large enough. Thus,

$$\mathbb{E}\ \left\|\frac{1}{M}\sum_{i=1}^M z_i \otimes z_i - id\right\| \le \mathbb{E}\ \left\|\frac{1}{M}\sum_{i=1}^M y_i \otimes y_i - id\right\| \cdot \|\mathbb{E}\, z \otimes z\| + \|\mathbb{E}\, z \otimes z - id\| \le \varepsilon.$$

□

The next application is connected to the approximation of a convex body by another one having a small number of contact points [R2]. Let K be a convex body in $\mathbb{R}^n$ such that the ellipsoid of minimal volume containing it is the standard Euclidean ball B_2^n. Then by the theorem of John, there exist $N \le (n+3)n/2$ points $z_1, \dots, z_N \in K, \quad \|x_i\| = 1$ and N positive numbers $c_1, \dots, c_N$ satisfying the following system of equations

$$id = \sum_{i=1}^N c_i\, z_i \otimes z_i \tag{2.3}$$

$$0 = \sum_{i=1}^N c_i\, z_i. \tag{2.4}$$

It was shown in [R1] for convex symmetric bodies and in [R2] in the general case, that the identity operator can be approximated by a sum of a smaller number of terms $x_i \otimes x_i$. We derive from Theorem 1 the following corollary, which improves Lemma 3.1 [R2].

Corollary 2.3. *Let $\varepsilon > 0$ and let K be a convex body in $\mathbb{R}^n$, so that the ellipsoid of minimal volume containing it is B_2^n. Then there exist*

$$M \le \frac{C}{\varepsilon^2} \cdot n \cdot \log \frac{n}{\varepsilon} \tag{2.5}$$

contact points $x_1, \dots, x_M$ and a vector u, $\|u\| \le \frac{C}{\sqrt{M}}$, so that the identity operator in $\mathbb{R}^n$ has the following representation

$$id = \frac{n}{M}\ \sum_{i=1}^M (x_i + u) \otimes (x_i + u) + S,$$

where

$$\sum_{i=1}^{M}(x_i+u)=0 \tag{2.6}$$

and

$$\|S:\ell_2^n\to\ell_2^n\|<\varepsilon.$$

Proof. Let (2.3) be a decomposition of the identity operator. Let y be a random vector in $\mathbb{R}^n$, taking values $\sqrt{n}z_i$ with probability $c_i/\sqrt{n}$. Then, by (2.3), y is in the isotropic position. Obviously, for all $1\le p<\infty$

$$(\mathbb{E}\ \|y\|^p)^{1/p}=\sqrt{n}.$$

So, taking M as in (2.5), we obtain that for sufficiently large C

$$\left\|\frac{1}{M}\sum_{i=1}^{M}y_i\otimes y_i-id\right\|\le\frac{\varepsilon}{2} \tag{2.7}$$

with probability greater than 3/4. Since by (2.4), $\mathbb{E}y=0$ and $\|y\|=\sqrt{n}$, we have

$$\left\|\sum_{i=1}^{M}y_i\right\|\le 2\sqrt{M} \tag{2.8}$$

with probability greater than 3/4. Take $y_1,\dots,y_M$ for which (2.7) and (2.8) hold and put

$$x_i=\frac{1}{\sqrt{n}}\cdot y_i,\qquad\qquad u=-\frac{1}{M}\sum_{i=1}^{M}x_i.$$

Then (2.6) is satisfied and

$$\left\|\frac{n}{M}\ \sum_{i=1}^{M}(x_i+u)\otimes(x_i+u)-id\right\|\le$$
$$\left\|\frac{n}{M}\ \sum_{i=1}^{M}x_i\otimes x_i-id\right\|+n\cdot\|u\otimes u\|\le\frac{\varepsilon}{2}+\frac{4\,n}{M}\le\varepsilon.\quad\square$$

Substituting Lemma 3.1 [R2] by Corollary 2.3 in the proof of Theorem 1.1 [R2] we obtain the following

Corollary 2.3. *Let B be a convex body in $\mathbb{R}^n$ and let $\varepsilon>0$. There exists a convex body $K\subset\mathbb{R}^n$, so that $d(K,B)\le 1+\varepsilon$ and the number of contact points of K with the ellipsoid of minimal volume containing it is less than*

$$M(n,\varepsilon)=\frac{C}{\varepsilon^2}\cdot n\cdot\log\frac{n}{\varepsilon}.$$

3. Proof of the Lemma

The proof of the Lemma is similar to that of Lemma 1 [R3]. For the reader's convenience we present here a complete proof.

Without loss of generality, we may assume that

$$\left\| \sum_{i=1}^M y_i \otimes y_i \right\| = 1.$$

Define a random process

$$V_x = \sum_{i=1}^M \varepsilon_i \langle x, y \rangle^2$$

for $x \in B_2^n$. We have to estimate

$$\mathbb{E} \sup_{x \in B_2^n} V_x. \tag{3.1}$$

Note that the process V_x has a subgaussian tail estimate

$$\mathcal{P}\{V_x - V_{\bar{x}} > a\} \le \exp\left(-C \frac{a^2}{\tilde{d}^2(x, \bar{x})} \right),$$

where C is an absolute constant and

$$\tilde{d}(x, \bar{x}) = \left(\sum_{i=1}^M \left(\langle x, y_i \rangle^2 - \langle \bar{x}, y_i \rangle^2 \right)^2 \right)^{1/2}.$$

The function $\tilde{d}$ is not a metric on B_2^n , since $\tilde{d}(x, \bar{x}) = 0$ does not imply $x = \bar{x}$. To avoid this obstacle, we shall estimate $\tilde{d}$ by a quasimetric d defined by

$$d(x, \bar{x}) = \left(\sum_{i=1}^M \langle x - \bar{x}, y_i \rangle^2 \left(\langle x, y_i \rangle^2 + \langle \bar{x}, y_i \rangle^2 \right) \right)^{1/2}.$$

Then for all , $\bar{x} \in B_2^n$

$$\tilde{d}(x, \bar{x}) \le \sqrt{2} \cdot d(x, \bar{x}),$$

so we may treat V_x as a subgaussian process with the quasimetric d. It can be easily shown that d satisfies a generalized triangle inequality

$$d(x, \bar{x}) \le 4 \cdot (d(x, z) + d(z, \bar{x})) \tag{3.2}$$

for all $x, \bar{x}, z \in B_2^n$.

Denote by $\mathcal{B}_\rho(x)$ a ball in the quasimetric d with center x and radius ρ. Then for any $x \in B_2^n$ and $\rho > 0$ we have

$$\operatorname{conv} \mathcal{B}_\rho(x) \subset \mathcal{B}_{4\rho}(x).$$

The proof of this fact is the same as that of Lemma 3 [R3], so we shall omit it. To estimate the expression (3.1), we apply the following version of the Majorizing measure theorem.

Theorem. *Let (T,d) be a quasimetric space. Let $(X_t)_{t\in T}$ be a collection of mean 0 random variables with the subgaussian tail estimate*

$$\mathcal{P}\left\{|X_t - X_{\bar{t}}| > a\right\} \leq \exp\left(-c\frac{a^2}{d^2(t,\bar{t})}\right),$$

for all $a>0$. Let $r>1$ and let k_0 be a natural number so that the diameter of T is less than r^{-k_0}. Let $\{\varphi_k\}_{k=k_0}^{\infty}$ be a sequence of functions from T to $\mathbb{R}^+$, uniformly bounded by a constant depending only on r. Assume that there exists $\sigma>0$ so that for any k the functions φ_k satisfy the following condition:

for any $s\in T$ and for any points $t_1,\ldots,t_N \in B_{r^{-k}}(s)$ with mutual distances at least r^{-k-1} one has

$$\max_{j=1,\ldots,N} \varphi_{k+2}(t_j) \geq \varphi_k(s) + \sigma\cdot r^{-k}\cdot\sqrt{\log N}. \tag{3.3}$$

Then

$$\mathbb{E}\sup_{t\in T} X_t \leq C(r)\cdot\sigma^{-1}.$$

This Theorem is a combination of the majorizing measure theorem of Fernique [L-T] and the general majorizing measure construction of Talagrand (Theorems 2.1 and 2.2 [T1] or Theorems 4.2, 4.3 and Proposition 4.4 [T2]).

Let

$$Q = \max_{i=1,\ldots,M}\|y_i\|.$$

Let r be a natural number and let k_0 and k_1 be the largest numbers so that

$$r^{-k_0} \geq Q$$
$$r^{-k_1} \geq \frac{Q}{\sqrt{n}}.$$

Then $k_1 - k_0 \leq (2\log r)^{-1}\log n$. Define now the functions $\varphi_k : B_2^n \to \mathbb{R}$ by

$$\varphi_k(x) = \min\{\|u\|^2 \;\Big|\; u\in \mathrm{conv}\mathcal{B}_{8r^{-k}}(x)\} + \frac{k-k_0}{\log M}, \qquad \text{if } k=k_0,\ldots,k_1,$$

$$\varphi_k(w) = 1 + \frac{1}{2\log r} + \sum_{l=k_1}^{k} r^{-l}\cdot\frac{\sqrt{n\cdot\log(1+4Qr^l)}}{Q\cdot\sqrt{\log M}}, \qquad \text{if } k>k_1.$$

The functions φ_k form a nonnegative nondecreasing sequence bounded by a constant depending only on r. Indeed, for $k\leq k_1$,

$$\varphi_k(x) \leq 1 + \frac{1}{2\log r}\cdot\frac{\log n}{\log M}.$$

For $k>k_1$ we have

$$\varphi_k(w) \leq 1 + \frac{1}{2\log r} + \sum_{l=k_1}^{\infty} r^{-l}\cdot\frac{\sqrt{n\cdot\log(1+4Qr^l)}}{Q\sqrt{\log M}} \leq$$

$$1 + \frac{1}{2\log r} + c(r)\cdot r^{-k_1}\cdot\sqrt{n}\cdot\frac{\sqrt{\log(1+4Qr^{k_1})}}{Q\sqrt{\log M}} \leq C(r).$$

Now let $x_1, \dots, x_N \in \mathcal{B}_{r^{-k}}(x)$ and suppose that

$$d(x_i, x_j) > r^{-k-1}$$

for any $i \neq j$. We have to prove that

$$\max_{j=1,\dots,N} \varphi_{k+2}(x_i) \geq \varphi_k(x) + \frac{C}{Q\sqrt{\log M}} \cdot \sqrt{\log N}. \tag{3.4}$$

Note that for $x, \bar{x} \in B_2^n$

$$\begin{aligned} d(x,\bar{x}) \leq & \sqrt{2} \cdot \max_{i=1,\dots,M} |\langle x - \bar{x}, y_i\rangle| \cdot \left(\sum_{i=1}^M \langle x, y_i\rangle^2 + \langle \bar{x}, y_i\rangle^2 \right)^{1/2} \leq \\ & \sqrt{2} \cdot \max_{i=1,\dots,M} |\langle x - \bar{x}, y_i\rangle| \cdot \left\| \sum_{i=1}^M y_i \otimes y_i \right\| \cdot \left(\|x\|^2 + \|\bar{x}\|^2 \right)^{1/2} \leq \\ & 2 \cdot \max_{i=1,\dots,M} |\langle x - \bar{x}, y_i\rangle|. \end{aligned}$$

Define a norm in $\mathbb{R}^n$ by

$$\|x\|_Y = \max_{i=1,\dots,M} |\langle x, y_i\rangle|.$$

If $k \geq k_1 - 2$ then (3.4) follows from a simple entropy estimate. Indeed, we have

$$\begin{aligned} N \leq & N(B_2^n, d, r^{-k-1}) \leq N(B_2^n, \|\cdot\|_Y, \frac{1}{2} r^{-k-1}) \leq \\ & N(B_2^n, \|\cdot\|, \frac{1}{2Q} r^{-k-1}) \leq (1 + 4Q \cdot r^{k+1})^n, \end{aligned}$$

since $\|\cdot\|_Y \leq Q \|\cdot\|$. Suppose now that $k < k_1 - 2$. For $j = 1, \dots, N$ denote z_j the point of $\mathrm{conv}\mathcal{B}_{8r^{-k-2}}(x_j)$ for which the minimum of $\|z\|^2$ is attained and denote u the similar point of $\mathcal{B}_{8r^{-k}}(x)$. Put

$$\theta = \max_{j=1,\dots,N} \|z_j\|^2 - \|u\|^2.$$

We have to show that

$$r^{-k} \cdot \left(c \cdot Q \cdot \sqrt{\log M} \right)^{-1} \cdot \sqrt{\log N} \leq \max_{j=1,\dots,N} \varphi_{k+2}(x_j) - \varphi_k(x) = \theta + \frac{2}{\log M}. \tag{3.5}$$

Since $d(x_i, x_j) \geq r^{-k-1}$, it follows from (3.2) that

$$d(z_i, z_j) \geq \frac{1}{2} r^{-k-1},$$

provided r is sufficiently large. From the other side,

$$d(x, z_j) \leq 4\Big(d(x, x_j) + d(x_j, z_j) \Big) \leq 8r^{-k}.$$

Since $\frac{z_j+u}{2} \in \mathrm{conv}\mathcal{B}_{8r^{-k}}(x)$, and $\|u\| \le \|z_j\|$, we have

$$\left\|\frac{z_j-u}{2}\right\|^2 = \frac{1}{2}\|z_j\|^2 + \frac{1}{2}\|u\|^2 - \left\|\frac{z_j+u}{2}\right\|^2 \le \|z_j\|^2 - \left\|\frac{z_j+u}{2}\right\|^2 \le \|z_j\|^2 - \|u\|^2,$$

so,

$$\|z_j - u\| \le 2\sqrt{\theta}. \tag{3.6}$$

Thus, N is bounded by the $\frac{1}{2}r^{-k-1}$-entropy of the set $K = u + 2\sqrt{\theta}B_2^n$ in the quasimetric d. To estimate this entropy we partition the set K into S disjoint subsets having diameter less than $\delta = \frac{1}{16}r^{-k-1}\theta^{-1/2}$ in the $\|\cdot\|_Y$ metric.

Let g be a Gaussian vector in $\mathbb{R}^n$, normalized by $\mathbb{E}\|g\|^2 = n$. Denote by $N(B,\Delta,\varepsilon)$ ε-entropy of the set B in the metric Δ. By dual Sudakov minoration [L-T] we have

$$\sqrt{\log S} \le \sqrt{\log N(2\sqrt{\theta}B_2^n, \|\cdot\|_Y, \delta)} \le$$
$$\frac{c}{\delta}\cdot 2\sqrt{\theta}\cdot \mathbb{E}\|g\|_Y \le C\cdot r^k\theta\cdot \mathbb{E}\max_{i=1,\dots,M}|\langle g, y_i\rangle| \le C\cdot r^k\theta\cdot Q\cdot\sqrt{\log M}. \tag{3.7}$$

If $S \ge \sqrt{N}$, we are done, because in this case (3.7) implies (3.5). Suppose that $S \le \sqrt{N}$. Then there exists an element of the partition containing at least $\sqrt{N}$ points z_j. Let $J \subset \{1,\dots,N\}$ be the set of the indices of these points. We have

$$\|z_j - z_l\|_Y \le \frac{1}{16}r^{-k-1}\cdot\theta^{-1/2} \tag{3.8}$$

for all $j,l \in J$.

For $j = 1,\dots,M$ denote

$$I_j = \{i \in \{1,\dots,M\} \mid |\langle z_j, y_i\rangle \ge 2|\langle u, y_i\rangle|\}.$$

Then (3.6) imlies that

$$\sum_{i\in I_j}\langle z_j, y_i\rangle^2 \le 2\sum_{i\in I_j}\langle z_j-u, y_i\rangle^2 + 2\sum_{i\in I_j}\langle u, y_i\rangle^2 \le 8\theta + \frac{1}{2}\sum_{i\in I_j}\langle z_j, y_i\rangle^2,$$

so,

$$\sum_{i\in I_j}\langle z_j, y_i\rangle^2 \le 16\theta. \tag{3.9}$$

Since $d(z_j, z_l) \ge \frac{1}{2}r^{-k-1}$, we have

$$\left(\frac{1}{2}r^{-k-1}\right)^2 \le \sum_{i=1}^M \langle z_j - z_l, y_i\rangle^2\cdot\left(\langle z_j, y_i\rangle^2 + \langle z_l, y_i\rangle^2\right) \le$$
$$\sum_{i=1}^M \langle z_j - z_l, y_i\rangle^2\cdot 4\langle u, y_i\rangle^2 +$$
$$\max_{i\in I_j}\langle z_j - z_l, y_i\rangle^2\cdot\sum_{i\in I_j}\langle z_j, y_i\rangle^2 + \max_{i\in I_l}\langle z_j - z_l, y_i\rangle^2\cdot\sum_{i\in I_l}\langle z_l, y_i\rangle^2.$$

Combining (3.8) and (3.9) we get that the last expression is bounded by

$$2 \cdot 16\theta \cdot \left(\frac{\theta^{-1/2}}{8} r^{-k-1} \right)^2 + 4 \sum_{i=1}^{M} \langle z_j - z_l, y_i \rangle^2 \cdot \langle u, y_i \rangle^2.$$

Define a norm $\|\cdot\|_{\mathcal{E}}$ by

$$\|x\|_{\mathcal{E}} = \left(\sum_{i=1}^{M} \langle x, y_i \rangle^2 \cdot \langle u, y_i \rangle^2 \right)^{1/2}.$$

Then, for all $j, l \in J,\ j \neq l$ we have

$$\|z_j - z_l\|_{\mathcal{E}} \geq \frac{1}{8} r^{-k-1}.$$

Applying again dual Sudakov minoration, we obtain

$$\begin{aligned} \sqrt{\log |J|} \leq & \sqrt{\log N(2\sqrt{\theta} B_2^n, \|\cdot\|_{\mathcal{E}}, \frac{1}{8} r^{-k-1})} \leq c r^k \cdot 2\sqrt{\theta} \cdot \mathbb{E} \, \|g\|_{\mathcal{E}} \leq \\ & c r^k \cdot 2\sqrt{\theta} \cdot \left(\mathbb{E} \sum_{i=1}^{M} \langle g, y_i \rangle^2 \cdot \langle u, y_i \rangle^2 \right)^{1/2} \leq \\ & C r^k \cdot 2\sqrt{\theta} \cdot \max_{i=1,\dots,M} \|y_i\| \cdot \left\| \sum_{i=1}^{M} y_i \otimes y_i \right\| \cdot \|u\| \leq C r^k \cdot 2\sqrt{\theta} \cdot Q. \end{aligned}$$

Since for all $\theta > 0$

$$2\sqrt{\theta} \leq \sqrt{\log M} \cdot \theta + \frac{1}{\sqrt{\log M}},$$

we get

$$\sqrt{\log N} \leq 2\sqrt{\log |J|} \leq C \cdot Q \cdot r^k \cdot \sqrt{\log M} \cdot \left(\theta + \frac{1}{\log M} \right),$$

so (3.5) is satisfied.

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