ADJUNCTION INEQUALITY FOR REAL ALGEBRAIC CURVES

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ABSTRACT. The zero set of a real polynomial in two variable is a curve in $\mathbb{R}^2$. For a generic choice of its coefficients this is a non-singular curve, a collection of circles and lines properly embedded in $\mathbb{R}^2$. What topological arrangements of these circles and lines appear for the polynomials of a given degree? This question arised in the 19th century in the works of Harnack and Hilbert and was included by Hilbert into his 16th problem. Several partial results were obtained since then (see [5], [1], [7], [8]). However the complete answer is known only for polynomials of degree 5 (see [6]) or less. The paper presents a new partial result toward the solution of the 16th Hilbert problem.

A consequence of the main theorem of the paper implies that the zero set of a polynomial of degree 7 can not be isotopic to a curve pictured on Fig.1 (contrary to the curve pictured on Fig.2). The realizibility of Fig.1 does not contradict to the previously known restrictions. The proof of the main theorem makes use of the proof by Kronheimer and Mrowka [3] of the Thom conjecture in $\mathbb{C}P^2$.

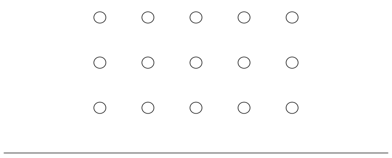

FIGURE 1. This picture is not isotopic to a curve of degree 7.

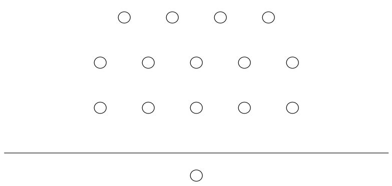

FIGURE 2. This picture is isotopic to a curve of degree 7.

1. STATEMENT OF THE THEOREM

Let f and g be two real homogeneous polynomials in two variables of degrees m and n, $m \geq n$, $m \equiv n \pmod 2$. Let

$$\mathbb{R}A = \{(x_0 : x_1 : x_2) \in \mathbb{R}P^2 \mid f(x_0, x_1, x_2) = 0\}$$
$$\mathbb{R}B = \{(x_0 : x_1 : x_2) \in \mathbb{R}P^2 \mid g(x_0, x_1, x_2) = 0\}$$

Research at MSRI supported in part by NSF grant #DMS 9022140.

be the corresponding curves in $\mathbb{R}P^2$. Let

$$\mathbb{C}A = \{(x_0 : x_1 : x_2) \in \mathbb{C}P^2 \mid f(x_0, x_1, x_2) = 0\}$$
$$\mathbb{C}B = \{(x_0 : x_1 : x_2) \in \mathbb{C}P^2 \mid g(x_0, x_1, x_2) = 0\}$$

be their complexifications in $\mathbb{C}P^2$. If the coefficients of f and g are generic then $\mathbb{C}A$ and $\mathbb{C}B$ are transverse smooth submanifolds of $\mathbb{C}P^2$. We call such curves *non-singular*. The real parts $\mathbb{R}A$ and $\mathbb{R}B$ of non-singular curves are transverse smooth submanifolds of $\mathbb{R}P^2$.

The complex conjugation involution $\operatorname{conj} : (x_0 : x_1 : x_2) \to (\bar{x}_0 : \bar{x}_1 : \bar{x}_2)$ acts on $\mathbb{C}P^2$, $\mathbb{C}A$ and $\mathbb{C}B$ fixing $\mathbb{R}P^2$, $\mathbb{R}A$ and $\mathbb{R}B$. Note that $\mathbb{C}A - \mathbb{R}A$ is either conneceted (then $\mathbb{R}A$ is called *of type II*) or consists of two components interchanged by conj (then $\mathbb{R}A$ is called *of type I*) since $\mathbb{C}A/\operatorname{conj}$ is connected and $\partial \mathbb{C}A/\operatorname{conj} = \mathbb{R}A$. If $\mathbb{R}A$ is of type I then following [7] we equip $\mathbb{R}A$ with *complex orientation*, i.e. with the boundary orientation induced from one of the components of $\mathbb{C}A - \mathbb{R}A$. Because of ambiguity in the choice of the component of $\mathbb{C}A - \mathbb{R}A$ this orientation is defined up to simultaneous reversion of the orientations of all the components of $\mathbb{R}A$.

Let $C \subset \mathbb{R}P^2$ be an oriented null-homologous topologically embedded (multicomponent) curve. Following [8] we define a locally constant function called the *index*

$$|\operatorname{ind}| : \mathbb{R}P^2 - C \to \mathbb{Z}$$

in the following way. The index of the non-orientable component of $\mathbb{R}P^2 - C$ is zero. The complement of the non-orientable component is a disjoint union of disks. A regular neighborhood N of each of these disks is $\mathbb{R}^2$ and in N there is a well-defined index of a point with respect to $C \cap N$ once we pick an orientation of N. We define $\operatorname{ind}|_{N-C}$ as the absolute value of the index in N.

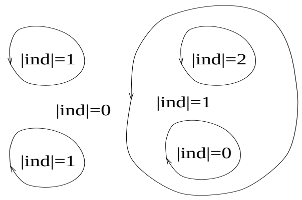

FIGURE 3. Function $|\operatorname{ind}|$.

For each locally constant function $F : \mathbb{R}P^2 - C \to \mathbb{R}$ we define

$$\int_{\mathbb{R}P^2} F d\chi = \sum_G F(G)\chi(G)$$

over all components G of $\mathbb{R}P^2 - C$ (χ denotes the Euler characteristic).

Let $\mathbb{R}A$ and $\mathbb{R}B$ be the curves of type I equipped with the complex orientations. We associate the following curve C to $\mathbb{R}A$ and $\mathbb{R}B$. Take $\mathbb{R}A \cup \mathbb{R}B$ and smooth all the points of intersection according to their orientations choosing any pattern of

Fig.4 (cf. smoothing in [4]). (Recall that $\mathbb{R}A$ and $\mathbb{R}B$ are transverse since we assume that coefficients of f and g are generic and C is homologous to zero since $m \equiv n \pmod 2$).

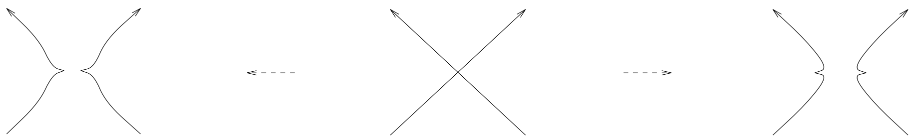

FIGURE 4. Smoothings of a double point.

Let G be a domain in $\mathbb{R}P^2$ such that ∂G is a component of C. Let c_{out} (c_{in}) be the number of the outward (inward) with respect to G cusps on the boundary of G. Let $\chi'(G) = \chi(G) - \frac{1}{2}(c_{\text{out}} - c_{\text{in}})$ (cf. Remark 1.3 of [4]). Define "the negative variation" of χ' by the following rules. For a connected G let $\chi'_-(G) = -\chi'(G)$ if $\chi'(G) < 0$ and $\chi'_-(G) = 0$ if $\chi'(G) \geq 0$. For a disconected G define $\chi'_-(G)$ as the sum of χ' over all components of G. Define

$$\Xi = \sum_{s=1}^{\infty} s\chi'_-\{|\operatorname{ind}(x)| > s\}.$$

Note that $\Xi = 0$ if $|\operatorname{ind}| < 2$ on all components of $\mathbb{R}P^2 - C$ of negative Euler characteristic. This will be the case for our applications. Let d be the number of the intersection points of $\mathbb{R}A$ and $\mathbb{R}B$.

Theorem 1.

$$\int\limits_{\mathbb{R}P^2} |\operatorname{ind}| d\chi \leq \frac{(m+n)^2}{4} - 3n + d + \Xi.$$

Note that there is ambiguity in choosing the complex orientations of $\mathbb{R}A$ and $\mathbb{R}B$ and in smoothing of double points of $\mathbb{R}A \cup \mathbb{R}B$. The inequality of the theorem holds for *any* choice of the complex orientation and smoothings.

Remark . If $\mathbb{R}A$ is a non-singular curve of type I of degree m then the curve obtained by taking s parallel copies of each component of $\mathbb{R}A$ is isotopic to a non-singular curve of type I of degree sm. Indeed, if f defines $\mathbb{R}A$ then

$$f(f - \epsilon h) \dots (f - (s-1)\epsilon h) = \delta p$$

defines the requred curve for a choice of real numbers ϵ, δ and polynomials h and p of degree m and sm.

This observation combined with the theorem yields additional restrictions on mutual position of two curves of type I. In particular, it allows to use the theorem in the case when the degrees of two curves are of opposite parity.

2. Proof of Theorem 1

Consider first the special case when $\mathbb{R}A \cap \mathbb{R}B = \emptyset$. Let A_+ be the component of $\mathbb{C}A - \mathbb{R}A$ which determines the chosen complex orientation of $\mathbb{R}A$ and let B_- be the component of $\mathbb{C}B - \mathbb{R}B$ which determines the orientation *opposite* to the chosen orientation of $\mathbb{R}B$. Let

$$F' = A_+ \cup B_- \cup L,$$

where $L = \bigcup_{s=1}^{\infty}\{|\operatorname{ind}| \geq s\}$. Note that F' makes an integer cycle such that the orientation induced from F' on A_+ is holomorphic and the orientation induced from F' on B_- is *anti*holomorphic.

Let F be the surface smoothly generically immersed into $\mathbb{C}P^2$ obtained by smoothing the corners and, then, perturbation of F'. The double points of F come from two sources, from $A_+ \cap B_-$ and from perturbation of $\bigcup_{s=1}^{\infty}\{|\operatorname{ind}| \geq s\}$. Note that the double points coming from A_+ are all negative since all points of intersection of $\mathbb{C}A$ and $\mathbb{C}B$ equipped with the holomorphic orientation are positive.

To obtain F out of F' we perturb every component $G' \subset \mathbb{R}P^2$ of $\{|\operatorname{ind}| \geq s\}$, $s > 1$, to G along a generic vector field v normal to G' but tangent to $\mathbb{C}A \cup \mathbb{C}B$ along $\partial G' \subset \mathbb{R}A \cup \mathbb{R}B$ (to keep F a cycle). The field iv is tangent to G' and to $\partial G'$. The zeroes of v and iv are of opposite index. Therefore (inductively by s) we can choose the perturbation field v so that G (equipped with the orientation induced from F) intersects the result of perturbation of $\{|\operatorname{ind}| \geq t\}$, $t < s$, in $|\chi(G')|$ points, which are negative if $\chi(G') > 0$ and positive if $\chi(G') < 0$. Therefore, the total number of positive double points of F is Ξ.

Note that

$$[F] = \frac{m-n}{2}[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2).$$

since $F \cup \operatorname{conj}(F)$ is homologous to $[\mathbb{C}A] - [\mathbb{C}B] = (m-n)[\mathbb{C}P^1]$ and conj acts as (-1) on $H_2(\mathbb{C}P^2)$. We need the following strengthening of the Thom conjecture [3]

$$-\chi(F) + 2\Xi \geq \frac{(m-n)^2}{4} - 3\frac{m-n}{2}. \tag{1}$$

This strengthening is implicitly contained in [3]. Indeed, it is proven there that if H is a surface smoothly embedded to $\mathbb{C}P^2 \# \bar{\mathbb{C}P}^2 \# \dots \# \bar{\mathbb{C}P}^2$ and $H.H \geq 0$ then $-\chi(H) \geq H.H + K.H$, where $K = (-3, 1, \dots, 1) \in H_2(\mathbb{C}P^2 \# \bar{\mathbb{C}P}^2 \# \dots \# \bar{\mathbb{C}P}^2)$ is the canonical class. We can get H out of F by blowing up all the double points of F. The intersection of F with a small sphere centered at a double point of F is the Hopf link (the difference between the positive and negative Hopf links disappears if we forget the orientation of the links). The resulting H intersects each exceptional divisor at two points, which are of the same sign (which is positive) for positive double points and of the opposite signs for negative double points. Therefore, $H.H = F.F - 4\Xi$ and $K.H = -3\frac{m-n}{2} + 2\Xi$ while $\chi(H) = \chi(F)$. This implies (1). Note that the assumption $m \geq n$ insures $H.[\omega] > 0$ and thus enables us to apply [3].

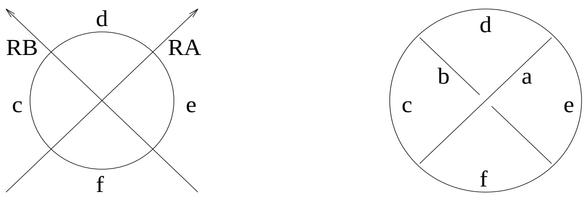

FIGURE 5. A double point in the real plane and its link in the complexification.

By additivity of χ

$$(2)\quad \chi(F)=\chi(A_+)+\chi(B_-)+\chi(L)= \\ \chi(A_+)+\chi(B_-)+\sum_{s=1}^{\infty}\chi(|\operatorname{ind}|\geq s)=\chi(A_+)+\chi(B_-)+\int\limits_{\mathbb{R}P^2}|\operatorname{ind}|d\chi.$$

But $\chi(A_+)=\frac{1}{2}\chi(\mathbb{C}A)=-\frac{m^2-3m}{2}$ and $\chi(B_-)=\frac{1}{2}\chi(\mathbb{C}B)=-\frac{n^2-3n}{2}$ by the adjunction formula for the holomorphic curves $\mathbb{C}A$ and $\mathbb{C}B$. Combining this with (1) and (2) we get

$$\int\limits_{\mathbb{R}P^2}|\operatorname{ind}|d\chi\leq\frac{m^2-3m}{2}+\frac{n^2-3n}{2}-\frac{(m-n)^2}{4}+3\frac{m-n}{2}+2\Xi=\frac{(m+n)^2}{4}-3n+2\Xi.$$

We return to the general case when $\mathbb{R}A$ and $\mathbb{R}B$ intersect in d points. We do the same thing still starting from $F'=A_+\cup B_-\cup\bigcup_{s=1}^{\infty}\{|\operatorname{ind}|\geq s\}$ but we have to pay special attention to smoothing of F' near $\mathbb{R}A\cap\mathbb{R}B$. The intersection of A_+, B_- and $\mathbb{R}P^2$ with a small sphere centered in a point of $\mathbb{R}A\cap\mathbb{R}B$ is pictured on Fig.5. It consists of 6 arcs, 2 of them, a and b come from the intersection with A_+ and B_- and 4 of them, c, d, e, f, come from the intersection with $\mathbb{R}P^2$. For perturbation of F' we need to choose such a framing on the circle $a\cup d\cup b\cup f$ that the result of perturbation of $a\cup d\cup b\cup f$ along this framing is not linked with $c\cup d\cup e\cup f$. The number of (full right) twists j of the framing along f has to be equal to the number of twists along d in order for these circles to be *homologically* unlinked. These circles are not linked if (and only if) $j=0$ or 1 (see Fig.6). This corresponds to the framing obtained by multiplication by i of the vector field tangent to the result of smoothing with two inwards or outwards cusps (cf. Fig.4). Gluing the disk bounded by $a\cup d\cup b\cup f$ along

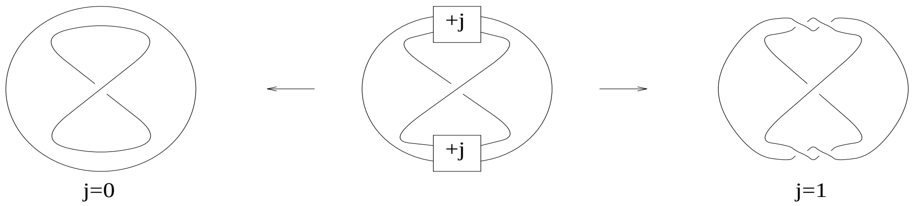

FIGURE 6. Choosing the framing along d and f.

the arcs f and d decreases the Euler characteristic of F by 1, so the number of the points of $\mathbb{R}A \cap \mathbb{R}B$ appears in the final formula.

3. Applications

By the Harnack inequality [2] an affine curve of degree $2k+1$ has no more then $k(2k-1)$ ovals.

Proposition 1. *If an affine curve of degree $2k+1$ consists of an arc and $k(2k-1)$ ovals with disjoint interiors then each component of the complement of the arc in the affine plane contains at least $\frac{(k-1)(k-2)}{2}$ ovals.*

Remark . This proposition is sharp in the following sense. For any $k > 0$ there exist affine curves of degree $(2k+1)$ which consist of $k(2k-1)$ ovals and an arc where the arc separates $\frac{(k-1)(k-2)}{2}$ ovals from $\frac{(3k-2)(k+1)}{2}$ ovals. Construction of such curves can be easily extracted from Harnack's construction of curves with $(k(2k-1)$ ovals [2].

Proof. We apply Theorem 1 to the union of the affine curve and the infinite line in $\mathbb{R}P^2$. The projectivization of the affine curve is of type I because the curve is extremal for the Harnack inequality (see [7]). Suppose the arc separates A ovals from $k(2k-1) - A$ ovals. Then for a proper choice of the orientation of the infinite line and the affine curve

$$\int\limits_{\mathbb{R}P^2} |\operatorname{ind}| d\chi \geq k(2k-1) - A + 1 - A$$

(the right hand side is the Euler characteristic of the region with $|\operatorname{ind}| = 1$, the only possible regions with $|\operatorname{ind}| > 1$ are the interiors of some of the A ovals (this depends on their orientation), $d = 1$ and $\Xi = 0$. Theorem 1 implies now that $1 + k(2k-1) - 2A \leq (k+1)^2 - 3 + 1$ and, therefore $A \geq \frac{(k-1)(k-2)}{2}$. $\square$

Corollary 1. *The curve pictured on Fig.1 is not isotopic to an affine algebraic curve of degree 7.*

Remark . Degree 7 is the smallest degree when such a restriction appears. For an affine curve of degree 5 which consists of an arc and some ovals there are no constraints on distribution of the ovals between the half-planes (as long as their total number is no more than 6).

For an even degree $2k$ Harnack's construction (for the maximal number of ovals) produces a curve $\mathbb{R}P^2$ which consists of a non-empty oval, $\frac{k^2-3k+2}{2}$ empty ovals inside the non-empty ovals and $\frac{3k^2-3k}{2}$ empty ovals outside of the non-empty oval (see Fig.8 for $k = 4$). Let $\mathbb{R}A \subset \mathbb{R}P^2$ be a curve of degree $2k$ with such an arrangement of the ovals.

Proposition 2. *Not more than* $(k-2)$ *out of the* $\frac{k^2-3k+2}{2}$ *interior ovals of* $\mathbb{R}A$ *may be contained inside an ellipse (a non-singular curve of degree 2) in* $\mathbb{R}P^2$ *not intersecting* $\mathbb{R}A$.

Remark . This proposition is also sharp. For the Harnack curves of degree $2k$ we may find ellipses enclosing $(k-2)$ of the interior ovals and not intersecting the curves.

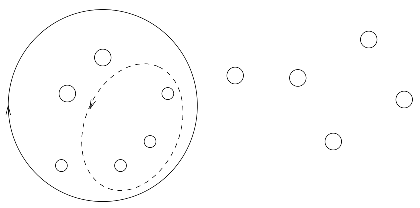

FIGURE 7. Orientations of the ellipse and the non-empty oval of $\mathbb{R}A$.

Proof. Let $L>0$ be the number of the interior ovals inside the ellipse. Apply Theorem 1 to $\mathbb{R}A$ and the ellipse oriented so that the non-empty oval of $\mathbb{R}A$ and the ellipse are oriented in the opposite way (see Fig.7). We get

$$\int\limits_{\mathbb{R}P^2} |\operatorname{ind}| d\chi \geq \frac{3k^2-3k}{2} - \frac{k^2-3k+2}{2} + 2L = k^2 - 1 + 2L,$$

$d=0$ and $\Xi=0$. Therefore, $k^2-1+2L \leq (k+1)^2-6$ and $L \leq k-2$. □

Corollary 2. *It is not possible to separate the 3 interior ovals of a curve of degree 8 with topological arrangement* $< 18 \sqcup 1 < 3 >>$ *from the rest of the ovals by a conic.*

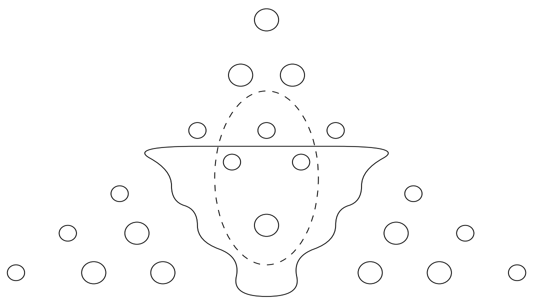

FIGURE 8. The Harnack curve of degree 8 and a conic.

Remark . Theorem 1 generalizes for curves on algebraic surfaces other than $\mathbb{R}P^2$. Instead of $m \equiv n \pmod 2$ we require that the union of the two curves is null-homologous in the surface (so that $|\operatorname{ind}|$ is well-defined). The formula then turns to

$$\int\limits_{\mathbb{R}P^2} |\operatorname{ind}| d\chi \leq \frac{([\mathbb{C}A]+[\mathbb{C}B])^2}{4} + K.[\mathbb{C}B] + d + \Xi,$$

where K is the canonical class of the complexification of the surface.

If the geometric genus of the surface is positive then we do not need to require that the degree of $\mathbb{C}A$ is greater than or equal to the degree of $\mathbb{C}B$.

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