ON THE HOMOLOGY COBORDISM GROUP OF HOMOLOGY 3-SPHERES

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ABSTRACT. In this paper we present our results on the homology cobordism group $\Theta^3_{\mathbb{Z}}$ of the oriented integral homology 3-spheres. We specially emphasize the role played in the subject by the gauge theory including Floer homology and invariants by Donaldson and Seiberg – Witten.

A closed oriented 3-manifold Σ is said to be an integral homology sphere if it has the same integral homology as the 3-sphere S^3. Two homology spheres Σ_0 and Σ_1 are homology cobordant, if there is a smooth compact oriented 4-manifold W with $\partial W = \Sigma_0 \cup -\Sigma_1$ such that $H_*(W, \Sigma_0; \mathbb{Z}) = H_*(W, \Sigma_1; \mathbb{Z}) = 0$. The set of all homology cobordism classes forms an abelian group $\Theta^3_{\mathbb{Z}}$ with the group operation defined by a connected sum. Here, the zero element is homology cobordism class of 3-sphere S^3, and the additive inverse is obtained by a reverse of orientation.

The Rochlin invariant μ is an epimorphism $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}_2$, defined by the formula

$$\mu(\Sigma) = \frac{1}{8}\ \mathrm{sign}(W) \mod 2,$$

where W is any smooth simply connected parallelizable compact manifold with $\partial W = \Sigma$. This invariant is well-defined due to well-known Rochlin theorem [R] which states that the signature $\mathrm{sign}(V)$ of any smooth simply connected *closed* parallelizable manifold V is divisible by 16.

We will focus our attention on the following problem concerning the group $\Theta^3_{\mathbb{Z}}$ and the homomorphism μ : does there exist an element of order two in $\Theta^3_{\mathbb{Z}}$ with non-trivial Rochlin invariant ? This is one of R. Kirby problems [K], Problem 4.4, and positive answer to this problem would imply [GS], in particular, that all closed topological n-manifolds are simplicially triangulable if $n \geq 5$ (this is not true in dimension 4 where a counterexample is due to A. Casson and M. Freedman, see e.g. [AK]).

Our approach to this problem is trying to lift the Rochlin homomorphism to integers. If we succeded in doing this, then all elements of finite order would have to lie in the kernel of μ, and the Kirby problem would have a negative solution. In fact, our goal is less ambitious – we define a lift of μ on a certain subgroup of $\Theta^3_{\mathbb{Z}}$. This implies that there are no solutions to the problem inside this subgroup. This is done in Section 1 with help of the $\bar{\mu}$–invariant introduced by W. Neumann and L. Siebenmann in 1978, see [N] and [Sb], for the so called plumbed homology spheres.

Section 2 is devoted to an interesting observation about Floer homology of homology 3-spheres homology cobordant to zero in the group $\Theta^3_{\mathbb{Z}}$. We manage to prove a two-periodicity in Floer homology groups of such homology spheres in a series of examples. This leads us to defining a new integral invariant ν which vanishes for all these examples. Unfortunately, Floer homology is too complicated an object to deal with, so one still cannot say much more about the homology cobordism invariance of ν in general.

In Section 3 we compare the invariants $\bar{\mu}$ and ν on the class of plumbed homology 3-spheres and discuss possible links to branched coverings and Jones polynomial.

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1. Homology cobordisms of plumbed homology 3–spheres

We first recall the definition of the invariant $\bar{\mu}$ by W. Neumann [N]. Note that our definition differs from the original one by factor 1/8.

Let Γ be a plumbing graph, that is a (not necessarily connected) graph with no cycles, each of whose vertices carries an integer weight e_i , $i = 1, \dots , s$. The matrix $A(\Gamma) = (a_{ij})_{i,j=1,\dots,s}$ with the entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge} \\ 0, & \text{otherwise} \end{cases}$$

is the intersection matrix for 4-dimensional manifold $P(\Gamma)$, obtained by plumbing D^2-bundles over 2-spheres according to Γ. This manifold is simply connected.

If Γ is a plumbing graph as above, then $M(\Gamma) = \partial P(\Gamma)$ is an integral homology sphere if and only if $\det A(\Gamma) = \pm 1$. For example, all Seifert fibered homology spheres $\Sigma(a_1, \dots , a_n)$ are of the form $\partial P(\Gamma)$ where Γ is a star-shaped graph, see [NR].

If $M(\Gamma)$ is a homology sphere, there exists a unique homology class $w \in H_2(P(\Gamma);\mathbb{Z})$ satisfying the following two conditions. First, w is *characteristic*, that is (dot represents intersection number)

$$w.x \equiv x.x \mod 2 \quad \text{for all } \; x \in H_2(P(\Gamma);\mathbb{Z}), \tag{1}$$

and second, all coordinates of w are either 0 or 1 in the natural basis of $H_2(P(\Gamma);\mathbb{Z})$. We call w *the integral Wu class* for $P(\Gamma)$. Due to W. Neumann [N], the integer $\mathrm{sign} P(\Gamma) - w.w$ only depends on $M(\Gamma)$ and not on Γ. This integer is divisible by 8, see [Sr], so one can define the Neumann-Siebenmann invariant by the formula

$$\bar{\mu}(M(\Gamma)) = \frac{1}{8}(\text{ sign } P(\Gamma) - w.w). \tag{2}$$

The class w is spherical, so its modulo 2 reduction is the usual Rochlin invariant μ, see [NR].

W. Neumann conjectured in [N] that the number $\bar{\mu}$ is in fact a homology cobordism invariant. This conjecture has been checked for all known examples of plumbed homology spheres homology cobordant to zero, including ones from the lists of A. Casson and J. Harer [CH] and R. Stern [St]. The list in [CH] includes Seifert fibered homology spheres $\Sigma(p, ps\pm 1, ps\pm 2)$, $p \geq 3$ odd, $s \geq 1$, and $\Sigma(p, ps-1, ps+1)$, $p \geq 2$ even, $s \geq 1$ odd. All Stern's examples are of the form $\Sigma(a_1, a_2, 2a_1a_2 \pm a_3)$ for certain $\Sigma(a_1, a_2, a_3)$ in the Casson-Harer list. Now, gauge theory enables us to prove more general results which weigh for positive answer to the conjecture.

Algebraic links. Our first result is concerned with the links of algebraic singularities. It was shown in [EN] that a plumbed homology sphere Σ is the link of an algebraic singularity if and only if there exists a plumbing graph Γ such that the manifold $P(\Gamma)$ with $\Sigma = \partial P(\Gamma)$ is negative definite. The simplest case is the Seifert fibered case: any Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ is the link of the singularity of $f^{-1}(0)$ where $f : \mathbb{C}^n \to \mathbb{C}^{n-2}$ is a map of the form

$$f(z_1, \dots, z_n) = \left(\sum_{k=1}^{n} b_{1,k} z_k^{a_k}, \dots, \sum_{k=1}^{n} b_{n-2,k} z_k^{a_k} \right)$$

with sufficiently general coefficient matrix $(b_{i,j})$, see [NR]. For instance,

$$\Sigma(p, q, r) = \{ z \in \mathbb{C}^3 \,|\, \|z\| = \varepsilon \text{ and } z_1^p + z_2^q + z_3^r = 0 \}$$

for $\varepsilon > 0$ small enough.

The next simplest case is the following: if p, q, r are pairwise relatively prime integers, as are p', q', r', then the homology sphere Σ obtained by splicing $\Sigma(p, q, r)$ and $\Sigma(p', q', r')$ along the singular fibers of degrees r and r', is the link of singularity if and only if $rr' > pp'qq'$, see [NW], § 4.

Theorem 1. *Let a homology sphere Σ be the link of an algebraic singularity. If Σ is homology cobordant to zero then $\bar{\mu}(\Sigma) \geq 0$.*

Proof. Since Σ is an algebraic link, one may assume that Σ is the boundary of a plumbed negative definite 4-manifold $P(\Gamma)$. Suppose that Σ bounds a smooth homology ball M. Let us consider the manifold $W = P(\Gamma) \cup_{\Sigma} (-M)$. This is a smooth closed oriented manifold whose intersection form is naturally isomorphic to the intersection form of $P(\Gamma)$, in particular, is negative definite. By S. Donaldson's Theorem 1 from [D], this form is diagonalizable over integers.

We use this fact to evaluate $\bar{\mu}(\Sigma)$. Since $\operatorname{sign} P(\Gamma) = -s$ where s is the number of vertices in the graph Γ, we only need to find the Wu class w. In the standard basis associated with plumbing, the matrix A of the intersection form of $P(\Gamma)$ takes the form $A = U^t(-E)U$ where $U \in \mathrm{SL}_s(\mathbb{Z})$ and E is the identity matrix. The defining relation (1) translates to

$$w^t U^t(-E)Ux \equiv x^t U^t(-E)Ux \mod 2 \quad \text{for all } \ x \in H_2(P(\Gamma);\mathbb{Z}),$$

or, equivalently,

$$(Uw)^t(-E)y \equiv y^t(-E)y \mod 2 \quad \text{for all } \ y \in H_2(P(\Gamma);\mathbb{Z}).$$

Therefore, Uw is characteristic for $-E$, in particular, all coordinates of Uw are odd. Now, we have

$$w.w = w^t U^t(-E)Uw = -(Uw)^t(Uw)$$

which is equal to the negative of the standard Euclidean square of vector Uw. Since all coordinates of Uw are odd, $w.w \le -s$, and therefore $\bar{\mu}(\Sigma) \ge 0$. □

Corollary. *If plumbed Σ is an algebraic link and $\bar{\mu}(\Sigma) < 0$, then Σ has infinite order in the group $\Theta^3_{\mathbb{Z}}$.* □

Example. For any relatively prime integers $p, q > 0$, one can easily see that $\bar{\mu}(\,\Sigma(p, q, pq-1)) < 0$. Therefore, all these homology spheres are of infinite order in $\Theta^3_{\mathbb{Z}}$. In particular, the Poincaré homology sphere $\Sigma(2,3,5)$ is of infinite order.

Example. The $\bar{\mu}$-invariant works not only for Seifert spheres, as it is seen from the following example. Let Σ be a splice [EN] of $\Sigma(4,7,9)$ and $\Sigma(2,3,25)$ along singular fibers of degrees 9 and 25. This manifold is an algebraic link. By using the additivity of $\bar{\mu}$ proven in [S5], we find that

$$\bar{\mu}(\Sigma) = \bar{\mu}(\Sigma(4,7,9)) + \bar{\mu}(\Sigma(2,3,25)) = -2 + 0 < 0.$$

Therefore, Σ has infinite order in $\Theta^3_{\mathbb{Z}}$.

Many Seifert homology spheres $\Sigma(a_1, \dots, a_n)$ having infinite order in the group $\Theta^3_{\mathbb{Z}}$ by the corollary above, can also be detected by the Fintushel-Stern invariant

$$R(a_1, \dots, a_n) = \frac{2}{a} - 3 + n + \sum_{i=1}^{n} \frac{2}{a_i} \sum_{k=1}^{a_i - 1} \cot\left(\frac{\pi a k}{a_i^2}\right) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi k}{a_i}\right)$$

where $a = a_1 \cdot \ldots \cdot a_n$. A theorem in [FS2] says that if $\Sigma(a_1, \dots, a_n)$ bounds a homology ball then $R(a_1, \dots, a_n) < 0$. There are however Seifert spheres which are not homology cobordant to zero and which can be detected by $\bar{\mu}$ and not by R, and vice versa.

Example. Both Seifert spheres $\Sigma(2,11,19)$ and $\Sigma(3,5,7)$ are not homology cobordant to zero. As to the former one, this follows from the fact that $\bar{\mu}(\Sigma(2,11,19)) = -1$ is negative (though $R(2,11,19) = -1$); the latter one has $\bar{\mu}(\Sigma(3,5,7)) = 0$ and the result follows from $R(3,5,7) = 1$.

Let $P(\Gamma)$ be a resolution of singularity whose link Σ is a homology 3–sphere. Let K be the Poincaré dual of the canonical class of a complex structure on $P(\Gamma)$, so it is characteristic. If Σ is homology cobordant to zero then the same argument as in Theorem 1 shows that the number $\operatorname{sign} P - K.K$ should be non-negative (in fact, this is true for *any* characteristic class K, not necessarily canonical).

On the other hand, one can compute $\operatorname{sign} P - K.K$ easily in terms of the Milnor fiber M of the singularity. Due to [Mo],

$$(3\operatorname{sign} + 2e)(M) - K'.K' = (3\operatorname{sign} + 2e)(P) - K.K$$

where e stands for Euler characteristic, and K' is the canonical class of M. Since manifold P is negative definite, we get

$$\operatorname{sign} P - K.K = (3\operatorname{sign} + 2e)(M) - K'.K' - 2.$$

One can get more precise result for Seifert manifolds.

Theorem 2. *If Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ is homology cobordant to zero then*

$$\lambda(\Sigma(a_1, \dots, a_n)) \le \frac{1}{12} \prod_{i=1}^{n} (a_i - 1)$$

where λ is the Casson invariant.

Proof. In the special case of $\Sigma(a_1, \dots, a_n)$, Milnor fiber M is simply-connected and its canonical class K' vanishes, see [Dv]. In addition,

$$\operatorname{rk} H_2(M) = \prod_{i=1}^{n} (a_i - 1), \text{ and } \operatorname{sign} M = -8\lambda(\Sigma(a_1, \dots, a_n)).$$

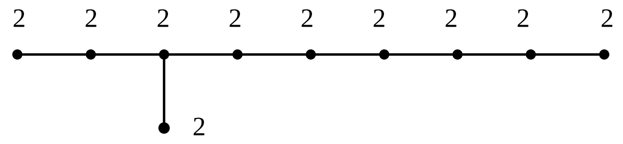
The first formula is due to J. Milnor [M], and the second one to Fintushel-Stern [FS1] and Neumann-Wahl [NW]. □

Example. For any relatively prime integers p, q and any positive integer k,

$$\lambda(\Sigma(p, q, 2pqk - 1)) = k(p^2 - 1)(q^2 - 1)/12.$$

Therefore, homology sphere $\Sigma(p, q, 2pqk - 1)$ is not homology cobordant to zero due to Theorem 2 though its $\bar{\mu}$–invariant vanishes. In fact, this could be also proven with help of the Fintushel-Stern R-invariant.

More constraints from even plumbing. Now, we have to deal with those plumbed homology spheres whose $\bar{\mu}$–invariant is positive, and therefore, they are not prohibited from bounding a homology ball by Theorem 1. A good example of this kind is $\Sigma(2, 3, 7)$ with $\bar{\mu} = 1$. This homology sphere bounds the following plumbing



Its intersection form is even and indefinite, hence is isomorphic over integers to the form $E_8 \oplus H$ where E_8 is the positive definite E_8–form, and H is the hyperbolic 2×2–matrix, see [Sr]. Suppose that a double of $\Sigma(2,3,7)$ is homology cobordant to zero. Then, by gluing the homology ball and double of the plumbing along $\Sigma(2,3,7)\#\Sigma(2,3,7)$, we get a closed smooth manifold with intersection form $2E_8 \oplus 2H$ which is prohibited by S. Donaldson's Theorem 2, see [D]. Therefore, $\Sigma(2,3,7)$ is not of the second order in $\Theta^3_{\mathbb{Z}}$.

The Seiberg-Witten gauge theory led M. Furuta [Fu], see also [A], to a proof of the 10/8-conjecture saying that if a simply-connected closed smooth 4-manifold has an even intersection form Q then

$$\frac{\operatorname{rank}(Q)}{|\operatorname{sign}(Q)|} > \frac{10}{8}.$$

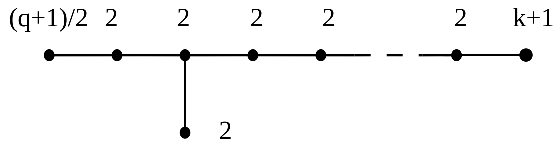
One can use this result to show that in fact none of $\Sigma(2,3,7)$ multiples can bound a contractible manifold. Because if any of them did, there would exist a closed smooth simply-connected manifold with rank/sign $= 10/8$.

One can use similar argument with even plumbings for many other Seifert fibered homology spheres with $\bar{\mu}$ positive. Unfortunately, the construction does not work for *all* homology spheres because the number of hyperbolics in even plumbing may be too large, and in order to get an even form with desirable quotient rank/sign one may need to surger some of them out (at the expense of losing the plumbing structure). In our paper [S2] we do this for Seifert fibered homology spheres $\Sigma(p,q,pqk \pm 1)$, $k > 0$ odd, which can be thought of as manifolds obtained by $(-1/k)$–surgery along a (p,q)–torus knot.

Theorem 3. *Let $p, q > 1$ be relatively prime integers, and $k > 0$ an odd integer. Then neither $\Sigma(p,q,pqk+1)$ nor any of its multiples can bound a smooth contractible 4–manifold.*

Note that $\bar{\mu}(\Sigma(p,q,pqk+1)) > 0$ for all p, q, k as in the theorem, and that a similar result for $\Sigma(p,q,pqk-1)$ followed from Theorem 1. Theorem 3 can not be generalized for $\Sigma(p,q,pqk+1)$ with k even; for instance, $\Sigma(2,3,13)$ is known to bound a contractible manifold [AK].

We refer for a complete proof of Theorem 3 to [S2], and only draw the final surgery diagrams in the simplest case of $p = 2$. If $q \equiv 3 \mod 4$, then $\Sigma(2,q,2qk+1)$ is the boundary of the following plumbing:



whose intersection form is isomorphic to $\frac{q+1}{4} \cdot E_8 \oplus H$. If $q \equiv 1 \mod 4$, some hyperbolics have to be surgered out. The homology sphere $\Sigma(2, q, 2qk+1)$ is then surgery on the link shown in Figure 1.

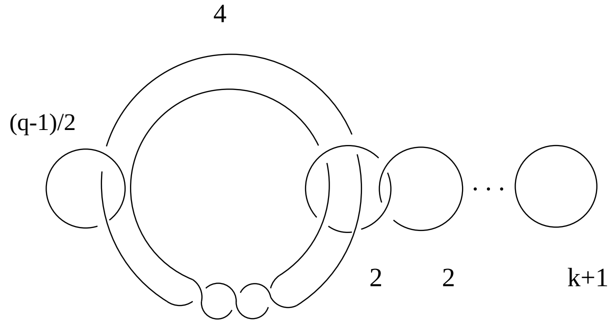


Figure 1

The corresponding intersection form is isomorphic to $\frac{q-1}{4} \cdot E_8 \oplus H$.

2. Floer homology and invariants of homology cobordism

Floer homology groups $I_n(\Sigma)$, $n = 0, \dots, 7$, are abelian groups, intrinsically associated to an oriented $\mathbb{Z}$-homology 3-sphere Σ. The definition of Floer homology in [F1] makes essential use of gauge theory on 3– and 4–dimensional manifolds, so they are generally difficult to compute.

We use the Floer exact triangle [F2] to compute $I_*(\Sigma)$ for several classes of integral homology spheres Σ unified by one property – they all are homology cobordant to zero. Among these homology spheres are: Mazur homology spheres [AK]; Seifert fibered homology spheres which bound smooth contractible manifolds [CH] and [St], homology spheres obtained by (± 1)-Dehn surgery on some slice knots. In short, the main results of our computations can be summarized as follows.

Theorem 4. *For all integral homology spheres Σ from the list above, Floer homology groups are 2–periodic, that is, $I_j(\Sigma) = I_{j+2}(\Sigma)$ for any j.*

Below we will give more specific description of the homology spheres in Theorem 4, but first we would like to use this theorem to justify introduction of a new invariant of oriented integral homology 3–spheres. For any such sphere Σ, we will define a number $\nu(\Sigma)$ keeping in mind the following conditions we would like it ideally to satisfy:

(1) ν is a $\mathbb{Z}$-homology cobordism invariant,

(2) ν is additive with respect to connected sums of $\mathbb{Z}$-homology spheres, and

(3) ν is a lifting of the Rochlin homomorphism μ , i.e. $\nu(\Sigma) \equiv \mu(\Sigma) \mod 2$.

Recall that according to C. Taubes [T] Casson λ–invariant equals one-half of the Euler characteristic of Floer homology,

$$\lambda(\Sigma) = \frac{1}{2} \sum_{n=0}^{7} (-1)^n \operatorname{rk}_{\mathbb{Q}} I_n(\Sigma). \tag{3}$$

We define our invariant ν in a manner similar to (3), i.e. as an algebraic sum of the ranks of Floer homology groups. In doing so, we observe that there is essentially only one choice of signs, besides that in (3), that makes the invariant sensitive to orientations (as it is required by conditions *(1)* and *(2)* above). We use this choice to define our invariant ν as follows:

$$\nu(\Sigma) = \frac{1}{2} \sum_{n=0}^{7} (-1)^{\frac{(n+1)(n+2)}{2}} \operatorname{rk}_{\mathbb{Q}} I_n(\Sigma). \tag{4}$$

This is an invariant of orientation preserving homeomorphism just because so is Floer homology. The most important problem, therefore, is to check its homology cobordism invariance, and Theorem 4 gives some evidence towards that.

Corollary. *The ν-invariant vanishes on all integral homology spheres homology cobordant to zero listed in Theorem 4.*

Concerning the conditions *(2)* and *(3)*, we mention that $\nu(\Sigma)$ is obviously equal to $\mu(\Sigma)$ modulo 2 for homology spheres whose Floer homology is 4–periodic. In general, this is an open problem. In [S3] we use Fukaya's spectral sequence in Floer homology [Fk] to show that $\nu(\Sigma)$ is additive with respect to connected sums under some restrictions on Floer homology of the factors.

Mazur homology spheres. By a Mazur homology sphere we mean an integral homology 3-spheres Σ obtained by the following construction. Let W be a compact contractible 4-manifold obtained by attaching a two-handle to $S^1 \times B^3$ along its boundary as in Figure 2.

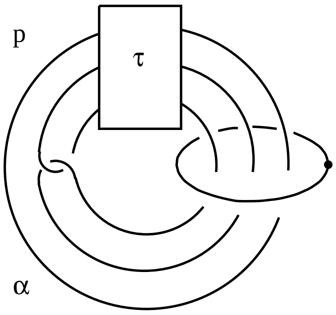


Figure 2

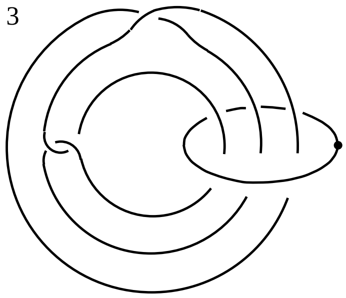


Figure 3

Here, the framing p is an integer number, and τ is an arbitrary tangle such that the component α of the link is connected. The manifold W is a straight generalization of the well known Mazur manifold [M] shown in Figure 3. The homology spheres Σ in question are the boundaries of such 4-manifolds. For example, the boundary of the manifold in Figure 3 is homeomorphic to Seifert fibered homology sphere $\Sigma(2,5,7)$, see [AK]. The following result is proven in [S3].

Theorem 5. *Let $\Sigma = \partial W$ be the boundary of a manifold W in Figure 2. Then, for any tangle τ and for any p, Floer homology groups $I_*(\Sigma)$ are two-periodic, that is $I_i(\Sigma) = I_{i+2}(\Sigma)$ for any i. Moreover, the groups $I_*(\Sigma)$ are independent of the framing p.*

This theorem implies in particular that the ν-invariant vanishes for all homology spheres as above. One can say more about $I_*(\Sigma)$ in the case when τ is a braid, see [S2].

Theorem 6. *If τ is a braid in Figure 1, then for any p, the group $I_i(\Sigma)$ is trivial if i is odd, and $I_i(\Sigma) = \mathbb{Z}$ if i is even, or vice versa, depending on the orientation.*

Casson-Harer and Stern homology spheres. Mazur manifolds are examples of contractible manifolds having handlebody decomposition with one 0-handle and one pair of 1- and 2-handles. In general, the geometric intersection of the 1- and 2-handles may be greater than or equal 3 (of course, the algebraic intersection still needs to be ± 1). The Casson-Harer homology spheres [CH] are of this sort. On the contrary, Stern homology spheres [St] bound contractible manifolds with more than one pair of 1- and 2-handles.

In [S3], we prove that Floer homology groups of all Seifert fibered homology spheres from [CH] and [St] known to bound contractible manifolds are 2–periodic. In our proof, we make use of Floer exact triangle applied to some special surgery on plumbed homology spheres that we introduce, and computations of Floer homology for Seifert manifolds in [FS1]. Here are some explicit computations.

Theorem 7. *Let p be any odd integer greater than 1, and $s \geq 1$ any integer. Then the Floer homology groups $I_j(\Sigma(p, ps \pm 1, ps \pm 2))$ are trivial for j odd, and isomorphic to a free abelian group of rank $s(p^2-1)(ps \pm 3)/48$ for any even j. In particular, $\nu(\Sigma(p, ps \pm 1, ps \pm 2)) = 0$.*

Slice knots. In [S3] we also compute Floer homology of homology spheres obtained by Dehn surgery on the knots $T(p,q)\#T(p,q)^*$ in S^3 where $T(p,q)$ is the left–handed (p,q)-torus knot, and $T(p,q)^*$ its mirror image. In the particular case of $(p,q) = (2, 2n+1)$, the Floer homology groups have been computed by A. Stipsicz and Z. Szabó in [SS] by different methods using the results of [KKR] on splitting spectral flow.

It is worth mentioning that for any relatively prime integers (p,q), the knot $T(p,q)\# T(p,q)^*$ is slice, and that both $(+1)$– and (-1)–surgeries on it bound contractible manifolds. As it follows from Theorem 8 below, the ν-invariants vanish in both cases.

Theorem 8. *Let $k = T(p,q)\#T(p,q)^*$ be a connected sum of the (p,q)-torus knot $T(p,q)$ with its mirror image $T(p,q)^*$. Let $M_n(p,q)$ be the homology sphere obtained by $(-1/n)$-surgery of S^3 along k, $n = 1, 2, \ldots$. Then $I_j(M_n(p,q))$ is trivial if j is odd, and isomorphic to a free abelian group of rank $n(p^2-1)(q^2-1)/24$ if j is even.*

In our proof, we first use Floer exact triangle to discover some symmetries in Floer homology of amphicheiral knots, and then apply a description of the representation space of the fundamental group of a splice of Seifert fibered homology spheres along singular fibers, see [S3].

3. Comparing the invariants

Our computations of both ν– and $\bar{\mu}$–invariants suggest that these invariants coincide for plumbed homology spheres. In comparing the two invariants, we use the well-known fact, see [EN] and [Sb], that any plumbed homology sphere can be obtained by pasting together graph links which are Seifert fibered homology spheres with a collection of Seifert fibers. This operation is called splicing [EN]. With respect to this operation, the $\bar{\mu}$-invariant is additive, see e.g. [S5]. In particular, if $\Sigma(a_1, \ldots, a_n)$ is a Seifert fibered homology sphere with Seifert invariants $a_1, \ldots, a_n$ and, for any $2 \leq j \leq n-2$, the integers $q = a_1 \cdots a_j$ and $p = a_{j+1} \cdots a_n$ are the products of the first j and the last $(n-j)$ Seifert invariants, respectively, then

$$\bar{\mu}(\Sigma(a_1, \ldots, a_n)) = \bar{\mu}(\Sigma(a_1, \ldots, a_j, p)) + \bar{\mu}(\Sigma(q, a_{j+1}, \ldots, a_n)),$$

$$\nu(\Sigma(a_1, \ldots, a_n)) = \nu(\Sigma(a_1, \ldots, a_j, p)) + \nu(\Sigma(q, a_{j+1}, \ldots, a_n)).$$

The formula for ν follows from [S1]. This implies that the problem of identifying $\bar\mu$ and ν reduces, at least on the class of Seifert fibered homology spheres, to the identification problem for Seifert spheres with just three singular fibers.

One can also compare the behaviour of the two invariants with respect to Dehn surgery on a singular fiber in a Seifert fibered homology sphere. Let $\Sigma(a_1,\dots,a_n)$ be a Seifert fibered homology sphere with Seifert invariants $a_1,\dots,a_n$, and $a = a_1\cdots a_{n-1}$ be the product of the first $(n-1)$ Seifert invariants. Then

$$\begin{aligned}\bar\mu(\Sigma(a_1,\dots,a_n)) &= \pm\,\bar\mu(\Sigma(a_1,\dots,a_{n-1},2a\pm a_n)).\\ \nu(\Sigma(a_1,\dots,a_n)) &= \pm\,\nu(\Sigma(a_1,\dots,a_{n-1},2a\pm a_n)),\end{aligned}$$

The first part of this result is proven in [N], the second in [S3]. As a corollary, both invariants $\bar\mu$ and ν vanish on infinite series of Seifert fibered homology spheres $\Sigma(a_1,\dots,a_{n-1},2ak\pm 1)$, $k\in\mathbb{Z}$, where $a = a_1\cdots a_{n-1}$.

Thus, in order to identify the two invariants for all Seifert fibered homology spheres, one only needs to do this for homology spheres of the form $\Sigma(p,q,r)$ for all pairwise relatively prime integers p,q,r such that $\max(p,q) < r < pq$. For example, all Casson-Harer homology spheres are of this sort. The invariants $\bar\mu$ and ν for these manifolds coincide (since they vanish). Our numerous attempts at proving this fact in general have led us to a few more special cases when it holds, as well as to some its reformulations. One of them seems to be worth stating.

It is well-known that a Seifert fibered homology sphere $\Sigma(a_1,\dots,a_n)$ is a doble cover over S^3 branched over Montesinos knot $K(a_1,\dots,a_n)$, see e.g. [BZ]. The $\bar\mu$–invariant can be defined in terms of the signature of this knot, as it was originally done by L. Siebenmann [Sb]. On the other hand, there exists a formula for Casson invariant of a double branched cover by D. Mullins [Mu]. These two facts together with formulae (3) and (4) enable us to reformulate our conjecture about the coincidence of $\bar\mu$ and ν for Seifert fibered homology spheres in the following equivalent form

$$\operatorname{rank}_{\mathbb{Q}} I_0(\Sigma(a_1,\dots,a_n)) = \pm\frac{1}{12}\cdot\frac{d}{dt}\,\ln V_K(-1)$$

where V_K is the Jones polynomial of the Montesinos knot $K = K(a_1,\dots,a_n)$.

References

[A] S. Akbulut, Lectures on Seiberg–Witten Invariants. *Alg-geom/9510012 15 Oct 1995*

[AK] S. Akbulut, R. Kirby, Mazur manifolds. *Michigan Math. J.* **26** (1979), 259–284

[AM] S. Akbulut, J. McCarthy, Casson's Invariant for Oriented Homology 3-Spheres. Princeton, 1990

[BZ] G. Burde, H. Zieschang, Knots. Walter de Gruyter, 1985

[CH] A. Casson, J. Harer, Some homology lens spaces which bound rational homology balls. *Pacific J. Math.* **96** (1981), 23–36

[Dv] I. Dolgachev, Weighted projective varieties. *Lecture Notes in Math. 956*, Springer-Verlag, Berlin and New-York, 1982, 34–71

[D] S. Donaldson, The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Diff. Geom.* **26** (1987), 397–428

[EN] D. Eisenbud, W. Neumann, Three-dimensional Link Theory and Invariants of Plane Curve Singularities. *Annals of Math. Studies, 110*, Princeton, 1985

[FS1] R. Fintushel, R. Stern, Instanton homology groups of Seifert fibered homology three spheres. *Proc. London Math.Soc.* **61** (1990), 109–137

[FS2] R. Fintushel, R. Stern, Pseudofree orbifolds. *Ann. Math.* **122** (1985), 335–364

[F1] A. Floer, An instanton-invariant for 3-manifolds. *Comm.Math.Phys.* **118** (1988), 215–240

[F2] A. Floer, Instanton homology and Dehn surgery. *The Floer memorial volume*, Birkhauser, 1995, 77–97

[Fk] K. Fukaya, Floer homology of connected sum of homology 3-spheres. Preprint, University of Tokyo

[Fu] M. Furuta, *Hand written lecture notes from ICTP*, 1995

[GS] D. Galewski, R. Stern, Classification of simplicial triangulations of topological manifolds. *Ann. Math.* **111** (1980), 1–34

[K] R. Kirby, Problems in low dimensional topology. *Proc. Symp. in Pure Math.* **32** (1978), 273–312

[KKR] P. Kirk, E. Klassen, D. Ruberman, Splitting the spectral flow and the Alexander matrix. *Comm. Math. Helv.* **69** (1994), 375–416

[M] J. Milnor, Singular Points of Complex Hypersurfaces. *Annals of Math. Studies, 91*, Princeton, 1968

[Mo] S. Morita, Almost complex manifolds and Hirzebruch invariants for isolated singularities in complex space. *Math. Ann.* **211** (1974), 245–260

[Mu] D. Mullins, The generalized Casson invariant for 2-fold branched covers of S^3 and the Jones polynomial. *Topology* **32** (1993), 419–438

[N] W. Neumann, An invariant of plumbed homology spheres. *Lecture Notes in Math. 788*, Springer-Verlag, Berlin and New-York, 1980, 125–144

[NR] W. Neumann, F. Raymond, Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. *Lecture Notes in Math. 664*, Springer-Verlag, Berlin and New-York, 1978, 163–196

[NW] W. Neumann, J. Wahl, Casson invariant of link of singularities, *Comm. Math. Helv.* **65** (1990), 58–78

[R] V. Rochlin, New results in the theory of 4-dimensional manifolds, *Dokl. Akad. Nauk SSSR*, **84** (1952), 221–224

[S1] N. Saveliev, Adding relations to instanton homology groups of Seifert fibered homology spheres. *Mat. Sbornik* **183** (1992), 125-140. – English transl. in: *Russian Acad. Sci. Sb. Math.* **77** (1994), 497–510

[S2] N. Saveliev, Dehn surgery along torus knots. Preprint, University of Michigan

[S3] N. Saveliev, Floer homology and invariants of homology cobordism. Submitted to *J. Diff. Geom.*

[S4] N. Saveliev, Floer homology and 3-manifold invariants. *Ph.D. Thesis, University of Oklahoma, 1995*

[S5] N. Saveliev, On homology cobordisms of plumbed homology spheres. Submitted to *Topology*

[Sr] J.-P. Serre, A Course in Arithmetic. Springer-Verlag, New-York, Heidelberg and Berlin, 1973

[Sb] L. Siebenmann, On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology 3-spheres. *Lecture Notes in Math. 788* , Springer-Verlag, Berlin and New-York, 1980, 172–222

[St] R. Stern, Some more Brieskorn spheres which bound contractible manifolds. *Notices Amer. Math. Soc.* **25** (1978), A448

[SS] A. Stipsicz, Z. Szabó, Floer homology groups of certain algebraic links. *Conference Proceedings and Lecture Notes in Geometry and Topology*, International Press, 1994, 173–185

[T] C. Taubes, Casson's invariant and gauge theory. *J. Diff. Geom.* **31** (1990), 547–599

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