A STANDARD FORM FOR INCOMPRESSIBLE SURFACES IN A HANDLEBODY

LINDA GREEN

ABSTRACT. Let Σ be a compact surface and let I be the unit interval. This paper gives a standard form for all 2-sided incompressible surfaces in the 3-manifold $\Sigma \times I$. Since $\Sigma \times I$ is a handlebody when Σ has boundary, this standard form applies to incompressible surfaces in a handlebody.

1. Introduction

Let M be a 3-dimensional manifold and let $X \subset M$ be a properly embedded surface. A *compression disk* for $X \subset M$ is an embedded disk $D \subset M$ such that $\partial D \subset X$, $\mathrm{int}(D) \subset (M \setminus X)$, and ∂D is an essential loop in X. The surface $X \subset M$ is *incompressible* if there are no compression disks for $X \subset M$ and no component of X is a sphere that bounds a ball. If $X \subset M$ is connected and 2-sided, then X is incompressible if and only if the induced map $\pi_1(X) \to \pi_1(M)$ is injective and X is not a sphere that bounds a ball. See, for example, [1, Chapter 6].

Let Σ be a compact surface and let I be the unit interval $[0, 1]$. The manifold $\Sigma \times I$ is foliated by copies of I, which can be thought of as vertical flow lines.

This paper shows that every properly embedded, 2-sided, incompressible surface in $\Sigma \times I$ can be isotoped to a standard form, called "near-horizontal position". A surface in near-horizontal position is transverse to the flow on $\Sigma \times I$ in the I direction, except at isolated intervals, where it coincides with flow lines. Near each of these intervals, Σ looks like a tiny piece of a helix, with the interval as its core. A surface in near-horizontal position can be described combinatorially by listing its boundary curves and the number of times it crosses each line in a certain finite collection of flow lines. When Σ is a compact surface with boundary, then $\Sigma \times I$ is a handlebody. So this paper applies, in particular, to incompressible surfaces in a handlebody.

2. Notation

Throughout this paper, if E is a topological space, then $|E|$ denotes the number of components of E. The symbol I denotes the unit interval $[0, 1]$. If M is a manifold, then ∂M refers to the boundary of M and $\mathrm{int}(M)$ refers to its interior. The symbol Σ refers to a compact surface, and p denotes the projection map $\Sigma \times I \to \Sigma$.

Research at MSRI is supported in part by NSF grant DMS-9022140.

The surface $X \subset M$ is a proper embedding if $\mathrm{int}(X) \subset \mathrm{int}(M)$, $\partial X \subset \partial M$, and the intersection of X with a compact subset of M is a compact subset of X. The map $G : X \times I \to M$ is a proper isotopy between $G|_{X\times 0}$ and $G|_{X\times 1}$ if for all $t \in I$, $G|_{X\times t}$ is a proper embedding. Unless otherwise stated, all surfaces contained in 3-manifolds are properly embedded and all isotopies are proper isotopies.

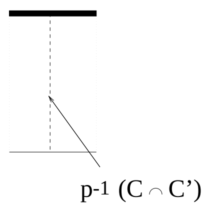
A connected surface $X \subset M$ is *boundary parallel* if X separates M and there is a component K of $M \setminus X$ such that (K, X) is homeomorphic to $(X \times I, X \times 0)$.

3. Definition of near-horizontal position

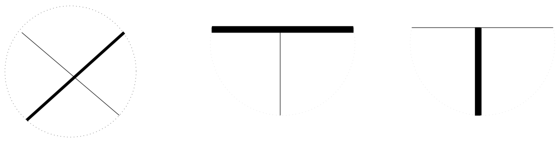
Let X be a surface in $\Sigma \times I$. Let $C \subset \Sigma$ be the union of loops and arcs $p(X \cap (\Sigma \times 1))$, let $C' = p(X \cap (\Sigma \times 0))$, and let $B = X \cap (\partial\Sigma \times I)$.

X is in *near-horizontal position* if

1. C and C' intersect transversely,
2. $p|_B : B \to \partial\Sigma$ is injective,
3. $p|_{X \setminus p^{-1}(C \cap C')}$ is a local homeomorphism, and
4. for any point $z \in C \cap C'$, there is a neighborhood $U \subset \Sigma$ of z such that $p^{-1}(U) \cap X$ either looks like the region



or else looks like a union of regions of the form



Here $p^{-1}(C) \cap X$ is drawn with thick lines and $p^{-1}(C') \cap X$ is drawn with thin lines. Pieces of the boundaries of neighborhoods that are not part of $p^{-1}(C) \cap X$ or $p^{-1}(C') \cap X$ are drawn with dotted lines. I will call the vertical line of $p^{-1}(C \cap C')$ in the first picture a *vertical twist line*.

4. Examples of surfaces in near-horizontal position

A surface X in near-horizontal position determines two unions of properly embedded arcs and loops in Σ: C and C' defined above, and a union of disjoint arcs of $\partial\Sigma$:

$p(B)$. By definition of near-horizontal position, C, C', and $p(B)$ have the following properties:

1. C and C' intersect transversely.
2. $\partial(p(B)) = \partial C \cup \partial C'$

Furthermore, a surface X in near-horizontal position determines a function

$$N: \text{ components of } X \setminus (C \cup C') \longrightarrow \text{ non-negative integers}$$

such that for each component r of $\Sigma \setminus (C \cup C')$ and each point $y \in \text{int}(r)$, $N(r)$ counts the number of times X intersects $y \times I$. It is easy to check that N has the following properties:

3. If r is a component of $\Sigma \setminus (C \cup C')$ with an edge in $p(B)$, then $N(r) = 1$.
4. If r_1 and r_2 are two components which meet along an arc of C or an arc of C', then $|N(r_1) - N(r_2)| = 1$.
5. If r_1, r_2, r_3, and r_4 meet at a common vertex, then either the set $\{N(r_1), N(r_2), N(r_3), N(r_4)\}$ contains 3 distinct numbers, or else $\{N(r_1), N(r_2), N(r_3), N(r_4)\} = \{0, 1\}$.

The second possibility occurs if and only if the common vertex is the projection of a vertical twist line.

Conversely, suppose that C and C' are unions of arcs and loops in Σ, that $p(B)$ is a union of disjoint arcs of $\partial\Sigma$, and that

$$N: \text{ components of } \Sigma \setminus (C \cup C') \longrightarrow \text{ non-negative integers}$$

is a numbering scheme satisfying conditions (1) - (5) above. Then the information $(C, C', p(B), N)$ determines a unique surface in near-horizontal position.

Figure 4 depicts a genus 1 surface with 4 boundary loops, in near-horizontal position in $\mathbb{R}^2 \times I$. The projection of the surface to $\mathbb{R}^2$ is drawn at left. Vertical twist lines, which project to points, are marked with dots. Loops of C are drawn with thick lines, and loops of C' are drawn with thin lines. The corresponding combinatorial description is given at right.

5. Putting incompressible surfaces in near-horizontal position

Theorem: *Suppose that $X \subset \Sigma \times I$ is a properly embedded, 2-sided, incompressible surface. Then X is isotopic to a surface in near-horizontal position.*

Proof of Theorem: Notice that it is possible to isotope X (leaving it fixed outside a neighborhood of ∂X), so that ∂X satisfies requirements 1 and 2 in the definition of near-horizontal position. Therefore, I will assume that C intersects C' transversely and that $p|_B$ is an embedding, where C, C', and B are defined as above.

Suppose that X contains components $X_1, X_2, \ldots X_n$ that are boundary parallel disks. Each X_i can be isotoped so that $p|_{X_i}$ is an embedding and so that $p(\partial X_i)$ is disjoint from $p(\partial X \setminus \partial X_i)$. Therefore, if $X \setminus (X_1 \cup X_2 \cup \cdots \cup X_n)$ can be isotoped to

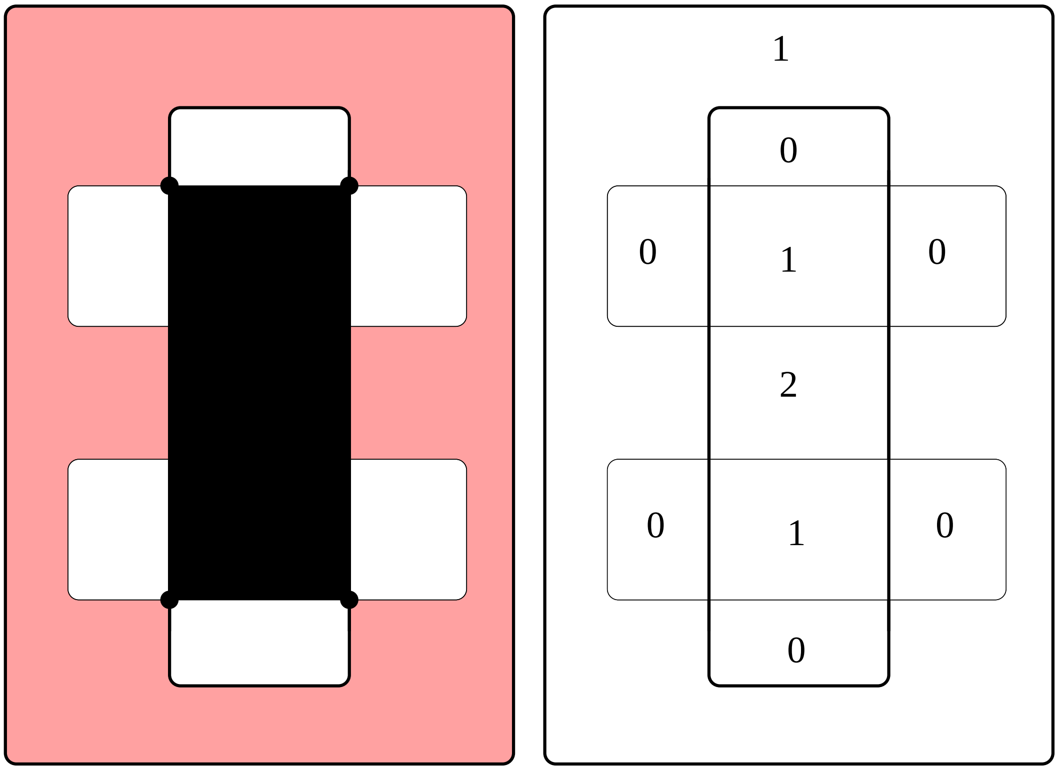


FIGURE 1. An example of a surface in near-horizontal position.

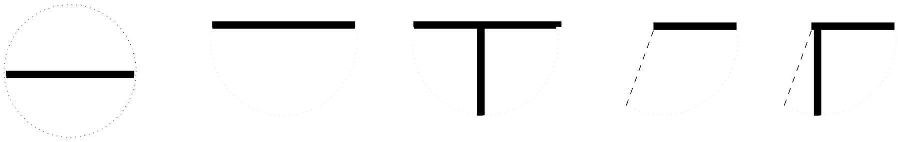
near-horizontal position, so can X. So without loss of generality, I can assume that X has no components that are boundary parallel disks. Since X is incompressible, this assumption insures that all loops in C and C' are essential in Σ.

All isotopies in the rest of the proof will leave ∂X fixed.

Let A be the union of vertical strips and annuli $C \times [0,1]$ and let $A' = C' \times [0,1]$. First, I define a notion of "pseudo-transverse" and a measure of complexity for surfaces that are pseudo-transverse to $A \cup A'$. Then I describe three moves which decrease this complexity. In Step 1, I isotope X so that it is pseudo-transverse to A and A'. In Step 2, I isotope X using the three moves as many times as possible. I then verify six claims about the position of X. In Step 3, I isotope X so that the projection map p is injective when restricted to any arc or loop of $X \cap (A \setminus A')$ and $X \cap (A' \setminus A)$. In Step 4, I complete the proof of the Theorem by isotoping X so that p is locally injective everywhere except at twist lines.

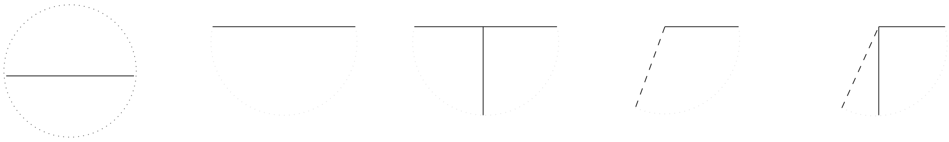
I will say that $X \subset \Sigma \times I$ is *pseudo-transverse* to $A \cup A'$ if the following three conditions hold:

1. For any point $z \in (C - C')$ there is a neighborhood $U \subset \Sigma$ of z such that $p^{-1}(U) \cap X$ is a disjoint union of regions of the form



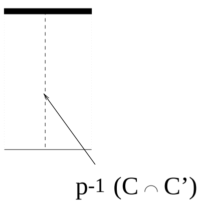
In the last two pictures above, the dashed lines represent pieces of B. One of these two pictures occurs if and only if $z \in \partial C$.

2. For any point $z \in (C' - C)$, there is a neighborhood U of z such that $p^{-1}(U) \cap X$ is a disjoint union of regions of the form

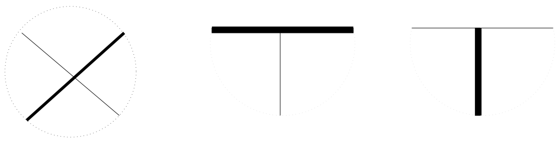


One of the last two pictures occurs if and only if $z \in \partial C'$.

3. For any point $z \in (C \cap C')$, there is a neighborhood U of z such that $p^{-1}(U) \cap X$ is either a single region of the form



or else a disjoint union of regions of the form



Suppose X is pseudo-transverse to $A \cup A'$. Define the complexity of X by

$$\zeta(X) = (|X \cap (A \cap A')|, \text{ rank } H_1(X \cap A) + \text{ rank } H_1(X \cap A'))$$

ordered lexicographically. Consider the following three moves.

Move 1: Suppose D is a disk of A such that $D \cap X = \partial D$ and $\partial D \cap A' = \emptyset$. Then D can be used to isotope X and decrease $\zeta(X)$. The isotopy leaves X in the class of pseudo-transverse embeddings. If the roles of A and A' are interchanged, an analogous move is possible.

Explanation of Move 1: Since X is incompressible in $\Sigma \times I$, ∂D bounds a disk D' in X. The set $D \cup D'$ forms a sphere in $\Sigma \times I$, which must be embedded since

$\text{int}(D) \cap X = \emptyset$. Since $\Sigma \times I$ is irreducible, the sphere bounds a ball, which can be used to isotope X relative to ∂X. If $\partial D \cap \partial X = \emptyset$, then D' can be pushed entirely off of ∂D, and one component of $X \cap A$ is eliminated. Components of $X \cap (A \cap A')$, components of $X \cap A'$, and additional components of $X \cap A$ may also be removed if $\text{int}(D') \cap (A \cup A') \neq \emptyset$, but no new components of any kind are added. Therefore, $\zeta(X)$ goes down. If, instead, $\partial D \cap \partial X \neq \emptyset$, then the isotopy of X relative to ∂X must leave $\partial D \cap \partial X$ fixed. But this isotopy still decreases the rank of $H_1(X \cap A)$ without increasing the rank of $H_1(X \cap A')$ or the number of components of $X \cap (A \cap A')$.

The explanation is analogous if the roles of A and A' are interchanged.

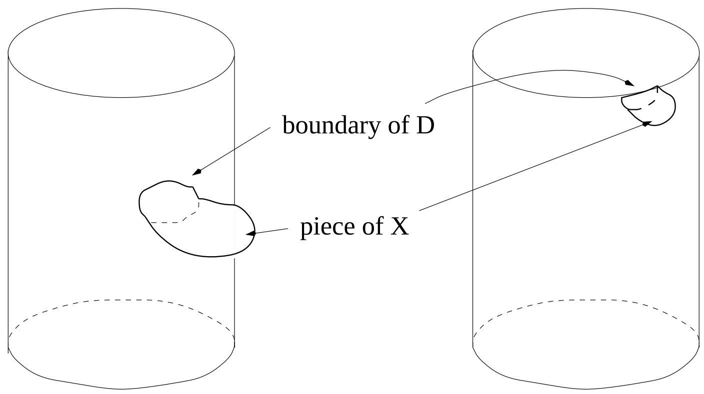


FIGURE 2. Move 1 disks.

MOVE 2: Suppose E is a disk in $\Sigma \times I$ whose boundary consists of two arcs α and σ. Suppose that $E \cap X = \sigma$ and that $E \cap A = E \cap A' = \alpha$. Then E can be used to isotope X relative to ∂X and decrease $\zeta(X)$. The isotopy will keep X in the class of pseudo-transverse embeddings.

EXPLANATION OF MOVE 2: Consider three cases depending on how many points of $\partial\sigma$ lie in ∂X. See Figures 3 and 4.

Case 1: Both endpoints of σ lie in $\text{int}(X)$. Then X can be isotoped in a neighborhood of σ so that it moves through E and slips entirely off of α. This isotopy decreases by two the number of components of $X \cap (A \cap A')$.

Case 2: One endpoint of σ lies in $\text{int}(X)$ and one endpoint lies in ∂X. Now X cannot be isotoped relative to ∂X entirely off α since the endpoint of $\partial\sigma$ in ∂X must remain fixed. But X can still be pushed off $\text{int}(\alpha)$ and off the free endpoint, lowering the number of components of $X \cap (A \cap A')$ by one.

Case 3: Both endpoints of σ lie in ∂X. Notice that one endpoint must lie in $X \cap (\Sigma \times 1)$ and one must lie in $X \cap (\Sigma \times 0)$, since α is a vertical line connecting them. In this case X can be isotoped relative to ∂X in a neighborhood of σ to move σ directly onto the vertical line α and produce a twist around this vertical

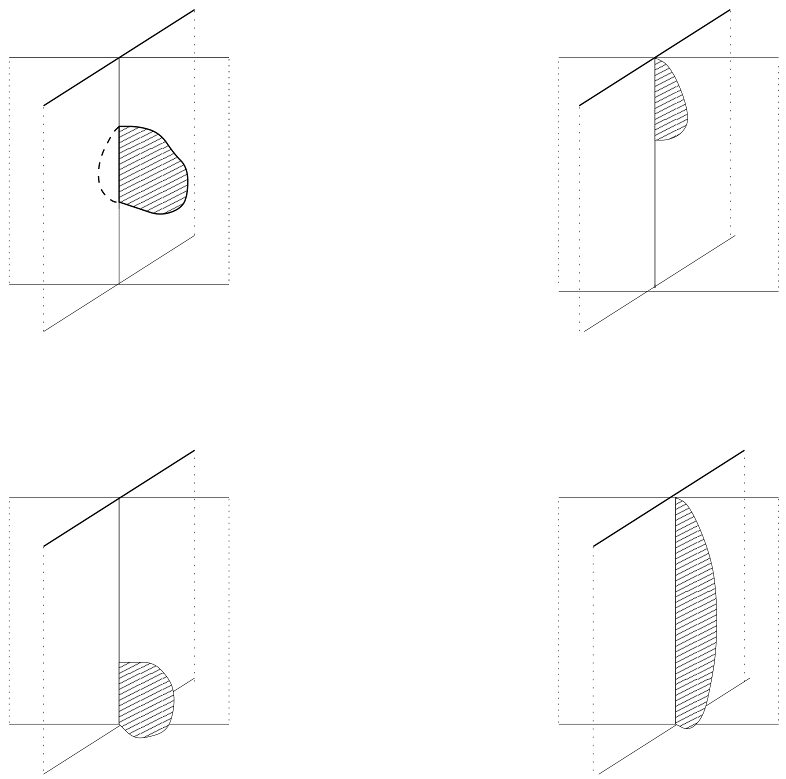


FIGURE 3. Move 2 disks.

line resembling picture (3)(i). This isotopy decreases the number of components of $X \cap (A \cap A')$ by one, since it transforms the two endpoints of $\partial\sigma$ into a single vertical line.

MOVE 3: Suppose W is an annulus of A such that $W \cap A' = \emptyset$ and ∂W is the boundary of an annulus of $X \setminus A$. Then W can be used to isotope X and decrease $\zeta(X)$. The isotopy leaves X in the class of pseudo-transverse embeddings. A similar move is possible if the roles of A and A' are interchanged.

EXPLANATION OF MOVE 3: Let K be the annulus of $X \setminus A$ such that $\partial K = \partial W$, and let L be the closure of the component of $\Sigma \times I \setminus A$ that contains K. $W \cup K$ forms a torus, which is embedded in L since $\text{int}(K) \cap A = \emptyset$. The torus $W \cup K$ compresses in L, because L is homotopic to a surface with boundary and therefore $\pi_1(L)$ cannot contain a $Z \times Z$ subgroup. Since L is irreducible and W lies on the boundary of L, $W \cup K$ actually bounds a solid torus in L, which can be used to isotope X relative to ∂X by pushing K through W. Notice that ∂W and ∂X share at most one component. If ∂W and ∂X are disjoint, then this isotopy removes at least two components of $X \cap A$. If ∂W and ∂X share a component, the the isotopy removes at least one component of $X \cap A$. In either case, the isotopy decreases the

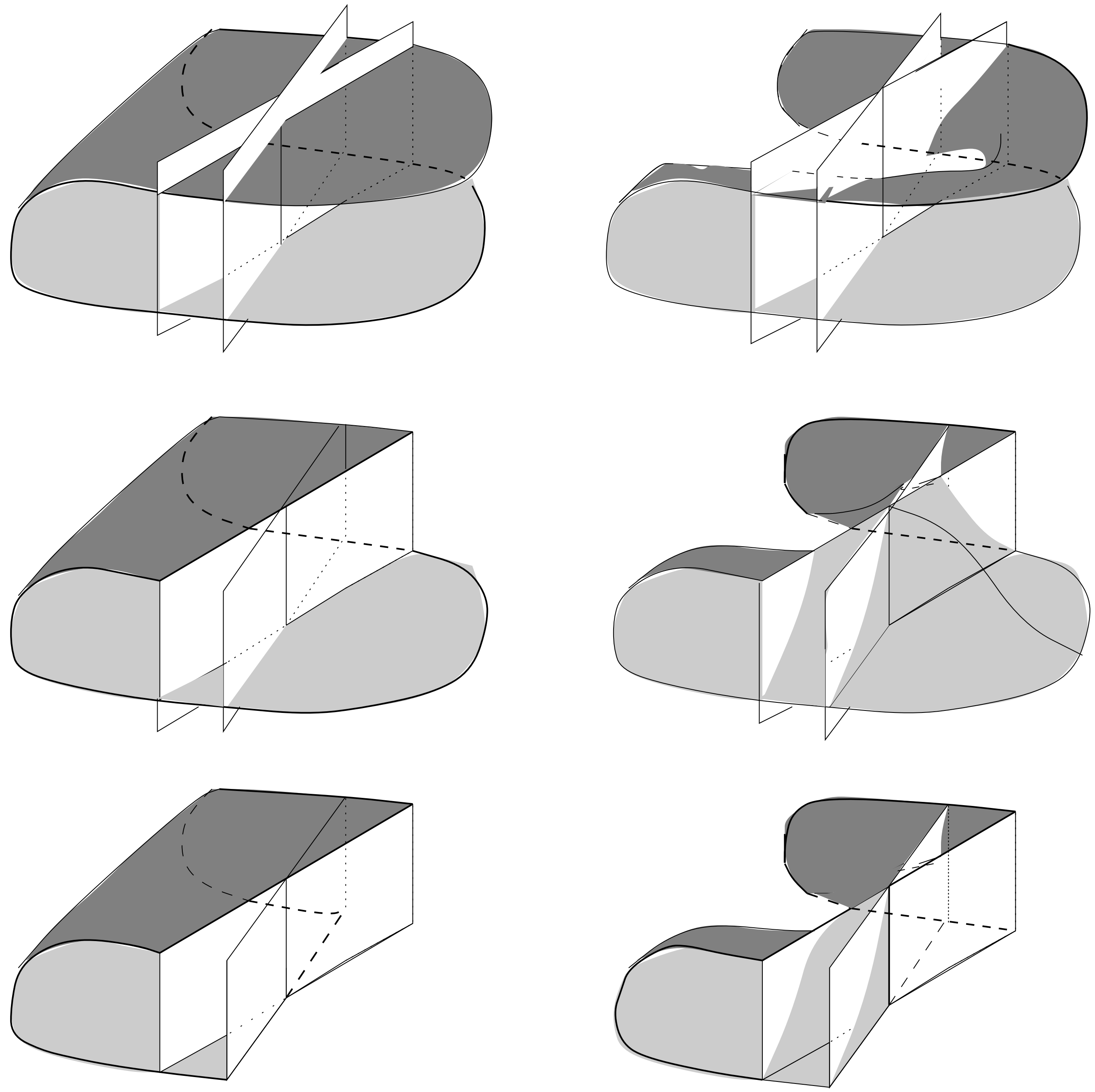


FIGURE 4. Using move 2 disks to isotope X.

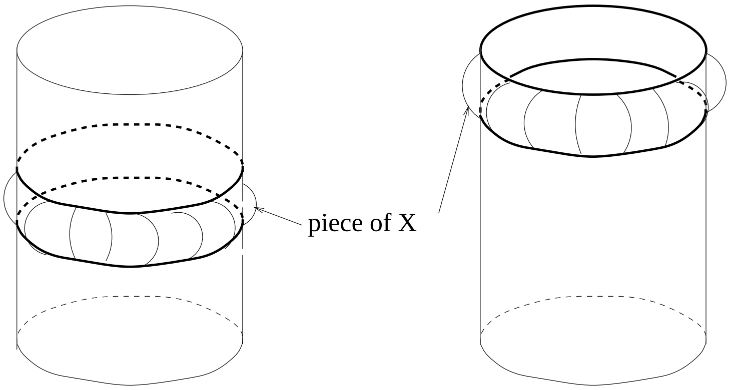


FIGURE 5. Move 3 annuli.

rank of $H_1(X \cap A)$ without increasing the rank of $H_1(X \cap A')$ or the number of components of $X \cap (A \cap A')$.

The explanation is analogous if the roles of A and A' are interchanged.

I will refer to the type of disk used in Move 1 as a move 1 disk, the type of disk used in Move 2 as a move 2 disk, and the type of annulus used in Move 3 as a move 3 annulus.

Step 1: Isotop X relative to ∂X so that it is pseudo-transverse to $A \cup A'$. This can be accomplished, for example, by making X honestly transverse to $A \cup A'$. Then only pictures (1)(i), (1)(ii), (1)(iv), (2)(i), (2)(ii), (2)(iv), (3)(ii), (3)(iii), and (3)(iv) in the definition of pseudo-transverse can occur.

Step 2: Suppose $\Sigma \times I$ contains a move 1 disk, a move 2 disk, or a move 3 annulus. Use it to isotope X. Repeat this step as often as necessary, until there are no more such disks or annuli. The process must terminate after finitely many moves, since each move decreases $\zeta(X)$.

At this stage, X already has a neat posture with respect to $A \cup A'$. In particular, the following claims hold, where K is any component of $X \setminus (A \cup A')$ and L is the component of $(\Sigma \times I) \setminus (A \cup A')$ that contains K.

Claim 1: Suppose that μ is an arc contained in $X \cap (A \setminus A')$ with both endpoints in $A \cap A'$. Then either the endpoints of μ go to distinct vertical lines of $(A \cap A')$ or else μ wraps all the way around an annulus of A. Likewise for arcs of $X \cap (A' \setminus A)$. See Figure 6.

Claim 2: For any point $z \epsilon (C \setminus C')$, there is a neighborhood U of z such that $p^{-1}(U) \cap X$ is a disjoint union of neighborhoods of the form:
Likewise for points of $(h(C) \setminus C)$. In other words, pictures (1)(iii), (1)(v), (2)(iii) and (2)(v) in the definition of pseudo-transverse do not occur.

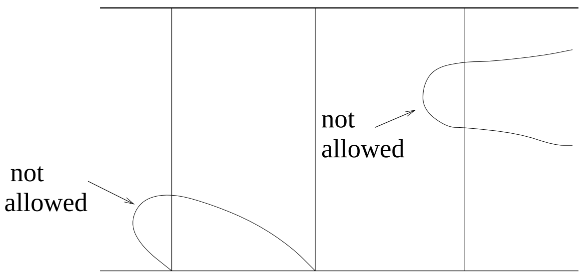


FIGURE 6. Endpoints of μ go to distinct vertical lines of $A \cap A'$.

Claim 3: Every circle of ∂K has non-zero degree in the cylinder of $\partial_v L$ that contains it. Here $\partial_v L$ refers to the vertical boundary of L, namely $\partial L \cap (A \cup A' \cup (\partial\Sigma \times I))$.

Claim 4: $\pi_1(K) \to \pi_1(L)$ is injective.

Claim 5: Either $\pi_1(K) \to \pi_1(L)$ is surjective, or else K is an annulus that is parallel to $\partial_v L$.

Claim 6: If K is an annulus, then the two loops of ∂K go to two distinct cylinders of $\partial_v L$.

Proof of Claim 1: Let μ be an arc contained in $X \cap (A \setminus A')$ and suppose both endpoints of μ lie in one vertical line of $A \cap A'$. Suppose that μ does not wrap all the way around an annulus of A, and let α be the segment of $A \cap A'$ that connects the endpoints of μ. Then $\alpha \cup \mu$ bounds a half-disk E in A. Notice that $E \cap A' = \alpha$. The set $E \cap X$ cannot contain any closed loops of $X \cap A$, since any such loop would bound a move 1 subdisk of E, which should have been removed in Step 2. But $E \cap X$ may contain other arcs besides μ with endpoints on α. (See Figure 7.) By replacing μ and E with an arc and subdisk closer to α if necessary, I can assume that $E \cap X = \mu$. Nudge E relative α off of A to get a new disk E' bounded by the arcs α and μ', where $\mu' \subset X$ and $\mathrm{int}(\mu') \subset \mathrm{int}(X)$. Since $E \cap X = \mu$ and $E \cap A' = \alpha$, I can assume that $E' \cap X = \mu'$ and $E' \cap A' = \alpha$. Also, $E' \cap A = \alpha$. So E' is a move 2 disk, in violation of Step 2. Thus, the endpoints of μ must lie in distinct components of $A \cap A'$ after all.

The same argument applies to arcs contained in $X \cap (A' \setminus A)$.

Proof of Claim 2: Suppose picture (1)(iii) or (1)(v) does occur. (The argument is similar if picture (2)(iii) or (2)(v) occurs.) Let α be the segment of $p^{-1}(U) \cap X \cap A$ drawn vertically in these pictures, extended in A until it first hits ∂X or $A \cap A'$. See

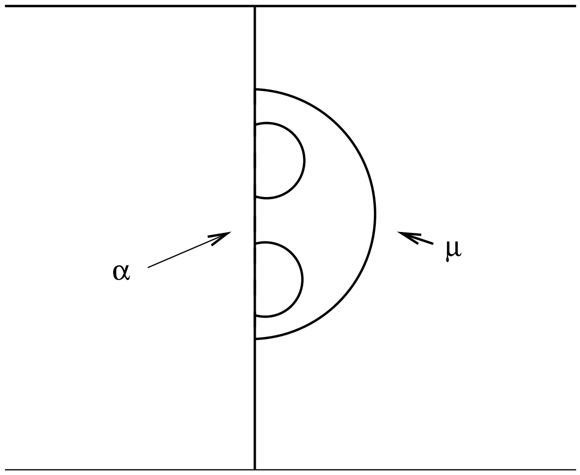


FIGURE 7. $E \cap X$ may contain additional arcs.

Figure 8. Label the first endpoint of α as $\partial_1\alpha$ and the second endpoint as $\partial_2\alpha$. So $\partial_1\alpha$ lies on $C \times 1$.

Suppose first that $\partial_2\alpha$ lies on ∂X but not on $A \cap A'$. Recall that $\partial X = (C \times 1) \cup (C' \times 0) \cup B$; therefore $(\partial X \cap A) \setminus (A \cap A') \subset ((C \times 1) \cap A) \cup (B \cap A)$. Since $p|_B$ is an embedding, it follows that $B \cap A \subset C \times 1$. So $\partial_2\alpha$ lies in $\partial X \cap (C \times 1))$. Thus, α cuts off an arc β of $C \times 1$ such that $\alpha \cup \beta$ bounds a move 1 disk. This disk should already have been removed in Step 2.

Next, suppose that $\partial_2\alpha$ lies on $A \cap A'$, and let β be the arc of ∂X such that $\partial_1\beta = \partial_1\alpha$ and $\partial_2\beta$ lies on the same vertical line of $A \cap A'$ as $\partial_2\alpha$, and so that $\alpha \cup \beta$ does not wrap all the way around an annulus of A. Then $\alpha \cup \beta$ is an arc contained in $X \cap A$ with both endpoints in the same vertical line which should not exist by Claim 1.

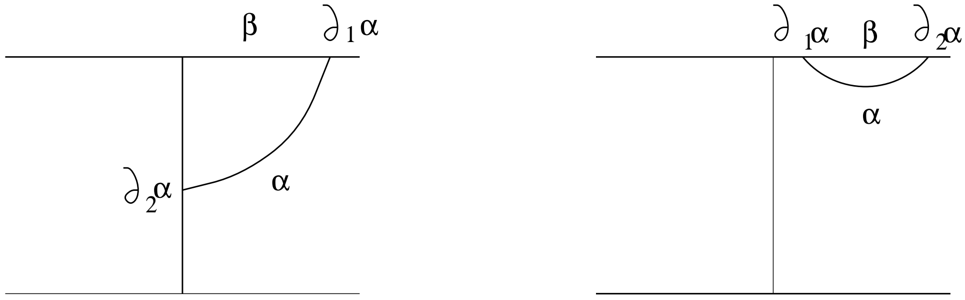


FIGURE 8. The arc α in the proof of Claim 2.

Proof of Claim 3: Let K be a component of $X \setminus (A \cup A')$ and let L be the component of $(\Sigma \times I) \setminus (A \cup A')$ that contains it. Let $\partial_v L$ denote the vertical boundary of L, that is, $\partial_v L = \partial L \cap (A \cup A' \cup (\partial\Sigma \times I))$. Suppose that a circle γ of ∂K gets sent to a cylinder G of $\partial_v L$ by degree 0. G may be an annulus of A or an annulus of A'. Or G may consist of rectangles of $A \setminus A'$ and $A' \setminus A$, joined together along vertical lines of $A \cap A'$ and possibly along vertical strips of $\partial\Sigma \times I$. Suppose first that γ is contained in a single component of $A \setminus A'$ or $A' \setminus A$. (This happens, in particular, if G is an annulus of A or A'.) Since γ has degree 0 in G, it must bound a disk in G, which I can assume has interior disjoint from X by replacing γ with an innermost loop if necessary. But this disk is a move 1 disk, so it should have been removed already in Step 2. See Figure 9.

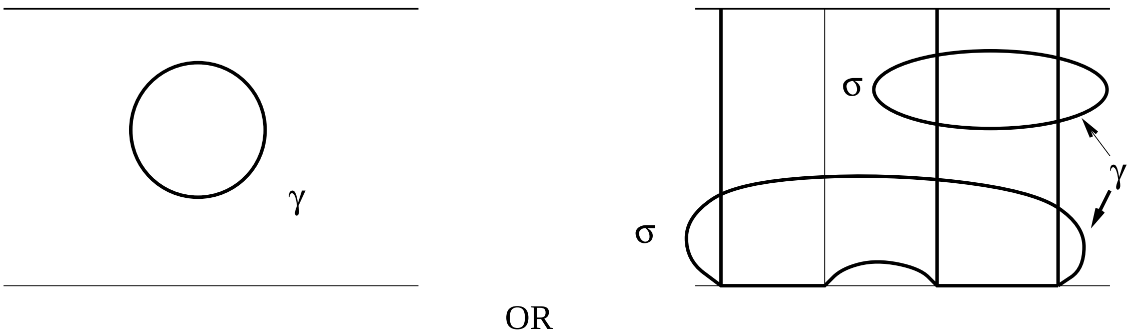


FIGURE 9. Suppose a circle γ of ∂K gets sent to a cylinder of $\partial_v L$ with degree 0.

Now suppose that γ is not contained in a single component of $A \setminus A'$ or $A' \setminus A$. Since $p|_B$ is an embedding, γ must be disjoint from $\partial\Sigma \times I$. Again, γ bounds a disk in G. The vertical lines of $A \cap A'$ cut the disk into subdisks. Consider an outermost subdisk and the arc $\sigma \subset \gamma$ that forms half its boundary. This arc σ is contained in $X \cap (A \setminus A')$ (or in $X \cap (A' \setminus A)$) and both its endpoints lie in the same vertical line. Furthermore, σ cannot wrap all the way around an annulus of A (or A'). Claim 1 says that such arcs do not exist.

Proof of Claim 4: The following diagram commutes, and the map $\pi_1(X) \to \pi_1(\Sigma \times I)$ is injective. So it will suffice to show that $\pi_1(K) \to \pi_1(X)$ is injective.

$$\begin{tikzcd}
\pi_1(K) \arrow[r] \arrow[dr] & \pi_1(L) \arrow[r] & \pi_1(\Sigma \times I) \\
 & \pi_1(X) \arrow[ur] &
\end{tikzcd}$$

If $\pi_1(K) \to \pi_1(X)$ does not inject, then some loop in ∂K must bound a disk D in $X \setminus K$. I claim that D contains at least one loop ω of $X \cap A$ or $X \cap A'$.

Since ∂D bounds D on one side and K on the other, ∂D must be disjoint from ∂X. So ∂D must intersect A or A'. Assume without loss of generality that X intersects

A. Possibly ∂D itself is contained in $X \cap A$. In this case set $\omega = \partial D$. Otherwise, take an arc of $\partial D \cap (X \cap A)$ and let ω be the component of $X \cap A$ that contains it. Notice that ω must lie in D, since it cannot intersect int(K). Since $D \cap \partial X = \emptyset$, ω is a closed loop rather than an arc.

Since $\omega \subset D$, ω is null-homotopic in $\Sigma \times I$. Since each annulus in A π_1-injects into $\Sigma \times I$, ω must bound a disk E in A. If $E \cap A' = \emptyset$, then E is a move 1 disk, which should already have been removed. If $E \cap A' \neq \emptyset$, then $\partial E \cap (A \setminus A')$ will contain an arc whose endpoints lie in the same vertical line of $A \cap A'$, as in the proof of Claim 3. But Claim 1 says that such arcs do not exist.

Proof of Claim 5: Let R be the surface $p(L)$. From Claim 4, the map $\pi_1(K) \to \pi_1(L)$ is injective. Therefore, it is possible to write $\pi_1(L)$ either as an amalgamated product or as an HNN extension over $\pi_1(K)$, depending on whether or not K separates L. In fact, K separates L, because the inclusion of $R \times 1$ into L induces an isomorphism of fundamental groups, which could not happen if $\pi_1(L)$ were an HNN extension. Let M_1 be the component of $L \setminus K$ that contains $R \times 1$, and let M_2 be the other component. Then the composition of maps

$$\pi_1(R \times 1) \to \pi_1(M_1) \to \pi_1(L)$$

is an isomorphism. So $\pi_1(M_1) \to \pi_1(L)$ is surjective. But

$$\pi_1(L) = \pi_1(M_1) *_{\pi_1(K)} \pi_1(M_2)$$

so the injection $\pi_1(K) \to \pi_1(M_2)$ must be an isomorphism. By the h-cobordism theorem [1, Theorem 10.2], $(M_2, K) \cong (K \times I, K \times 1)$.

Now there are two possibilities: either $R \times 0 \subset M_1$ or $R \times 0 \subset M_2$. If $R \times 0 \subset M_1$, then $\partial M_2 \setminus K$ is a subset of $\partial_v L$. So K is parallel to a subsurface of $\partial_v L$, and therefore must be a disk or an annulus. By Claim 3, K must be a boundary parallel annulus.

If $(R \times 0) \subset M_2$, then it follows as above that the composition

$$\pi_1(R \times 0) \to \pi_1(M_2) \to \pi_1(L)$$

is an isomorphism. Therefore, $\pi_1(M_2) \to \pi_1(L)$ is surjective. Also, $\pi_1(M_2) \to \pi_1(L)$ is injective since

$$\pi_1(L) = \pi_1(M_1) *_{\pi_1(K)} \pi_1(M_2)$$

So $\pi_1(M_2) \to \pi_1(L)$ is an isomorphism. In addition, $\pi_1(K) \to \pi_1(M_2)$ is an isomorphism, from above. Therefore, $\pi_1(K) \to \pi_1(L)$ is an isomorphism, and the claim is proved.

Proof of Claim 6: Let K be any annulus component of $X \setminus (A \cup A')$, let L be the corresponding component of $(\Sigma \times I) \setminus (A \cup A')$, and suppose that both circles of ∂K go to the same cylinder G of ∂L. Since both circles have degree ± 1 in $\partial_v L$, they bound an annulus W in $\partial_v L$. Notice that W is disjoint from $\partial \Sigma \times I$, since by assumption, $p|_B : B \to \partial \Sigma$ is an embedding. If W is entirely contained in A or A', then W is a move 3 annulus, which should have been removed in Step 2. So W must consist of alternating rectangles of A and A'.

The union $W \cup K$ forms a torus, which is embedded in L since $\mathrm{int}(K) \cap \partial_v L = \emptyset$. The torus $W \cup K$ compresses in L, because L is homotopic to a surface with boundary and therefore $\pi_1(L)$ cannot contain a $Z \times Z$ subgroup. Since L is irreducible and W lies on the boundary of L, $W \cup K$ bounds a solid torus T in L.

I will construct a move 2 disk as follows. Start with a vertical arc α of $W \cap (A \cap A')$. Connect its endpoints with an embedded arc σ of K so that $\alpha \cup \sigma$ is null homotopic in T. This is possible because each component of ∂K generates $\pi_1(T)$, so I can replace a poor choice of σ by one that wraps around ∂K an additional number of times and get a good choice of σ. Now $\alpha \cup \sigma$ bounds an embedded disk E in T, which I can assume has interior disjoint from X by replacing it with a subdisk if necessary. In addition, $E \cap A = E \cap A' = \alpha$, so E is a move 2 disk. But Step 2 already eliminated all disks of this form.

Step 3: Recall that $p : \Sigma \times I \to \Sigma$ is the projection map. I will isotope X relative to ∂X so that for any arc or loop γ of $X \cap (A \setminus A')$ or $X \cap (A' \setminus A)$ $p|_\gamma$ is a local homeomorphism onto its image. I will consider arcs first and loops next.

Take any arc μ of $X \cap (A \setminus A')$. If $\mathrm{int}(\mu) \cap \partial X \neq \emptyset$, then $\mu \subset \partial X$ by Claim 2. The map p is already injective on arcs of $\partial X \cap A$ and $\partial X \cap A'$, so I can leave μ alone. If $\mathrm{int}(\mu) \cap \partial X = \emptyset$, then isotope μ relative to $\partial \mu$ to an embedded arc μ' in $A \setminus A'$ such that $p|_{\mu'}$ is a homeomorphism onto its image. This is possible because by Claim 1, either μ wraps all the way around an annulus of A or else the endpoints of μ lie in distinct vertical lines of $A \cap A'$. The isotopy can be done in such a way that μ' does not intersect any other arcs of $X \cap (A \setminus A')$ that may lie in the same vertical rectangle. Perturb X in a neighborhood of μ to extend the isotopy on μ.

Pick another arc of $X \cap (A \setminus A')$ and repeat the procedure. When all the arcs of $X \cap (A \setminus A')$ have been pulled taut, continue with arcs of $X \cap (A' \setminus A)$.

Next, consider any loop λ of $X \cap A$ that does not intersect A'. By Claim 3, the loop λ has degree 1 in A, so it can be isotoped to a loop λ' such that $p|_\lambda$ an homeomorphism onto its image. As before, the isotopy can be done in such a way that λ' does not intersect any other loops of $X \cap A$, and the isotopy can be extended to a neighborhood of λ in X. Loops of $X \cap A'$ that are disjoint from A can be isotoped similarly.

I claim that at this stage, for any component K of $X \setminus (A \cup A')$, $p|_{\partial K}$ is a local homeomorphism except along vertical lines of $K \cap (A \cap A')$. Every point of ∂K is either a point on the interior of an arc of $\partial X \cap (\partial \Sigma \times I)$, a point of $X \cap A \cap (\partial \Sigma \times I)$ or $X \cap (A' \cap (\partial \Sigma \times I))$, a point of $A \cap A'$, or a point on the interior of an arc or loop of $X \cap (A \setminus A')$ or $X \cap (A' \setminus A)$. The map $p|_{\partial K}$ is a local homeomorphism near the first type of point by assumption. It is a local homeomorphism near the second type of point by Claim 2. It is a local homeomorphism near the third type of point (away from vertical twist lines) because X is pseudo-transverse. Finally, $p|_{\partial K}$ is a local homeomorphism near the fourth type of point by the work done in Step 3.

Step 4: I will finish isotoping X so that p is locally injective except at twist lines. I will do this by using a fact about maps between surfaces.

Fact: *Let $f : (G, \partial G) \to (H, \partial H)$ be a map such that $f|\partial G$ is a local homeomorphism and $f_* : \pi_1(G) \to \pi_1(H)$ is injective. Then there is a homotopy $f_\tau : (G, \partial G) \to (H, \partial H)$, with $\tau \in I$, $f_0 = f$, and $f_\tau|_{\partial G} = f_0|_{\partial G}$ for all τ, such that either (1) or (2) holds:*

1. *G is an annulus or Mobius band and $f_1(G) \subset \partial H$, or*
2. *$f_1 : G \to H$ is a covering map*

The case when G is a disk is easy to verify; all other cases are covered by [1, Theorem 13.1].

Pick a component K of $X \setminus (A \cup A')$, and let L be the corresponding component of $(\Sigma \times I) \setminus (A \cup A')$. Assume first that ∂K does not contain any vertical arcs of $A \cap A'$. By Claim 4, the map $\pi_1(K) \to \pi_1(L)$ is injective. Since $p_* : \pi_1(L) \to \pi_1(p(L))$ is an isomorphism, the composition $(p|_K)_* : \pi_1(K) \to \pi_1(p(L))$ is injective. Furthermore, by the discussion following Step 3, $p|\partial K : \partial K \to \partial p(L)$ is a local homeomorphism on ∂K. Therefore, there is a homotopy $f_\tau : K \to p(L)$ with $f_0 = p|_K$ and $f_\tau|_{\partial K} = p|_{\partial K}$ such that either K is an annulus or Mobius band and $f_1(K) \subset \partial p(L)$, or f_1 is a covering map.

If K is a Mobius band and $f_1(K) \subset \partial p(L)$, then $f_0(\partial K)$ is a degree 2 loop in $\partial p(L)$ which is impossible since $\partial K \subset \partial_v L$ is embedded. By Claim 6, it not possible for K to be an annulus and $f_1(K)$ a subset of $\partial p(L)$. Therefore, f_1 must be a covering map. By Claims 5 and 6, the map $\pi_1(K) \to \pi_1(L)$ is surjective. So the map $(p_K)_* : \pi_1(K) \to \pi_1(p(L))$ is surjective. Therefore, f_1 must be a homeomorphism.

If ∂K contains vertical arcs of $A \cap A'$, then $p|_{\partial K}$ is still very close to a local homeomorphism – in fact, if $\hat{K}$ is the surface obtained by collapsing each vertical arc of $\partial K \cap (A \cap A')$ to a point, then $p|_K$ factors through a map $\hat{p} : \hat{K} \to p(L)$ such that $\hat{p}|_{\partial \hat{K}}$ is a local homeomorphism. So it is still possible to homotope $p|_K$ relative to ∂K to a map f_1 such that $f_1|_{\mathrm{int}(K)}$ is a homeomorphism.

The homotopy f_τ of K relative to ∂K in $p(L)$ induces a homotopy of K relative to ∂K in L which keeps the vertical coordinate of each point of K constant and changes its horizontal coordinate according to f_τ. At the end of the homotopy, the new surface K' itself will be embedded in $\Sigma \times I$, since $p|_{K'}$ is a homeomorphism.

Homotope X as described above for every other component of $X \setminus (A \cup A')$. At this stage, each component of $X \setminus (A \cup A')$ is embedded in $\Sigma \times I$. But two components K_1' and K_2' in the same piece L of $(\Sigma \times I) \setminus (A \cup A')$ might intersect each other. If that happens, an additional homotopy of K can be tacked on to clear up the problem, as follows.

Let $K_1 \subset L$ and $K_2 \subset L$ be two components of $X \setminus (A \cup A')$ before the homotopy of Step 4 and let K_1' and K_2' be these components after the homotopy. Suppose that K_1' intersects the component K_2'. Since K_1 and K_2 are disjoint, the component of $L \setminus K_1$ that meets $\Sigma \times 0$ either contains all of K_2 or else contains no part of K_2. Therefore, either all loops of K_1 lie above the corresponding loops of K_2 or else they all lie below the corresponding loops. Since the homotopies of K_1 and K_2 did not move ∂K_1 and ∂K_2, the same statement holds for loops of $\partial K_1'$ and $\partial K_2'$. Therefore, it is possible to alter the vertical coordinates of K_1' and K_2' to make the two surfaces parallel, so that K_1' lies entirely above K_2' or entirely below K_2'. Therefore all components of $X \setminus (A \cup A')$ can be assumed disjoint after the homotopy of Step 4.

In its final position, X is embedded in $\Sigma \times I$. A theorem of Waldhausen [2, Corollary 5.5] states that if G and H are incompressible surfaces embedded in an irreducible 3-manifold, and there is a homotopy from G to H that fixes ∂G, then there is an isotopy from G to H that fixes ∂G. Therefore, the above sequence of homotopies can be replaced by an isotopy.

6. Remarks

It is not true that every surface in near-horizontal position is incompressible. For example, the surface in Figure 4 compresses in $\mathbb{R}^2 \times I$. I hope to find simple conditions for when a surface in near-horizontal position is incompressible. A special case is considered in [**?**].

References

[1] John Hempel. *3-Manifolds*. Number 86 in Annals of Mathematics Studies. Princeton University Press, 1976.

[2] F. Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Annals of Mathematics*, 87:56–88, 1968.

MSRI, 1000 Centennial Drive, Berkeley, CA 94720
E-mail address: `lindag@msri.org`