# CANNON-THURSTON MAPS FOR TREES OF HYPERBOLIC METRIC SPACES

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ABSTRACT. Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition. Let $v$ be a vertex of $T$. Let $(X_v, d_v)$ denote the hyperbolic metric space corresponding to $v$. Then $i : X_v \to X$ extends continuously to a map $\hat{i} : \widehat{X_v} \to \widehat{X}$. This generalizes a Theorem of Cannon and Thurston. The techniques are used to give a new proof of a result of Minsky: Thurston's ending lamination conjecture for certain Kleinian groups. Applications to graphs of hyperbolic groups and local connectivity of limit sets of Kleinian groups are also given.

## 1. INTRODUCTION

Let $G$ be a hyperbolic group in the sense of Gromov [13]. Let $H$ be a hyperbolic subgroup of $G$. We choose a finite symmetric generating set for $H$ and extend it to a finite symmetric generating set for $G$. Let $\Gamma_H$ and $\Gamma_G$ denote the Cayley graphs of $H$, $G$ respectively with respect to these generating sets. By adjoining the Gromov boundaries $\partial\Gamma_H$ and $\partial\Gamma_G$ to $\Gamma_H$ and $\Gamma_G$, one obtains their compactifications $\widehat{\Gamma_H}$ and $\widehat{\Gamma_G}$ respectively.

We'd like to understand the extrinsic geometry of $H$ in $G$. Since the objects of study here come under the purview of coarse geometry, asymptotic or 'large-scale' information is of crucial importance. That is to say, one would like to know what happens 'at infinity'. We put this in the more general context of a hyperbolic group $H$ acting freely and properly discontinuously on a proper hyperbolic metric space $X$. Then there is a natural map $i : \Gamma_H \to X$, sending the vertex set of $\Gamma_H$ to the orbit of a point under $H$, and connecting images of adjacent vertices in $\Gamma_H$ by geodesics in $X$. Let $\widehat{X}$ denote the Gromov compactification of $X$.

A natural question seems to be the following:

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**Question:** Does the continuous proper map $i : \Gamma_H \to X$ extend to a continuous map $\hat{i} : \widehat{\Gamma_H} \to \widehat{X}$ ?

Questions along this line have been raised by Bonahon [5]. Related questions in the context of Kleinian groups have been studied by Bonahon [6], Floyd [10] and Minsky [19]. In, [7], [10] or [19], explicit metrics were used. So though some of their results can be thought of as 'coarse', the techniques of proof are not. In [20], coarse techniques were used to answer the above question affirmatively for $X = \Gamma_G$, where $G$ is a hyperbolic group and $H$ a normal subgroup of $G$. This in turn was a generalization of a theorem of Cannon and Thurston [7]. In this paper, we extend the techniques of [20] to cover examples arising from trees of hyperbolic metric spaces satisfying an extra technical condition introduced by Bestvina and Feighn in [3]: the *quasi-isometrically embedded* condition. [See Section 3 of this paper or [3] for definitions.]

**Definition:** *Let $X$ and $Y$ be hyperbolic metric spaces and $i : Y \to X$ be an embedding. A* **Cannon-Thurston map** *$\hat{i}$ from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of $i$. Such a continuous extension will occassionally be called a Cannon-Thurston map for the pair $(Y, X)$. If $Y = \Gamma_H$ and $X = \Gamma_G$ for a hyperbolic subgroup $H$ of a hyperbolic group $G$, a Cannon-Thurston map for $(\Gamma_H, \Gamma_G)$ will occassionally be referred to as a Cannon-Thurston map for $(H, G)$.*

It is easy to see that such a continuous extension, if it exists, is unique.

The main theorem of this paper is :

**Theorem 3.8:** *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition. Let $v$ be a vertex of $T$. Let $(X_v, d_v)$ denote the hyperbolic metric space corresponding to $v$. Then $i : X_v \to X$ extends continuously to a map $\hat{i} : \widehat{X_v} \to \widehat{X}$.*

A direct consequence of Theorem 3.8 above is the following:

**Corollary 3.9:** *Let $G$ be a hyperbolic group acting cocompactly on a simplicial tree $T$ such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let $H$ be the stabilizer of a vertex or edge of $T$. Then there exists a Cannon-Thurston map from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$.*

In [3], Bestvina and Feighn give sufficient conditions for a graph of hyperbolic groups to be hyperbolic. Vertex and edge subgroups are thus natural examples of hyperbolic subgroups of hyperbolic groups.

Essentially all previously known examples of non-quasiconvex hyperbolic subgroups of hyperbolic groups arise this way. Theorem 3.8 shows that these have Cannon-Thurston maps.

Another consequence of Theorem 3.8 above is:

**Theorem 4.7:** Let $\Gamma$ be a geometrically tame Kleinian group, such that $\mathbb{H}^3/\Gamma = M$ has injectivity radius uniformly bounded below by some $\epsilon > 0$. Then there exists a continuous map from the Gromov boundary of $\Gamma$ (regarded as an abstract group) to the limit set of $\Gamma$ in $\mathbb{S}^2_\infty$.

Theorem 4.7 was independently proven by Klarreich [14], using different techniques.

After some further work and using a theorem of Minsky [18], we are able to give a different proof of another result of Minsky [19] : Thurston's Ending Lamination Conjecture for geometrically tame manifolds with freely indecomposable fundamental group and a uniform lower bound on injectivity radius.

**Theorem 4.13** [19]: *Let $N_1$ and $N_2$ be homeomorphic hyperbolic 3-manifolds with freely indecomposable fundamental group. Suppose there exists a uniform lower bound $\epsilon > 0$ on the injectivity radii of $N_1$ and $N_2$. If the end invariants of corresponding ends of $N_1$ and $N_2$ are equal, then $N_1$ and $N_2$ are isometric.*

## 2. Preliminaries

We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [13]. For details, see [8], [11]. Let $(X, d)$ be a hyperbolic metric space. The **Gromov boundary** $X$, denoted by $\partial X$, is the collection of equivalence classes of geodesic rays $r : [0, \infty) \to \Gamma$ with $r(0) = x_0$ for some fixed $x_0 \in X$, where rays $r_1$ and $r_2$ are equivalent if $sup\{d(r_1(t), r_2(t))\} < \infty$. Let $\widehat{X}$=$X \cup \partial X$ denote the natural compactification of $X$ topologized the usual way(cf.[11] pg. 124).

The **Gromov inner product** of elements $a$ and $b$ relative to $c$ is defined by

$$(a, b)_c = 1/2[d(a, c) + d(b, c) - d(a, b)].$$

**Definitions:** *A subset $X$ of $\Gamma$ is said to be* $k$**-quasiconvex** *if any geodesic joining $a, b \in X$ lies in a $k$-neighborhood of $X$. A subset $X$ is* **quasiconvex** *if it is $k$-quasiconvex for some $k$. A map $f$ from one metric space $(Y, d_Y)$ into another metric space $(Z, d_Z)$ is said to be a* $(K, \epsilon)$**-quasi-isometric embedding** *if*

$$\tfrac{1}{K}(d_Y(y_1, y_2)) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \epsilon$$

*If $f$ is a quasi-isometric embedding, and every point of $Z$ lies at a uniformly bounded distance from some $f(y)$ then $f$ is said to be a* **quasi-isometry**. *A $(K,\epsilon)$-quasi-isometric embedding that is a quasi-isometry will be called a $(K,\epsilon)$-quasi-isometry.*

*A $(K,\epsilon)$-***quasigeodesic** *is a $(K,\epsilon)$-quasi-isometric embedding of a closed interval in $\mathbb{R}$. A $(K,0)$-quasigeodesic will also be called a $K$-quasigeodesic.*

Let $(X,d_X)$ be a hyperbolic metric space and $Y$ be a subspace that is hyperbolic with the inherited path metric $d_Y$. By adjoining the Gromov boundaries $\partial X$ and $\partial Y$ to $X$ and $Y$, one obtains their compactifications $\widehat{X}$ and $\widehat{Y}$ respectively.

Let $i : Y \to X$ denote inclusion.

**Definition:** *Let $X$ and $Y$ be hyperbolic metric spaces and $i : Y \to X$ be an embedding. A* **Cannon-Thurston map** *$\hat{i}$ from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of $i$.*

The following lemma says that a Cannon-Thurston map exists if for all $M > 0$ and $y \in Y$, there exists $N > 0$ such that if $\lambda$ lies outside an $N$ ball around $y$ in $Y$ then any geodesic in $X$ joining the end-points of $\lambda$ lies outside the $M$ ball around $i(y)$ in $X$. For convenience of use later on, we state this somewhat differently. The proof is similar to Lemma 2.1 of [20]

**Lemma 2.1.** *A Cannon-Thurston map from $\widehat{Y}$ to $\widehat{X}$ exists if the following condition is satisfied:*

*There exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ and for all geodesic segments $\lambda$ lying outside an $N$-ball around $y_0 \in Y$ any geodesic segment in $\Gamma_G$ joining the end-points of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0) \in X$.*

*Proof:* Suppose $i : Y \to X$ does not extend continuously . Since $i$ is proper, there exist sequences $x_m$, $y_m \in Y$ and $p \in \partial Y$, such that $x_m \to p$ and $y_m \to p$ in $\widehat{Y}$, but $i(x_m) \to u$ and $i(y_m) \to v$ in $\widehat{X}$, where $u, v \in \partial X$ and $u \neq v$.

Since $x_m \to p$ and $y_m \to p$, any geodesic in $Y$ joining $x_m$ and $y_m$ lies outside an $N_m$-ball $y_0 \in Y$, where $N_m \to \infty$ as $m \to \infty$. Any bi-infinite geodesic in $X$ joining $u, v \in \partial X$ has to pass through some $M$-ball around $i(y_0)$ in $X$ as $u \neq v$. There exist constants $c$ and $L$ such that for all $m > L$ any geodesic joining $i(x_m)$ and $i(y_m)$ in $X$ passes through an $(M+c)$-neighborhood of $i(y_0)$. Since $(M+c)$ is a constant not depending on the index $m$ this proves the lemma. □

## 3. Trees of Hyperbolic Metric Spaces

We start with a notion closely related to one introduced in [3].

**Definition:** *A tree (T) of hyperbolic metric spaces satisfying the q(uasi) i(sometrically) embedded condition is a metric space $(X, d)$ admitting a map $P : X \to T$ onto a simplicial tree $T$, such that there exist $\delta$,$\epsilon$ and $K > 0$ satisfying the following:*

1. *For all vertices $v \in T$, $X_v = P^{-1}(v) \subset X$ with the induced path metric $d_v$ is a $\delta$-hyperbolic metric space. Further, the inclusions $i_v : X_v \to X$ are uniformly proper, i.e. for all $M > 0$, $v \in T$ and $x, y \in X_v$, there exists $N > 0$ such that $d(i_v(x), i_v(y)) \leq M$ implies $d_v(x, y) \leq N$.*
2. *Let $e$ be an edge of $T$ with initial and final vertices $v_1$ and $v_2$ respectively. Let $X_e$ be the pre-image under $P$ of the mid-point of $e$. Then $X_e$ with the induced path metric is $\delta$-hyperbolic.*
3. *There exist maps $f_e : X_e \times [0, 1] \to X$, such that $f_e|_{X_e \times (0,1)}$ is an isometry onto the pre-image of the interior of $e$ equipped with the path metric.*
4. *$f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ are $(K, \epsilon)$-quasi-isometric embeddings into $X_{v_1}$ and $X_{v_2}$ respectively. $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ will occassionally be referred to as $f_{v_1}$ and $f_{v_2}$ respectively.*

$d_v$ and $d_e$ will denote path metrics on $X_v$ and $X_e$ respectively. $i_v$, $i_e$ will denote inclusion of $X_v$, $X_e$ respectively into $X$.

The main theorem of this paper can now be stated:

**Theorem: 3.8** Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Let $v$ be a vertex of $T$. Then $i_v : X_v \to X$ extends continuously to $\hat{i_v} : \widehat{X_v} \to \widehat{X}$.

Some aspects of the proof of the main theorem of this section are similar to the proof of the main theorem of [20]. Given a geodesic segment $\lambda \subset X_v$, we construct a quasi-convex set $B_\lambda \subset X$. It follows from the construction that if $\lambda$ lies outside a large ball around $y_0 \in X_v$, $B_\lambda$ lies outside a large ball around $i_v(y_0) \in X$, i.e. for all $M \geq 0$ there exists $N \geq 0$ such that if $\lambda$ lies oustside the $N$-ball around $y_0 \in X_v$, $B_\lambda$ lies outside the $M$-ball around $i_v(y_0) \in X$. Combining this with Lemma 2.1 above, the proof of Theorem 3.8 is completed.

For convenience of exposition, $T$ shall be assumed to be rooted, i.e. equipped with a base vertex $v_0$. Let $v \neq v_0$ be a vertex of $T$. Let $v_-$ be the penultimate vertex on the geodesic edge path from $v_0$ to $v$. Let $e$ denote the directed edge from $v_-$ to $v$. Define $\phi_v : f_{e_-}(X_{e_-} \times \{0\}) \to f_{e_-}(X_{e_-} \times \{1\})$ by

$$\phi_v(f_{e_-}(x\times\{0\}) = f_{e_-}(x\times\{1\}) \text{ for } x \in X_{e_-}.$$

Let $\mu$ be a geodesic in $(X_{v_-})$, joining $i_{v_-}^{-1}(a), i_{v_-}^{-1}(b) \in i_{v_-}^{-1}\cdot f_{e_-}(X_{e_-}\times\{0\})$. $\Phi_v(\mu)$ will denote a geodesic in $X_v$ joining $\phi_v(a)$ and $\phi_v(b)$. Let $X_{v_0} = Y$.

For convenience of exposition, we shall modify $X, X_v, X_e$ by quasi-isometric perturbations. Given a geodesically complete metric space $(Z, d)$ of bounded geometry, choose a maximal disjoint collection $\{N_1(z_\alpha)\}$ of disjoint 1-balls. Then by maximality, for all $z \in Z$ there exist $z_\alpha$ in the collection such that $d(z, z_\alpha) < 2$. Construct a graph $Z_1$ with vertex set $\{z_\alpha\}$ and edge set consisting of distinct vertices $z_\alpha$, $z_\beta$ such that $d(z_\alpha, z_\beta) \leq 4$. Then $Z_1$ equipped with the path-metric is quasi-isometric to $(Z, d)$. All metric spaces in this section will henceforth be assumed to be graphs of edge length 1 and maps between them will be assumed to be cellular.

We start with a general lemma about hyperbolic metric spaces. This follows easily from the fact that local quasigeodesics in a hyperbolic metric space are quasigeodesics [11]. If $x, y$ are points in a hyperbolic metric space, $[x, y]$ will denote a geodesic joining them.

**Lemma 3.1.** *Given $\delta > 0$, there exist $D, C_1$ such that if $a, b, c, d$ are vertices of a $\delta$-hyperbolic metric space $(Z, d)$, with $d(a, [b, c]) = d(a, b)$, $d(d, [b, c]) = d(c, d)$ and $d(b, c) \geq D$ then $[a, b] \cup [b, c] \cup [c, d]$ lies in a $C_1$-neighborhood of any geodesic joining $a, d$.*

Given a geodesic segment $\lambda \subset Y$, we now construct a quasi-convex set $B_\lambda \subset X$.

**Construction of $B_\lambda$**

Choose $C_2 \geq 0$ such that for all $e \in T$, $f_e(X_e\times\{0\})$ and $f_e(X_e\times\{1\})$ are $C_2$-quasiconvex in the appropriate vertex groups. Let $C{=}C_1{+}C_2$, where $C_1$ is as in Lemma 3.1.

For $Z \subset X_v$, let $N_C(Z)$ denote the $C$-neighborhood of $Z$, that is the set of points at distance less than or equal to $C$ from $Z$.

*Step 1 :* Let $\mu \subset X_v$ be a geodesic segment in $(X_v, d_v)$. Then $P(\mu) = v$. For each edge $e$ incident on $v$, but not lying on the geodesic (in $T$) from $v_0$ to $v$, choose $p_e$, $q_e \in N_C(\mu)\cap f_v(X_e)$ such that $d_v(p_e, q_e)$ is maximal. Let $v_1, \cdots, v_n$ be terminal vertices of edges $e$ for which $d_v(p_e, q_e) > D$, where $D$ is as in Lemma 3.1 above. Observe that there are only finitely many $v_i$'s as $\mu$ is finite. Define

$$B^1(\mu) = i_v(\mu)\cup\bigcup_{k=1\cdots n}\Phi_{v_i}(\mu)$$

Note that $P(B^1(\mu)) \subset T$ is a finite tree.

*Step 2 :* Step 1 above constructs $B^1(\lambda)$ in particular. We proceed inductively. Suppose that $B^m(\lambda)$ has been constructed such that the convex hull of $P(B^m(\lambda)) \subset T$ is a finite tree. Let $\{w_1, \cdots, w_n\} = P(B^m(\lambda)) \backslash P(B^{m-1}(\lambda))$. (Note that $n$ may depend on $m$, but we avoid repeated indices for notational convenience.) Assume further that $P^{-1}(v_k) \cap B^m(\lambda)$ is a path of the form $i_{v_k}(\lambda_k)$, where $\lambda_k$ is a geodesic in $(X_{v_k}, d_{v_k})$. Define

$$B^{m+1}(\lambda) = B^m(\lambda) \cup \bigcup_{k=1 \cdots n} (B^1(\lambda_k))$$

where $B^1(\lambda_k)$ is defined in Step 1 above.

Since each $\lambda_k$ is a finite geodesic segment in $\Gamma_H$, the convex hull of $P(B^{m+1}\lambda)$ is a finite subtree of $T$. Further, $P^{-1}(v) \cap B^{m+1}(\lambda)$ is of the form $i_v(\lambda_v)$ for all $v \in P(B^{m+1}(\lambda))$. This enables us to continue inductively. Define

$$B_\lambda = \cup_{m \geq 0} B^m_\lambda.$$

Note finally that the convex hull of $P(B_\lambda)$ in $T$ is a locally finite tree $T_1$.

**Quasiconvexity of $B_\lambda$**

We shall now show that there exists $C' \geq 0$ such that for every geodesic segment $\lambda \subset Y$, $B_\lambda \subset X$ is $C'$-quasiconvex. To do this we construct a retraction $\Pi_\lambda$ from (the vertex set of) $X$ onto $B_\lambda$ and show that there exists $C_0 \geq 0$ such that $d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d_X(x, y)$. Let $\pi_v : X_v \to \lambda_v$ be a nearest point projection of $X_v$ onto $\lambda_v$. $\Pi_\lambda$ is defined on $\bigcup_{v \in T_1} X_v$ by

$$\Pi_\lambda(x) = i_v \cdot \pi_v(x) \text{ for } x \in X_v.$$

If $x \in P^{-1}(T \setminus T_1)$ choose $x_1 \in P^{-1}(T_1)$ such that $d(x, x_1) = d(x, P^{-1}(T_1))$ and define $\Pi'_\lambda(x) = x_1$. Next define $\Pi_\lambda(x) = \Pi_\lambda \cdot \Pi'_\lambda(x)$.

The following Lemma says nearest point projections in a $\delta$-hyperbolic metric space do not increase distances much.

**Lemma 3.2.** *Let $(Y, d)$ be a $\delta$-hyperbolic metric space and let $\mu \subset Y$ be a geodesic segment. Let $\pi : Y \to \mu$ map $y \in Y$ to a point on $\mu$ nearest to $y$. Then $d(\pi(x), \pi(y)) \leq C_3 d(x, y)$ for all $x, y \in Y$ where $C_3$ depends only on $\delta$.*

*Proof:* Let $[a, b]$ denote a geodesic edge-path joining vertices $a, b$. Recall that the Gromov inner product $(a, b)_c = 1/2[d(a, c) + d(b, c) - d(a, b)]$. It suffices by repeated use of the triangle inequality to prove the Lemma when $d(x, y) = 1$. Let $u, v, w$ be points on $[x, \pi(x)]$, $[\pi(x), \pi(y)]$ and $[\pi(y), x]$ respectively such that $d(u, \pi(x)) = d(v, \pi(x))$, $d(v, \pi(y)) =$

$d(w, \pi(y))$ and $d(w, x) = d(u, x)$. Then $(x, \pi(y))_{\pi(x)} = d(u, \pi(x))$. Also, since $Y$ is $\delta$-hyperbolic, the diameter of the inscribed triangle with vertices $u, v, w$ is less than or equal to $2\delta$ (See [1]).

$$\begin{aligned} d(u,x) + d(u,v) &\geq d(x, \pi(x)) = d(u,x) + d(u, \pi(x)) \\ \Rightarrow d(u, \pi(x)) &\leq d(u,v) \leq 2\delta \\ \Rightarrow (x, \pi(y))_{\pi(x)} &\leq 2\delta \end{aligned}$$

Similarly, $(y, \pi(x))_{\pi(y)} \leq 2\delta$.

$$\begin{aligned} &\text{i.e. } d(x, \pi(x)) + d(\pi(x), \pi(y)) - d(x, \pi(y)) \leq 4\delta \\ &\text{and } d(y, \pi(y)) + d(\pi(x), \pi(y)) - d(y, \pi(x)) \leq 4\delta \end{aligned}$$

Therefore,

$$\begin{aligned} & 2d(\pi(x), \pi(y)) \\ &\leq 8\delta + d(x, \pi(y)) - d(y, \pi(y)) + d(y, \pi(x)) - d(x, \pi(x)) \\ &\leq 8\delta + d(x,y) + d(x,y) \\ &\leq 8\delta + 2 \end{aligned}$$

Hence $d(\pi(x), \pi(y)) \leq 4\delta + 1$. Choosing $C_3 = 4\delta + 1$, we are through. □

**Lemma 3.3.** *Let $(Y, d)$ be a $\delta$-hyperbolic metric space. Let $\mu$ be a geodesic segment in $Y$ with end-points $a, b$ and let $x$ be any vertex in $Y$. Let $y$ be a vertex on $\mu$ such that $d(x, y) \leq d(x, z)$ for any $z \in \mu$. Then a geodesic path from $x$ to $y$ followed by a geodesic path from $y$ to $z$ is a $k$-quasigeodesic for some $k$ dependent only on $\delta$.*

*Proof:* As in Lemma 3.2 let $u, v, w$ be points on edges $[x, y]$, $[y, z]$ and $[z, x]$ respectively such that $d(u, y) = d(v, y)$, $d(v, z) = d(w, z)$ and $d(w, x) = d(u, x)$. Then $d(u, y) = (z, x)_y \leq 2\delta$ and the inscribed triangle with vertices $u, v, w$ has diameter less than or equal to $2\delta$ (See [1]). $[x, y] \cup [y, z]$ is a union of 2 geodesic paths lying in a $4\delta$ neighborhood of a geodesic $[x, z]$. Hence a geodesic path from $x$ to $y$ followed by a geodesic path from $y$ to $z$ is a $k-$quasigeodesic for some $k$ dependent only on $\delta$. □

**Lemma 3.4.** *Suppose $(Y, d)$ is a $\delta$-hyperbolic metric space. If $\mu$ is a $(k_0, \epsilon_0)$-quasigeodesic in $Y$ and $p, q, r$ are 3 points in order on $\mu$ then $(p, r)_q \leq k_1$ for some $k_1$ dependent on $k_0$, $\epsilon_0$ and $\delta$ only.*

*Proof:* $[a, b]$ will denote a geodesic path joining $a, b$. Since $p, q, r$ are 3 points in order on $\mu$, $[p, q]$ followed by $[q, r]$ is a $(k_0, \epsilon_0)$-quasigeodesic in the $\delta$-hyperbolic metric space $Y$. Hence there exists a $k_1$ dependent

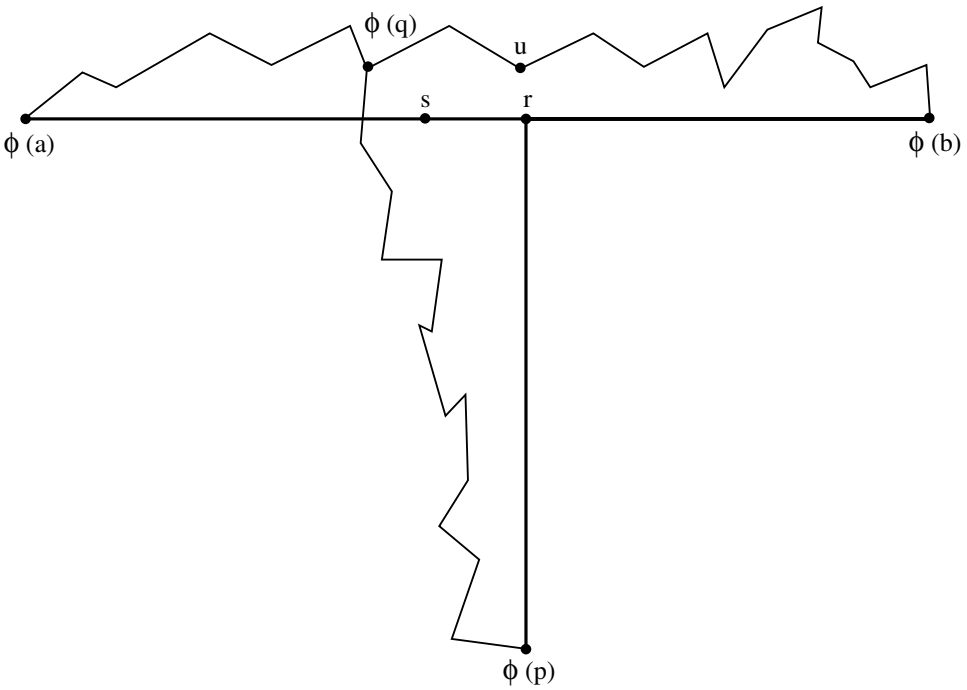


FIGURE 1

on $k_0$, $\epsilon_0$ and $\delta$ alone such that $d(q,[p,r]) \leq k_1$. Let $s$ be a point on $[p,r]$ such that $d(q,s) = d(q,[p,r]) \leq k_1$. Then

$$\begin{aligned}(p,r)_q &= 1/2(d(p,q)+d(r,q)-d(p,r)) \\ &= 1/2(d(p,q)+d(r,q)-d(p,s)-d(r,s)) \\ &\leq d(q,s) \leq k_1.\Box\end{aligned}$$

**Lemma 3.5.** *Suppose $(Y,d)$ is $\delta$-hyperbolic. Let $\mu_1$ be some geodesic segment in $Y$ joining $a,b$ and let $p$ be any vertex of $Y$. Also let $q$ be a vertex on $\mu_1$ such that $d(p,q) \leq d(p,x)$ for $x \in \mu_1$. Let $\phi$ be a $(K,\epsilon)$ - quasiisometry from $Y$ to itself. Let $\mu_2$ be a geodesic segment in $Y$ joining $\phi(a)$ to $\phi(b)$ for some $g \in S$. Let $r$ be a point on $\mu_2$ such that $d(\phi(p),r) \leq d(\phi(p),x)$ for $x \in \mu_2$. Then $d(r,\phi(q)) \leq C_4$ for some constant $C_4$ dependent only on $K,\epsilon$ and $\delta$.*

*Proof:* Since $\phi(\mu_1)$ is a $(K,\epsilon)-$ quasigeodesic joining $\phi(a)$ to $\phi(b)$, it lies in a $K'$-neighborhood of $\mu_2$ where $K'$ depends only on $K,\epsilon,\delta$. Let $u$ be a vertex in $\phi(\mu_1)$ lying at a distance at most $K'$ from $r$. Without loss of generality suppose that $u$ lies on $\phi([q,b])$, where $[q,b]$ denotes the geodesic subsegment of $\mu_1$ joining $q,b$. [See Figure 1.]

Let $[p,q]$ denote a geodesic joining $p,q$. From Lemma 3.3 $[p,q] \cup [q,b]$ is a $k$-quasigeodesic, where $k$ depends on $\delta$ alone. Therefore $\phi([p,q]) \cup \phi([q,b])$ is a $(K_0,\epsilon_0)$-quasigeodesic, where $K_0, \epsilon_0$ depend on $K, k, \epsilon$. Hence, by Lemma 3.4 $(\phi(p),u)_{\phi(q)} \leq K_1$, where $K_1$ depends on $K$, $k$, $\epsilon$ and $\delta$ alone. Therefore,

$$\begin{aligned}
&(\phi(p),r)_{\phi(q)} \\
&= 1/2[d(\phi(p),\phi(q)) + d(r,\phi(q)) - d(r,\phi(p))] \\
&\leq 1/2[d(\phi(p),\phi(q)) + d(u,\phi(q)) + d(r,u) \\
&\qquad\qquad -d(u,\phi(p)) + d(r,u)] \\
&= (\phi(p),u)_{\phi(q)} + d(r,u) \\
&\leq K_1 + K'
\end{aligned}$$

There exists $s \in \mu_2$ such that $d(s,\phi(q)) \leq K'$

$$\begin{aligned}
(\phi(p),r)_s &= 1/2[d(\phi(p),s) + d(r,s) - d(r,\phi(p))] \\
&\leq 1/2[d(\phi(p),\phi(q)) + d(r,\phi(q)) - d(r,\phi(p))] + K' \\
&= (\phi(p),r)_{\phi(q)} + K' \\
&\leq K_1 + K' + K' \\
&= K_1 + 2K'
\end{aligned}$$

Also, as in the proof of Lemma 3.2 $(\phi(p),s)_r \leq 2\delta$

$$\begin{aligned}
d(r,s) &= (\phi(p),s)_r + (\phi(p),r)_s \\
&\leq K_1 + 2K' + 2\delta \\
d(r,\phi(q)) &\leq K_1 + 2K' + 2\delta + d(s,\phi(q)) \\
&\leq K_1 + 2K' + 2\delta + K'
\end{aligned}$$

Let $C_4 = K_1 + 3K' + 2\delta$. Then $d(r,\phi(q)) \leq C_4$ and $C_4$ is independent of $a, b, p$. $\square$

$d_T$ will denote the metric on $T$. We are now in a position to prove:

**Theorem 3.6.** *There exists $C_0 \geq 0$ such that $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d_G(x,y)$ for $x, y$ vertices of $\Gamma_G$.*

*Proof:* It suffices to prove the theorem when $d_G(x,y) = 1$.

*Case (a):* $x, y \in P^{-1}(v)$ for some $v \in T_1$. From Lemma 3.2, there exists $C_3$ such that $d_v(\pi_v \cdot i_v^{-1}(x), \pi_v \cdot i_v^{-1}(y)) \leq C_3$. Since embeddings of $X_v$ in $X$ are cellular, $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_3$.

*Case (b):* $x \in P^{-1}(w)$ and $y \in P^{-1}(v)$ for some $v, w \in T_1$.

Since $d_G(x,y)=1$, $v,w$ are adjacent in $T_1$. Assume, without loss of generality, $w=v_-$.

Recall that

$$B_\lambda \cap P^{-1}(v) = i_v(\lambda_v)$$
$$B_\lambda \cap P^{-1}(w) = i_w(\lambda_w)$$

Also, $\lambda_v = \Phi_v(\mu_w)$, for some geodesic $\mu_w$ contained in $X_w$, such that end-points of $\mu_w$ lie in a $C$-neighborhood of $\lambda_w$.

Let $z \in (X_w)$ denote a nearest point projection of $i_w^{-1}(x)$ onto $\mu_w$. Then, by Lemma 3.5,

$$d(i_w(z), \Pi_\lambda \cdot \phi_v(x)) \leq d(i_w(z), \phi_v \cdot i_w(z)) + d(\phi_v \cdot i_w(z), \Pi_\lambda \cdot \phi_v(x)) \leq 1 + C_4.$$

Since, $d(x,y) = 1 = d(x, \phi_v(x))$ and $i_v$'s are uniformly proper embeddings, there exists $C_5 > 0$ such that $d_v(\phi_v(x), y) \leq C_5$ and $d(\Pi_\lambda(\phi_v(x)), \Pi_\lambda(y)) \leq C_3 C_5$.

Since the end-points of $\mu_w$ lie in a $C$-neighborhood of $\lambda_w$, there exists $C_6$, depending on $\delta$ and $C$ such that $d(z, \Pi_\lambda(x)) \leq C_6$.

Finally, by the triangle inequality,

$$d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_6 + 1 + C_4 + C_3 C_5 = C_7 (say)$$

*Case (c):* $P([x,y])$ is not contained in $T_1$.

Since $d(x,y) = 1$ $P(x)$ and $P(y)$ belong to the closure $T_2$ of the same component of $T \backslash T_1$. Then $P \cdot \Pi'_\lambda(x) = P \cdot \Pi'_\lambda(y) = v$ for some $v \in T$.

Also $d(\Pi_\lambda(x), \Pi_\lambda(y)) = d(\Pi_\lambda \cdot \Pi'_\lambda(x), \Pi_\lambda \cdot \Pi'_\lambda(y))$

Let $x_1 = \Pi'_\lambda(x)$ and $y_1 = \Pi'_\lambda(y)$.

Let $D$ and $C_1$ be as in Lemma 3.1. If $d(\Pi_\lambda(x_1), \Pi_\lambda(y_1)) \geq D$, let

$$\begin{aligned} {i_v}^{-1}(x_1) &= u_1, \\ {i_v}^{-1}(\Pi_\lambda(x_1)) &= u_2, \\ {i_v}^{-1}(y_1) &= v_1, \\ {i_v}^{-1}(\Pi_\lambda(y_1)) &= v_2. \end{aligned}$$

Then by Lemma 3.1 $[u_1,u_2] \cup [u_2,v_2] \cup [v_2,v_1]$ is a quasigeodesic lying in a $C_1$-neighborhood of $[u_1,v_1]$.

Also, $x_1, y_1 \in i_v(X_v)$. Since the image of an edge space in a vertex space is $C_2$-quasiconvex, there exist $e \in T$ and $x_2, y_2 \in f_e(X_e \times \{0\})$ such that $d(x_2, \Pi_\lambda(x_1)) \leq C_1 + C_2 = C$ and $d(y_2, \Pi_\lambda(y_1)) \leq C_1 + C_2 = C$.

By construction $d(\Pi_\lambda(x_2), \Pi_\lambda(y_2)) \leq D$. (Else the edge $P([x,y])$ of $T$ would be in $T_1$). Therefore,

$$\begin{aligned} d(\Pi_\lambda(x), \Pi_\lambda(y)) &= d(\Pi_\lambda(x_1), \Pi_\lambda(y_1)) \\ &\leq 2C + D + 2C \\ &= 4C + D \end{aligned}$$

Choosing $C_0 = \max\{C_3, C_7, 4C + D\}$, we are through. □

To complete the proof of our main Theorem, we need a final Lemma.

**Lemma 3.7.** *There exists $A > 0$, such that if $a \in P^{-1}(v) \cap B_\lambda$ for some $v \in T_1$ then there exists $b \in i(\lambda)$ with $d(a, b) \leq Ad_T(Pa, Pb)$.*

*Proof:* Let $\mu$ be a geodesic path from $v_0$ to $v$ in $T$. Order the vertices on $\mu$ so that we have a finite sequence $v_0 = y_0, y_1, \cdots, y_n = v$ such that $d_T(y_i, y_{i+1}) = 1$. and $d_T(v_0, v) = n$. Recall further, $P(B_\lambda) = T_1$. Hence $y_i \in T_1$.

Recall that $B_\lambda$ is of the form $\bigcup_{v \in T_1} i_v(\lambda_v)$.

It suffices to prove that there exists $A > 0$ independent of $v$ such that if $p \in i_{y_j}(\lambda_{y_j})$, there exists $q \in i_{y_{j-1}}(\lambda_{y_{j-1}})$ with $d(p, q) \leq A$.

By construction, $\lambda_{y_j} = \Phi_{y_j}(\mu)$ for some geodesic $\mu$ in $X_{y_{j-1}}$ such that end-points of $\mu$ lie in a $C$-neighborhood of $\lambda_{y_{j-1}}$. Since $\phi_{y_j}$ is a quasi-isometry, there exists $C_1$ such that $p$ lies in a $C_1$ neighborhood of $\phi_{y_j}(q_0)$ for some $q_0 \in \mu$. Therefore, $d(q_0, p) \leq 1 + C$.

Also, since end-points of $\mu$ lie in a $C$-neighborhood of $\lambda_{y_{j-1}}$, there exists $q \in i_{y_j}(\lambda_{y_{j-1}})$ with $d(q_0, q) \leq C_2$ where $C_2$ depends only on $\delta$ and $C$. Choosing $A = 1 + C + C_2$, we are through. □

The main theorem of this paper follows:

**Theorem 3.8.** *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Let $v$ be a vertex of $T$. Then $i_v : X_v \to X$ extends continuously to $\hat{i_v} : \widehat{X_v} \to \widehat{X}$.*

*Proof:* Without loss of generality, let $v_0 = v$ be the base vertex of $T$. To prove the existence of a Cannon-Thurston map, it suffices to show (from Lemma 2.1) that for all $M \geq 0$ and $x_0 \in X_v$ there exists $N \geq 0$ such that if a geodesic segment $\lambda$ lies oustside the $N$-ball around $x_0 \in X_v$, $B_\lambda$ lies outside the $M$-ball around $i_v(x_0) \in X$.

To prove this, we show that if $\lambda$ lies outside the $N$-ball around $x_0 \in X_v$, $B_\lambda$ lies outside a certain $M(N)$-ball around $i_v(x_0) \in X$, where $M(N)$ is a proper function from $\mathbb{N}$ into itself.

Since $X_v$ is properly embedded in $X$ there exists $f(N)$ such that $i_v(\lambda)$ lies outside the $f(N)-$ball around $x_0$ in $X$ and $f(N) \to \infty$ as $N \to \infty$.

Let $p$ be any point on $B_\lambda$. There exists $y \in i_v(\lambda)$ such that $d(y,p) \leq Ad_T(Py, Pp)$ by Lemma 3.7. Therefore,

$$\begin{aligned} d(x_0,p) &\geq d(x_0,y) - Ad_T(Py,Pp) \\ &\geq f(N) - Ad_T(P(x_0),Pp) \end{aligned}$$

By our choice of metric on $X$,

$$d(x_0,p) \geq d_T(P(x_0),Pp)$$

Hence

$$\begin{aligned} d(x_0,p) &\geq max(f(N) - Ad_T(P(x_0),Pp), d_T(P(x_0),Pp)) \\ &\geq \frac{f(N)}{A+1} \end{aligned}$$

¿From Theorem 3.6 there exists $C'$ independent of $\lambda$ such that $B_\lambda$ is a $C'$-quasiconvex set containing $i_v(\lambda)$. Therefore any geodesic joining the end-points of $i_v(\lambda)$ lies in a $C'$-neighborhood of $B_\lambda$.

Hence any geodesic joining end-points of $i_v(\lambda)$ lies outside a ball of radius $M(N)$ where

$$M(N) = \tfrac{f(N)}{A+1} - C'$$

Since $f(N) \to \infty$ as $N \to \infty$ so does $M(N)$. □

The following is a direct consequence of Theorem 3.8 above.

**Corollary 3.9.** *Let $G$ be a hyperbolic group acting cocompactly on a simplicial tree $T$ such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let $H$ be the stabilizer of a vertex or edge of $T$. Then there exists a Cannon-Thurston map for $(H,G)$.*

## 4. Geometrically Tame Kleinian Groups

In this section we apply Theorem 3.8 to geometrically tame Kleinian groups.

The *convex core* of a hyperbolic 3-manifold $N$ (without cusps) is the smallest convex submanifold $C(N) \subset N$ for which inclusion is a homotopy equivalence. If $C(N)$ has finite volume, $N$ is said to be *geometrically finite*. There exists a compact 3-dimensional submanifold $M \subset N$, the *Scott core* [21] whose inclusion is a homotopy equivalence. The ends of $N$ are in one-to-one correspondence with the components of $N - M$ or, equivalently, the components of $\partial M$. We say that an end of $N$ is *geometrically finite* if it has a neighborhood missing $C(N)$. An end of $N$ is *simply degenerate* if it has a neighborhood homeomorphic to $S \times \mathbb{R}$, where $S$ is the corresponding component of $\partial M$, and if there is a sequence of pleated surfaces homotopic in this neighborhood to the

inclusion of $S$, and exiting every compact set. $N$ is called *geometrically tame* if all of its ends are either geometrically finite or simply degenerate. In particular, $N$ is homeomorphic to the interior of $M$. For a more detailed discussion of pleated surfaces and geometrically tame ends, see [23] or [18].

Let $inj_N(x)$ denote the injectivity radius at $x \in N$. For the purposes of this section, we shall assume that there exists $\epsilon_0 > 0$ such that $inj_N(x) > \epsilon_0$ for all $x \in N$. In order to apply Theorem 3.8 we need some preliminary Lemmas.

Let $E$ be a simply degenerate end of $N$. Then $E$ is homeomorphic to $S \times [0, \infty)$ for some closed surface $S$ of genus greater than one.

**Lemma 4.1.** [23] *There exists $D_1 > 0$ such that for all $x \in N$, there exists a pleated surface $g : (S, \sigma) \to N$ with $g(S) \cap B_{D_1}(x) \neq \emptyset$.*

The following Lemma follows easily from the fact that $inj_N(x) > \epsilon_0$:

**Lemma 4.2.** [6],[23] *There exists $D_2 > 0$ such that if $g : (S, \sigma) \to N$ is a pleated surface, then $dia(g(S)) < D_2$.*

The following Lemma due to Minsky [18] follows from compactness of pleated surfaces.

**Lemma 4.3.** [18] *Fix $S$ and $\epsilon > 0$. Given $a > 0$ there exists $b > 0$ such that if $g : (S, \sigma) \to N$ and $h : (S, \rho) \to N$ are homotopic pleated surfaces which are isomorphisms on $\pi_1$ and $inj_N(x) > \epsilon$ for all $x \in N$, then*

$$d_N(g(S), h(S)) \leq a \Rightarrow d_{Teich}(\sigma, \rho) \leq b,$$

*where $d_{Teich}$ denotes Teichmuller distance.*

**Lemma 4.4.** *There exist $K, \epsilon$ and a homeomorphism $h$ from $E$ to the universal curve over a Lipschitz path in Teichmuller space, such that $h$ is a $(K, \epsilon)$-quasi-isometry.*

**Proof:** We can assume that $S \times \{0\}$ is mapped to a pleated surface $S_0 \subset N$ under the homeomorphism from $S \times [0, \infty)$ to $E$. We shall construct inductively a sequence of 'equispaced' pleated surfaces $S_i \subset E$ exiting the end. Assume that $S_0, \cdots, S_n$ have been constructed such that:

1. If $E_i$ be the non-compact component of $E \backslash S_i$, then $S_{i+1} \subset E_i$.
2. Hausdorff distance between $S_i$ and $S_{i+1}$ is bounded above by $3(D_1 + D_2)$.
3. $d_N(S_i, S_{i+1}) \geq D_1 + D_2$.
4. From Lemma 4.3 and condition (2) above there exists $D_3$ depending on $D_1$, $D_2$ and $S$ such that $d_{Teich}(S_i, S_{i+1}) \leq D_3$

Next choose $x \in E_n$, such that $d_N(x, S_n) = 2(D_1 + D_2)$. Then by Lemma 4.1, there exists a pleated surface $g : (S, \tau) \to N$ such that $d_N(x, g(S)) \leq D_1$. Let $S_{n+1} = g(S)$. Then by the triangle inequality and Lemma 4.2, if $p \in S_n$ and $q \in S_{n+1}$,

$$D_1 + D_2 \leq d_N(p, q) \leq 3(D_1 + D_2).$$

This allows us to continue inductively. The Lemma follows. □

Observe that $\widetilde{E}$ is quasi-isometric to a tree (in fact a ray) of hyperbolic metric spaces by setting $T = [0, \infty)$, with vertex set $\{n : n \in \mathbb{N} \cup \{0\}\}$, edge set $\{[n-1, n] : n \in \mathbb{N}\}$, $X_n = \widetilde{S_n} = X_{[n-1,n]}$. Further, by Lemma 4.3 this tree of hyperbolic metric spaces satisfies the quasi-isometrically embedded condition. We shall now describe $\widetilde{C(N)}$ as a tree of hyperbolic metric spaces. Assume $M \subset C(N)$ and $\partial M = \{F_1, \cdots, F_n\}$ where $F_i$ are pleated surfaces in $N$ cutting off ends $E_i$.

**Lemma 4.5.** [3] *$\pi_1(N)$ is hyperbolic in the sense of Gromov. Also, if $i : E \to N$, denotes inclusion, then $i_*\pi_1(E)$ is a quasiconvex subgroup of $\pi_1(N)$.*

**Remark:** In fact there exists a geometrically finite hyperbolic manifold homeomorphic to $N$. This is part of Thurston's monster theorem. See [16] for a different proof of the fact. Also, the limit set of a geometrically finite manifold is locally connected [2]. This shall be of use later.

Note that $\widetilde{M} \subset \widetilde{N}$ is quasi-isometric to the Cayley graph of $\pi_1(N)$. Hence, $\widetilde{M}$ is a hyperbolic metric space. Let $\widetilde{F_i} \subset \widetilde{N}$ represent a lift of $F_i$ to $\widetilde{N}$. Then, by Lemma 4.5 above, $\widetilde{F_i}$ is a word-hyperbolic metric space. If $\widetilde{E_i}$ is a lift of $E_i$ containing $\widetilde{F_i}$ then from our previous discussion, $\widetilde{E_i}$ is a ray of hyperbolic metric spaces. Since there are only finitely many ends $E_i$, we have thus shown:

**Lemma 4.6.** *The hyperbolic metric space $\widetilde{C(N)}$ is quasi-isometric to a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Further, we can choose a base vertex $v_0$ of $T$ such that $X_{v_0}$ is homeomorphic to $\widetilde{M}$.*

Applying Theorem 3.8, we get

**Theorem 4.7.** *Let $\Gamma$ be a geometrically tame Kleinian group, such that $\mathbb{H}^3/\Gamma = M$ has injectivity radius uniformly bounded below by some $\epsilon > 0$. Then there exists a continuous map from the Gromov boundary of $\Gamma$ (regarded as an abstract group) to the limit set of $\Gamma$ in $\mathbb{S}^2_\infty$.*

The above theorem has been independently proven by Klarreich [14] using different techniques.

**Lemma 4.8.** *Let $N$ be a geometrically tame 3-manifold with $inj_N(x) > \epsilon_0 > 0$ for all $x \in N$. Then the Gromov boundary of $\pi_1(N)$ is locally connected.*

**Proof:** This follows from the fact that there exists a geometrically finite manifold $M = \mathbb{H}^3/\Gamma$ homeomorphic to $N$ [16] and that for such an $M$, the limit set of $\Gamma$ is locally connected [2]. $\square$

Since a continuous image of a compact locally connected set is locally connected, Lemma 4.8 and Theorem 4.7 give:

**Corollary 4.9.** *Let $N = \mathbb{H}^3/\Gamma$ be a geometrically tame 3-manifold with $inj_N(x) > \epsilon_0$ for all $x \in N$. Then the limit set of $\Gamma$ is locally connected.*

Lemma 4.4 shows that there exists a quasi-isometry from a lift $\widetilde{E}$ of an end to the universal cover of a universal curve over a Lipschitz path $\sigma$ in $Teich(S)$. We show further that $\sigma$ is a Teichmuller quasigeodesic.

It is well known that geodesics in hyperbolic metric spaces diverge exponentially. The following proposition 'quasi-fies' this statement:

**Proposition 4.10.** *Given $\delta$, $A_0 \geq 0$ there exist $\beta$>1, $B > 0$, $K \geq 1$ and $\epsilon \geq 0$ such that if $[x, y], [y, z]$ and $[z, w]$ are geodesics in a $\delta$-hyperbolic metric space $(X, d)$ with $(x, z)_y \leq A_0$, $(y, w)_z \leq A_0$ and $d(y, z) \geq B$ then any path joining $x$ to $w$ and lying outside a $D$-neighborhood of $[y, z]$ has length greater than or equal to $\beta^D d(y, z)$, where $D = min\{(d(x, [y, z]) - 1), (d(w, [y, z]) - 1)\}$.*

**Lemma 4.11.** *$\sigma$ is a Teichmuller quasigeodesic.*

**Proof:** Let $S_0 = \partial E$ be a pleated surface containing a closed geodesic $l$ of $N$. This can always be arranged by taking a simple closed geodesic sufficiently far out in $E$ and mapping in a pleated surface containing it [23]. Construct a sequence of equispaced pleated surfaces as in Lemma 4.4. $\widetilde{E}$ is quasi-isometric to a ray of hyperbolic metric spaces $X$, with vertex set $\{n : n \in \mathbb{N} \cup \{0\}\}$, edge set $\{[n-1, n] : n \in \mathbb{N}\}$, $X_n = \widetilde{S_n} = X_{[n-1,n]}$.

Fix $x_0$ in $S_0$. Inductively, define $x_n$ to be the image of $x_{n-1}$ under the Teichmuller map from $S_{n-1}$ to $S_n$. Let $r$ denote a quasi-isometric embedding of $[0, \infty)$ sending $[n, n+1]$ to the shortest geodesic from $x_n$ to $x_{n+1}$. Then $r$ is a quasigeodesic in $E$. Let $[a, b]$ be a lift of $l$ to $\widetilde{E}$. Let $\lambda \subset X$ be the image of $[a, b]$ under a quasi-isometric homeomorphism

$h$ between $\widetilde{E}$ and $X$, sending $\widetilde{S_n}$ to $X_n$. Construct $B_\lambda \subset X$, as in the previous section. Lifts of $r$ through $a, b$ diverge exponentially.

¿From Theorem 3.6, $B_\lambda$ is quasiconvex and hence a hyperbolic metric space with the inherited metric. Let $r_1$, $r_2$ be the images of these lifts through the end-points of $\lambda$. Then $r_1$, $r_2$ are $(K, \epsilon)$-quasigeodesics diverging exponentially in $X$. Assume, after reparametrization if necessary, $r_1(n)$, $r_2(n) \in X_n$. Let $d_n$ denote the path metric on $X_n$. Then by Proposition 4.10, there exist $C_1 > 1, n \geq 1$ such that $d_{N+n}(r_1(N+n), r_2(N+n)) \geq C_1 d_N(r_1(N), r_2(N))$ for all $N \geq 0$. Hence there exists $C_2$ such that $d_{Teich}(S_N, S_{N+n}) \geq C_2$ for all $N \geq 0$. Since $\sigma$ was shown to be Lipschitz in Lemma 4.4, this proves that $\sigma$ is a Teichmuller quasigeodesic. □

So far arguments have been coarse. At this stage, we need to quote a part of the main theorem of [18].

**Theorem 4.12.** [18] *If $N$ is a geometrically tame hyperbolic 3-manifold with indecomposable fundamental group, such that there exists $\epsilon_0 > 0$ with $inj_N(x) > \epsilon_0$ for all $x \in N$, then each simply degenerate end $E$ of $N$ gives rise to a unique Teichmuller ray $r$, such that every pleated surface in $E$ lies at a uniformly bounded distance from $r$. Further, $r$ depends only on the corresponding ending lamination.*

That $r$ depends only on the corresponding ending lamination was proven by Masur [15].

Combining Lemma 4.11 and Theorem 4.12 we have a new proof of the main theorem of [19] : the ending lamination theorem for 3-manifolds with freely indecomposable fundamental group and a uniform lower bound on injectivity radius.

**Theorem 4.13.** *Let $N_1$ and $N_2$ be homeomorphic hyperbolic 3-manifolds with freely indecomposable fundamental group. Suppose there exists a uniform lower bound $\epsilon > 0$ on the injectivity radii of $N_1$ and $N_2$. If the end invariants of corresponding ends of $N_1$ and $N_2$ are equal, then $N_1$ and $N_2$ are isometric.*

**Proof:** From Lemma 4.11, corresponding simply degenerate ends $E_{i1}$, $E_{i2}$ of $N_1$ and $N_2$ are homeomorphic via quasi-isometries to universal curves over Teichmuller quasi-geodesics $l_{i1}$ and $l_{i2}$ lying in bounded neighbourhoods of Teichmuller geodesics $l_i$. Hence corresponding ends are homeomorphic via quasi-isometries to each other. Therefore $N_1$, $N_2$ are homeomorphic by a quasi-isometry. Finally, by [22] $N_1$ and $N_2$ are isometric.

## 5. Examples

Let $H$ be a hyperbolic subgroup of a hyperbolic group $G$.

**Definition :** [12] [9] *If $i : \Gamma_H \to \Gamma_G$ be an embedding of the Cayley graph of $H$ into that of $G$, then the distortion function is given by*

$$disto(R) = R^{-1} Diam_{\Gamma_0}(\Gamma_0 \cap B(R)),$$

*where $B(R)$ is the ball of radius $R$ around $1 \in \Gamma_G$.*

All previously known examples of non-quasiconvex hyperbolic subgroups of hyperbolic groups exhibit exponential distortion. We construct in this section some examples exhibiting greater distortion. Some of these will be shown to have Cannon-Thurston maps. For the rest, existence of Cannon-Thurston maps is not yet known. Further, we shall describe certain examples of free subgroups of $PSL_2\mathbb{C}$ and show that they exhibit arbitrarily large distortion. The existence of Cannon-Thurston maps for some of these is not yet known.

Our starting point for constructing distorted subgroups of hyperbolic groups is the following Lemma of Bestvina, Feighn and Handel [4]:

**Lemma 5.1.** [4] *There exists a hyperbolic group $G$ such that $1 \to F \to G \to F \to 1$ is exact, where $F$ is free of rank 3.*

Let $F_1 \subset G$ denote the normal subgroup. Let $F_2 \subset G$ denote a section of the quotient group. Let $G_1, \cdots, G_n$ be $n$ distinct copies of $G$. Let $F_{i1}$ and $F_{i2}$ denote copies of $F_1$ and $F_2$ respectively in $G_i$. Let

$$G = G_1 *_{H_1} G_2 * \cdots *_{H_{n-1}} G_n$$

where each $H_i$ is a free group of rank 3, the image of $H_i$ in $G_i$ is $F_{i2}$ and the image of $H_i$ in $G_{i+1}$ is $F_{(i+1)1}$. Then $G$ is hyperbolic. This follows inductively from the fact that the image of $H_i$ in $G_i$ is quasiconvex in $G_1 *_{H_1} G_2 * \cdots *_{H_{i-1}} G_i$ and the main combination theorem of [3].

Let $H = F_{11} \subset G$. Then the distortion of $H$ is superexponential for $n > 1$. In fact, it can be readily checked that the distortion function is an iterated exponential of height $n$.

Note further that $G_1 *_{H_1} G_2$ can be regarded as a graph of groups with one vertex and three edges, where the vertex group is $G_1$ and edge groups are isomorphic to $F$. Then from Corollary 3.9, the pair $(G_1, G_1 *_{H_1} G_2)$ has a Cannon-Thurston map. Proceeding inductively and observing that a composition of Cannon-Thurston maps is a Cannon-Thurston map, we see that $(H, G)$ has a Cannon-Thurston map.

The next class of examples are not known to have Cannon-Thurston maps:

Our starting point is again Lemma 5.1. Let $a_1, a_2, a_3$ be generators of $F_1$ and $b_1, b_2, b_3$ be generators of $F_2$. Then

$$G = \{a_1, a_2, a_3, b_1, b_2, b_3 : b_i^{-1} a_j b_i = w_{ij}\}$$

where $w_{ij}$ are words in $a_i$'s. We add a letter $c$ conjugating $a_i$'s to 'sufficiently random' words in $b_j$'s to get $G_1$. Thus,

$$G_1 = \{a_1, a_2, a_3, b_1, b_2, b_3, c : b_i^{-1} a_j b_i = w_{ij}, c^{-1} a_i c = v_i\},$$

where $v_i$'s are words in $b_j$'s satisfying a small-cancellation type condition to ensure that $G_1$ is hyperbolic. See [13], pg. 151 for details on addition of 'random' relations.

It can be checked that these examples have distortion function greater than any iterated exponential.

The above set of examples were motivated largely by examples of distorted cyclic subgroups in [12], pg. 67.

So far, there is no satisfactory way of manufacturing examples of hyperbolic subgroups of hyperbolic groups exhibiting arbitrarily high distortion. It is easy to see that a subgroup of sub-exponential distortion is quasiconvex [12]. Not much else is known. For instance, one does not know if $A^{n^2}$ can appear as a distortion function.

The situation is far more satisfactory in the case of Kleinian groups. The following class of examples appears in work of Minsky [17]:

Let $S$ be a hyperbolic punctured torus so that the two shortest geodesics $a$ and $b$ are orthogonal and of equal length. Let $S_0$ denote $S$ minus a neighborhood of the cusp. Let $N_\delta(a)$ and $N_\delta(b)$ be regular collar neighborhoods of $a$ and $b$ in $S_0$. For $n \in \mathbb{N}$, define $\gamma_n = a$ if $n$ is even and equal to $b$ if $n$ is odd. Let $T_n$ be the open solid torus neighborhood of $\gamma_n \times \{n + \frac{1}{2}\}$ in $S_0 \times [0, \infty)$ given by

$$T_n = N_\delta(\gamma_n) \times (n, n+1)$$

and let $M_0 = (S_0) \times [0, \infty) \backslash \bigcup_{n \in \mathbb{N}} T_n$.

Let $a(n)$ be a sequence of positive integers greater than one. Let $\hat{\gamma_n} = \gamma_n \times \{n\}$ and let $\mu_n$ be an oriented meridian for $\partial T_n$ with a single positive intersection with $\hat{\gamma_n}$. Let $M$ denote the result of gluing to each $\partial T_n$ a solid torus $\hat{T_n}$, such that the curve $\hat{\gamma_n}^{a(n)} \mu_n$ is glued to a meridian. Let $q_{nm}$ be the mapping class from $S_0$ to itself obtained by identifying $S_0$ to $S_0 \times m$, pushing through $M$ to $S_0 \times n$ and back to $S_0$. Then $q_{n(n+1)}$ is given by $\Phi_n = D_{\gamma_n}^{a(n)}$, where $D_c^k$ denotes Dehn twist along $c$, $k$ times. Matrix representations of $\Phi_n$ are given by

$$\Phi_{2n} = \begin{pmatrix} 1 & a(2n) \\ 0 & 1 \end{pmatrix}$$

and

$$\Phi_{2n+1} = \begin{pmatrix} 1 & 0 \\ a(2n+1) & 1 \end{pmatrix}$$

Recall that the metric on $M_0$ is the restriction of the product metric. $\hat{T_n}$'s are given hyperbolic metrics such that their boundaries are uniformly quasi-isometric to $\partial T_n \subset M_0$. Then from [17], $M$ is quasi-isometric to the complement of a rank one cusp in the convex core of a hyperbolic manifold $M_1 = \mathbb{H}^3/\Gamma$. Let $\sigma_n$ denote the shortest path from $S_0 \times 1$ to $S_0 \times n$. Let $\overline{\sigma_n}$ denote $\sigma_n$ with reversed orientation. Then $\tau_n = \sigma_n \gamma_n \overline{\sigma_n}$ is a closed path in $M$ of length $2n+1$. Further $\tau_n$ is homotopic to a curve $\rho_n = \Phi_1 \cdots \Phi_n(\gamma_n)$ on $S_0$. Then

$$\Pi_{i=1\cdots n} a(i) \leq l(\rho_n) \leq \Pi_{i=1\cdots n}(a(i)+2)$$

Hence

$$\Pi_{i=1\cdots n} a(i) \leq (2n+1) disto(2n+1) \leq \Pi_{i=1\cdots n}(a(i)+2)$$

Since $M$ is quasi-isometric to the complement of the cusp of a hyperbolic manifold and $\gamma_n$'s lie in a complement of the cusp, the distortion function of $\Gamma$ is of the same order as the distortion function above. In particular, functions of arbitraily fast growth may be realised. This answers a question posed by Gromov [12] pg. 66.

Manifolds with unbounded $a(n)$'s are not known to have Cannon-Thurston maps.

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