ASYMPTOTIC GEOMETRY AND CONFORMAL TYPES OF CARNOT–CARATHÉODORY SPACES

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Abstract. An intrinsic definition in terms of conformal capacity is proposed for the conformal type of a Carnot–Carathéodory space (parabolic or hyperbolic). Geometric criteria of conformal type are presented. They are closely related to the asymptotic geometry of the space at infinity and expressed in terms of the isoperimetric function and the growth of the area of geodesic spheres. In particular, it is proved that a sub-Riemannian manifold admits a conformal change of metric that makes it into a complete manifold of finite volume if and only if the manifold is of conformally parabolic type. Further applications are discussed, such as the relation between local and global invertibility properties of quasiconformal immersions (the global homeomorphism theorem).

1. Carnot–Carathéodory Metric and Sub-Riemannian Manifolds

A Carnot–Carathéodory (C-C) metric on a manifold appears when one is allowed to travel not along arbitrary paths but along *distinguished* ones on the manifold.

The main example of this kind is *polarized Riemannian manifolds*.

A *polarization* of a manifold M is a subbundle H of the tangent bundle TM. The polarization H distinguishes a set of directions (tangent vectors) in M which are usually called *horizontal*.

A piecewise smooth curve in M is called *distinguished*, *admissible*, or *horizontal* with respect to H if the tangent vectors to this curve are horizontal, i.e., belong to H.

The Carnot–Carathéodory (C-C) metric $d_{H,g}(x_1, x_2)$ associated to the polarization H of the Riemannian manifold (M, g) is defined as the infimum of the g-lengths of the distinguished curves joining the points x_1 and x_2.

Thus

$$d_{H,g}(x_1, x_2) := \inf\,(g\text{-length of } H\text{-horizontal curves between } x_1 \text{ and } x_2).$$

This distance obviously satisfies the axioms of a metric, provided that every two points in M can be connected by a distinguished curve.

This connectivity property takes place if the Lie brackets $[[H, H] \dots]$ of H span the tangent bundle TM.

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This is the case, e.g., for contact subbundles and for general non-holonomic (completely non-integrable) distributions.

(About Carnot–Carathéodory spaces we refer to [Gro], where one can find also a broad bibliography.)

Notice that to define the Carnot–Carathéodory metric on (M, g) one needs a Riemannian structure only on the polarization $H \subset TM$ (but not on the whole tangent bundle TM).

A smooth manifold M with a polarization $H \subset TM$ and a Riemannian structure g on H is refered to as the *sub-Riemannian manifold* (M, H, g).

We assume that the polarization H on M is *generic* in the sense that H is a smooth distribution of subspaces tangent to M, of equal dimension, and such that Lie brackets (commutators of all degrees) of H span the tangent bundle TM.

We also assume that the polarization H is *equiregular*, i.e., that the dimension of the subspace in T_xM generated by the commutators of fixed degree does not depend on $x \in M$. Then the tangent bundle TM is filtered by smooth subbundles

$$H = H_1 \subset H_2 \subset \ldots \subset H_j \subset \ldots \subset H_k = TM \tag{1}$$

such that H_j is spanned by the j-th degree commutators of the fields in H.

For example, this is the case if M is a Lie group and H is left (or right) translation-invariant non-integrable field of tangent spaces, obtained by translation of the corresponding subspace $t_0M \subset T_0M$ tangent to M at the neutral element 0 of the group. Here t_0M is supposed to generate whole the Lie algebra of the group.

The Heisenberg group H^n with natural (but very unusual) translation-invariant metric structure is, probably, the most important example of this kind.

As the simplest example ('flat' in an approrpiate sense), H^n plays the same role relative to general strictly pseudoconvex CR-manifolds, as $\mathbb{R}^n$ relative to Riemannian manifolds.

(The structure of the Heisenberg group H^n of dimension $n = 2l + 1$ is the same as the structure of the unit sphere S^n in $\mathbb{C}^{l+1}$. Mostow observed the close relation between rigidity properties of homogeneous spaces and quasiconformal mappings of the corresponding sphere at infinity [Mos]. Complete framework for a theory of quasiconformal mappings on the Heisenberg group is presented in [KR2].)

2. Hausdorff Measure on Carnot–Carathéodory Space

Consider a sub-Riemannian manifold (M^n, H, g) i.e., the smooth n-dimensional manifold M^n with 'horizontal' subbundle $H \subset TM^n$ and a Riemannian metric g on H.

We suppose the subbundle H is regular in the sense defined above. Then H and g induce the *Carnot–Carathéodory metric* $d_{H,g}$ on M^n and turn M^n into *Carnot–Carathéodory manifold* (with horizontal curves as distinguished ones).

Now one can consider the Hausdorff measure of any degree with respect to this Carnot–Carathéodory metric.

It is important to notice that the *metric* (or the *Hausdorff*) *dimension* m of the Carnot–Carathéodory space $(M^n, d_{H,g})$ is usually greater than the topological dimension n of the manifold M^n.

Namely,

$$m = \sum_{j=1}^{k} j \cdot \text{ rank } (H_j/H_{j-1}),$$

where $H = H_1 \subset \ldots \subset H_j \subset \ldots \subset H_k = TM$ is the commutator filtration defined in (1) (we assume $H_0 = \emptyset$).

The inequality $n < m$ is related to the shape of the Carnot–Carathéodory ball. Carnot–Carathéodory metric is highly non-isotropic and non-homogeneous in contrast to the usual Euclidean one.

Nevertheless, the Hausdorff measure (volume) induced by Carnot–Carathéodory metric may coincide with the usual Lebesque measure on a Riemannian manifold or with the invariant Haar measure on a Lie group.

For instance, the Haar measure on the Heisenberg group H^n coincides with the Lebesque measure on it ($H^n \approx \mathbb{R}^n$), as well as with the Hausdorff measure induced by the above Carnot–Carathéodory metric on H^n.

At the same time the metric (Hausdorff) dimension m of the Heisenberg group (or any domain in it) with respect to the Carnot–Carathéodory metric is equal to $n+1$.

The corresponding Carnot–Carathéodory Hausdorff dimension of a smooth hypersurface that bounds a domain in H^n is equal to n.

A change of the Riemannian structure g on the horizontal bunde H of the sub-Riemannian manifold (M^n, H, g) results in locally quasiisometric change of the Carnot–Carathéodory metric and thus it does not change the Hausdorff dimension and many other characteristics of the Carnot–Carathéodory space. This is why one often uses a shorter notation (M^n, H) for the Carnot–Carathéodory structure on the manifold M^n.

3. Horizontal Gradient

The Riemannian structure g on the horizontal subbundle H is, nevertheless, very useful for the following definition.

The *horizontal gradient* ∇f (or $\text{grad}_{H,g} f$) of a function f on M^n is defined as the unique horizontal vector such that

$$\langle \nabla f, X \rangle_g = Xf$$

for all horizontal vectors X.

Here $\langle , \rangle g$ is the inner product with respect to g and Xf is the Lie derivative of f along X (that is supposed to exist).

The *horizontal normal unit vector* to a hypersurface $\{f = 0\}$ is defined by

$$\mathbf{n} := \frac{1}{\|\nabla f\|}\nabla f.$$

This is the horizontal normal pointing outward for the domain $\{f < 0\}$.

The vector $\mathbf{n}(x)$ is undefined at points $x \in M^n$ where $\nabla f = 0$. However, such points form a submanifold of essentially lower dimension since the horizontal subbundle H is completely non-integrable. We shall use this remark below without additional comments.

The sub-Riemannian structure (M^n, H, g) on the manifold may be considered (see e.g., [KR1]) as a limit of the following Riemannian structures (M^n, g_τ).

Consider the decomposition $TM^n = H \oplus H'$. Fix n independent vector fields $X_1, \dots, X_d, \dots X_n$ such that $X_1, \dots, X_d$ span H and are orthonormal with respect to the Riemannian structure g on H, and $X_{d+1}, \dots, X_n$ span H'.

We introduce the Riemannian structure g_τ on M^n by the condition that the vectors $\tau X_{d+1}, \dots, \tau X_n$ $(\tau > 0)$ also form an orthonormal system.

The one parameter family of Riemannian metrics g_τ tends to the singular metric g as $\tau \to 0$ in the sense that the length l_τ of any curve $\gamma \subset M^n$ measured with respect to g_τ tends to the Carnot–Carathéodory length l of γ with respect to the initial sub-Riemannian structure (H, g) on M^n.

This remark allows one to clarify the Carnot–Carathéodory counterparts of some clasical notions and relations that we use below.

For instance, the horizontal gradient is the limit of Riemannian g_τ-gradients as $\tau \to 0$.

The classical Fubini-type integral formula

$$\int f\,dv = \int_{\mathbb{R}} dt \int_{\{u=t\}} f\;\frac{d\sigma}{|\nabla u|} \tag{2}$$

(where the domain of integration is foliated by the level surfaces $\{u = t\}$ of a function u) remains valid for sub-Riemannian manifolds with respect to the induced Hausdorff volume and area measures $dv, d\sigma$ and the horizontal grandient ∇u, respectively.

In the special case when $f \equiv 1$ and u is the distance to the fixed point $O \in M^n$, we obtain the standard relation

$$V'(r) = S(r) \tag{3}$$

between the volume of the r-ball and the area of its boundary.

The formula (2) also imply the general relation

$$dV = \sigma\,dl \tag{4}$$

between the volume variation of a regular domain and the area σ of its boundary under the horizontally-normal variation of the boundary.

The formula (4) is essentially local: it remains valid for the local variation of the boundary as well. In this case σ should be replaced by the area $d\sigma$ of the variated part of the boundary.

Thus,

$$dv = d\sigma \ dl \tag{5}$$

where dl is the oriented Carnot–Carathéodory length of the local infinitesimal displacement in the direction horizontally normal to the variated germ of the hypersurface. (One may consider (5) as the volume formula for the cylinder.)

Notice that the volume and area integrals in the left- and right-hand sides of the formula (2) are meaningful even when the function under the integration is not well-defined on some set of volume zero or area zero respectively.

For instance, the Carnot–Carathéodory sphere is not a smooth hypersurface. Thus the horizontal normal is not well-defined somewhere. But the singularities form a set of area zero.

4. Capacity

In order to characterize conformal types of Riemannian manifolds we used in [ZK1] some conformal invariants. Namely, the capacity of an annulus (ring) and the modulus (extremal length) of a family of manifolds (curves). We need similar special conformal invariants for the sub-Riemannian case. However, we start with describing the general notion of capacity (in the spirit of [Maz]) adopted for sub-Riemannian manifolds.

Consider a smooth n-dimensional manifold M^n with a regular horizontal subbundle $H \subset TM^n$ and a Riemannian metric g on H. Thus, (M^n, H, g) is a sub-Riemannian manifold.

Let m be its Hausdorff dimension with respect to the induced Carnot–Carathéodory metric, and $dv, d\sigma$ are the volume and area elements, i.e., elements of Hausdorff m-measure for domains and $(m{-}1)$-measure for hypersurfaces in M^n.

Let $\Phi(x, \xi)$ be a nonnegative continuous function on the tangent bundle TM^n which is positive homogeneous of the first degree with respect to $\xi \in T_xM^n$.

Let C be a compact set in a domain $D \subset M^n$ and $A = A(C, D)$ be the set $\{u\}$ of smooth nonnegative functions with compact support in D such that $u \equiv 1$ in a neighborhood of C. We refer to them as *admissible functions* for the pair (C, D).

The (p, Φ)-*capacity* of the set C relative to the domain D is defined as follows

$$(p, \Phi) - \operatorname{cap}(C, D) := \inf_{u \in A} \int_D \Phi^p(x, \nabla u) \ dv(x), \tag{6}$$

where the infimum is taken over all admissible functions; ∇u is the horizontal gradient; $dv(x)$ is the volume element of Hausdorff m-measure induced by the Carnot–Carathéodory metric of (M^n, H, g).

We need this general notion of capacity only for a special case of C, D, p and for $\Phi(x, \nabla u) = |\nabla u|$. Nevertheless, it is reasonable to mention some useful capacity relations in their general form.

One can rewrite the definition (6) of the capacity in the following form for $p > 1$:

$$(p,\Phi)-\operatorname{cap}(C,D) = \inf_{u\in A}\left(\int_0^1 \frac{dt}{\left(\int_{\{u=t\}} \Phi^p(x,\nabla u)\frac{d\sigma}{|\nabla u|}\right)^{1/p-1}}\right)^{1-p}, \tag{7}$$

(see [Maz, section 2.2.2]). Here the inner integration is over the level surface $\{u = t\}$ of the function u and $d\sigma$ is the element of area, i.e., of Hausdorff $(m-1)$-measure induced on the level hypersurface by the Carnot–Carathéodory metric.

This new representation (7) leads to the estimates of capacity we need below.

Set

$$P(v) := \inf_{G:|G|\geq v} \int_{\partial G} \Phi(x, \mathbf{n}(x))\ d\sigma, \tag{8}$$

where the infimum is taken over all domains $G \subset M^n$ with regular boundary ∂G and such that the volume $|G|$ of G is not less than v; $\mathbf{n}(x)$ is the unit vector horizontally normal to ∂G at the point $x \in \partial G$ directed towards the interior of G.

(The integral in (8) is well-defined even if ∂G fails to be smooth on some subset of Hausdorff $(m-1)$-measure zero.)

Thus $P(v)$ in (8) may be considered as a generalized isoperimetric function. For $\Phi(x, \xi) := |\xi|$ the function $P(v)$ is the ordinary isoperimetric function of the manifold. Indeed, in this case for any regular domain G the following relation is fulfilled

$$P(v) \leq S, \tag{9}$$

where v is the volume (Hausdorff m-measure) of G and S is the area (Hausdorff $(m-1)$-measure) of the boundary ∂G.

Recall that a function $P : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be the *isoperimetric function* on a manifold (equipped with volume and area measures) if for every domain G with regular boundary ∂G the relation (9) holds.

If the relation (9) holds for a special family of domains they say that P is the isoperimetric function for this family.

For example, we need below such an isoperimetric function for the large Carnot–Carathéodory balls that form an exhaustion of a sub-Riemannian manifold.

At the moment we need function P in (8) only for domains G bounded by level surfaces of a function u admissible for the pair (C, D).

By means of this function and of representation (7) of the capacity one obtains the following inequality

$$(p,\Phi)-\operatorname{cap}(C,D) \geq \left(\int_0^{|D|} \frac{dv}{P^{p/p-1}(v)}\right)^{1-p}, \tag{10}$$

where $|D|$ is the volume of the domain D and P is defined by the formula (8) applied to $G = \{u \leq t\}$.

The proof of this inequality for a sub-Riemannian manifold is the same as for the case of a Riemannian one (see [Maz, section 2.2.2]).

Now turn to the special cases we need.

Let (M^n, H, g) be the sub-Riemannian manifold which is not compact, but complete with respect to the natural Carnot–Carathéodory metric.

Fix a point $0 \in M^n$ that will play the role of the origin. Let $B(r)$ be the ball of Carnot–Carathéodory radius r centered at 0, $v(r)$ the volume (Hausdorff m-measure) of $B(r)$, $S(r)$ the area (Hausdorff $(m-1)$-measure) of the sphere $\partial B(r)$, $R_a^b := B(b)\backslash \bar{B}(a)$ the annulus (or ring) domain whose boundary components are spheres of radius a and b $(a < b)$ respectively.

For $\Phi(x, \xi) := |\xi|$ define

$$\operatorname{cap}_p R_a^b := (p, \Phi) - \operatorname{cap}(\bar{B}(a), B(b)). \tag{11}$$

For this function $\Phi(x, \xi) = |\xi|$ and for the special choice $u(x) = (b-a)^{-1}(r(x) - a)$ of the admissible function, where $r(x)$ is the Carnot–Carathéodory distance of the point x to the origin 0, by means of representation (7) of the capacity we obtain the following estimate for the p-capacity of the annulus R_a^b:

$$\operatorname{cap}_p R_a^b \leq \left(\int_a^b \frac{dr}{S^{\frac{1}{p-1}}(r)} \right)^{1-p}. \tag{12}$$

Along with (10) applied to the isoperimetric function $P(v)$ of geodesic balls $B(r)$, we finally obtain the following estimates

$$\left(\int_{v(r_1)}^{v(r_2)} P^{\frac{p}{p-1}} \right)^{1-p} \leq \operatorname{cap}_p R_{r_1}^{r_2} \leq \left(\int_{r_1}^{r_2} S^{-\frac{1}{p-1}} \right)^{1-p}, \tag{13}$$

where $v(r_1), v(r_2)$ are volumes of $B(r_1)$ and $B(r_2)$ respectively and $p > 1$ as in (10) and (12).

Remind that in our special case

$$\operatorname{cap}_p R_a^b := \inf \int_{M^n} |\nabla u|^p(x)\, dv(x), \tag{14}$$

where the infimum is taken over the smooth functions such that $u \equiv 0$ in the neighborhood of $B(a)$ and $u \equiv 1$ in the neighborhood of $M^n \backslash B(b)$, while $dv(x)$ is the element of the volume (Hausdorff m-measure) generated by the Carnot–Carathéodory metric of the sub-Riemannian manifold (M^n, H, g).

Conformal change $\lambda^2 g$ of the Riemannian tensor g on the bundle H produces local rescaling of the Carnot–Carathéodory lengths element by factor λ, of horizontal gradient by factor λ^{-1}, and of Hausdorff m-measure element by factor λ^m.

Thus, for $p = m$ the integral in (14) is invariant with respect to conformal changes of the sub-Riemannian structure and the Carnot–Carathéodory metric.

We get the following conformally invariant capacity of the annulus (ring, or condenser) R_a^b in the Carnot–Carathéodory space (M^n, H, g):

$$\operatorname{cap}_m R_a^b = \inf \int_{M^n} |\nabla u|^m(x)\, dv(x), \tag{15}$$

where m is the Hausdorff dimension of the Carnot–Carathéodory space (M^n, H, g), $dv(x)$ is the element of the induced Hausdorff m-measure on M^n, ∇u is the horizontal gradient of the function u, and infimum is taken over all admissible functions u mentioned above.

It is well known (see e.g., [Vai]) that the conformal capacity of a spherical condenser R_a^b in $\mathbb{R}^n$ (thus $m = n$) is equal to

$$\operatorname{cap}_n R_a^b = \omega_{n-1} \left(\ln \frac{b}{a}\right)^{-n+1}, \tag{16}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

For the case of the Heisenberg group H^n (instead of $\mathbb{R}^n$), equipped with the natural $n{-}1$-dimensional polarization H and the Carnot-Carathéodory metric, the conformal capacity $(m = n + 1)$ of the spherical (with respect to Carnot-Carathéodory metric) condenser R_a^b is equal (see [KR1]) to

$$\operatorname{cap}_{n+1} R_a^b = \omega_{n-1} \left(\ln \frac{b}{a}\right)^{-n}, \tag{17}$$

where ω_{n-1} is as above.

After these preparations one can carry over to sub-Riemannian manifolds the main notions and results described in [ZK1] for Riemannian manifolds.

Below we formulate the notions and results in full detail, sometimes with comments, but we omit the proofs parallel to those in [ZK1].

5. Ahlfors–Gromov Lemma

Most of standard geometric characteristics of a space are not conformally invariant. For instance, the unit ball $B^n \subset \mathbb{R}^n$ is flat with respect to Euclidean metric, but it admits a conformally Euclidean metric of constant negative curvature (the Poincaré metric) and at the same time admits a conformal (stereographic) projection to the sphere $S^n \subset \mathbb{R}^{n+1}$ of positive constant curvature.

Nevertheless, some of geometric relations (that are actually homogeneous-like or so) preserve or change in a controllable way under conformal changes of the initial metric.

Recall the following useful Ahlfors–Gromov lemma that we formulated now for sub-Riemannian manifolds.

Lemma. *If two sub-Riemannian manifolds (M^n, H, g) and $(M^n, H, \tilde{g})$ are conformally equivalent, then*

$$\int_{\tilde{v}(r_0)}^{\tilde{v}(r_1)} \tilde{P}^{\frac{m}{1-m}} \geq \int_{r_0}^{r_1} S^{\frac{1}{1-m}}$$

where m is the common Hausdorff dimension of the Carnot–Carathéodory spaces under consideration; $\tilde{P}$ is any isoperimetric function of $(M^n, H, \tilde{g})$; $\tilde{v}(r_0), \tilde{v}(r_1)$ are volumes in $(M^n, H, \tilde{g})$ of any two concentric Carnot–Carathéodory balls $B(r_0), B(r_1)$ of the Carnot–Carathéodory space (M^n, H, g); and $S = S(r)$ is the area (Hausdorff $(m-1)$-measure) of the sphere $\partial B(r)$ in (M^n, H, g).

The proof of the lemma for a sub-Riemannian manifold is parallel to that for the Riemannian case (see e.g., [ZK1]).

The proof shows that Ahlfors–Gromov lemma remains valid even if $\tilde{P}$ is not a universal $\tilde{g}$-isoperimetric function but is a $\tilde{g}$-isoperimetric function for g-balls only.

6. Conformal Types of Sub-Riemannian Manifolds

Consider a noncompact manifold M^n endowed with a horizontal subbundle $H \subset TM^n$ and a Riemannian metric g on H that induces the Carnot–Carathéodory metric on M^n. Let m be the Hausdorff dimension of a sub-Riemannian manifold (M^n, H, g) with respect to the induced Carnot–Carathéodory metric.

Let C be a nondegenerate compact set in M^n, say, a ball. We are interested in the conformal capacity $\operatorname{cap}_m(C, M^n)$.

The manifold M^n, or more precisely, the sub-Riemannian manifold (M^n, H, g), is called *conformally parabolic* or *conformally hyperbolic* if $\operatorname{cap}_m(C, M^n) = 0$ or $\operatorname{cap}_m(C, M^n) > 0$ respectively.

The capacity relations described in the definition do not depend on the choice of the compact set C and reflect some conformally invariant properties of the manifold 'at infinity'.

In other words, for the manifold of conformally parabolic type one has

$$\lim_{b \to +\infty} \operatorname{cap}_m R_a^b = 0,$$

and for conformally hyperbolic manifold

$$\lim_{b \to +\infty} \operatorname{cap}_m R_a^b > 0$$

independently of $a > 0$.

The formula (16) shows that the standard Euclidean space $\mathbb{R}^n$ is of conformally parabolic type.

If we consider the ordinary hyperbolic space modeled on the Euclidean unit ball, we conclude by means of (16) that the Lobachevsky space of constant negative curvature is of conformally hyperbolic type.

The formula (17) shows that the Heisenberg group as a sub-Riemannian manifold is of conformally parabolic type.

The situation changes if we consider the same group equipped with a translation-invariant Riemannian structure.

Notice, that the in both these cases the Heisenberg group H^n by itself, endowed with the word-length metric, induces isoperimetric inequalities with the same isoperimetric function $P(v) = v^{\frac{n+1}{n}}$.

But the Hausdorff dimension m of the manifold (or rather Carnot–Carathéodory space) H^n is different in the two cases under consideration.

In the former (sub-Riemannian) case $m = n + 1$ while in the latter (Riemannian) one $m = n$.

The left-hand-side of (13) with $p = n$ and $P(v) = v^{\frac{n+1}{n}}$ shows now that the Heisenberg group H^n equipped with the translation-invariant Riemannian metric is indeed a manifold of conformally hyperbolic type.

Note that the Carnot–Carathéodory metric in many respects is more adequate for the Heisenberg group than the translation-invariant Riemannian one. The Carnot–Carathéodory metric on H^n is also translation-invariant but, in addition, this metric follows the similitudes of the Heisenberg group. It changes by a factor under such homogeneous transformations.

These two metrics on the Heisenberg group are not conformal or quasiconformal equivalent even locally.

By the way, every metric space that admits self similitudes (such as $\mathbb{R}^n$ with Euclidean metric or H^n with Carnot–Carathéodory metric) must be of conformally parabolic type.

7. Asymptotic Geometry and Conformal Types of Sub-Riemannian Manifolds

As it was mentioned above the conformal type of a manifold depends on the behavior of the manifold at infinity.

We present now an explicit geometric version of this general claim.

We say that a certain property or relation is *realizable in a class of metrics* if it is valid for some metric of the class.

For instance, any sub-Riemannian manifold (M^n, H, g) can be realized as a complete one in the conformal class of its metric.

Below under conformal change of the metric on sub-Riemannian manifold (M^n, H, g) we mean a conformal change $\tilde{g} = \lambda^2 g$ of the Riemannian structure on the horizontal bundle H. It results in the conformal change of the corresponding Carnot–Carathéodory metric.

Now we are in a position to formulate the following theorem.

Theorem. *Let (M^n, H, g) be a noncompact sub-Riemannian manifold of the Hausdorff dimension m with respect to the induced Carnot–Carathéodory metric.*

The manifold is of conformally parabolic type if and only if any of the following equivalent conditions is realizable in the class of complete metrics conformally equivalent to the initial one

(i) $\mathrm{vol}_m(M^n) < \infty$,
(ii) $\int^{\infty} S^{\frac{1}{1-m}}(r)\, dr = \infty$,
(iii) $\int^{\infty} (\frac{r}{v(r)})^{\frac{1}{m-1}}\, dr = \infty$,
(iv) $\liminf_{r\to\infty} \frac{v(r)}{r^m} < \infty$.

Here, as above, $v(r)$ and $S(r)$ are the volume and area (i.e., the Hausdorff m-measure and $(m-1)$-measure) respectively of the Carnot–Carathéodory ball $B(r)$ and its boundary sphere $\partial B(r)$; $\mathrm{vol}_m(M^n)$ is the Hausdorff m-measure of the whole manifold M^n.

The proof is similar to that for the Riemannian manifolds and we omit it (see [ZK1]).

8. Canonical Forms of Isoperimetric Function

In this section we supplement two statements related to possible variations of the isoperimetric function under conformal changes of the Carnot–Carathéodory metric.

Consider a g-spherical exhaustion of a sub-Riemannian manifold (M^n, H, g), i.e., a system of Carnot–Carathéodory balls $B(r)$ of varying radius r and fixed center $0 \in M^n$. The system is invariant under the *spherically conformal change* $\tilde{g} = \lambda^2(r)g$ of the Riemannian metric g on H. Let m be the Hausdorff dimension of (M^n, H, g) with respect to the induced Carnot–Carathéodory metric.

We denote by $v(r)$, $S(r)$ and $\tilde{v}(r)$, $\tilde{S}(r)$ the volume (Hausdorff m-measure) of the ball $B(r)$ and the area (Hausdorff $(m-1)$-measure) of the sphere $\partial B(r)$ considered in spaces (M^n, H, g) and $(M^n, H, \tilde{g})$ respectively. Let P be a nonnegative function on a sub-Riemannian manifold (M^n, H, g). Consider the class of metrics spherically conformally equivalent to the given one along with the corresponding spherical exhaustion of the manifold.

Proposition 1. *The function P is an isoperimetric function and, moreover, the maximal one (i.e., $P(\tilde{v}) = \tilde{S}$) for sufficiently large balls of the spherical exhaustion in a certain metric from this class if and only if the integrals*

$$a)\ \int^{\bullet} P^{\frac{m}{1-m}} \quad , \qquad b)\ \int^{\bullet} S^{\frac{1}{1-m}}$$

converge or diverge simultaneously at the upper bound of the domains of integrands.

Moreover, the new Carnot–Carathéodory metric (induced by $\tilde{g} = \lambda^2(r)g$) is complete if and only if $\int^{\bullet} P^{-1} = \infty$.

Proposition 2. *Let (M^n, H, g) be a sub-Riemannian manifold of the Hausdorff dimension m with respect to the Carnot–Carathéodory metric on M^n induced by the Riemannian structure g on the horizontal subbundle $H \subset TM^n$.*

Consider the class of metrics conformally equivalent to the given one on (M^n, H, g) along with the geodesic exhaustion of M^n, i.e., the exhaustion by balls with respect to the corresponding Carnot–Carathéodory metric.

Then in this class of conformal metrics and of the corresponding geodesic exhaustion the maximal isoperimetric function (for large balls) can be reduced to the following canonical forms

$$P(\tilde{v}) = \tilde{v}^{\frac{m-1}{m}} \qquad or \qquad P(\tilde{v}) = \tilde{v}$$

according to whether the sub-Riemannian manifold (M^n, H, g) is of conformally parabolic or conformally hyperbolic type respectively.

The Proposition 2 is in a sense incomplete. Roughly speaking, any function P may occur on the conformally parabolic manifold as the maximal isoperimetric function for large balls of the geodesic exhaustion corresponding to a certain metric conformally equivalent to the initial one. Besides, it is sometimes possible to get the canonical form of the isoperimetric function even by the spherically conformal change of the initial metric.

At last, there is no indication to what extent the Proposition 2 is invertible.

We omit proofs of Proposition 1 and Proposition 2. They are similar to the ones for Riemannian manifolds (cf. [ZK1]). Notice, however, that Proposition 2 can be essentially developed in spirit of the paper [ZK2].

9. Concluding Remarks

Asymptotic geometry or, in other words, behavior of the manifold (space) at infinity sometimes plays a decisive role in global problems. For instance, it is responsible for the existence of special solutions of operators on the manifold. In the case of the Laplace operator the problem is often closely related to (and for two dimensional manifolds coincides with) the problem of the manifold type discussed above.

Conformal type of the manifold arises as we consider global transformations of the manifold or the questions related to the mappings of one manifold into another.

Liouville-type theorems are examples of this kind. In the classical form the Liouville theorem claims that there is no bounded non-constant entire function. In other words, there is no holomorphic mapping of the plane into a disk. This phenomenon of nonexistence is of rather general nature and it holds for a much broader class of mappings. It is related to different conformal types of the Euclidean plane and the disk (or the hyperbolic space). The former is conformally parabolic, while the latter is conformally hyperbolic.

We complete the paper with one more example.

The global homeomorphism theorem (GHT) is the following specifically multidimensional phenomenon: *any locally invertible quasiconformal mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is globally invertible provided* $n \geq 3$.

The theorem essentially remains valid for mapings $f : M^n \to N^n$ of Riemannian manifolds (M^n, g_M), (N^n, g_N) provided $n \geq 3$, $\pi_1(N^n) = 0$, and (M^n, g_M) is of conformally parabolic type.

There are serious evidence to expect the validity of the GHT for sub-Riemannian manifolds as well, and with the same condition of conformal parabolicity on (M^n, H, g).

The initial idea of the proof [Zo1], which was used in further generalizations and developments of the GHT (see [Zo2]), seems to be working for the sub-Riemannian case. Indeed, the topological part of the proof holds for sub-Riemannian manifolds as well. Thus, it remains to prove some capacity estimates equivalent to lemmas 1 and 2 of [Zo1].

For instance, it is sufficient to prove the following.

Star Lemma. *Let D be a domain in the Carnot–Carathéodory space (M^n, H, g) star-like relative to a point $0 \in D$, i.e., D is formed by rays or by their intervals originating from 0 and horizontally normal to Carnot–Carathéodory spheres centered at 0. The finite intervals end at points of the part $\triangle$ of the boundary ∂D 'visible' from 0.*

Let Γ_ϵ be the family of all curves in D which join $\triangle$ with $(\epsilon > 0)$-neighborhood $B(\epsilon)$ of 0. If $\operatorname{mod}_m \Gamma_\epsilon = 0$ then

(i) *projection of $\triangle$ along these rays to the sphere $\partial B(\epsilon)$ does not contain connected components different from a point;*
(ii) *∂D does not locally divide the manifold M^n provided $n \geq 3$.*

Here m is the Hausdorff dimension of the sub-Riemannian manifold (M^n, H, g) considered.

For our purposes we may suppose the manifold to be of conformally parabolic type.

Conformal modulus $\operatorname{mod}_m \Gamma$ of the family Γ of curves in (M^n, H, g) is defined as follows

$$\operatorname{mod}_m \Gamma := \inf \int_{M^n} \rho^m(x)\ dv(x),$$

where the infimum is taken over all Borel measurable nonnegative functions such that $\int_\gamma \rho \geq 1$ for each curve $\gamma \in \Gamma$.

One readily shows (cf. the arguments above concerning the conformal capacity) that $\operatorname{mod}_m \Gamma$ is a conformal invariant.

It is closely related to the conformal capacity. For instance,

$$\operatorname{cap}_m R_a^b = \operatorname{mod}_m \Gamma_a^b,$$

where Γ_a^b is the family of all curves that join boundary components of the spherical condenser R_a^b.

Notice that the Star lemma formulated above for the sub-Riemannian case is true for Riemannian manifolds. In the latter case it can be proved by means of the slightly modified method that was used in the proof of lemmas 1 and 2 in [Zo1].

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