CERTAIN APPLICATIONS OF THE BURNSIDE RINGS AND GHOST RINGS IN THE REPRESENTATION THEORY OF FINITE GROUPS (II)

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ABSTRACT. Using the Burnside ring theoretic methods a new setting and a complete description of the Artin exponent $A(G)$ of finite p-groups was obtained in [4]. In this paper, we compute $A(G)$ for any finite group G – hence providing the global version of [4].

1. INTRODUCTION

This is a continuation of the paper [4], where we obtained a new setting for the Artin exponent of a finite group and described these exponents for finite p-groups. In this note we show that the restriction made in [4] that G is a p-group can be remove. In otherwords the Artin exponent for any finite group G depends on the value of a p-subgroup of G. We claim that the Artin exponent $A(G\ ,\ \mathcal{U})$ of a finite G is equal to 1 if and only if G is cyclic whereas $A(G\ ,\ \mathcal{U})$ difers by p from the order of a p-subgroup of G or is equal to the order of a p-subgroup of G if G is noncylic and theorem 4.2 dictates which. If the featuring p-subgroups of G is Dihedral, Quoternion or semi-dihedral, then $A(G\ ,\ \mathcal{U})$ is 2 or 4 and theorem 4.4 dictates which.
In section 3 we show that we can reduce our problem essentially to the case where G is a finite p-groups. For the convenience of the reader, we recall in section 2 some results in Burnside ring theory and the useful results in [4]. Finally in section 4, we described the Artin exponent for finite group G.

Some notation: G in this paper denotes a finite group. Let 1_G denote the trivial subgroup of G; p, a prime number; $< T >$, subgroup of G generated by T; $< T >_N$, the normal subgroup of G generated by T; g_h, the g conjugate of h; $[g\ ,\ h]$, the commutator of g and $h-$ in particular $[S\ ,\ T] :=< [s\ ,\ t] \mid s \in S,\ t \in T >$. Let $G^{sub/\sim}$ denote a set of representatives of the conjugacy classes of subgroups of G and $Z(G)$ denote the center of G. Let $G_s := \{g \in G \mid gs = s\ \forall\ s \in S\}$, the stabilizer subgroup of s.

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2. Preliminaries

Let G be a finite group. The set of isomorphism classes of finite left G-sets form a commutative semi ring $B^+(G)$ with respect to disjoint union and cartesian product. Using the Grothendieck ring construction on $B^+(G)$ one obtains the corresponding Grothendieck ring $B(G)$ associated with $B^+(G)$. This resulting ring is called the Burnside ring $B(G)$ of G. So the Burnside ring is the Grothendieck ring of finite G-sets which is generated as an algebra over Z by the isomorphism classes of finite left G-sets $S_1\ S_2\ \ldots$ subject to the relations $S_1 - S_2 = 0$ if $S_1 \cong S_2$; $S_1 + S_2 - (S_1 \sqcup Y) = 0$ and $S_1 \cdot S_2 - (S_1 \times S_2) = 0$. In otherwords, the elements of the Burnside ring are the virtual G-sets, that is the formal difference $S_1 - S_2$ of isomorphism classes of finite G-sets $S_1,\ S_2$. Furthermore one has that $S_1 - S_2 \ = \ S_1' - S_2' \iff S_3 \sqcup S_1 \sqcup S_2' \ \cong \ S_3 \sqcup S_2 \sqcup S_1'$ where $S_1',\ S_2'$ and S_3 are G-sets. This of course is equivalent with $S_1 \sqcup S_2' \ \cong \ S_2 \sqcup S_1'$. For any subgroup $U \leq G$ of G the map $\chi_U :\ S \ \longrightarrow \ \#\{s \in S \ \mid \ U \leq G_s\}$, from S onto the number of U invariant elements in S, induces a homomorphism from $B(G)$ into Z – the integers. Using standard arguments concerning the Burnside ring the following statements are easily verified (see [2]).

(a) All homomorphism from $B(G)$ into Z is of the form χ_U defined above.

(b) For any subgroup $U,\ V \leq G$ one has that $\chi_U \ = \ \chi_V$ if and only if $U \sim V$ (U is G-conjugate to V).

(c) The product map $\chi := \prod_{i=1}^{k} \chi_i : B(G) \longrightarrow \prod_{i=1}^{k} Z$ of k different homomorphisms is injective and maps $B(G)$ onto a subring of finite index $\prod_{i=1}^{k} (N_G(U_i \ : \ U_i)$ of $\prod_{i=1}^{k} Z$ – this way identifying $\prod_{i=1}^{k} Z$ with the integral closure $\tilde{B}(G)$ of $B(G)$ in its total quotient ring $Q \otimes_z B(G) \cong \prod_{i=1}^{k} Q$ (see [2],[5]). The product $\prod_{i=1}^{k} Z$ is called the *Ghost ring* of G.

It is well known (see [2]) that there are canonical set of congruence relation where by one can characterize the image $\chi(B(G))$ as a subgroup of $\prod_{i=1}^{k} Z$. More exactly, if $U \trianglelefteq V \leq G$ then for a G-set X the Burnside lemma applied with respect to the V/U-set X^U, implies that $\sum_{i=i}^{k} \chi_{<\overline{v}>}(X) \equiv 0(mod\ (V : U))$, where $< \overline{v} >$ denotes the subgroup of $V \leq G$ generated by the coset $\overline{v} \subseteq V$.

Lemma 2.1. *Identifying $B(G)$ with its image in $\tilde{B}(G)$ with respect to the map $\prod_{U \in G^{sub/\sim}} \chi_U$ one has that $\{n \in Z \mid n\tilde{B}(G) \subseteq B(G)\} \ = \ |G|Z$.*

Because for $U,\ V \leq G$, one has $\chi_U = \chi_V$ if and only if U and V are G-conjugate and for $x,\ y \in B(G)$, one has $\chi_U(x) = \chi_U(y)$ for all $U \leq G$ if and only if $x = y$ it follows that we can identify each $x \in B(G)$ with the associated map (also denoted by x) $U \longrightarrow \chi_U(x)$, from the set of all subgroups of G into Z. Therefore we can consider $B(G)$ in a canonical way as a subring of the ghost ring $\tilde{B}(G)$. For more detail on the Burnside rings see [2]. We recall the following simple facts (see any standard book on group theory).

Definition 2.2. *Let G be a 2-group with generators g and h. We say that:*
(A) *G is quaternion* **(Q)** *if $g,\ h$ have the relations*

$$g^{2^n} = 1, \quad h^2 = g^{2^{n-1}}, \quad hgh^{-1} = g^{-1}.$$

(B) *G is dihedral* **(D)** *if $g,\ h$ have the relations*

$$g^{2^n} = 1, \quad h^2 = 1, \quad hgh^{-1} = g^{-1}.$$

(C) *G is semi-dihedral* **(SD)** *if $g,\ h$ have the relations*

$$g^{2^n} = 1, \quad h^2 = 1, \quad hgh^{-1} = g^{-1+2^{n-1}}.$$

Lemma 2.3. *Let G be a noncyclic abelian p-group ($p \neq 2$) and $U \trianglelefteq G$, $|U| = p$. Then there exists a normal subgroup $V \trianglelefteq G$ of G of order p^2, containing U and isomorphic to $C_p \times C_p$.*

Lemma 2.4. *Let G be a finite p-group and U a subgroup of G of order less than p^β. For $g \in G$ we have that $< g\ ,\ U >$ is cyclic of order p^β if and only if $U \subseteq < g >$.*

Lemma 2.5. *Let G be a finite p-group and let U and H be subgroups of G such that U is a cyclic normal subgroup with $U \subseteq H$. Let $\mathcal{C}(H)$ denote the set of all cyclic subgroups V of H with $U \leq V$ and $(V : U) = p$, and let $\mathcal{C}'(H) = \mathcal{C}'_U(H)$ denote the subset of $\mathcal{C}(H)$, consisting of those $V \in \mathcal{C}(H)$ which are normal in H. We have:*
(a.) $|\mathcal{C}(H)| \ \equiv \ |\mathcal{C}'(H)|$ (mod p).
(b.) $H' = H'_U := \{h \in H \mid [h\ ,\ H] \subseteq U\}$ is a normal subgroup of H.
(c.) $\mathcal{C}'(H) = \mathcal{C}(H')$
(d.) If $G' \subseteq H \trianglelefteq G$ then $|\mathcal{C}(H)| \ = \ |\mathcal{C}(G')|$(mod p).

Lemma 2.6. *With the above notation, one has that if $|U| = p \neq 2$ and $[G\ ,\ G] \subseteq U$, then the following are eqivalent:*
(i) $|\mathcal{C}(G)| = 1$
(ii) $|\mathcal{C}(G)| \not\equiv 0(mod\ \ p)$
(iii) G is cyclic

Lemma 2.7. *If G is abelian, then the following are equivalent:*
(i) $|\mathcal{C}(G)| = 1$
(ii) $|\mathcal{C}(G)| \not\equiv 0(mod\ \ p)$
(iii) G is cyclic

Lemma 2.8. *If $|U| = p \neq 2$, then the following are eqivalent:*
(i) $|\mathcal{C}'(G)| = 1$
(ii) $|\mathcal{C}'(G)| \not\equiv 0(mod\ \ p)$
(iii) G' is cyclic

Lemma 2.9. *With the definitions and assumptions of (3.1), if G is a 2-group, one has*
(a.) $|\mathcal{C}(H)| \equiv 1(2) \Rightarrow Z(G)$ *is cyclic.*
(b.) If $[G\ ,\ G] \subseteq U$ and $|\mathcal{C}(H)|$ is odd, then G is cyclic or nonabelian of order 8.
(c.) If $H \leq G$ with $[H\ ,\ H] \subseteq U \subseteq H$ and $|\{V \in \mathcal{C}(H) \mid V \subseteq H\}| \equiv 1(mod\ 2)$, then H is cyclic or nonabelian of order 8.
(d.) If $|\mathcal{C}(H)| \equiv 1(mod\ 2)$, then either H' is cyclic or G is nonabelian of order 8.
(e.) If $|\mathcal{C}(H)| \equiv 1(mod\ 2)$, then G contains a cyclic normal subgroup of index 2.

In [4], we characterized the elements in $\tilde{B}(G)$ which are in $B(G)$ in the following way:

Definition and Theorem 2.10. *For $\mathcal{U}$ a family of cyclic subgroups of G. Define $e_{\mathcal{U}} \in \tilde{B}(G)$ by $e_{\mathcal{U}}(U) = 1$ if U is contained in $\mathcal{U}$ and 0 otherwise. Then one has that $|G| \cdot e_{\mathcal{U}}(U) \in B(G)$. Furthermore define $A(G\ ,\ \mathcal{U}) := min(n \in N \mid n \cdot e_{\mathcal{U}} \in B(G))$. The integer $A(G\ ,\ \mathcal{U})$ is said to be the Artin exponent of G. One also has that $A(G\ ,\ \mathcal{U})$ divides $|G|$ (see [4]).*

In [4] we also gave a Burnside ring theoretic proof of the following well known results of Lam (see [3]).

Theorem 2.11. *Let G be a finite p-group. One has that $A(G\ ,\ \mathcal{U}) = 1$ if and only if G is cyclic.*

Theorem 2.12. *Let G be a noncyclic finite p-group $(p \neq 2)$ of order p^{α}. Then one has $A(G\ ,\ \mathcal{U}) = p^{\alpha - 1}$.*

Theorem 2.13. *If $p = 2$ and G is a noncyclic 2-group of order 2^{α} then $A(G) = 2^{\alpha-1}$, excluding the cases where G is the Quaternion or Dihedral in which cases one has that $A(G\ ,\ \mathcal{U}) = 2$.*

These results provides the complete computation of $A(G\ ,\ \mathcal{U})$ where G is a p-group.

3. Reduction to p-Groups

Let p be a prime and consider Z_p, the localization of Z at p, that is $Z_p := \{a/b \mid a \in Z,\ b \in Z - pZ\} \subseteq Q$. Put $B_p(G) := Z_p \otimes_z B(G)$, and $\tilde{B}_p(G) := Z_p \otimes_z \tilde{B}(G)$. One has that $Z_p \cdot \chi(B(G)) \cong B_p(G) := Z_p \otimes_z B(G)$ coincides with the subgroup of $\tilde{B}_p(G)$ for which the relations $\sum n \cdot e_{\mathcal{U}} \equiv 0(mod\,(V_p : U))$, with V_p representing a subgroup of G between U and V for which V_p/U is the Sylow p-subgroup of V/U (see [2]). Therefore $e_{\mathcal{U}}$ is characterized as a subgroup of $\prod_{i=1}^{k} Z$ by those relations, which runs through all the prime divisors of $|G|$. Hence we have;

Lemma 3.1. *An element $n \cdot e_{\mathcal{U}} \in \tilde{B}(G)$ is contained in $B(G)$ if and only if for every $U \trianglelefteq V \leq G$ with $(V : U)$ a prime power one has $\sum_{vU \in V/U} n \cdot e_{\mathcal{U}}(< v\ ,\ U >) \equiv 0(mod\ (V : U))$, where $e_{\mathcal{U}}(< v\ ,\ U >) := \#\{vU \in V/U \mid\ < v\ ,\ U > \in\ \mathcal{U}\}$.*

Corollary 3.2. *For $U \trianglelefteq V \leq G$, assume that $(V : U) = p^{\alpha}$ for some prime p. Consider the decomposition $U = U_p \times U_{p'}$ where $|U_p| = p^{\beta}$ is a power of p and $|U_{p'}|$ is prime to p. Let V_p denote the Sylow subgroup of V. Then one has*
(a) $(V : U) = p^{\alpha}$ implies that $V = V_p \propto U_{p'}$ and $V = V_p \times U_{p'}$ if and only if $U \leq Z(V)$, the center of V. (Here $\propto$, resp. $\times$ denotes the semidirect product, resp. direct product).
(b) $(V : U) = (V_p : U_p)$.
(c) If $c(U\ ,\ V) := \#\{vU \in V/U \mid\ < v\ ,\ U > \text{ is cyclic }\}$ and $c(U_p\ ,\ V_p) := \#\{vU_p \in V_p/U_p \mid\ < v\ ,\ U_p > \text{ is cyclic }\}$ then $c(U\ ,\ V) = c(U_p\ ,\ V_p)$ if $V = V_p \times U_{p'}$, that is $U \leq Z(V)$.

Remark 3.3. *Observe that for $vU \in V/U$ the group $< v\ ,\ U >$ is cyclic only if $v \in C_V(U) := \{w \in V \mid wu = uw \text{ for all } u \in U\}$. This implies that $(V : C_V(U))$ must always divide the number "n" in order for $(V : U)$ to divide $n \cdot \mathbf{c}(U\ ,\ V)$. So we have the following strategy for computing the Artin exponent. For each p subgroup V of G, consider the set of all cyclic normal subgroups U of V with U contained in the centre of V and compute the minimum of those positive integers n for which $(V : U)$ divides $n \cdot c(U\ ,\ V)$. Finally, it is easy to see that if $U \leq Z(V)$, for a p-group V such that U cyclic one has that, $\mathbf{c}(U\ ,\ V) = \mathbf{c}(U/U^p\ ,\ V/U^p)$, where U^p denotes the unique subgroup of U of index p. So we can restrict ourselves to the case where $|U| = p$.*

We now summarize with the following,

Theorem 3.4. *Let G be a finite group. Then $A(G\ ,\ \mathcal{U}) = A(G_p\ ,\ \mathcal{U})$, where G_p is a p-subgroup of G.*

4. Global Discussion

For this discussion it suffices to consider the following cases;

Case 1: G is cyclic.

The first thing to note is that the proof of Theorem(2.8) in [4] does not depend on the fact that G is a p-group. For the sake of explicitness we state the following global version of the theorem. That is we remove the condition that G is a p-group.

Theorem 4.1. *Let G be a finite group. For $n \in N$ and a subgroup $U \leq G$, let $x_n \in \tilde{B}(G)$ be defined by*

$$x_n = \begin{cases} n, & \textit{if } U \textit{ is cyclic} \\ 0, & \textit{otherwise} \end{cases}$$

Then $A(G\ ,\ \mathcal{U}) = 1$ if and only if G is cyclic.

Proof: We check by congruences. Assume $U \trianglelefteq V \leq G$ such that U cyclic of order p and U is contained in the center of V, where $|V| = p^\alpha$, (p a prime). Then $x \in B(G)$ implies $\sum_{vU \in V/U} x(< v\ ,\ U >) = n \cdot \#\{vU \in V/U \mid\ < v\ ,\ U >$ is cyclic$\}$. To compute $A(G\ ,\ \mathcal{U})$, we let $V = G$, that is $|G| = p^\alpha$. These assumption remains valid for the balance of this paper. Now if $n = 1$, then one has that $\#\{gU \in G/U \mid\ < g\ ,\ U >$ is cyclic $\} \equiv 0(mod\ (G : U))$. So $(G : U) \mid \#\{gU \in G/U \mid\ < g\ ,\ U >$ is cyclic $\}$. This implies $|G| \mid \#\{g \in G \mid\ < g\ ,\ U >$ is cyclic $\}$. So for all $g \in G$, one has that $< g\ ,\ U >$ is cyclic. This implies that G is cyclic since if G is noncyclic then from (2.3) there exists a normal subgroup $W \trianglelefteq G$ such that W contains U and $W \cong C_p \times C_p \cong U \times C_p = U\times < g >$ for some $g \in G$. That is $< g\ ,\ U > = W$ for some $g \in G$. So G is cyclic.

The converse is obvious. □

Case 2: G is noncylic.

Theorem 4.2. *With the assumptions of Theorem 4.1 one has that if G is noncyclic then*

$$A(G\ ,\ \mathcal{U}) := \begin{cases} p^\alpha, & \textit{if } G' := \{g \in G \mid [g\ ,\ U] \subseteq U\} \textit{ is cyclic} \\ p^{\alpha-1}, & \textit{if otherwise.} \end{cases}$$

Proof: Since G is noncyclic, then one obviously has that $|G|$ does not divide $\#\{g \in G \mid \ < g\ ,\ U > \text{ is cyclic}\}$. If G is abelian recall from lemma 2.3 that there exist a normal subgroup W of order p^2 of the form
$\{W \trianglelefteq G \mid U \ \leq\ W \cong C_p \times C_p\}$ and indeed we have (see lemma 2.5 and 2.7) that $\mathcal{C}(W)\ := \#\{C\ \leq\ W \mid C \text{ is cyclic } C \supseteq U\} \equiv 0(mod\ p)$. This implies $A(G\ ,\ \mathcal{U}) = p^{\alpha-1}$.

If G be nonabelian and G' is noncyclic then repeating the above argument (by setting $G = G'$, one has that $A(G\ ,\ \mathcal{U})\ =\ p^{\alpha-1}$. If G' is cyclic then it is easy to derive (from 2.8) that $\#\{g \in G \mid\ < g\ ,\ U > \text{ is cyclic }\} \not\equiv 0(mod\ p)$, in which case $A(G\ ,\ \mathcal{U})\ =\ p^{\alpha}$. □

We now compute $A(G\ ,\ \mathcal{U})$ for the special groups Q, D, and SD.

Corollary 4.3. *Keeping the assumptions of (4.1) one has that if G is any of Q, D, or SD then*

$$A(G\ ,\ \mathcal{U}) := \begin{cases} 4, & \text{if } G' \text{ is cyclic} \\ 2, & \text{if otherwise.} \end{cases}$$

Proof: Using the notation in section 3 one has here that if $\mathcal{C}(G) \equiv 0(mod\ 2)$ then by a repetition of the above arguments one has that $A(G\ ,\ \mathcal{U}) = 2^{\alpha-1}$. If $\mathcal{C}(G) \not\equiv 0(mod\ 2)$ then, from lemma 2.9, G is cyclic or a nonabelian group of order 8. But by our assumption G is noncyclic and so G must be a nonabelian group of order 8. It is now easy to see (by simple computation using lemma 2.9 repeatedly) that $A(G) = 4$ when G' is cyclic and $A(G) = 2$ otherwise. □

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