A TOPOLOGICAL METHOD TO COMPUTE SPECTRAL FLOW

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ABSTRACT. This paper describes a topological method to compute the spectral flow of a family of twisted Dirac operators, it includes two detailed examples. Briefly, a formula of Atiyah, Patodi and Singer expresses the spectral flow in terms of Chern-Simons invariants and rho invariants. The first step is to construct a flat cobordism to a new bigger 3-manifold. The advantage of the new connection is that it is in the path component of a reducible connection. The second step is to calculate the effect of these operations on the invariants. The final step is an application of the G-signature theorem to compute the invariants.

The spectral flow of a family of operators is a generalization of the signature of a self-adjoint finite dimensional operator. If A_t is a family of operators with a real, discrete, spectrum, then the spectral flow of A_t is the number of eigenvalues that move from negative to positive minus the number which move from positive to negative. Recalling that the number of positive eigenvalues minus the number of negative eigenvalues, we see that

$$SF(A_t) = \text{ Sign } A_1 - \text{ Sign } A_0$$

for finite dimensional operators.

In this paper, we extend to spectral flow the method that we used to compute Chern–Simons invariants of flat SU_2 connections in a previous paper [A]. After outlining the method, we work out two examples in detail. Briefly, a formula of Atiyah, Patodi and Singer expresses the spectral flow in terms of Chern–Simons invariants and rho invariants [APS]. To compute the rho invariant of a flat SU_2 connection on one 3-manifold, we construct a flat cobordant connection on a new, bigger 3-manifold. It is straightforward to compute the difference between the rho invariant of a connection and a flat cobordant connection. The advantage of the new connection is that it is in the path component of a reducible connection in the space of flat connections. It is possible to compute the change in the rho invariant along a path of flat connections. Finally, the reducible connection may be assumed to have finite holonomy so its rho invariant may be computed with the G-signature theorem.

We begin by describing a collection of self-adjoint elliptic operators over any 3-manifold and stating the Atiyah–Patodi–Singer formula for the spectral flow. Let A be a special unitary connection on a vector bundle over a 3-manifold, M.

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Definition. $B(A) : \Gamma(\Lambda^0 \oplus \Lambda^2 M \otimes \mathbb{C}^n) \to \Gamma(\Lambda^0 \oplus \Lambda^2 M \otimes \mathbb{C}^n)$;

$$\alpha \mapsto i^{\deg\,\alpha}(*d_A\alpha - d_A * \alpha)$$

is *the Chern–Simons Hessian.*

The unitary connections that we consider will be constructed in the same way. Start with a homomorphism from $\pi_1(M)$ to SU_2. This induces a representation into U_3 by composition with the map:

$$\begin{aligned}&\mathrm{Ad} : SU_2 \to U_3;\\&(\mathrm{Ad}\ A)(X \otimes z) = AXA^{-1} \otimes z.\end{aligned}$$

Here we are identifying $\mathbb{C}^3$ with $su_2 \otimes \mathbb{C}$. Finally, the canonical flat connection on the universal covering of M induces a natural connection on the twisted product, $\tilde{M} \times_{\pi_1(M)} U_3$. Given two SU_2 representations, α_0 and α_1 we construct the corresponding U_3 connections, A_0 and A_1, and the path of connections

$$A_t = (1-t)A_0 + tA_1.$$

By the spectral flow from α_0 to α_1, we will mean the spectral flow of the family of operators $B(A_t)$. When one of the operators A_0 or A_1 has a non-trivial kernel, we need to pick some convention to count the zero modes. We count the zero modes of A_0 as positive eigenvalues and the zero modes of A_1 as negative eigenvalues. This is the convention used by Fintushel and Stern [FS1], [FS2]. After the first example, we will explain why this is a natural convention.

As an aside, we will explain why the spectrum of $B(A_t)$ is discrete. For any constant, λ, $B(A_t) - \lambda$ is an elliptic operator. It follows from elliptic regularity and Rellich's Lemma that $B(A_t) - \lambda$ is Fredholm. This in turn implies that the spectrum of $B(A_t) - \lambda$ does not accumulate near zero.

More than this is true. Atiyah, Patodi and Singer show that the function

$$\eta_{B(A_t)}(s) = \sum_{\lambda \in\ \mathrm{Spec}(B(A_0))} \lambda^{-s}\ \mathrm{Sign}\ \lambda$$

is an analytic function of s for Re $s \gg 1$. By analytic continuation, they show that $\eta_{B(A_t)}(0)$ may be defined as a finite number [APS]. This eta invariant is an extension of the signature to a large class of infinite rank operators. Let θ denote the connection constructed from the trivial representation.

Definition. The *rho invariant* of a representation, α, is

$$\rho_M(\alpha) = \eta_{B(A)} - \eta_{B(\theta)}$$

where A is the connection constructed from α. The *Chern–Simons* invariant of a representation is

$$CS_M(\alpha) = \int_W -\frac{1}{4\pi^2}\ \mathrm{Tr}(\Lambda^2 F_{\bar{A}})$$

where W is any 4-manifold with boundary M, and $\bar{A}$ is any SU_2 connection which restricts to the SU_2 connection induced from the representation on the boundary.

The main theorem of Atiyah, Patodi and Singer may be stated in two different ways depending on the boundary conditions. If $D : \Gamma(E) \to \Gamma(F)$ is an elliptic first order operator that restricts to the form

$$D = \frac{\partial}{\partial t} + B$$

on a metric product collar of the boundary of the base manifold, then one way to put boundary conditions on the operator is to restrict the domain to sections of E that have $Ps|_{\partial X} = 0$ where P is the spectral projection of B corresponding to non-negative eigenvalues. An alternate way to put boundary conditions on D is to introduce a non-compact base manifold, $\hat{X} = X \cup_{\partial X} ([0,\infty) \times \partial X)$ with a product metric on the end, and first extend D to an operator over $\hat{X}$, say $\hat{D}$, then restrict the domain and range to L^2-sections. In either case Atiyah, Patodi and Singer prove that the index of D is an integral of curvature terms plus a correction term depending on the dimension of the kernel of B and the eta invariant of B [APS]. We will call the first index the APS index and the second the L^2 index. A path of connections on M describes a connection, $\bar{A}$, on $[0,t] \times M$. The APS theorem gives us the following formula for the APS index of the signature operator twisted by $\bar{A}$:

$$\operatorname{Index}(d_{\bar{A}} + d^*_{\bar{A}}|_{\Lambda^+}) = 16(CS_M(A_0) - CS_M(A_t)) - h(A_0) - h(A_t) + \rho_M(A_0) - \rho_M(A_t).$$

Here $h(A) = \dim\ker B(A)$. Assuming that there is a consistent way to split the spectrum of $B(A_t)$ into a finite part and an infinite part. The spectral flow of $B(A_t)$ may be expressed in terms of the finite part of the eta invariants and the $h(B(A_t))$. The above index formula can then be arranged to set an integer valued function of the spectral flow and the L^2 index and $h(B(A_t))$ equal to a continuous function of t. Evaluating the continuous function at $t = 0$ gives zero. Finally, solving for the spectral flow gives:

Lemma 1.

$$SF(\alpha_0, \alpha_1) = 8(CS_M(\alpha_1) - CS_M(\alpha_0)) + \frac{1}{2}(\rho_M(\alpha_1) - \rho_M(\alpha_0) - h(\alpha_1) - h(\alpha_0)).$$

The general case of this result follows by cutting the path of connections into parts where the spectrum may be split into a finite part and an infinite part. This argument may be found in a paper by Kirk, Klassen and Ruberman [KKR].

The Chern–Simon terms in the above formula may be computed by the method in [A]. The connection corresponding to a representation is flat, so the sequence

$$\Gamma(\Lambda^0 M \otimes \mathbb{C}^3) \xrightarrow{d_A} \Gamma(\Lambda^1 M \otimes \mathbb{C}^3) \xrightarrow{d_A} \Gamma(\Lambda^2 M \otimes \mathbb{C}^3) \xrightarrow{d_A} \Gamma(\Lambda^3 M \otimes \mathbb{C}^3)$$

is a complex. Hodge theory implies that $h(\alpha) = \dim H^0(M;\alpha) \oplus H^2(M;\alpha)$. By Poincaré duality we see that $h(\alpha) = \dim H^0 \oplus H^1$. Since the 2-skeleton of M is the

2-skeleton of a $K(\pi_1(M), 1)$, we see that computing $h(\alpha)$ is reduced to an algebraic computation in group cohomology. In fact, $H^0(\pi_1(M); SU_2) = SU_2^{\pi_1(M)}$, thus

$$\dim H^0 = \begin{cases} 0 & \text{if } \alpha \text{ is irreducible} \\ 1 & \text{if } \alpha \text{ is abelian but not central} \\ 3 & \text{if } \alpha \text{ is central.} \end{cases}$$

and $\dim H^1 = \dim Z^1 - 3 + \dim H^0$ where

$$Z^1 = \{f : \pi_1(M) \to su_2 \mid f(gh) = f(g) + \alpha(g)f(h)\alpha(g^{-1})\}$$

is the space of cocycles. This leaves the rho invariants which may be computed with a refined version of the same three steps which are used to compute Chern–Simons invariants.

The first step in computing the rho invariant is to construct a flat cobordism to a new representation and compute the change in the rho invariant. The standard cobordism to use is $W^4 = [0,1] \times M \cup_{S^1 \times D^2} V$, where $V = [0,1] \times S^1 \times D^2 / \sim (0, \lambda, z) \quad \sim (1, \bar{\lambda}, \bar{z})$ and $S^1 \times D^2 \to S^1 \times [1/3, 2/3]^2 \to V;\ \ \lambda, s, t \mapsto (s, \lambda, exp(2\pi i t))$. The difference between a pair of L^2 indices gives the following formula.

Lemma 2. *If* $\alpha : \pi_1(W^4) \to SU_2$*, then*

$$\rho_{\partial W}(\alpha|_{\partial W}) = 3 \text{ Sign } W - \text{ Sign } Q_\alpha$$

where Sign W *is the signature of the intersection form on* $\ker(H^2(W) \to H^2(\partial W))$ *and* Sign Q_α *is similar but with twisted coefficients.*

The group $\ker(H^2(W; \alpha) \to H^2(\partial W; \alpha))$ is called the L^2-cohomology of W, and is denoted by $H^2_{L^2}(W)$. The following lemma computes the L^2 cohomology of the standard cobordism.

Lemma 3. $H^2_{L^2}(W; \alpha)$ *injects in coker* $(H^1(M; \alpha) \oplus H^1(V, \alpha) \to H^1(S^1 \times D^2; \alpha))$.

Proof. In the diagram,

$$\begin{array}{ccccc} H^1([0,1] \times M) \oplus H^1(V) \to H^1(S^1 \times D^2) \to & H^2(W) & \to & H^2([0,1] \times M) \oplus H^2(V) \\ & \downarrow & /// & \downarrow \\ & H^2(\partial W) & \to & H^2(M), \end{array}$$

the top row is exact and the map on the right is an injection. $\square$

The next step is to construct a path of representations from the new representation to a reducible representation and compute the change in the rho invariant across this path. There is a general method to compute the spectral flow along a path of flat connections due to Kirk and Klassen [KK2]. We do not, however, need this formula, because we may pick a path with special properties.

Lemma 4. *Let α_t be a family of representations so that,*

$$\dim H^0(M;\alpha_t) = \begin{cases} 0 & \mathit{if}\ t>0 \\ 1 & \mathit{if}\ t=0 \end{cases}$$

and,

$$\dim H^1(M;\alpha_t) = \begin{cases} h & \mathit{if}\ t>0 \\ h+1 & \mathit{if}\ t=0, \end{cases}$$

then $\rho_M(\alpha_1) = \rho_M(\alpha_0)$.

Proof. We will use the formula in Lemma 1. First compute,

$$\begin{aligned} CS(\alpha_1) - CS(\alpha_0) &= -\tfrac{1}{4\pi^2}\int_{[0,1]\times M} \operatorname{Tr}(\Lambda^2 F_{\bar{A}}) \\ &= -\tfrac{1}{4\pi^2}\int_{[0,1]\times M} \operatorname{Tr}(\Lambda^2 (F_{\alpha_t} + dt \times \tfrac{\partial\alpha_t}{\partial t})) \\ &= -\tfrac{1}{4\pi^2}\int_{[0,1]\times M} \operatorname{Tr}(\Lambda^2 dt \times \tfrac{\partial\alpha_t}{\partial t}) = 0. \end{aligned}$$

Now, write

$$\Gamma((\Lambda^0 \oplus \Lambda^2)M \otimes \mathbb{C}^3) \cong \Gamma(\Lambda^0 M \otimes \mathbb{C}^3) \oplus \ker(d_A *) \oplus \operatorname{Im}(d_A *).$$

We can do this because, d_A* is self-adjoint. This decomposes $B(A)$ as:

$$B(A) = \begin{bmatrix} 0 & -*d_A & 0 \\ *d_A & 0 & 0 \\ 0 & 0 & d_A* \end{bmatrix}.$$

It follows that the two eigenvalues that approach zero as $t \to 0$ have opposite signs. Just check that when $\begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue λ, $\begin{bmatrix} u \\ -v \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue $-\lambda$. From this it is apparent that $SF(\alpha_0,\alpha_1) = -h-1$. Plugging into Lemma 1 will now give the result. □

It is sometimes useful to compute the change in rho invariants along a path of reducible connections. In the typical case, the dimension of $\ker B(A_t)$ will be a constant, the spectral flow will be determined by the zero modes, and the rho invariant will be constant.

The final step in the computation of the rho invariant and thus the spectral flow, is to use the G-signature theorem to compute the rho invariant of the reducible representation. If G is a finite group which acts effectively on a 4-manifold W, and $g \in G$, then we can define a g-signature.

Definition. $\operatorname{Sign}(g,W) = \operatorname{Tr}(g^*|_{H^2_+(W)}) - \operatorname{Tr}(g^*|_{H^2_-(W)})$.

A group element may have a surface of fixed points modeled on, $g\cdot(z,w) = (z, e^{i\psi}w)$ for $(z,w) \in \mathbb{C}^2$ or have isolated fixed points modeled on $g\cdot(z,w) = (e^{i\theta_1}z, e^{i\theta_2}w)$.

Define a local g-signature by

$$L(g,W)=\sum_{F,p}F\cdot F\csc^2(\psi(F)/2)-\cot(\frac{\theta_1(p)}{2})\cot(\frac{\theta_2(p)}{2}).$$

The g-signature theorem states that,

$$\mathrm{Sign}(g,W)=L(g,W).$$

If G acts freely on a 3-manifold, M, then there is a 4-manifold with boundary several copies of M which the action extends over. We can see this by using the transfer map to prove that $\bar{H}_*(BG)$ is torsion for finite groups. In fact, by a direct geometric construction, we will see that we may take $\partial W=M$, when G is finite cyclic. In this setting define a signature defect by

$$\sigma_M(g)=L(g,W)-\ \mathrm{Sign}(g,W).$$

This is independent of W by the g-signature theorem. If a representation, factors through a finite group,

$$\begin{array}{ccc}\alpha:\pi_1(M) & \rightarrow & SU_2\\ & \searrow \quad \nearrow & \\ & G & \end{array}$$

Then there is a formula for the rho invariant in terms of the signature defect. By an argument from [A], we may assume that every reducible representation has this form.

Lemma 5.

$$\rho_M(\alpha)=\frac{1}{|G|}\sum_{g\neq 1}\sigma_{\hat{M}}(g)(\mathit{Tr}\ \mathit{Ad}\ \alpha(g)-3).$$

($\hat{M}$ is the cover of M coming from $\pi_1(M)\rightarrow G$.)

The proof of this may be found in [APS]. There is also an expression for the signature defects involving the rho invariants. The rho invariant may be considered to be a generalization of an invariant for finite covering spaces to infinite covering spaces. The main idea of this paper is to go backward and reduce the computation of this invariant to the case of finite coverings. This method works for any representation that is flat cobordant to a representation in the path component of a reducible representation. It works the best when the path may be chosen to have the property in Lemma 3. Every representation on every graph manifold has this property. Also some representations on some hyperbolic manifolds have this property. It is possible that every representation on every 3-manifold has this property.

Now that we have described a method to compute the spectral flow, we will work out two examples. The first example is a representation on the Poincaré homology sphere, $\Sigma(2,3,5)$. The second example is a representation on a hyperbolic manifold.

The spectral flow for $\Sigma(2,3,5)$ could be computed with Lemma 1 and Lemma 4. It could also be computed with the Fintushel–Stern method by considering the mapping cylinder down to the base orbifold. Even so, it is a simple example that well illustrates all of the important aspects of our method.

Since the fundamental group of $\Sigma(2,3,5)$ is fairly simple,

$$\pi_1(\Sigma(2,3,5)) = \langle Q_1, Q_2, Q_3, H \mid [H, Q_i], Q_1^2 H, Q_2^3 H, Q_3^5 H, HQ_1Q_2Q_3\rangle,$$

it is not hard to find all SU_2 representations. There are three representations; the trivial one, and two irreducible representations. The fundamental representation is given by

$$H \mapsto -1, Q_1 \mapsto i, Q_2 \mapsto g\exp(i\pi/3)g^{-1}, Q_3 \mapsto h\exp(i\pi/5)h^{-1},$$

where g and h are elements chosen to solve the final relation. Here, we are identifying SU_2 with Sp_1 and writing elements as unit quaternions.

We will compute the spectral flow from the trivial representation to the fundamental representation. The main part of the computation is computing the rho invariant. The first step in the computation of the rho invariant is to compute the change across the cobordism.

Since $\Sigma(2,3,5)$ is a Seifert fiber space, we can see the cobordism at the level of base orbifolds.

FIGURE 1. The cobordism.

The base orbifold of the cobordism is a 2-sphere with three cone points cross with an interval glued along a disk to a solid Klein bottle. A handlebody decomposition of the part of the cobordism over the solid Klein bottle is drawn in Figure 2.

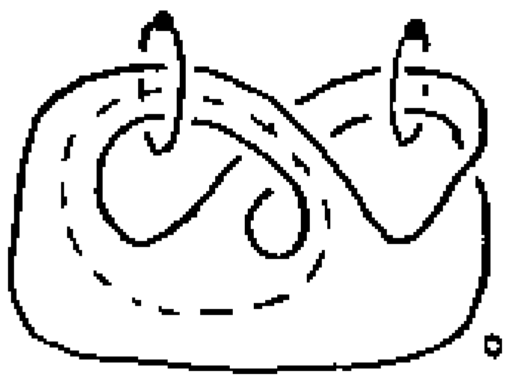

FIGURE 2. The standard part of the cobordism.

A neighborhood of the dotted curve above is identified with the regular fiber in $\Sigma(2,3,5) \times \{1\} \subseteq \Sigma(2,3,5) \times [0,1]$. This regular fiber is drawn as a dotted circle in Figure 3.

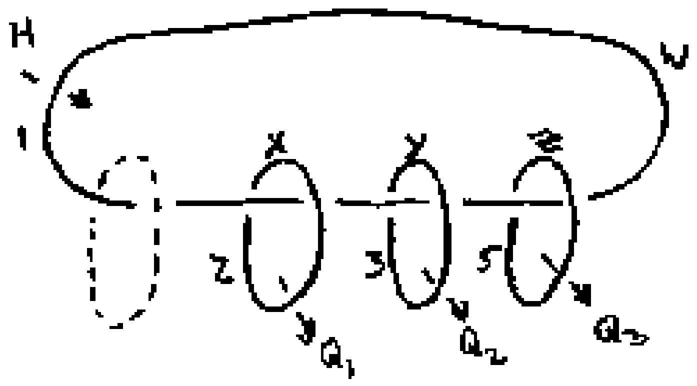

FIGURE 3. A regular fiber.

By Lefschitz duality, we may compute twisted signatures in either L^2-cohomology or L^2-homology. By Lemma 3, the elements of the second L^2-homology are all the union of a surface with boundary in one half of the cobordism with a surface with boundary in the other half. The surface in the first half of the cobordism is drawn in Figure 4.

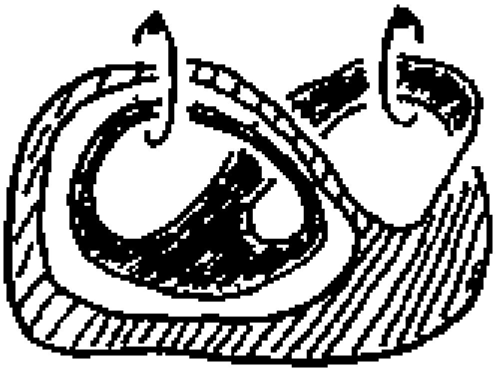

FIGURE 4. The surface.

The core of the two handle is also part of the surface. For the other half of the surface, note that the boundaries of the 2-handles labeled x, y, z and w are given by:

$$\begin{aligned}\partial x &= H + 2Q_1\\ \partial y &= H + 3Q_2\\ \partial z &= H + 5Q_3\\ \partial w &= H + Q_1 + Q_2 + Q_3.\end{aligned}$$

After solving, we see that there is a surface with boundary the regular fiber, given by

$$15x + 10y + 6z - 30w.$$

The surface has no self intersections in the standard half of the cobordism and has -30 intersections in the $[0,1] \times \Sigma(2,3,5)$ part of the cobordism. This implies that Sign $W = -1$.

When we compute the twisted signature, the surface in the standard half will be the same as it was in the untwisted case because the representation is trivial on that half.

The homology with twisted coefficients of a manifold, W, may be computed from the universal cover, $\tilde{W}$. The twisted homology is just the homology of the complex:

$$\to C_n(\tilde{W}) \otimes_{\mathbb{Z}\pi_1(W)} su_2 \to C_{n-1}(\tilde{W}) \otimes_{\mathbb{Z}\pi_1(W)} su_2 \to .$$

The universal cover may be identified with a space of equivalence classes of paths starting at a fixed base point. A map from a cell to the universal cover is determined by the induced map into W together with one path from the base point to the cell. The group $\pi_1(W)$ acts on $C_*(\tilde{W})$ by deck transformations and it acts on SU_2 by the adjoint representation. In Figure 5, we have drawn a picture of a chain with boundary j times the regular fiber.

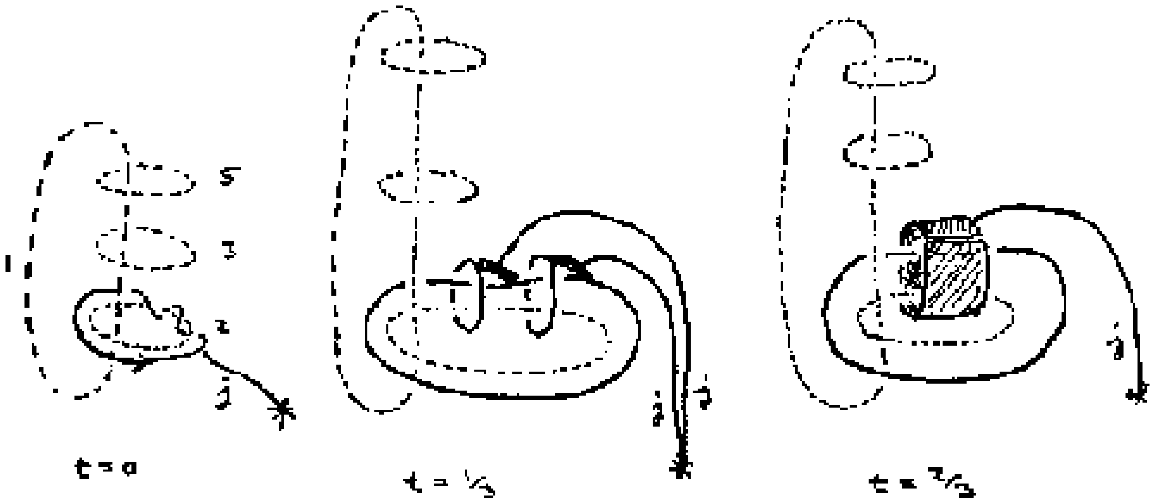

FIGURE 5. A twisted cocycle.

The figures are slices $\{t\} \times \Sigma(2,3,5)$. The surgery description of $\Sigma(2,3,5)$ is drawn as a dashed set of curves. The solid lines represent slices of the cells representing the chain. At the $t=0$ slice, the 2-cell is a disk on the outside of the 2-handle. The solid curve is the boundary of this disk. It is oriented by the induced orientation. Until $t = 1/3$ this curve traces out an isotopy. At $t = 1/3$ two more 2-cells appear. By taking one pair of sides of each of the new rectangles, we get the slice at $t = 1/6$. By taking the other sides, we get the slice at $t = 1/2$. It is important to understand the labels on these cells. Following the original path from the base point might lead one to expect that the path to the top new 2-cell should wrap once around the meridian of the 2-framed component. At first this is the case, but we change the path by an element of $\pi_1(W)$ and act on the label by the inverse element. In this case we compute the new label by $\alpha(Q_1^{-1})j\alpha(Q_1) = -iji = -j$. At $t = 2/3$ the final 2-cell appears, canceling the two meridians and leaving j times the regular fiber. If the coefficients were not twisted, the two meridians would have the wrong orientations and would not cancel. Drawing a parallel copy of this 2-chain shows that it has self intersection zero. After drawing two more chains, we can see that the twisted signature is trivial.

We will see that the fundamental representation extends over the cobordism, thus Lemma 2 implies that

$$\rho_M(\alpha) - \rho_{\Sigma(2,3,5)}(\alpha) = 3 \cdot \text{ Sign } W - \text{ Sign } Q_\alpha = -3.$$

The next step in the computation of the rho invariant is to construct a path of representations on the new manifold. The fundamental group of the new manifold is:

$$\pi_1(M) = \langle H, Q_1, Q_2, Q_3, A_1, A_2 \,|\, [H, Q_i], A_k H A_k^{-1} H, Q_1^2 H, Q_2^3 H, Q_3^5 H, H Q_1 Q_2 Q_3 A_1^2 A_2^2 \rangle.$$

The group of the cobordism is: $\pi_1(M)/\langle A_1A_2\rangle$. Define a path of representations on M by

$$H \mapsto -1, A_1 \mapsto \exp((t^2-1)\pi i/60), A_2 \mapsto 1, Q_1 \mapsto i, Q_2 \mapsto g_t \exp(\pi i/3)g_t^{-1},$$
$$Q_3 \mapsto g_t \exp(-\pi i/3)g_t^{-1}\exp(\pi i/2 + (1-t^2)\pi i/30).$$

At $t = 1$ this representation extends over the cobordism and induces the fundamental representation on $\Sigma(2,3,5)$.

To compute the group cohomology along this path of representations we must compute the space of cocycles, $Z^1(M)$. By the cocycle condition, we see that $f(1) = 0$ and $f(g^{-1}) = -g^{-1}f(g)$ for any cocycle, f. It follows that a cocycle is determined by its values on a set of generators for $\pi_1(M)$. Let $f(H) = h$ and so on. With this conversion, we see that any cocycle must satisfy the following equations, derived from the relations in the group.

$$\begin{aligned}
0 &= h + Hq_i - Q_ih - q_i = (1-Q_i)h,\\
0 &= h + Ha_k + HA_kh - a_k = (1+A_k)h,\\
0 &= H(Q_3^4 + Q_3^3 + Q_3^2 + Q_3^1 + 1)q_3 + h = \tfrac{Q_3^5-1}{Q_3-1}q_3 + h\\
0 &= \tfrac{Q_2^3-1}{Q_2-1}q_2 + h\\
0 &= \tfrac{Q_1^2-1}{Q_1-1}q_1 + h\\
0 &= h + q_1 + Q_1q_2 + Q_1Q_2q_3 + Q_1Q_2Q_3(1+A_1)a_1 + Q_1Q_2Q_3A_1^2(1+A_2)a_2.
\end{aligned}$$

With the understanding that the lower case letters represent purely imaginary quaternions and that multiplication by a group element just conjugates the quaternion by the image of the group element in $SU_2 \cong Sp_1$, we see that this is a linear system of 27 equations and 18 unknowns. By solving, we see that $\dim Z^1(M) = 9$ independent of t. This shows that this path of representations satisfies the hypothesis of Lemma 3 with $h = 6$. This $h = 6$ represents the formal dimension of the moduli space of representations, so there are many paths connecting the irreducible representation with a reducible representation. A similar computation shows that $\dim H^0(\Sigma(2,3,5);\alpha) = \dim H^1(\Sigma(2,3,5);\alpha) = 0$.

It only remains to compute the rho invariant of the reducible representation. The reducible representation, $\alpha_0 : \pi_1(M) \to SU_2$ induces a $120 = 2^3\cdot 3\cdot 5$-fold cover of M which is a circle bundle over a non-orientable surface, $\hat{M} \to M$. At the level of base orbifolds, the cover has $2^2\cdot 3\cdot 5$ branch points of order 2, $2^3\cdot 5$ branch points of order 3, and $2^3\cdot 3$ branch points of order 5. A good toy model of the desk transformations in this cover is given in Figure 6.

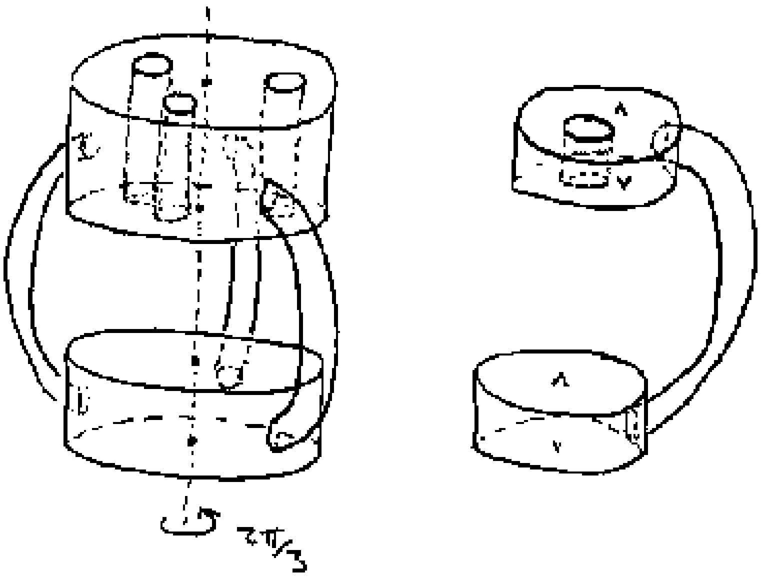

FIGURE 6. A branched cover.

In this case, the action of the group of deck transformations extends over the disk bundle, W, with boundary, $\partial W = \hat{M}$. Clearly W is homotopy equivalent to the non-orientable base orbifold. Thus $H^2(W) = 0$, so $H^2_{L^2}(W) = 0$, and $\mathrm{Sign}(g, W) = 0$. The image of α_0 is generated by $\exp(-\pi i/60)$. The adjoint action is given by:

$$\mathrm{Ad}(e^{i\theta}) \cdot i = i, \mathrm{Ad}(e^{i\theta}) \cdot j = e^{2i\theta} j, \mathrm{Ad}(e^{i\theta}) \cdot k = e^{2i\theta} k.$$

Thus, $\mathrm{Tr}(\mathrm{Ad}(e^{i\theta})) - 3 = 2(\cos 2\theta - 1)$.

The fixed points of order 5 come from $\exp(-\pi i/60)^{2^3\cdot 3\cdot n} = \exp(-\frac{2\pi n}{5} i)$. These elements correspond to sending $Q_3 \mapsto \exp(\pi i/5)^{2^3\cdot 3\cdot n} = \exp(\frac{4\pi n}{5} i)$. From this we see that the order 5 fixed points have rotation angles; $\theta_1 = \frac{4\pi n}{5}$, $\theta_2 = \frac{4\pi n}{5}$. It follows that the contribution to the rho invariant from the fixed points of order 5 is

$$-\frac{2^3\cdot 3}{120}\sum_{n=1}^{4} \cot(\frac{2\pi n}{5})\cot(\frac{2\pi n}{5}) \cdot 2 \cdot (\cos(-\frac{4\pi n}{5}) - 1) = \frac{2^4 \cdot 3^2}{120}.$$

In the same way, we can add up the contributions from all of the fixed points to get:

$$\rho_M(\alpha_0) = \frac{28}{15}.$$

Putting this together with Lemma 2 and Lemma 3 gives, $\rho_{\Sigma(2,3,5)}(\alpha) = \frac{28}{15} + 3 = \frac{73}{15}$.

In [A] we computed $CS_{\Sigma(2,3,5)}(\alpha) = \frac{1}{120}$. Filling in Lemma 1 gives the final answer: $SF(\theta, \alpha) = 1$.

In order to understand the meaning of this, extend the path of operators, $B(A_t)$ to the whole real line by $B(A_t) = B(A_0)$ for $t < 0$ and $B(A_t) = B(A_1)$ for $t > 1$.

Now solve the differential equation,

$$\frac{du}{dt} + B(A_t) \cdot u = 0,$$

by splitting u into eigenfunctions. The solution will decay (or grow) like $e^{-\lambda_0 t}$ as $t \to -\infty$ and like $e^{-\lambda_1 t}$ as $t \to \infty$. In other words, the solution decays on the ends of $\mathbb{R} \times \Sigma(2,3,5)$ if and only if $\lambda_0 < 0$ and $\lambda_1 > 0$. The above differential equation is the linearized version of the anti-self duality equations. A moment's reflection indicates that the spectral flow from θ to α is the dimension of the moduli space of anti-self dual connections on $\mathbb{R} \times \Sigma(2,3,5)$ with boundary values θ as $t \to -\infty$ and α as $t \to \infty$. Starting with the standard instanton on $\mathbb{R} \times S^3$, we can construct a one parameter family of ASD connections on $\mathbb{R} \times \Sigma(2,3,5)$. First, mod out by the binary icosahedral group then translate that solution in the $\mathbb{R}$ direction. David Austin has shown that these solutions make up the entire moduli space of finite energy ASD connections on $\mathbb{R} \times \Sigma(2,3,5)$ with the given boundary conditions [Au].

We will now work through a hyperbolic example. This manifold in Figure 7 is hyperbolic according to the computer program Snappea [W].

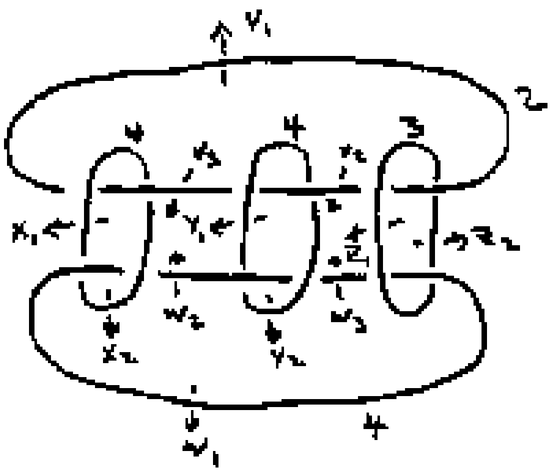

FIGURE 7. The manifold, M.

The fundamental group is generated by Wittinger generators,

$$V_1, V_2, V_3, W_1, W_2, W_3, X_1, X_2, Y_1, Y_2, Z_1, Z_2$$

with relations coming from the crossings and the longitudes of the surgeries. The crossing relation from the upper left is, $V_3 = X_1 V_1 X_1^{-1}$. Using the crossing relations, it is possible to see that $\pi_1(M)$ is generated by V_1, W_1, X_1, Y_1, and Z_1. The surgery relations may also easily be read from Figure 7. For example the surgery relation arising from the V component is, $V^2 ZYX = 1$. It is an elementary exercise to check that the assignments $V \mapsto -1$, $W \mapsto -1$, $X \mapsto i\exp(-5/3\pi k)$, $Y \mapsto i$, and $Z \mapsto \exp(2\pi k/3)$ satisfy all of the crossing relations and all of the surgery relations. The above assignment, therefore, defines a representation, $\alpha : \pi_1(M) \to Sp_1$. We will compute the spectral flow from the trivial representation to α.

The first step is to construct the flat cobordism, W, described in Lemma 2. The new boundary component, M_1, is drawn in Figure 8.

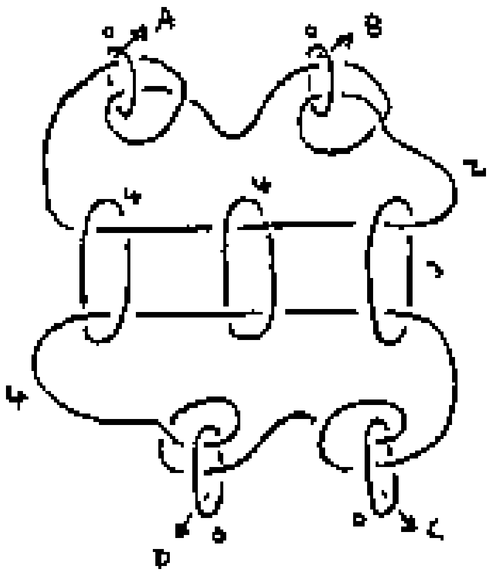

FIGURE 8. A flat cobordant manifold, M_1.

The fundamental group for M_1 may be obtained from the fundamental group of M by adding four new generators: A, B, C, and D, four new relations: $AVA^{-1}V$, $BVB^{-1}V$, $CWC^{-1}W$, $DWD^{-1}W$ and altering two of the surgery relations to

$$V^2ZYXA^2B^2$$

and

$$W^4XV^{-1}XVX^{-1}Z^{-1}V^{-1}ZYZ^{-1}VC^2D^2.$$

The induced representation on M_1, α_1, is clearly in the path component of a reducible representation, α_1' on M_1. To see this, just conjugate $\alpha_1(X)$ and $\alpha_1(Z)$ to i and $\exp(2\pi i/3)$, set $\alpha_1'(B) = \alpha_1'(D) = 1$ and solve for $\alpha_1'(A)$ and $\alpha_1'(C)$ with the altered surgery relations.

The reducible representation is in the path component of the trivial representation in the representation space of the link complement. We may, therefore, compute the Chern–Simons invariant with the Kirk and Klassen formula [KK1], [A]. For each meridian pick a "linear" path from the trivial element of Sp_1 and a positively oriented longitude. As an example, send X to $\exp(2\pi i a_X(t))$ where $a_X(t) = \frac{1}{4}t$ and notice that the longitude is: $\lambda_X = X^4VW$, so that $b_X(t) = 4a_X(t) + a_V(t) + a_W(t) = 2t$. Plugging in all of the meridians and longitudes gives:

$$CS_M(\alpha) \equiv -1/3 \mod \mathbb{Z}.$$

Elementary linear algebra will now be sufficient to compute the twisted cohomology groups and the change in the rho invariant from M, α to M_1, α_1'. By inspection, α is irreducible and θ is trivial, so

$$\dim H^0(M;\theta) = 3 \text{ and } \dim H^0(M;\alpha) = 0.$$

The group, $H^1(M;\theta) \cong H_1(M;\theta) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)] \otimes_{\mathbb{Z}} \mathbb{R}^3 \cong H_1(M;\mathbb{R}) \otimes \mathbb{R}^3$ as will be verified when we compute Sign W. According to Lemma 3, the only contribution to Sign W comes from $\operatorname{coker}(H^1(M) \oplus H^1(2V) \to H^1(2S^1 \times D^2))$, or by the universal coefficient theorem from $\ker(H_1(2S^1 \times D^2) \to H_1(M) \oplus H_1(V))$. The group, $H_1(2S^1 \times D^2)$ is generated by V and W. Both V and W map to zero in $H_1(2V)$ as in Figure 4. In $H_1(M)$, we have

$$\begin{aligned}
\partial a &= 4X + V + W\\
\partial b &= 4Y + V + W\\
\partial c &= 3Z + V - W\\
\partial d &= 2V + X + Y + Z
\end{aligned}$$

and

$$\partial e = 4W + X + Y - Z.$$

From these equations we may solve for each of V, W, X, Y and Z in terms of boundaries so that $H_1(M;\mathbb{R}) = 0$. In particular, $V = \partial(\frac{1}{66}(57d - 18c + 3e - 15a - 15b))$, so we get a 2-dimensional homology class associated to v, namely $\frac{1}{66}(57d - 18c + 3e - 15a - 15b) - \frac{1}{2}f$, where f is the surface in Figure 4. From this we may compute $V \cdot V$ given $c \cdot c = 3$ etc. After computing $W \cdot W$ and $V \cdot W$, we see that

$$\text{Sign } W = 2.$$

A direct computation will show that

$$\dim H^1(M;\alpha) = 3 \text{ and recall that } \dim H^1(M;\theta) = 0.$$

Furthermore, dim $\operatorname{coker}(H^1(M;\alpha) \oplus H^1(V;\alpha) \to H^1(2S^1 \times D^2;\alpha)) = 3$ and the coker is generated by $V - W$. The corresponding labeled surfaces in W are a union of a pair of thin pants and a pair of fat pants labeled with an i around the X-component, also the same surface labeled with either a j or a k around the Y-component. See Figure 9.

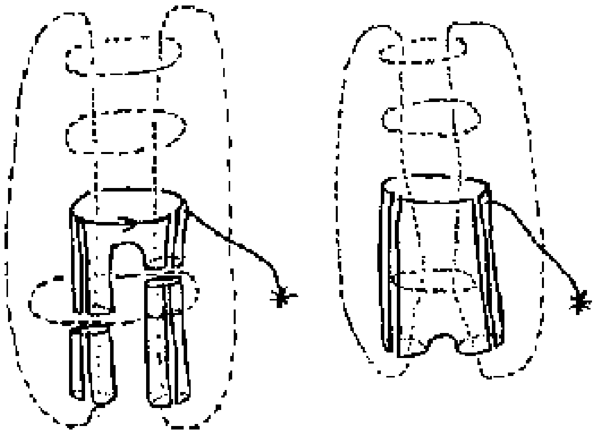

FIGURE 9. Thin pants, fat pants.

This shows that

$$\text{Sign } Q_\alpha = 0.$$

We will do a similar computation in a bit more detail later.

More linear algebra will verify that the path of representations from α_1' to α_1 satisfies the hypothesis of Lemma 3. Combining Lemma 2 and Lemma 3 gives:

$$\rho_M(\alpha) = \rho_{M_1}(\alpha_1') - 6.$$

After computing $\rho_{M_1}(\alpha_1')$, we will know all of the terms in the formula for the spectral flow. We could directly construct a 4-manifold which equivariantly bounds the cover of M_1 induced from α_1' and compute $\rho_{M_1}(\alpha_1')$ with Lemma 4, but it is easier to first perform a sequence of flat cobordisms and a trivial deformation.

One natural way to build a flat cobordism is to attach a 2-handle to $I \times M$ along a curve in the kernel of the representation. Lemma 3 is also valid for this type of cobordism, because the only fact about V used in the proof was that the second cohomology group with the appropriate coefficients is trivial. For the first cobordism, W_1, just attach a 0-framed two handle to the meridian, B. For W_2 attach a 0-framed two handle to D. Always call the new boundary component of W_n, M_{n+1}. The third cobordism is constructed by sliding the W handle over the V handle in M_3 and then attaching a 0-framed two handle to the meridian of the V-handle. See Figure 10.

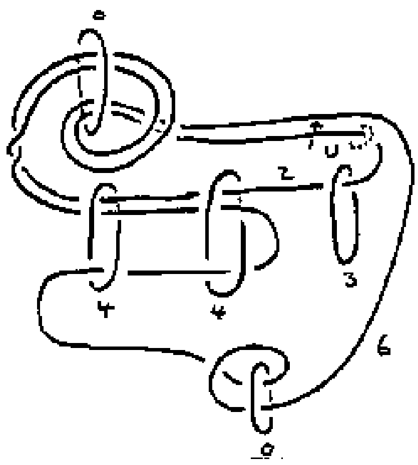

FIGURE 10. The cobordism W_3.

In a surgery description, the linking matrix is the same as the intersection matrix of the 4-manifold. Thus when we slide handle W over handle V the matrix changes as though we replaced W by $W+V$ in the basis of a quadratic form. The image under the representation of the handle that we slide does not change. However, the meridian to V now represents VW^{-1} in π_1. Thus the loop labeled U in Figure 10 is in the kernel of the representation.

Sliding the Y-handle over the X-handle and attaching a 0-framed 2-handle to the meridian of X produces the next cobordism, W_4. See Figure 11.

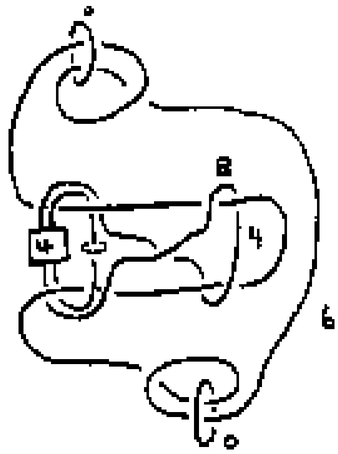

FIGURE 11. The cobordism W_4.

At this stage we will deform the representation along the path $A \mapsto \exp(2\pi i(\frac{1}{12}t - \frac{1}{2}))$, $C \mapsto \exp(2\pi i(-\frac{1}{12}t))$ leaving all other meridians constant. It is easy to see that all of these representations are abelian but not central. In fact, one can check that the dimension of the group cohomology is constant along this path of representations. The proof of Lemma 4 shows that

$$\rho_{M_5}(\alpha_5') = \rho_{M_5}(\alpha_5).$$

Continuing in this manner, we kill the C component with the W_5 cobordism, then slide the A component over the W-component and kill the W-component with the W_6 cobordism. This leaves us with the manifold and representation pictured in Figure 12.

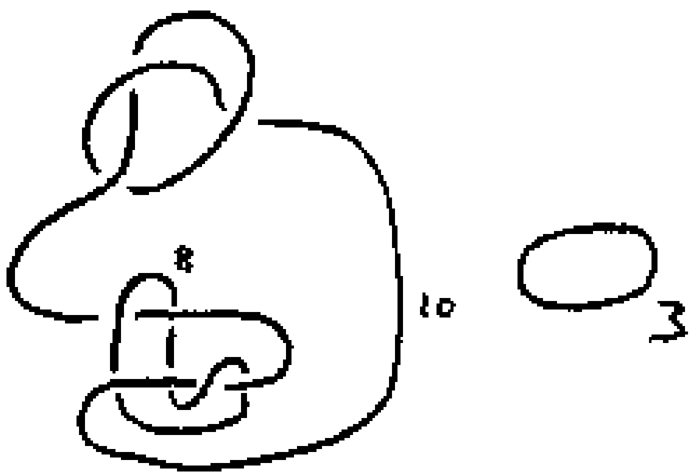

FIGURE 12. The manifold, M_7.

In order to apply Lemma 2 to compute the change in the rho invariant in this sequence of flat cobordisms, we need to compute the signatures and twisted signatures of the cobordisms. The ordinary signature may be computed from the linking matrix of the left end of the cobordism. If the attaching map of the new 2-handle is trivial in $H_1(M;\mathbb{R})$, then the computation is exactly the same as the computation of sign W above. If the attaching map is non-trivial, then the signature is zero. The W_5 cobordism is an example of this second situation. The linking matrix of the given surgery description of M_5 is:

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 6 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

in the basis $\{A, W, C, Y\}$. From this it follows that $H_1(M;\mathbb{R}) \cong \mathbb{R}$ generated by $[C]$. Thus, the kernel is trivial. The result of these computations is:

$$\begin{array}{lll} \text{Sign } W_1 = 0 & \text{Sign } W_2 = 0 & \text{Sign } W_3 = 0 \\ \text{Sign } W_4 = 1 & \text{Sign } W_5 = 0 & \text{Sign } W_6 = -1. \end{array}$$

The twisted signatures may be computed in basically the same way. Since the coefficients are reducible, $H_1(M_k;\alpha) \cong H_1(M_k;\mathbb{R}) \oplus H_1(M_k;\tilde{\mathbb{C}})$. It follows that $\mathrm{Sign}^Q W_k =$ Sign $W_k +$ $\mathrm{Sign}^{\mathbb{C}} W_k$. The group $\ker(H_1(S^1 \times D^2;\tilde{\mathbb{C}}) \to H_1(M_k;\tilde{\mathbb{C}}))$ is trivial for $k = 1, 2, 4$ and 5, thus $\mathrm{Sign}^Q W_k =$ Sign W_k for these values of k. If $\alpha(z) \in S^1$ then $\alpha(z)$ acts trivially on the $i\mathbb{R}$ part of Sp_1 and acts with weight two on the $\mathbb{C}j$ part, since

$$e^{i\theta} \cdot zj = e^{i\theta} z j e^{-i\theta} = e^{2i\theta} zj.$$

The group, $H_1(M_3;\tilde{\mathbb{C}})$ is generated by the generators of $\pi_1(M_3)$ with relations given by the Fox derivatives of the relators. The relation coming from the upper right crossing in Figure 10 is: $UZU^{-1}Z^{-1} = 1$. Taking the Fox derivative gives:

$$(1 - e^{4\pi i/3})u = u + \alpha_3(U)^2 z - \alpha(UZU^{-1})^2 u - z = 0.$$

A representative generating $H_2^{L^2}(W_3)$ is drawn in Figure 13. It is clear that it has trivial self intersection, so that $\mathrm{Sign}^{\mathbb{C}} W_3 = 0$.

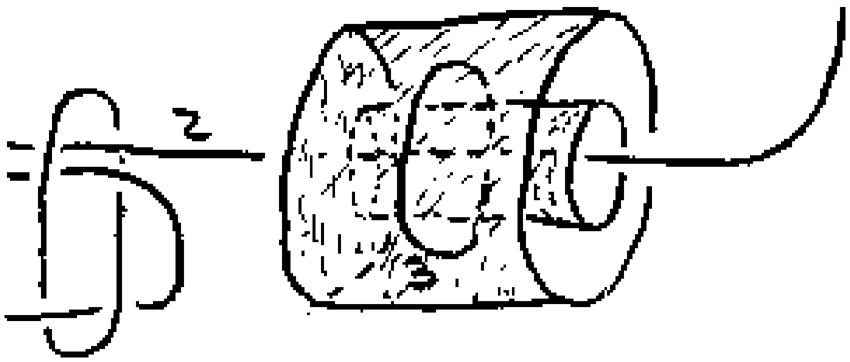

FIGURE 13. A representative for U.

The signature, $\mathrm{Sign}^{\mathbb{C}} W_6$ may be computed by the method above. The generator of $H_2^{L^2}(W_6;\tilde{\mathbb{C}})$ is fairly complicated and the picture is messy. There is an alternative way to compute this signature. First note that we are considering $H_2^{L^2}(W_6;\tilde{\mathbb{C}})$ as a real vector space. This means that $\mathrm{Sign}^{\tilde{\mathbb{C}}} W_6 = 2\ \mathrm{Sign}_\alpha W_6$, where $\mathrm{Sign}_\alpha W_6$ is the signature, considering $H_2^{L^2}(W_6;\tilde{\mathbb{C}})$ as a complex vector space. Inventing formula 2.5 from [APS] gives:

$$\mathrm{Sign}_\alpha X = \frac{1}{|G|} \sum_{g \in G} \mathrm{Sign}(g, \tilde{X}) \chi_\alpha(g).$$

In our situation, $G = \mathbb{Z}_2$, $X = W_6$ and $\tilde{X}$ is the 2-fold cover of W_6 induced from the representation. By restricting the intersection form to the positive and negative eigenspaces of the generator of $\mathbb{Z}_2$, we get:

$$\mathrm{Sign}(1, \tilde{X}) = \mathrm{Sign}(\tilde{X}) = \mathrm{Sign}\ Q|_{E_+} + \mathrm{Sign}\ Q|_{E_-}$$

and

$$\operatorname{Sign}(-1, \tilde{X}) = \operatorname{Sign} Q|_{E_+} - \operatorname{Sign} Q|_{E_-}.$$

Furthermore, a transfer argument shows that $\operatorname{Sign} Q|_{E_+} = \operatorname{Sign} W_6$, giving:

$$\operatorname{Sign}_\alpha W_6 = \operatorname{Sign} \tilde{X} - \operatorname{Sign} W_6 = -2 - (-1) = -1.$$

The cover, $\tilde{X}$ is pictured in Figure 14.

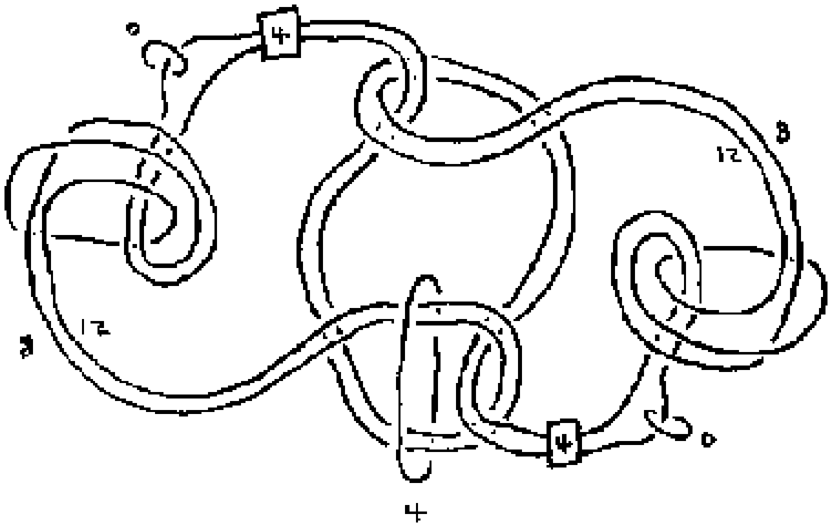

FIGURE 14. The cover $\tilde{X}$.

Summarizing:

$$\begin{array}{lll} \operatorname{Sign}^Q W_1 = 0 & \operatorname{Sign}^Q W_2 = 0 & \operatorname{Sign}^Q W_3 = 0 \\ \operatorname{Sign}^Q W_4 = 1 & \operatorname{Sign}^Q W_5 = 0 & \operatorname{Sign}^Q W_6 = -3 \end{array}$$

so that

$$\rho_{M_7}(\alpha_1') = \rho_{M_7}(\alpha_7) - 2.$$

by Lemma 2.

We are now in a position to finish the computation. Since $M_7 = M_8 \# L(3,-1)$,

$$\rho_{M_7}(\alpha_7) = \rho_{M_8}(\alpha_7|_{M_8}) + \rho_{L(3,-1)}(\alpha_7|_{L(3,-1)}).$$

The induced cover of the lens space is S^3 which equivariantly bounds D^4 with one fixed point. Lemma 4, therefore, implies that $\rho_{L(3,-1)}(\alpha_7|_{L(3,-1)}) = -2/3$. In order to compute $\rho_{M_8}(\alpha_7|_{M_8})$, we need to construct a 4-manifold with a $\mathbb{Z}_4$ action so that the quotient map on the boundary is exactly the covering of M_8 induced from the representation.

There are two general methods to construct such a 4-manifold. The first method would be to use Kirby moves to produce a surgery description of M_8 so that the meridian of one unknotted component is sent to a generator and the meridians of all other components are in the kernel of the representation. The unknotted component alone would produce a lens space. The cover of this lens space is S^3 which bounds D^4. By gluing 2-handles to the lifts of the remaining components, we will produce the desired 4-manifold. The second method is described in the work of Casson and Gordon [CG]. It is this second method that we will use here. Start by performing handle slides until all meridians are sent to the generator. End the construction by taking the branched cover of the surgery description branched along a pushed in Seifert surface glued to the cores of all of the 2-handles.

One apparent difference between our computations and the Casson–Gordon computations is the difference between using L^2-cohomology or ordinary homology to compute signatures. There really is no difference. By duality, we could compute in L^2-homology, $H_2^{L^2}(W) \equiv \operatorname{Im}(H_2(W) \to H_2(W, \partial W))$. If F_1 is any class in $H_2(W)$ which is trivial in $H_2^{L^2}(W)$, then F_1 is represented by a surface in ∂W. Using a collar of ∂W, any other surface may be pushed into the interior of W. Thus the intersection number with F is zero and F is in the null space of the intersection form on $H_2(W)$.

Recapping from Casson and Gordon, if $\langle \tau \mid \tau^m = 1 \rangle = \mathbb{Z}_m$ acts on X^4, then

$$\begin{aligned} \operatorname{Sign}(\tau^s, X) &= \textstyle\sum_{r=0}^{m-1} \exp(2\pi i r s/m) \operatorname{Sign}(Q|_{E_r}) \\ &= \operatorname{Sign}(X/\mathbb{Z}_m) + \textstyle\sum_{r=1}^{m-1} \exp(2\pi i r s/m) \operatorname{Sign}(Q|_{E_r}). \end{aligned}$$

Here E_r is the $\exp(2\pi i r/m)$-eigenspace of τ and the second line is true by a transfer argument. The transfer argument is just that $[\alpha] \in E_0$ implies that $[\alpha] = [\frac{1}{m}\sum_{r=0}^{m-1}(\tau^r)^*\alpha]$, but $1/m\sum_{r=0}^{m-1}(\tau^r)^*\alpha$ is equivariant and may therefore be pushed forward. This constructs an isomorphism, $E_0(X) \to H(X/\mathbb{Z}_m)$. Casson and Gordon use a Mayer–Vietoris argument to show that the signatures of the other eigenspaces may be computed from the cyclic cover of the complement of a pushed in Seifert surface. If $\{x_k\}$ is a basis for the first homology of the Seifert surface, then

$$\{\sum_{s=0}^{m-1} \exp(2\pi i r s/m) \tau_*^s x_k^*\}$$

is a basis for E_r. In this basis, $Q|_{E_r} = (1 - \exp(-2\pi i r/m))A + (1 - \exp(2\pi i r/m))A^*$, where A is the Seifert form.

Figure 15 shows the manifold, M_8 after the appropriate handle slides have been performed.

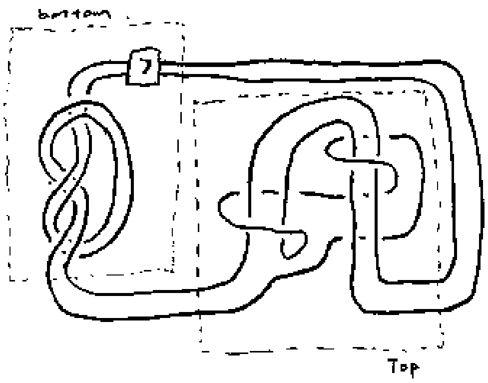

FIGURE 15. The manifold, M_8.

Figures 16 and 17 show two parts of the Seifert surface so that the Seifert form may be computed.

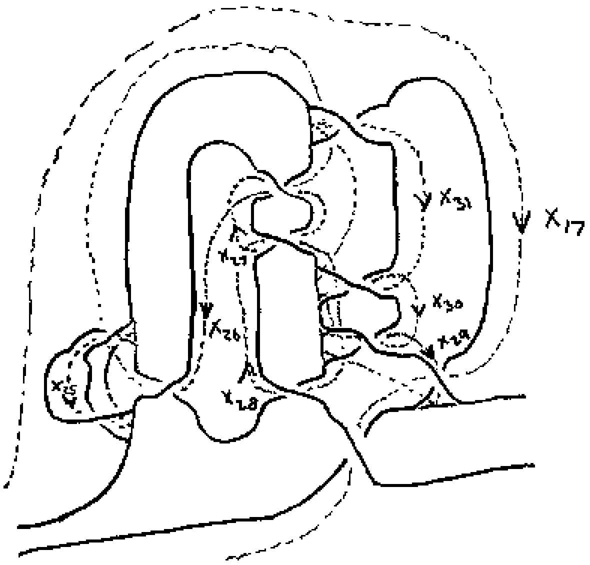

FIGURE 16. Top half of Seifert surface.

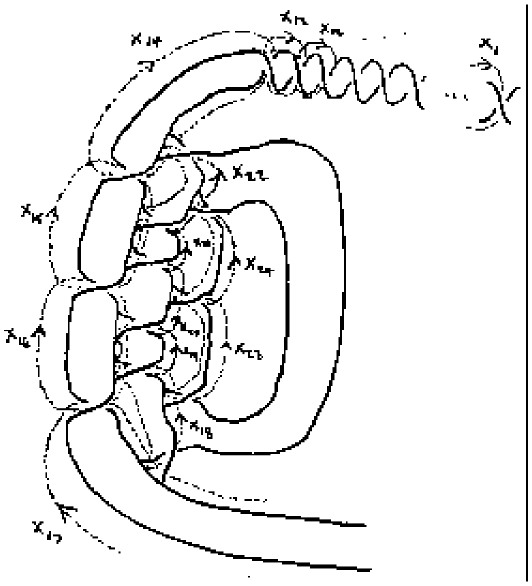

FIGURE 17. Bottom half of Seifert surface.

$$
\mathbf{A}=\begin{bmatrix}
-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&-1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
$$

Plugging into the preceding formulae for g-signatures, and then Lemma 3, gives:

$$\rho_{M_8}(\alpha_7|_{M_8}) = 150.$$

To compute the signatures, we just entered "N[Eigenvalues[Q_{E_r}]]" into *Mathematica* and counted the number of positive and negative eigenvalues [Wo]. There are in fact very good numerical methods for computing signatures. See [?] for example.

Combining all of the previous computations with Lemma 1 completes the computation to give:

$$SF_M(\theta, \alpha) = 65.$$

To conclude, we have explained a method to compute the spectral flow. This method works for a large class of representations on a large class of manifolds. Namely, it works for any representation that is flat cobordant-flat deformation equivalent to a reducible representation. It is conceivable that this method works for any representation on any 3-manifold. The recent work of Reznikov gives some slight evidence supporting this possibility.

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References

[APS] M. Atiyah, V. Patodi and I. Singer, *Spectral asymmetry and Riemannian geometry I, II, III*, Math. Proc. Camb. Phil. Soc. **77,78,79** (1975), 43–69, 405–432, 71–99.

[A] D. Auckly, *Topological methods to compute Chern–Simons invariants*, Math. Proc. Camb. Phil. Soc. **115** (1994), 229–251.

[Au] D. Austin, *Equivariant Floer groups for binary polyhedral spaces*, Math. Ann. **302** (1995) no 2, 295–322.

[CG] A. Casson and C. Gordon, *On Slice Knots in dimension three*, Proc. Symp. Pure Math. **32** (1978), 39–53.

[FS1] R. Fintushel and R. Stern, *Pseudofree orbifolds*, Ann. of Math. **122** (1985), 335–364.

[FS2] R. Fintushel and R. Stern, *Instanton homology of Seifert-fibered homology spheres*, Proc. Lond. Math. Soc. (3) **61** (1990), 109–137.

[KK1] P. Kirk and E. Klassen, *Chern–Simons invariants of* 3*-manifolds and representation spaces of knot groups*, Math. Ann. **287** (1990), 347–367.

[KK2] P. Kirk and E. Klassen, *Computing spectral flow via cup products*, J. Diff. Geom. **40** (1994), 505–562.

[KKR] P. Kirk, E. Klassen and D. Ruberman, *Splitting the spectral flow and the Alexander matrix*, Comment. Math. Helv. **69** (1994) no 3, 375–416.

[R] A. Rezinikov, *Rationality of secondary classes*, J. Diff. Geom. to appear.

[W] J. Weeks, "Snappea", software available from http://geom.umn.edu/software/download/snappea.html.

[Wo] S. Wolfram, Mathematica, A system for doing mathematics by computer, Addison–Wesley, 1988.

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