QUANTIZATION OF THE ALGEBRA OF CHORD DIAGRAMS

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Abstract. In this paper we define an algebra structure on the vector space $L(\Sigma)$ generated by links in the manifold $\Sigma \times [0,1]$ where Σ is an oriented surface. This algebra has a filtration and the associated graded algebra $L_{Gr}(\Sigma)$ is naturally a Poisson algebra. There is a Poisson algebra homomorphism from the algebra of chord diagrams $ch\,(\Sigma)$ on Σ to $L_{Gr}(\Sigma)$.

We show that multiplication in $L\,(\Sigma)$ provides a geometric way to define a deformation quantization of the algebra of chord diagrams, provided there is a universal Vassiliev invariant for links in $\Sigma \times [0,1]$. The quantization descends to a quantization of the moduli space of flat connections on Σ and it is universal with respect to group homomorphisms. If Σ is a compact with free fundamental group we construct a universal Vassiliev invariant.

Contents

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1. Introduction

The study of finite type (or Vassiliev) invariants of knots and links [Vas90], [Vas92] leads to the notion of chord diagrams [BN95]. All the known quantum invariants of links are determined by finite type invariants [BL93].

In [AMR96] we extended the notion of a chord diagram to arbitrary oriented Riemann surfaces Σ and showed that these chord diagrams are an effective tool to clarify the study of the algebra of functions on the moduli space of flat connections on Σ as started by Goldman in [Gol86]. The algebra of chord diagrams $ch\,(\Sigma)$ is a graded Poisson algebra, its component of degree zero is exactly the algebra of loops considered in [Gol86]. There is a natural Poisson algebra homomorphism from the Poisson algebra of (coloured) chord diagrams to $\mathcal{F}\left(\mathcal{M}_\Sigma^G\right)$, the functions on the moduli space of flat G-connections, which for many simple groups G is known to be surjective [AMR96],[Mat]. Therefore $ch\,(\Sigma)$ can be viewed as a universal moduli space.

The present paper is the first in a series devoted to the study of the quantum versions of the structures used in [AMR96]. We show that there is a universal invariant of links in $\Sigma \times [0,1]$ with values in chord diagrams, where Σ is a punctured surface. Using this we construct a quantization of the algebra of chord diagrams on Σ using the (noncommutative) multiplication of links in $\Sigma \times [0,1]$. We show that (in contrast to the "obvious" linear quantization of $ch\,(\Sigma)$) our quantization descends to a quantization of the moduli space $\mathcal{M}_\Sigma^G$ of flat G-connections on Σ.

Other approaches to the quantization of the moduli space of flat connections include skein modules as in [Tur91],[Oza90] and the related quantization of the SL_2-moduli space of [BFKB96], geometric quantization [Ati90], [ADPW91], [Fal93], [Hit90] and the combinatorial approach of [AGS95a], [AGS95b], [AS95]. We will comment on relations to these constructions in a forthcoming publication.

Chord diagrams also arose in Yang-Mills theory on Minkowski space [RT95] and in quantum gravity [Bae94].

We have been greatly helped by discussions with Dylan Thurston and Pol Vanhaecke as well as by Thang Le, Pierre Vogel, Akira Yoshioka and by Dror Bar-Natan's Mathematica package for calculating with chord diagrams, available at
http://www.ma.huji.ac.il/~drorbn, cf. [BN93].

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2. Algebra of links in $\Sigma \times [0,1]$

2.1. **Filtered and graded rings.** By a filtered ring we mean a ring F with a descending filtration $F = F_0 \supseteq F_1 \supseteq \dots$ indexed by the natural numbers, such that the multiplication respects the filtration $F_m F_n \subseteq F_{m+n}$. We write $F_\infty = \bigcap_{n\geq 0} F_n$

and $F' = F/F_\infty$. A map of filtered rings $f : F \to G$ has to respect the filtrations: $f(F_n) \subseteq G_n$.

Each filtered ring defines a graded ring $F_{Gr} = \bigoplus_{n\geq 0} F^{(n)}$ with graded components $F^{(n)} = F_n/F_{n+1}$. A map of filtered rings f induces a map f_{Gr} of the associated graded rings. The associative multiplication on F induces an associative multiplication on F_{Gr}: For $x \in F_n, y \in F_m$ define the product of their classes $[x] \in F^{(n)}, [y] \in F^{(m)}$ as the class of the product:

$$[x].[y] := [xy] \in F^{(n+m)} \tag{1}$$

If f is a ring homomorphism, so is f_{Gr}. Conversely, given a graded ring $G = \bigoplus_{n\geq 0} G^{(n)}$ defines a filtered ring G^F with filtered components $G^F_m = \bigoplus_{n\geq m} G^{(n)}$ and another filtered ring $\overline{G}$ with filtered components $\overline{G}_m = \prod_{n\geq m} G^{(n)}$. This is the topological completion of G^F in the topology whose basis of open sets is given by translates of the filtered components. Clearly $F' \subseteq \overline{F_{Gr}}$.

Lemma 1. *If $F = F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots$ is a filtered ring such that the associated graded ring $F_{Gr} = \bigoplus_{n=0}^{\infty} F_n/F_{n+1}$ is commutative, then the bracket*

$$\{[x],[y]\} := [xy - yx] \in F^{(n+m+1)} \tag{2}$$

for $x \in F_n, y \in F_m$ defines a Poisson bracket on F_{Gr}.

2.2. **Multiplication of links.** If M_1 and M_2 are two 3-manifolds, $X \hookrightarrow \partial M_1$ and $X^- \hookrightarrow \partial M_2$, then we denote by $M_1 \cup_X M_2$ the result of gluing M_1 and M_2 along X. Denote by $L(\Sigma)$ the ring spanned by (framed) links in $\Sigma \times [0,1]$ (by a link we mean an isotopy class of smooth imbeddings $(S^1)^{\cup k} \hookrightarrow \Sigma \times [0,1]$), with multiplication in $L(\Sigma)$ defined by the following composition of maps:

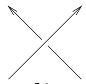
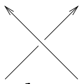
$$L_1 \otimes L_2 \mapsto L_1 \times L_2 \mapsto L_1 \cup L_2$$

where $L_1 \times L_2 \subset (\Sigma \times [0,1/2]) \times (\Sigma \times [1/2,1])$ and $L_1 \cup L_2 \subset (\Sigma \times [0,1/2]) \cup_{\Sigma \times \{1/2\}} (\Sigma \times [1/2,1]) = \Sigma \times [0,1]$.

Proposition 2. *This multiplication determines on $L(\Sigma)$ the structure of an associative (in general noncommutative) ring with the empty link being the unit element.*

Proof: Obvious. □

2.3. **Filtration.** Let $L \subset \Sigma \times [0,1]$ be a link and $D_L \subset \Sigma$ some link diagram of L, so that D_L is (an isotopy class of) a regular projection of L to Σ. As usual we distinguish vertices of two types: For each vertex $v \in D_L$ we introduce an oriented crossing number $\epsilon(v)$ as the sign of the vertex

$+$ is  , $-$ is 

If the diagram D_L has vertices $v_1, \ldots, v_n$ with corresponding oriented crossing numbers $\epsilon_1, \ldots, \epsilon_n$ we also denote it by $D_L^{\epsilon_1,\ldots,\epsilon_n}$ when we wish to emphasize the types of the crossings. We may regard L as an equivalence class $[D_L]$ of diagrams that are related by Reidemeister moves.

Introduce the following operation ∇: Choose a set of crossings $v_{i_1}, \ldots, v_{i_m}$ of D_L and set

$$\nabla_{v_{i_1},\ldots,v_{i_m}} D_L := \sum_{\epsilon_{i_1},\ldots,\epsilon_{i_m}=\pm 1} \epsilon_{i_1} \ldots \epsilon_{i_m} [D_L^{\epsilon_1,\ldots,\epsilon_n}]$$

This maps the link diagram D_L to a linear combination of links whose regular projections are obtained from D_L by switching crossings. If $v_1, \ldots, v_m$ is a list of all crossing of D_L then we denote $\nabla_{v_{i_1},\ldots,v_{i_m}} D_L$ simply by ∇D_L.

Now let $L_m(\Sigma) \subseteq L(\Sigma)$ be the span of all elements of the form $\nabla_{v_{i_1},\ldots,v_{i_m}} D_L$ where D_L runs over all possible link diagrams.

Proposition 3. 1. *The filtration $L(\Sigma) \supset L_1(\Sigma) \supset L_2(\Sigma) \ldots$ is compatible with the algebra structure.*
2. *If $x \in L_n$ and $x' \in L_m$ then $x.x' - x'.x \in L_{m+n+1}$.*

Proof :

1. We have to prove that $L_n(\Sigma).L_m(\Sigma) \subset L_{n+m}(\Sigma)$. Indeed, on generators, if $x = \nabla_{v_1,\ldots,v_n} D_L, x' = \nabla_{w_1,\ldots,w_m} D_{L'}$ are in general position, then $x.x' = \nabla_{v_1,\ldots,v_n,w_1,\ldots,w_m} D_{L.L'}$.
2. We know that $x.x' - x'.x = \nabla_{v_1,\ldots,v_n,w_1,\ldots,w_m} D_{L.L'} - \nabla_{v_1,\ldots,v_n,w_1,\ldots,w_m} D_{L'.L}$ and one can obtain $D_{L'.L}$ from $D_{L.L'}$ by a sequence of crossing changes away from $v_1, \ldots, v_n, w_1, \ldots, w_m$. This clearly implies the assertion.

□

It follows from proposition 3(2) that the multiplication on L_{Gr} is commutative. By continuity it extends to $\overline{L_{Gr}(\Sigma)}$.

2.4. **Finite type invariants.** Recall that every link invariant extends in the obvious way to an element of $L(\Sigma)^*$, we identify the invariant with this extension. A link invariant v is called a finite type invariant (or Vassiliev invariant) if $v|_{L_n(\Sigma)} = 0$ for some n. The degree of v is the smallest such n minus 1.

Set $L_\infty(\Sigma) := \bigcap_{n\in\mathbb{N}} L_n(\Sigma)$. We will be interested mostly in the algebra $L'(\Sigma) := L(\Sigma)/L_\infty(\Sigma)$. The question whether $L(\Sigma) = L'(\Sigma)$ is difficult, in fact it is equivalent to the main open problem in the theory of finite type invariants: whether finite type invariants are sufficient to determine link types.

Proposition 4. *Finite type invariants distinguish links in $\Sigma \times [0,1]$ iff $L_\infty(\Sigma) = \emptyset$.*

Proof: If $L_\infty(\Sigma) = \emptyset$, then for any pair of distinct links L_1, L_2 there is some n such that $[L_1 - L_2] \neq 0 \in L(\Sigma)/L_n(\Sigma)$. Therefore there is an element v of $(L(\Sigma)/L_n(\Sigma))^*$, i.e. an invariant of type $\leq n$, such that $v(L_1) - v(L_2) \neq 0$.

Conversely, assume $\sum_{i=0}^n \lambda_i L_i \in L_\infty(\Sigma)$. By induction on n one constructs a finite type invariant v such that $v(L_i) \neq v(L_j)$ for $i \neq j$. Using that any polynomial in v is again a finite type invariant and that $\det\left(v^k(L_l)\right)_{k,l} \neq 0$, we can clearly find a finite type invariant $w = p(v)$ with $\sum_{i=0}^n \lambda_i w(L_i) \neq 0$. Hence $\sum_{i=0}^n \lambda_i L_i \neq 0$ in $L(\Sigma)/L_{\deg w}(\Sigma)$ which implies $\sum_{i=0}^n \lambda_i L_i \notin L_{\deg w}(\Sigma) \supseteq L_\infty(\Sigma)$.□

Remark 5. *We could not find this result in the literature, even for $\Sigma = S^2$.*

3. Chord diagrams and universal finite type invariants

3.1. **Chord diagrams.** Now let us give an explicit description of the vector spaces $L_n(\Sigma)$. Recall the following definitions from [AMR96]:

Definition 6. *A* chord diagram *is a graph consisting of disjoint oriented circles $S_i, i \in \{1, ..., n\}$ and disjoint arcs $C_j, j \in \{1, ..., m\}$ such that:*

1. *the endpoints of the arcs are distinct*
2. $\cup_j \partial C_j = (\cup_i S_i) \cap (\cup_j C_j)$

The arcs are called chords, *the circles S_i are called the* core components *of the diagram.*

Definition 7. *Given a closed oriented surface Σ, a* geometrical chord diagram *on Σ is a smooth map from a chord diagram D to Σ, mapping the chords to points. A* chord diagram on Σ *is a class of geometric chord diagrams modulo homotopy.*

Note that diffeomorphism classes of chord diagrams correspond to chord diagrams on S^2.

Definition 8. *By a* generic chord diagram *(on Σ) we will mean a geometrical chord diagram on Σ such that all circles are immersed, and with all double points transverse.*

Clearly every chord diagram on Σ contains generic chord diagrams. A generic chord diagram on Σ with images of chords at points $v_1, \ldots, v_m$ will be denoted by $D(v_1, \ldots, v_m)$.

Consider the complex vector space V_Σ with the basis given by the set of chord diagrams on Σ and the subspace W generated by the 4T-relations [BN95].

Definition 9. *The algebra $ch(\Sigma) := V_\Sigma / W$ is called the* algebra of chord diagrams *on Σ.*

It has a natural ring structure with multiplication given by union of chord diagrams, with unit the empty diagram. It is isomorphic to the polynomial ring on the space of connected chord diagrams.

These rings are graded by the number of chords

$$ch\,(\Sigma) = \bigoplus_{n \geq 0} ch^{(n)}\,(\Sigma)$$

and we have an associated filtered space with filtered components $ch_m\,(\Sigma) := \bigoplus_{n \geq m} ch^{(n)}\,(\Sigma)$ and completion $\overline{ch\,(\Sigma)} = \prod_{n \geq 0} ch^{(n)}\,(\Sigma)$.

To any element $D \in ch^{(n)}\,(\Sigma)$ we can associate an element $\lambda\,(D) \in L\,(\Sigma)\,/L_{n+1}\,(\Sigma)$ by setting

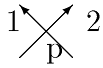
$$\lambda\,(D) := \nabla_{v_{i_1},\ldots,v_{i_n}} D_L \mod L_{n+1}\,(\Sigma)$$

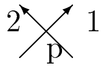
for any link L that projects to the diagram D, where $\{v_{i_1}, \ldots, v_{i_n}\}$ is the set of chords of D. This defines a graded linear map $\lambda : ch(\Sigma) \to L_{Gr}(\Sigma)$.

Given a Lie group G with invariant inner product one can generalize all of the above in a trivial way to define an algebra $ch^G\,(\Sigma)$ of coloured chord diagrams where each core component is coloured by a finite dimensional representation V of G.

We will also need the notion of *chord tangles* which is just that of a chord diagram on $\mathbb{R} \times [0,1]$ except that we allow the S_i to be neatly imbedded intervals rather than circles (so that the image of $\cup S_i$ forms a tangle rather than a link). These chord tangles will be related to tangles as chord diagrams are to links. Chord tangles with suitable numbers and orientations of endpoints can be composed in the obvious way.

3.2. **The Poisson structure.** Recall from [AMR96] that $ch\,(\Sigma)$ has a natural Poisson structure given as follows: Assume $D_1 \cup D_2$ is a generic chord diagram. For $p \in D_1 \cap D_2$ we define the oriented intersection number by $\epsilon_{12}(p) := \begin{cases} + & \text{for} \\ - & \text{for} \end{cases}$





where 1 and 2 indicate components of the corresponding diagrams.

For each for $p \in D_1 \cap D_2$ we define $D_1 \cup_p D_2$ to be the chord diagram on Σ given by joining $D_1^{-1}(p) \in$ and $D_2^{-1}(p)$ by a chord. Under the above assumptions, for chord diagrams D_1, D_2 we define their Poisson bracket to be

$$\{[D_1], [D_2]\} := \sum_{p \in D_1 \cap D_2} \epsilon_{12}(p)[D_1 \cup_p D_2] \tag{3}$$

Proposition 10. *The map $\lambda : ch\,(\Sigma) \to L_{Gr}\,(\Sigma)$ is a homomorphism of graded Poisson rings.*

Proof: Given two chord diagrams D_1, D_2 in generic position with $D_1 \cap D_2 = \{p_{n+m+1}, \ldots, p_{n+m+k}\}$ the Poisson structures compare as follows:

$$\begin{aligned}
\{\lambda(D_1), \lambda(D_2)\} &= [\nabla_{v_1,\ldots,v_n} D_{L_1}, \nabla_{w_1,\ldots,w_m} D_{L_2}] \bmod L_{n+1+m+1}(\Sigma) \\
&= \nabla_{v_1,\ldots,v_n,w_1,\ldots,w_m}(D_{L_1.L_2} - D_{L_2.L_1}) \\
&= \nabla_{v_1,\ldots,v_n,w_1,\ldots,w_m}\left(\sum_{i=1}^{k} \varepsilon_i \nabla_{p_i} D_{L_1.L_2}\right) \bmod L_{n+m+2}(\Sigma) \\
&= \sum_{i=1}^{k} \varepsilon_i \nabla_{v_1,\ldots,v_n,w_1,\ldots,w_m,p_i} D_{L_1.L_2} \\
&= \lambda(\{D_1, D_2\})
\end{aligned}$$

where we used

$$\begin{aligned}
D_{L_1.L_2} - D_{L_2.L_1} &= D_{L_1.L_2}^{\varepsilon_1,\ldots,\varepsilon_{n+m},\varepsilon_{n+m+1},\ldots,\varepsilon_{n+m+k}} - D_{L_1.L_2}^{\varepsilon_1,\ldots,\varepsilon_{n+m},-\varepsilon_{n+m+1},\ldots,-\varepsilon_{n+m+k}} \\
&= \sum_{i=1}^{k} \Big(D_{L_1.L_2}^{\varepsilon_1,\ldots,\varepsilon_{n+m+i-1},\varepsilon_{n+m+i},-\varepsilon_{n+m+i+1},\ldots,-\varepsilon_{n+m+k}} \\
&\quad - D_{L_1.L_2}^{\varepsilon_1,\ldots,\varepsilon_{n+m+i-1},-\varepsilon_{n+m+i},-\varepsilon_{n+m+i+1},\ldots,-\varepsilon_{n+m+k}} \Big) \\
&= \sum_{i=1}^{k} \varepsilon_i \nabla_{p_i} D_{L_1.L_2}^{\varepsilon_1,\ldots,\varepsilon_{n+m+i},-\varepsilon_{n+m+i+1},\ldots,-\varepsilon_{n+m+k}} \\
&= \sum_{i=1}^{k} \varepsilon_i \nabla_{p_i} D_{L_1.L_2} \bmod L_{n+m+2}(\Sigma)
\end{aligned}$$

□

This Poisson structure trivially extends to the case of coloured chord diagrams. It is closely related to the Poisson structure on the moduli space of flat connections on Σ. The following is one of the main results of [AMR96]:

Theorem 11. *There is a Poisson algebra homomorphism from $ch^G(\Sigma)$ to the Poisson algebra $\mathcal{F}(\mathcal{M}^G)$ of functions on the moduli space of flat G-connections on Σ. This homomorphism is universal with respect to Lie group homomorphisms that preserve the invariant inner product.*

This Poisson algebra homomorphism known to be surjective for many interesting groups ([AMR96],[Mat]). We will see later that this allows us to obtain quantizations of the algebra of functions on moduli space from our quantization of chord diagrams.

3.3. **Vassiliev-Kontsevich invariants.** Recall that for the 2-sphere there exists the notion of a *universal Vassiliev invariant* (or universal Vassiliev-Kontsevich invariant

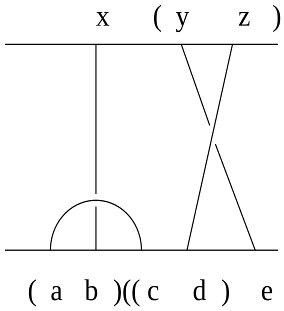


FIGURE 1. A possible non-associative tangle with upper endpoints $x(yz)$ and lower endpoints $(ab)((cd)e)$

[Kon],[BN95]). It can be regarded as an invariant of links in the cylinder $S^2 \times [0,1]$ that takes values in the algebra $\overline{ch\left(S^2\right)}$. If L is a link in $S^2 \times [0,1]$ then this can be written as a formal sum over all chord diagrams

$$V\left(L\right) = \sum_{D \in ch(S^2)} \langle D, L \rangle D.$$

Here $\langle D, L \rangle$ should be certain intersection numbers which can be written in many different ways (see [Kon, BT94, T95, PV94]). Similarly, for any compact oriented surface Σ we expect that there is an analogous universal invariant of links in $\Sigma \times [0,1]$ with values in $ch(\Sigma)$,

$$V\left(L\right) = \sum_{D \in ch(\Sigma)} \langle D, L \rangle D,$$

having the following important property: $V\left(L_n\left(\Sigma\right)\right) \subseteq \bigoplus_{m \geq n} ch^{(m)}\left(\Sigma\right)$ and for any chord diagram $D \in ch^{(k)}\left(\Sigma\right)$

$$V\left(\lambda\left(D\right)\right) = D \mod ch^{(k+1)}\left(\Sigma\right) \tag{4}$$

The existence of a universal Vassiliev invariant is also known as the *fundamental theorem for Vassiliev invariants* [BNS96]. The approach that we will use follows the construction of [BN93] closely: Given a link L, the link projection is decomposed into a composition of elementary tangles, then to each tangle one associates a series of string chord diagrams (in the terminology of [AMR96]) and the invariant $V\left(L\right)$ is the composition of the string chord diagrams, i.e. a series of chord diagrams in $\overline{ch\left(\Sigma\right)}$.

We recall the construction from [BN93, definition 2.5]. A *non-associative tangle* is a tangle where the upper and lower endpoints are parenthesized, see figure 1.

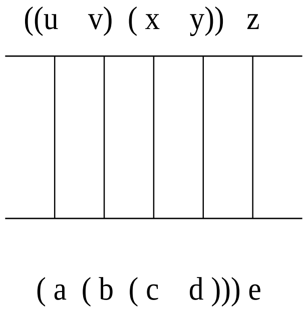


FIGURE 2. A possible associativity morphism $(ab)((cd)e) \to ((uv)(xy))z$

Clearly, nonassociative tangles with suitable bracketings and orientations of endpoints can be composed. Every non-associative tangle can be decompose into elementary ones: crossings (denoted G2 in Bar-Natan's paper), local maxima and minima (G3), and trivial tangles id, where only one parenthesis is moved in the form $(A\,(BC)) \longleftrightarrow ((AB)\,C)$ (*associativity morphisms*, G1), see for example figure 2. Associativity morphisms are often drawn so that the location of parenthesis is indicated by the distance between strands in the tangle. An example of this can be seen in figure 5.

It follows from Drinfeld's work [Dri89], that one can associate formal series of chord tangles Φ, R, e, i to the elementary nonassociative tangles G1,G2,G3 such that the following holds: If $U\,(T_i)$ denotes the series corresponding to the elementary tangle T_i and a given nonassociative tangle T is decomposed into elementary ones $T = \prod_{i=1}^{n} T_i$ then

Theorem 12 (Drinfeld,Kontsevich,Bar-Natan)**.** *The series of chord tangles* $\prod_{i=1}^{n} U\,(T_i)$ *is an invariant of* T*.*

This implies the fundamental theorem of Vassiliev invariants for $S^2 \times [0,1]$ (by letting T have no endpoints, hence no parenthesis). Analogously we can show

Theorem 13. *If* Σ *is a compact Riemann surface with nonempty boundary, then there exists a universal Vassiliev-Kontsevich invariant.*

Proof: Without loss of generality we can assume Σ is connected and not equal to $\mathbb{D}^2$. Given a link in $\Sigma \times [0,1]$, we can cut Σ along a finite number of neatly embedded intervals I_j, to obtain a decomposition $\Sigma = \cup_i \Sigma_i^S \cup \cup_k \Sigma_k^H$ where the Σ_i^S's are squares and the Σ_k^H are hexagons, with the following properties: We can choose a bracketing of all the ordered sets $L \cap I_j$ such that

1.: For each square Σ_i^S, $L_i^S = \Sigma_i^S \cap L \subset \Sigma_i^S \times [0,1]$ with the bracketing of the top and bottom of L_i^S is an elementary non-associative tangle of type G1, G2, G3 or a trivial non-associative tangle G4 as in figure 3, where the bracketing is the same at the top and at the bottom.

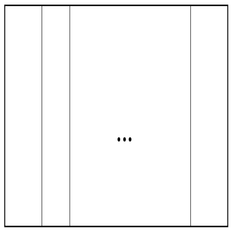


FIGURE 3. The trivial square non-associative tangle G4

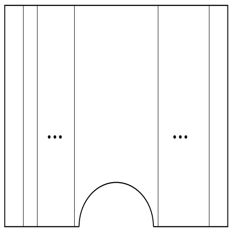


FIGURE 4. The hexagon non-associative tangle G5

2.: Each hexagon Σ_k^H, $L_k^H = \Sigma_k^H \cap L \subset \Sigma_k^H \times [0,1]$ with the bracketing of the 3 ordered sets $L_k^H \cap I_j$ looks like figure 4, i.e. there is no braiding, association, maxima or minima for L_k^H and the bracketing respect the decomposition of the bottom of Σ_k^H into two components. The type of such a configuration we will denote by G5.

See figure 5 for an example.

Now we choose a Drinfeld associator as in [BN93]. Then we construct

$$V : L(\Sigma) \to \overline{ch(\Sigma)}$$

as follows: We associate to the surface a formal series of chord diagrams as follows: G1-G3 are mapped as in [BN93, section 3] whereas G4 and G5 are map to themselves. Since we have no natural way of distinguishing the top and the bottom of the rectangles, we need to make sure that the associations of G1, G2 and G3 are equivariant with respect to 180° rotation. The only real issue here is the associator. However, by Prop. 3.1. in [LM94] we know associators with this symmetry property exits. Alternatively, one can choose the intervals I_j, such that a consistent choice of top and bottom on all squares and hexagons can be made. In this case we do not need the symmetry assumption on the associator.

Claim: The map V is an invariant of the link L.

This holds since

1. Isotopies of links can be assumed to be generic with respect to the decomposition of the surface into squares and hexagons. So isotopies are sequences of isotopies

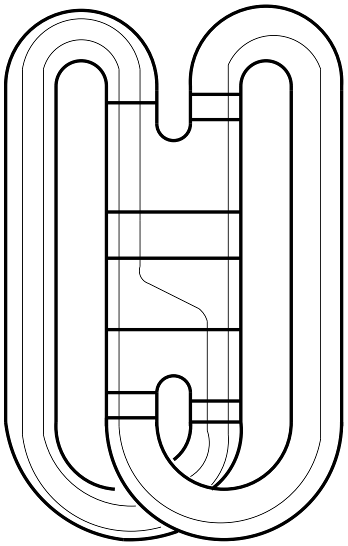


FIGURE 5. A possible decomposition of a diagram on the punctured torus

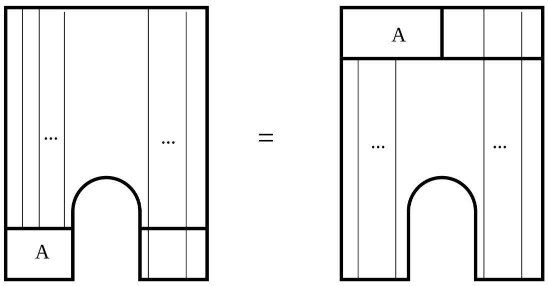


FIGURE 6. Naturality of the hexagon, R10

in compositions of morphisms, hence leaves V invariant by the relations R1-R8 of [BN93], and of moves R10, naturality of the hexagons, see figure 6, which obviously also leaves V invariant.

2. Any two choices of bracketings on the same underlying link give the same invariant since we can assume by R10 that the bracketings agree on the hexagons which again reduces the problem to one inside compositions of morphisms.

Finally, equation (4) follows from the form of R, R^{-1} as in the case of quantum group invariants, cf. [BN95, theorem 5].□

Corollary 14. *If Σ is a compact Riemann surface with nonempty boundary, the space $L^{(n)}(\Sigma)$ is isomorphic to $ch^{(n)}(\Sigma)$, the space of link diagrams of order n.*

Proof: The map $\lambda : ch(\Sigma) \to L_{Gr}(\Sigma)$ is surjective by definition of L_n and injective since V_{Gr} is a left inverse by (4).
□

Remark 15. *This contains Goryunov's theorem for the solid torus (see e.g.* [Gor95]*) as a special case.*

Remark 16. *Compare with* [Lin94, theorem 0.1] *where it is shown that for a homotopy $\mathbb{R}^3$ the theory of Vassiliev invariants is the same as for the usual $\mathbb{R}^3$.*

Corollary 17. *The homomorphism $\lambda : ch(\Sigma) \to L_{Gr}(\Sigma)$ in proposition 10 is an isomorphism of graded Poisson rings.*

4. Quantization

Given a Poisson algebra $\mathcal{A}$ with Poisson bracket $\{.,.\}$, a *deformation quantization* of $\mathcal{A}$ is a $\mathbb{C}[[h]]$-linear associative product $*$ on $\mathcal{A}[[h]]$ such that $x*y$ mod h is the given product on $\mathcal{A}$ and $x*y - y*x = h\{x,y\} \bmod h^2$ (see [BFFLS77]). The product can be written in the form $x*y = \sum_{n=0}^{\infty} \pi_n(x,y)\, h^n$. We will use the terms quantization, deformation quantization and star product interchangeably. We recall that two such star products are equivalent, if there is a $\mathbb{C}[[h]]$-linear automorphism of $\mathcal{A}[[h]]$, which induces the identity mod h and which is a homomorphism with respect to the two star products.

In the first subsection we discuss the classical way of quantizing linear Poisson structures. This applies to the algebra of chord diagrams, but we show that the resulting star-product does not descend to a star product on the moduli space of flat SL_2-connections on a Riemann surface. Following this discussion we give a geometric construction of a quantization of the algebra of chord diagrams and verify for Gl_n and SL_n that this latter quantization does descend to a deformation quantization of moduli space.

4.1. Quantization of chord diagrams as a linear Poisson algebra.

Proposition 18. *Chord diagrams that are connected as a topological space are closed under the Poisson bracket. The algebra of all chord diagrams is isomorphic to the polynomial algebra over the set of connected chord diagrams.*

Proof: Obvious from the definitions. □

The proposition says that the algebra of chord diagrams is a linear Poisson algebra, i.e. a Poisson algebra that is obtained from a Lie algebra $\mathfrak{g}$ (connected chord diagrams in our case) by extending the Lie bracket as a derivation to the symmetric algebra $S\mathfrak{g}$ over the given Lie algebra. Linear Poisson structure can always be quantized in the following way:

Consider the algebra $T\mathfrak{g}[[h]]/\langle x \bigotimes y - y \bigotimes x - h[x,y]\rangle$. This is an associative algebra and as a linear space it is isomorphic to $S\mathfrak{g}[[h]]$, it defines a deformation quantization $*$ of $S\mathfrak{g}[[h]]$.

Lemma 19. *For $x, y \in \mathfrak{g}$ we have $x * y = xy + \frac{h}{2}\{x, y\}$ and $x^2 * y = x^2 y + h\{x^2, y\} + \frac{h^2}{6}\{x, \{x, y\}\}$.*

Proof: Direct computation. □

Using this one can show that e.g. for SL_2 this star-product does not descend to a star product on the moduli space of flat connections:

Example 20. *On the 4-punctured sphere let $x_i, i = 1, \ldots, 4$ denote the generators of the fundamental group, (1) a trivial loop and $(x_{i_1} \ldots x_{i_n})$ the loop given as the corresponding product of generators. The Cayley-Hamilton theorem implies that the linear combination of chord diagrams $(x_1x_2x_1x_2) - (x_1x_2)(x_1x_2) + (1)$ is in the kernel of the map to moduli space. Taking star product with (x_1x_3) we get*

$$\begin{aligned}((x_1x_2x_1x_2) - (x_1x_2)(x_1x_2) + (1)) * (x_1x_3)& \\ = ((x_1x_2x_1x_2) - (x_1x_2)(x_1x_2) + (1))(x_1x_3)& \\ + h\{((x_1x_2x_1x_2) - (x_1x_2)(x_1x_2)), (x_1x_3)\}& \\ + \frac{h^2}{6}\{(x_1x_2)(x_1x_2), \{(x_1x_2)(x_1x_2), (x_1x_3)\}\}&\end{aligned}$$

The first two lines are zero since the ideal is a Poisson ideal and $\{(1), (x_1x_3)\} = 0$ but the last expression does not map to zero on $\mathcal{M}^{SL_2}$, hence the star product does not respect the kernel of the quotient map.

4.2. Universal quantization of the algebra of chord diagrams.

By universality of the quantization we mean the following: The quantization descends to the algebra of functions on moduli space in a functorial way with respect to group homomorphisms that respect the invariant inner product on the groups. In this subsection we construct the quantization, in the next subsection we discuss the universality property. The construction uses the following general facts about filtered algebras:

Proposition 21. *Let $G = \bigoplus_{n=0}^{\infty} G^{(n)}$ and $H = \bigoplus_{n=0}^{\infty} H^{(n)}$ be graded spaces with associated filtered spaces $\overline{G}$ and $\overline{H}$ respectively. If $\Lambda : \overline{H} \to \overline{G}$ is linear map respecting the filtrations such that the associated graded map $\Lambda_{Gr} : H \to G$ is an isomorphism, then Λ is an isomorphism.*

Recall from lemma 1 that for a filtered algebra F such that the associated graded algebra F_{Gr} is commutative F_{Gr} has a natural structure of a Poisson algebra. For a graded algebra G we have the projection $x \in G \longmapsto x^{(i)} \in G^{(i)}$, which obviously extends to the completion $\overline{G}$. Likewise, if we have a $u \in F_i$, we denote the project of u onto $F^{(i)}$ by $u^{(i)}$.

Theorem 22. *Let G be a graded Poisson algebra and F a filtered algebra such that the associated graded algebra is commutative. Assume that $\Lambda : F \to \overline{G}$ is a linear* *filtered map such that $\Lambda_{Gr} : F_{Gr} \to G$ is a Poisson algebra isomorphism. Then setting*

$$x_1 * x_2 := h^{-\deg(x_1 x_2)} \sum_{i=0}^{\infty} \left(\Lambda\left(\Lambda^{-1}(x_1).\Lambda^{-1}(x_2)\right)\right)^{(i)} h^i$$

for $x_i \in G^{(\deg x_i)}$ defines a star product on $G[[h]]$.

Proof: We have to show that the product is associative and that

$$x_1 * x_2 = x_1 x_2 \bmod h \tag{5}$$

$$x_1 * x_2 - x_2 * x_1 = h\{x_1, x_2\} \bmod h^2 \tag{6}$$

Associativity holds because the pullback $x_1 \circ x_2 := \Lambda\left(\Lambda^{-1}(x_1).\Lambda^{-1}(x_2)\right)$ of an associative multiplication is an associative multiplication and the coefficients of h^p in $(x_1 * x_2) * x_3$ and $x_1 * (x_2 * x_3)$ are just the coefficients of $(x_1 \circ x_2) \circ x_3$ and $(x_1 \circ x_2) \circ x_3$ respectively in $G^{(\deg(x_1 x_2 x_3)+p)}$.

To see the equations relating the star product to the given product and the Poisson bracket we compute $\bmod h^2$

$$\begin{aligned} & x_1 * x_2 \\ \equiv\ & (\Lambda(\Lambda^{-1}(x_1).\Lambda^{-1}(x_2)))^{(\deg(x_1x_2))} + (\Lambda(\Lambda^{-1}(x_1).\Lambda^{-1}(x_2)))^{(\deg(x_1x_2)+1)} h \\ =\ & \Lambda_{Gr}(\Lambda_{Gr}^{-1}(x_1)\Lambda_{Gr}^{-1}(x_2)) + (\Lambda(\Lambda^{-1}(x_1).\Lambda^{-1}(x_2)))^{(\deg(x_1x_2)+1} h \\ =\ & x_1x_2 + (\Lambda(\Lambda^{-1}(x_1).\Lambda^{-1}(x_2)))^{(\deg(x_1x_2)+1)} h \bmod h^2 \end{aligned}$$

and

$$\begin{aligned} & x_1 * x_2 - x_2 * x_1 \\ \equiv\ & (\Lambda(\Lambda^{-1}(x_1).\Lambda^{-1}(x_2) - \Lambda^{-1}(x_2).\Lambda^{-1}(x_1)))^{(\deg(x_1x_2)+1)} h \\ =\ & \Lambda_{Gr}((\Lambda^{-1}(x_1).\Lambda^{-1}(x_2) - \Lambda^{-1}(x_2).\Lambda^{-1}(x_1))^{(\deg(x_1x_2)+1)}) h \\ =\ & \{x_1, x_2\} h \bmod h^2. \end{aligned}$$

We used here that Λ_{Gr} is a Poisson homomorphism and the definition of the Poisson bracket on F_{Gr}. □

Remark 23. *Fixing $\lambda = \Lambda_{Gr}^{-1}$ we see that any two star products on $G[[h]]$ obtained this way are equivalent.*

Now let us return to the algebra of chord diagrams. Recall from corollary 17 that the map $\lambda = V_{Gr}^{-1} : ch(\Sigma) \to L_{Gr}$ is an isomorphism of graded Poisson rings. Extend $V : L' \to \overline{ch(\Sigma)}$ by continuity to $\overline{V} : \overline{L'} \to \overline{ch(\Sigma)}$, then $\overline{V}$ satisfies the assumptions of theorem 22.

The star product on $ch(\Sigma)$ obtained from theorem 22 is given as follows: Write $V(L) = \sum_{D \in ch(\Sigma)} \langle D, L \rangle D = \sum_{V(L)^{(i)} \in ch^{(i)}(\Sigma)} V(L)^{(i)}$ and define a map F from $L'(\Sigma \times [0,1])$ to $ch(\Sigma)[[h]]$ by

$$F(L) := \sum_{D \in ch(\Sigma)} \langle D, L \rangle D h^{\deg D} = \sum V(L)^{(i)} h^i$$

Now the multiplication on $ch(\Sigma)[[h]]$ reads

$$D_1 * D_2 := h^{-\deg(D_1 D_2)} F\left(V^{-1}(D_1) . V^{-1}(D_2)\right) \tag{7}$$

Theorem 24. *The product (7) defines a deformation quantization of $ch(\Sigma)$.*

Remark 25. *Instead of $\overline{V}$ we could have used $\overline{V_{Gr}}$ and we would have obtained an isomorphic product.*

Remark 26. *For the case of closed surfaces computer calculations using Dror Bar-Natan's mathematica package* [BN93] *suggest that the generalization of the above construction is nontrivial. Notice that in the work of Alekseev, Grosse and Schomerus on quantization of moduli space the case of closed surfaces also presented additional difficulties, in fact they formulate their result for closed surfaces only as a conjecture* [AS95].

4.3. **Quantization of $\mathcal{M}^G$.** Here we show that (in contrast to the quantization in section 4.1) the quantization of the algebra of chord diagrams given above descends to the moduli space of flat G-connections. The proof rests on the following locality result for the universal Vassiliev invariants constructed above:

Theorem 27. *Suppose a subspace W in $ch(\Sigma)$ is defined by local ("skein") relations, then it is a (two-sided) ideal with respect to the star multiplication.*

Proof: This follows from the fact that the map U in theorem 12 is multiplicative by definition. Since U is multiplicative, so is U^{-1} (because $U(U^{-1}(S) U^{-1}(T)) = ST$ implies $U^{-1}(S) U^{-1}(T) = U^{-1}(ST)$).

Let $D_1 \in W$ and D_2 be two chord diagrams, $\mathbb{D}$ some disk on the surface containing all chords and crossings, d_i the chord tangle obtained by intersecting D_i with $\mathbb{D}$, $L = V^{-1}(D_1) V^{-1}(D_2)$ and $l = L \cap \mathbb{D}$. Recall that $V(L)$ is given by 'closing up' $U(l)$ on the surface, analogously $V^{-1}(D_i)$ is given by 'closing up' $U^{-1}(d_i)$.

Since the relation R is local we can assume that $d_1 = \tilde{d}_1 R \bar{d}_1$ and $l = \tilde{l}(U^{-1}R \otimes \mathrm{id}) \bar{l}$ for suitable (chord) tangles $\tilde{d}_1, \bar{d}_1, \tilde{l}, \bar{l}$. This implies

$$U\left(U^{-1}(D_1) U^{-1}(D_2)\right) = U\left(\tilde{l}\right) U\left(U^{-1} R \otimes \mathrm{id}\right) U\left(\bar{l}\right) = U\left(\tilde{l}\right)(R \otimes \mathrm{id}) U\left(\bar{l}\right)$$

which shows that $V(V^{-1}(D_1) V^{-1}(D_2))$ is in W. Checking that the powers of h work out correctly shows that $D_1 * D_2 \in W$ so that W is a left ideal. The same proof shows that it is also a right ideal. □

Next we show that the star product is compatible with the passage to the loop algebra (cf. [Gol86],[AMR96, proposition 5]):

Corollary 28. *The star product (7) is compatible with the ideal in* $ch(\Sigma)$ *defined at the end of section 3 of* [AMR96].

Proof: The relations defining the ideal are local on the surface. □

We still have to describe the ideal in the loop algebra which will map to 0 on moduli space. All functions on M_n^k, invariant under the diagonal GL_n-action, are traces of products. The relations between them are given by the trace identities for $n \times n$ matrices of [Pro76, theorem 4.5]. The ideal of relations is generated by the fundamental trace identities which have the form $\sum_\sigma a_\sigma \mathrm{tr}\mu_\sigma (A_1 \otimes \cdots \otimes A_n)$ where $\sigma = (i_1 \dots i_k)(j_1 \dots j_h) \dots (t_1 \dots t_e)$ is a permutation, the A_i are matrices and

$$\mu_\sigma (A_1 \otimes \cdots \otimes A_n) = (A_{i_1} \dots A_{i_k}) \otimes (A_{j_1} \dots A_{j_h}) \otimes \cdots \otimes (A_{t_1} \dots A_{t_e})$$

In terms of chord diagrams this is again a local relation as in the lemma.

The moduli space GL_n^k / GL_n has as additional invariant functions only those of the form $\det^{-1} A_i$ which can be written as traces of representations of monomials in the A_i^{-1}. The only new relations will be $\det^{-1} A_i \det A_i = 1$. Analogous statements hold for SL_n^k / SL_n.

This means that all relations defining the kernel of the map $ch(\Sigma) \to \mathcal{F}\left(\mathcal{M}^G\right)$ are local on Σ. Therefore the above theorem implies

Theorem 29. *The star product (7) descends to a star product on* $\mathcal{M}^G$ *for* $G = GL(n, \mathbb{C}), SL(n, \mathbb{C})$.

A discussion of other Lie groups and relations to other attempts at quantizing the moduli space will appear elsewhere. Here we only note that we are led to the following

Conjecture 30. *The star product (7) descends to a star product on* $\mathcal{M}^G$ *for any simple group* G.

Remark 31. *Recall that the Poisson homomorphism from the algebra of chord diagrams to the moduli space of flat connections is universal in the sense that the following diagram commutes*

$$\begin{array}{ccc}
ch(\Sigma) & \xrightarrow{id} & ch(\Sigma) \\
\uparrow & & \uparrow \\
ch(\Sigma)^{G_2} & \xrightarrow{\phi^*} & ch(\Sigma)^{G_1} \\
\downarrow f & & \downarrow f \\
\mathcal{F}(\mathcal{M}_\Sigma^{G_2}) & \xrightarrow{(\phi_*)^*} & \mathcal{F}(\mathcal{M}_\Sigma^{G_1})
\end{array}$$

for a given group homomorphism $\phi : G_1 \to G_2$ respecting the inner products on G_1, G_2. The homomorphisms in the diagram are described in [AMR96]. *Clearly the quantized version has an analogous universality property*

$$\begin{array}{ccc} ch(\Sigma)[[h]] & \xrightarrow{id} & ch(\Sigma)[[h]] \\ \uparrow & & \uparrow \\ ch(\Sigma)^{G_2}[[h]] & \xrightarrow{\phi^*} & ch(\Sigma)^{G_1}[[h]] \\ \downarrow f & & \downarrow f \\ \mathcal{F}(\mathcal{M}_\Sigma^{G_2})[[h]] & \xrightarrow{(\phi_*)^*} & \mathcal{F}(\mathcal{M}_\Sigma^{G_1})[[h]] \end{array}$$

with respect to the star products.

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