HYPERBOLIC REINHARDT DOMAINS IN $\mathbb{C}^2$ WITH NONCOMPACT AUTOMORPHISM GROUP

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Abstract. We give an explicit description of hyperbolic Reinhardt domains $D \subset \mathbb{C}^2$ such that: **(i)** D has C^k-smooth boundary for some $k \geq 1$, **(ii)** D intersects at least one of the coordinate complex lines $\{z_1 = 0\}$, $\{z_2 = 0\}$, and **(iii)** D has noncompact automorphism group. We also give an example that explains why such a setting is natural for the case of hyperbolic domains and an example that indicates that the situation in $\mathbb{C}^n$ for $n \geq 3$ is essentially more complicated than that in $\mathbb{C}^2$.

0. Introduction and Results

Let D be a Kobayashi-hyperbolic domain in $\mathbb{C}^n$, $n \geq 2$ (see [Ko] for terminology). Denote by $\text{Aut}(D)$ the group of holomorphic automorphisms of D. The group $\text{Aut}(D)$ with the topology of uniform convergence on compact subsets of D (the compact-open topology) is in fact a Lie group (see [Ko]).

The present paper is motivated by results characterizing a domain by its automorphism group (see e.g. [R], [W], [BP1], [BP2]). More precisely, we assume that $\text{Aut}(D)$ is noncompact in the compact-open topology. Most of the known results deal with the case of bounded domains (see, however, [B]). In the present paper we consider possibly unbounded hyperbolic domains. Our thesis is that (unbounded) hyperbolic domains have some of the geometric characteristics of bounded domains. In particular, they are tractable for our studies. But they also exhibit new automorphism group action phenomena, and are therefore of special interest. We present some of these new features in this work.

Here we assume that D is a Reinhardt domain, i.e. a domain which the standard action of the n-dimensional torus $\mathbb{T}^n$ on $\mathbb{C}^n$,

$$z_j \mapsto e^{i\phi_j} z_j, \qquad \phi_j \in \mathbb{R}, \quad j = 1, \dots, n, \tag{1}$$

leaves invariant. In [FIK] we gave a complete classification of smoothly bounded Reinhardt domains with noncompact automorphism group, and in [IK] we extended this result to Reinhardt domains with boundary of any finite smoothness C^k, $k \geq 1$. One of the main steps for obtaining these classifications was to show that the noncompactness of $\text{Aut}(D)$ is equivalent to that of $\text{Aut}_0(D)$, the connected component of the identity in $\text{Aut}(D)$. We will now explain this point in more detail, as it will provide some motivation for the results of the present paper.

Following [Sh], we denote by $\text{Aut}_{\text{alg}}((\mathbb{C}^*)^n)$ the group of algebraic automorphisms of $(\mathbb{C}^*)^n$, i.e. the group of mappings of the form

$$z_i \mapsto \lambda_i z_1^{a_{i1}} \dots z_n^{a_{in}}, \quad i = 1, \dots, n, \tag{2}$$

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where $\lambda_i \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}$, $a_{ij} \in \mathbb{Z}$, and $\det(a_{ij}) = \pm 1$. For a hyperbolic Reinhardt domain $D \subset \mathbb{C}^n$, denote by $\mathrm{Aut}_{\mathrm{alg}}(D)$ the subgroup of $\mathrm{Aut}(D)$ that consists of algebraic automorphisms of D, i.e. automorphisms induced by mappings from $\mathrm{Aut}_{\mathrm{alg}}((\mathbb{C}^*)^n)$. It is shown in [Kr] that $\mathrm{Aut}(D) = \mathrm{Aut}_0(D) \cdot \mathrm{Aut}_{\mathrm{alg}}(D)$, where the " $\cdot$ " denotes the composition operation in $\mathrm{Aut}(D)$. Therefore if one can show that, for a hyperbolic Reinhardt domain D, $\mathrm{Aut}_{\mathrm{alg}}(D)$ is finite up to the action of $\mathbb{T}^n$ (see (1)), then the noncompactness of $\mathrm{Aut}(D)$ is equivalent to that of $\mathrm{Aut}_0(D)$ (see Proposition 1.1 in [FIK] for the case of bounded domains). Next, as is shown in [Kr], $\mathrm{Aut}_0(D)$ admits an explicit description if D is mapped into its normalized form by a mapping of the form (2). This normalized form was the main tool that we used in [FIK], [IK].

Unfortunately, as the following example shows, for the case of hyperbolic Reinhardt domains the group $\mathrm{Aut}_{\mathrm{alg}}(D)$ may be essentially infinite, and therefore the scheme used in [FIK], [IK], fails.

Example 1. Consider the Reinhardt domain $D \subset \mathbb{C}^2$

$$D = \left\{ \sin\left(\log\frac{|z_1|}{|z_2|}\right) < \log|z_1 z_2| < \sin\left(\log\frac{|z_1|}{|z_2|}\right) + \frac{1}{2} \right\}. \tag{3}$$

The boundary of D is clearly C^∞-smooth. The group $\mathrm{Aut}_{\mathrm{alg}}(D)$ is not finite up to the action of $\mathbb{T}^2$, since it contains all the mappings

$$\begin{aligned} z_1 &\mapsto e^{\sqrt{2}\pi k} z_1, \\ z_2 &\mapsto e^{-\sqrt{2}\pi k} z_2, \end{aligned}$$

for $k \in \mathbb{Z}$. This also shows, of course, that $\mathrm{Aut}(D)$ is noncompact.

To see that D is hyperbolic, consider the mapping $f\colon D \to \mathbb{C}$, $f(z_1, z_2) = z_1 z_2$. It is easy to see that f maps D onto the annulus $A = \left\{ e^{-1} < |z| < e^{\frac{3}{2}} \right\}$ which is a hyperbolic domain in $\mathbb{C}$. The domains

$$\begin{aligned} A_1 &= \left\{ e^{-\frac{1}{4}} < |z| < e^{\frac{1}{2}} \right\}, \\ A_2 &= \left\{ e^{-1} < |z| < e^{-\frac{1}{8}} \right\}, \\ A_3 &= \left\{ e^{\frac{1}{4}} < |z| < e^{\frac{3}{2}} \right\} \end{aligned}$$

obviously cover A, and each of the preimages $D_j = f^{-1}(A_j)$, $j = 1, 2, 3$, is hyperbolic since D_j is contained in a union of bounded pairwise nonintersecting domains. It then follows (see [PS]) that D is hyperbolic. ∎

It should be noted here that the domain (3) does not intersect the coordinate complex lines $\{z_1 = 0\}$, $\{z_2 = 0\}$ (note that, for any hyperbolic Reinhardt domain in $\mathbb{C}^n$ not intersecting the coordinate hyperplanes, one has $\mathrm{Aut}(D) = \mathrm{Aut}_{\mathrm{alg}}(D)$ [Kr]). As the following proposition shows, in complex dimension $n = 2$, the sort of pathology described in Example 1 above cannot occur if the domain intersects at least one of the coordinate complex lines.

Proposition A. *Let $D \subset \mathbb{C}^2$ be a hyperbolic Reinhardt domain with C^1-smooth boundary, and let D intersect at least one of the coordinate complex lines $\{z_j = 0\}$, $j = 1, 2$. Then $Aut_{\rm alg}(D)$ is finite up to the action of $\mathbb{T}^2$.*

In particular, for such a domain D, $Aut(D)$ is noncompact if and only if $Aut_0(D)$ is noncompact.

The above proposition allows us to use the description of $\mathrm{Aut}_0(D)$ from [Kr] to obtain the following classification result.

Theorem. *Let $D \subset \mathbb{C}^2$ be a hyperbolic Reinhardt domain with C^k-smooth boundary, $k \geq 1$, and let D intersect at least one of the coordinate complex lines $\{z_j = 0\}$, $j = 1, 2$. Assume also that $Aut(D)$ is noncompact. Then D is biholomorphically equivalent to one of the following domains:*

(i) $\left\{|z_1|^2 + |z_2|^{\frac{1}{\alpha}} < 1\right\}$,

where either $\alpha < 0$, or $\alpha = \dfrac{1}{2m}$ for some $m \in \mathbb{N}$, or $\alpha \neq \dfrac{1}{2m}$ for any $m \in \mathbb{N}$ and $0 < \alpha < \dfrac{1}{2k}$;

(ii) $\left\{|z_1| < 1, (1 - |z_1|^2)^\alpha < |z_2| < R(1 - |z_1|^2)^\alpha\right\}$,

where $1 < R \leq \infty$ and $\alpha < 0$;

(iii) $\left\{e^{\beta|z_1|^2} < |z_2| < Re^{\beta|z_1|^2}\right\}$, *where $1 < R \leq \infty$, $\beta \in \mathbb{R}$, $\beta \neq 0$.*

If $k < \infty$ and ∂D is not C^∞-smooth, then D is biholomorphically equivalent to a domain of the form (i) with $\alpha \neq \frac{1}{2m}$ for any $m \in \mathbb{N}$ and $0 < \alpha < \frac{1}{2k}$.

In case (i) the equivalence is given by dilations and a permutation of the coordinates; in cases (ii) and (iii) the equivalence is given by a mapping of the form

$$
\begin{aligned}
z_{\sigma(1)} &\mapsto \lambda z_{\sigma'(1)} z^a_{\sigma'(2)},\\
z_{\sigma(2)} &\mapsto \mu z^{\pm 1}_{\sigma'(2)},
\end{aligned}
$$

where $\lambda, \mu \in \mathbb{C}^*$, $a \in \mathbb{Z}$ and σ, σ' are permutations of $\{1, 2\}$.

In the following example we show that, in complex dimension $n \geq 3$, an explicit classification result analogous to the above theorem does not exist if we do not impose extra conditions on the domain D.

Example 2. Consider the domain $D \subset \mathbb{C}^3$ given by

$$
\begin{aligned}
D = \Big\{ z : \phi(z) \equiv |z_1|^2 + (1 - |z_1|^2)^2 |z_2|^2 \rho\Big(|z_2|^2(1 - |z_1|^2), |z_3|^2(1 - |z_1|^2)\Big) \\
+ (1 - |z_1|^2)^2 |z_3|^2 - 1 < 0 \Big\},
\end{aligned}
\tag{4}
$$

where $\rho(x_1, x_2)$ is a C^∞-smooth function on $\mathbb{R}^2$ such that $\rho(x_1, x_2) > c > 0$ everywhere, and the partial derivatives of ρ are nonnegative for $x_1, x_2 \geq 0$.

To show that ∂D is smooth, we calculate

$$
\begin{aligned}
\frac{\partial \phi}{\partial z_1} &= \overline{z_1}\Biggl(1-(1-|z_1|^2)\Bigl(2|z_2|^2\rho+(1-|z_1|^2)|z_2|^4\frac{\partial \rho}{\partial x_1}\\
&\qquad\qquad +(1-|z_1|^2)|z_2|^2|z_3|^2\frac{\partial \rho}{\partial x_2}+2|z_3|^2\Bigr)\Biggr),\\
\frac{\partial \phi}{\partial z_2} &= (1-|z_1|^2)^2\overline{z_2}\left(\rho+(1-|z_1|^2)|z_2|^2\frac{\partial \rho}{\partial x_1}\right),\\
\frac{\partial \phi}{\partial z_3} &= (1-|z_1|^2)^2\overline{z_3}\left((1-|z_1|^2)|z_2|^2\frac{\partial \rho}{\partial x_2}+1\right).
\end{aligned}
\tag{5}
$$

It follows from (5) that not all the partial derivatives of ϕ can vanish simultaneously at a point of ∂D. Indeed, if $\frac{\partial \phi}{\partial z_3}(p)=0$ at some point $p \in \partial D$ then, at p, either $|z_1|=1$ or $z_3=0$. If $|z_1|=1$, then clearly $\frac{\partial \phi}{\partial z_1}(p)\neq 0$. If $|z_1|\neq 1$, $z_3=0$, and, in addition, $\frac{\partial \phi}{\partial z_2}(p)=0$, then $z_2=0$, and therefore $|z_1|=1$, which is a contradiction. Therefore, ∂D is C^∞-smooth.

To show that D is hyperbolic, consider the holomorphic mapping $f(z_1,z_2,z_3)=z_1$ from D into $\mathbb{C}$. Clearly, f maps D onto the unit disc $\Delta=\{|z|<1\}$, which is a hyperbolic domain in $\mathbb{C}$. Further, the discs $\Delta_r=\{|z|<r\}$ for $r<1$ form a cover of Δ, and $f^{-1}(\Delta_r)$ is a bounded open subset of D for any such r. Thus, as in Example 1 above, we see that D is hyperbolic (see [PS]).

Further, $\mathrm{Aut}(D)$ is noncompact since it contains the automorphisms

$$
(z_1,z_2,z_3)\longmapsto\left(\frac{z_1-a}{1-\overline{a}z_1},\ \frac{(1-\overline{a}z_1)z_2}{\sqrt{1-|a|^2}},\ \frac{(1-\overline{a}z_1)z_3}{\sqrt{1-|a|^2}}\right)
\tag{6}
$$

for $|a|<1$. ∎

Examples similar to Example 2 can be constructed in any complex dimension $n \geq 3$. They indicate that, most probably, there is no reasonable classification of smooth hyperbolic Reinhardt domains with noncompact automorphism group for $n \geq 3$ even in the case when the domains intersect the coordinate hyperplanes. Indeed, in Example 2 we have substantial freedom in choosing the function ρ.

We note that the boundary of domain (4) contains a complex hyperplane $z_1=\alpha$ for any $|\alpha|=1$. It may happen that, by imposing the extra condition of the finiteness of type in the sense of D'Angelo [D'A] on the boundary of the domain, one would eliminate the difficulty arising in Example 2 and obtain an explicit classification. It also should be observed that any point of the boundary of domain (4) with $|z_1|=1$, $z_2=z_3=0$ is an orbit accumulation point for $\mathrm{Aut}(D)$ (see (6)); therefore, it is plausible that one needs the finite type condition only at such points (cf. the Greene/Krantz conjecture for bounded domains [GK]).

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Proof of Proposition A

We consider two cases.

Case 1. Suppose first that D intersects each of the coordinate complex lines $\{z_j = 0\}$, $j = 1, 2$. Then any element $F \in \mathrm{Aut}_{\mathrm{alg}}(D)$ has the form

$$\begin{aligned} z_1 &\mapsto \lambda z_{\sigma(1)} \\ z_2 &\mapsto \mu z_{\sigma(2)} \end{aligned} \tag{7}$$

where $\lambda, \mu \in \mathbb{C}^*$ and σ is a permutation of the set $\{1, 2\}$.

First let $\sigma = \mathrm{id}$. We assume that mapping (7) is not of the form (1); hence either $|\lambda| \neq 1$, or $|\mu| \neq 1$. By passing to the inverse mapping if necessary, we can also assume that $|\lambda| < 1$, or $|\mu| < 1$.

Let $|\lambda| < 1$. Take a point $p \in D$ of the form $p = (c, 0)$ and apply the k^{th} iteration F^k of F to it: $F^k(p) = (\lambda^k c, 0)$. Since $|\lambda| < 1$, it follows that $\lambda^k c \to 0$ as $k \to \infty$, and therefore $(0, 0) \in \overline{D}$. Since ∂D is C^1-smooth, we actually obtain that $(0, 0) \in D$. Therefore, for some $\epsilon > 0$, the disc $\Delta_\epsilon = \{|z_1| < \epsilon, z_2 = 0\}$ lies in D. By applying the k^{th} iteration of F^{-1} to Δ_ϵ and letting $k \to \infty$, we obtain (since $|\lambda^{-k}| \to \infty$) that the domain D contains the entire complex line $\{z_2 = 0\}$ and therefore cannot be hyperbolic. The case of $|\mu| < 1$ is treated similarly. Hence, $|\lambda| = |\mu| = 1$, and F is of the form (1).

Suppose now that $\sigma(1) = 2$, $\sigma(2) = 1$. We will show that there exists no more than one automorphism of the form (7) with this σ (up to mappings of the form (1)). Let F_1, F_2 be two such automorphisms, with F_j for $j = 1, 2$ given by

$$\begin{aligned} z_1 &\mapsto \lambda_j z_2, \\ z_2 &\mapsto \mu_j z_1, \end{aligned}$$

where $\lambda_j, \mu_j \in \mathbb{C}^*$. Then, for the composition $F_1 \circ F_2^{-1}$, we find that

$$\begin{aligned} z_1 &\mapsto \frac{\lambda_1}{\lambda_2} z_1, \\ z_2 &\mapsto \frac{\mu_1}{\mu_2} z_2. \end{aligned}$$

Hence, by the preceding argument, $|\lambda_1| = |\lambda_2|$ and $|\mu_1| = |\mu_2|$; therefore F_1 differs from F_2 by a mapping of the form (1).

Case 2. Let D intersect only one of the coordinate complex lines, say $\{z_1 = 0\}$. Then any element of $\mathrm{Aut}_{\mathrm{alg}}(D)$ either has the form

$$\begin{aligned} z_1 &\mapsto \lambda z_1 z_2^a, \\ z_2 &\mapsto \mu z_2, \end{aligned} \tag{8}$$

or the form

$$\begin{aligned} z_1 &\mapsto \lambda z_1 z_2^a, \\ z_2 &\mapsto \mu z_2^{-1}, \end{aligned} \tag{9}$$

where $\lambda, \mu \in \mathbb{C}^*$, $a \in \mathbb{Z}$. We will show that there is at most one element of $\mathrm{Aut}_{\mathrm{alg}}(D)$ of each of the forms (8) and (9).

Let F_j, $j = 1, 2$, be two automorphisms of the form (8) given by

$$\begin{aligned} z_1 &\mapsto \lambda_j z_1 z_2^{a_j}, \\ z_2 &\mapsto \mu_j z_2, \end{aligned}$$

where $\lambda_j, \mu_j \in \mathbb{C}^*$, $a_j \in \mathbb{Z}$. Then for $F = F_1 \circ F_2^{-1}$ we see that

$$\begin{aligned} z_1 &\mapsto \frac{\lambda_1}{\lambda_2} \mu_2^{a_2 - a_1} z_1 z_2^{a_1 - a_2}, \\ z_2 &\mapsto \frac{\mu_1}{\mu_2} z_2. \end{aligned}$$

Let $D_0 = D \cap \{z_1 = 0\}$. Then, since ∂D is C^1-smooth, $\mathrm{dist}(D_0, \{z_2 = 0\}) > 0$. Clearly, F preserves D_0. Suppose now that $|\mu_1| < |\mu_2|$. Then, by considering the images of D_0 under iterations of F, we see that $\mathrm{dist}(D_0, \{z_2 = 0\}) = 0$ which contradicts the smoothness of ∂D. Similarly, if $|\mu_1| > |\mu_2|$, then by applying iterations of F^{-1} to D_0 we obtain the same contradiction. Therefore, $|\mu_1| = |\mu_2|$.

By composing F_2 with a mapping of the form (1), we can now assume that $\mu_1 = \mu_2 = \mu$ and therefore F is given by

$$\begin{aligned} z_1 &\mapsto \frac{\lambda_1}{\lambda_2} \mu^{a_2 - a_1} z_1 z_2^{a_1 - a_2}, \\ z_2 &\mapsto z_2. \end{aligned}$$

The k^{th} iteration of F then has the form

$$\begin{aligned} z_1 &\mapsto \left(\frac{\lambda_1}{\lambda_2}\right)^k z_1 \left(\frac{z_2}{\mu}\right)^{k(a_1 - a_2)} \\ z_2 &\mapsto z_2. \end{aligned}$$

We now observe that there exist $\epsilon > 0$ and a disc $\tilde{\Delta} \subset \mathbb{C}$ such that the bidisc $\{|z_1| < \epsilon, z_2 \in \tilde{\Delta}\}$ lies in D. Let $\Delta_{\epsilon,c} = \{|z_1| < \epsilon, z_2 = c\}$ for $c \in \tilde{\Delta}$. If for some $c \in \tilde{\Delta}$ we have $\left|\frac{\lambda_1}{\lambda_2}\right| \cdot \left|\frac{c}{\mu}\right|^{(a_1 - a_2)} > 1$ then, by applying the iterations F^k to $\Delta_{\epsilon,c}$ and letting $k \to \infty$, we see that the complex line $\{z_2 = c\}$ belongs entirely to D; this conclusion contradicts the hyperbolicity of D. Similarly, if for some $c \in \tilde{\Delta}$, $\left|\frac{\lambda_1}{\lambda_2}\right| \cdot \left|\frac{c}{\mu}\right|^{(a_1 - a_2)} < 1$, then applying iterations of F^{-1} to $\Delta_{\epsilon,c}$ yields the same contradiction. Therefore, $\left|\frac{\lambda_1}{\lambda_2}\right| \cdot \left|\frac{c}{\mu}\right|^{(a_1 - a_2)} \equiv 1$ in $\tilde{\Delta}$, and hence $a_1 = a_2$, $|\lambda_1| = |\lambda_2|$. Thus F_1 differs from F_2 by a mapping of the form (1).

The case of mappings of the form (9) is treated analogously. This completes the proof of the proposition. ∎

2. Proof of Theorem

We will use the following description of $\mathrm{Aut}_0(D)$ from [Kr]. Any hyperbolic Reinhardt domain in $\mathbb{C}^n$ can—by a biholomorphic mapping of the form (2)—be put into a normalized form G written as follows. There exist integers $0 \le s \le t \le p \le n$ and $n_i \ge 1$, $i = 1, \dots, p$, with $\sum_{i=1}^p n_i = n$, and real numbers α_i^j, $i = 1, \dots, s$, $j = t+1, \dots, p$, such that if we set $z^i = \left(z_{n_1+\dots+n_{i-1}+1}, \dots, z_{n_1+\dots+n_i}\right)$, $i = 1, \dots, p$, then $\tilde{G} := G \bigcap \left\{z^i = 0,\ i = 1, \dots, t\right\}$ is a hyperbolic Reinhardt domain in $\mathbb{C}^{n_{t+1}} \times \cdots \times \mathbb{C}^{n_p}$, and G can be written in the form

$$
\begin{aligned}
G = \Bigg\{ & |z^1| < 1, \dots, |z^s| < 1, \\
& \left(\frac{z^{t+1}}{\prod_{i=1}^s \left(1 - |z^i|^2\right)^{\alpha_i^{t+1}} \prod_{j=s+1}^t \exp\left(-\beta_j^{t+1} |z^j|^2\right)}, \dots, \right. \\
& \left. \frac{z^p}{\prod_{i=1}^s \left(1 - |z^i|^2\right)^{\alpha_i^p} \prod_{j=s+1}^t \exp\left(-\beta_j^p |z^j|^2\right)} \right) \in \tilde{G} \Bigg\},
\end{aligned}
\tag{10}
$$

for some real numbers β_j^k, $j = s+1, \dots, t$, $k = t+1, \dots, p$. A normalized form can be chosen so that $\mathrm{Aut}_0(G)$ is given by the following formulas:

$$
\begin{aligned}
z^i &\mapsto \frac{A^i z^i + b^i}{c^i z^i + d^i}, \quad i = 1, \dots, s, \\
z^j &\mapsto B^j z^j + e^j, \quad j = s+1, \dots, t, \\
z^k &\mapsto C^k \frac{\prod_{j=s+1}^t \exp\left(-\beta_j^k \left(2 \overline{e^j}^T B^j z^j + |e^j|^2\right)\right) z^k}{\prod_{i=1}^s (c^i z^i + d^i)^{2\alpha_i^k}}, \quad k = t+1, \dots, p,
\end{aligned}
$$

where

$$
\begin{aligned}
& \begin{pmatrix} A^i & b^i \\ c^i & d^i \end{pmatrix} \in SU(n_i, 1), \quad i = 1, \dots, s, \\
& B^j \in U(n_j), \quad e^j \in \mathbb{C}^{n_j}, \quad j = s+1, \dots, t, \\
& C^k \in U(n_k), \quad k = t+1, \dots, p.
\end{aligned}
$$

The above classification implies that $\mathrm{Aut}_0(G)$ is noncompact only if $t > 0$.

Now let $n = 2$. Clearly there are the following possibilities for a hyperbolic Reinhardt domain $\tilde{G} \subset \mathbb{C}$ (see (10)):

(i) $\tilde{G} = \{|z_2| < R\}$, $\quad 0 < R < \infty$;
(ii) $\tilde{G} = \{r < |z_2| < R\}$, $\quad 0 < r < R \le \infty$;
(iii) $\tilde{G} = \{0 < |z_2| < R\}$, $\quad 0 < R < \infty$.

This observation allows us to list all normalized forms of hyperbolic Reinhardt domains in $\mathbb{C}^2$ as follows

$$\begin{aligned} G &= \left\{|z_1| < 1, |z_2| < R(1-|z_1|^2)^\alpha\right\}, \\ &\quad 0 < R < \infty, \quad \alpha \in \mathbb{R}, && (11) \\ G &= \left\{|z_1| < 1, r(1-|z_1|^2)^\alpha < |z_2| < R(1-|z_1|^2)^\alpha\right\}, \\ &\quad 0 < r < R \le \infty, \quad \alpha \in \mathbb{R}, && (12) \\ G &= \left\{|z_1| < 1, 0 < |z_2| < R(1-|z_1|^2)^\alpha\right\}, \\ &\quad 0 < R < \infty, \quad \alpha \in \mathbb{R}, && (13) \\ G &= \left\{re^{\beta|z_1|^2} < |z_2| < Re^{\beta|z_1|^2}\right\}, \\ &\quad 0 < r < R \le \infty, \quad \beta \in \mathbb{R}, \quad \beta \neq 0, && (14) \\ G &= \left\{0 < |z_2| < Re^{\beta|z_1|^2}\right\}, \\ &\quad 0 < R < \infty, \quad \beta \in \mathbb{R}, \quad \beta \neq 0. && (15) \end{aligned}$$

We are now going to select only those among the normalized forms (11)–(15) that can be the images of domains with C^k-smooth boundary under normalizing mappings of the form (2). We will treat each of cases (11)–(15) separately.

Domain of type (11). Observe first that, since the domain G contains the origin, the normalizing mapping is linear (actually, it is given by dilations and a permutation of the coordinates). Therefore, the domain G is a normalized form of a Reinhardt domain with C^k-smooth boundary iff ∂G is also C^k-smooth. Hence $\alpha \neq 0$ (for otherwise G is a bidisc). If $\alpha > 0$, then G has a C^k-smooth boundary iff either $\alpha = \frac{1}{2m}$, for some $m \in \mathbb{N}$, or $\alpha \neq \frac{1}{2m}$ for any $m \in \mathbb{N}$ and $\alpha < \frac{1}{2k}$. If $\alpha < 0$, then ∂G is C^∞-smooth. It is also clear that, for $k < \infty$, ∂G has C^k-smooth, but not C^∞-smooth, boundary iff $\alpha \neq \frac{1}{2m}$ for any $m \in \mathbb{N}$, and $0 < \alpha < \frac{1}{2k}$.

Domain of type (12). First of all, if $\alpha < 0$, then ∂G is C^∞-smooth. It is also clear that, if $\alpha = 0$, then G cannot be a normalized form of any Reinhardt domain with everywhere C^k-smooth boundary for $k \geq 1$.

Assume now that $\alpha > 0$, and suppose first that $R < \infty$. Then ∂G is C^∞-smooth everywhere except at the points where $|z_1| = 1$, $z_2 = 0$. By applying the transformation

$$\begin{aligned} z_1 &\mapsto z_1, \\ z_2 &\mapsto \frac{1}{z_2}, \end{aligned} \tag{16}$$

we produce the following domain with C^∞-smooth boundary

$$\left\{|z_1| < 1, \frac{1}{R}(1-|z_1|^2)^{-\alpha} < |z_2| < \frac{1}{r}(1-|z_1|^2)^{-\alpha}\right\}.$$

Let $\alpha > 0$ and $R = \infty$. We claim that in this case G cannot be a normalized form of a Reinhardt domain with everywhere C^k-smooth boundary for $k \geq 1$. Indeed, this is easy to see if we notice that the general form (up to permutation of the components) of any mapping (2) that is biholomorphic on G is as follows

$$\begin{aligned} z_1 &\mapsto \lambda z_1 z_2^a, \\ z_2 &\mapsto \mu z_2^{\pm 1}, \end{aligned} \tag{17}$$

where $\lambda, \mu \in \mathbb{C}^*$, $a \in \mathbb{Z}$.

It is also easy to see that, for $k < \infty$, no domain (12) can be a normalized form of a Reinhardt domain with C^k-smooth, but not C^∞-smooth, boundary.

Domain of type (13). By transformation (16), G is mapped into a domain of the form (12) corresponding to the case $R = \infty$, so it can be treated as above.

Domain of type (14). The boundary ∂G of G is C^∞-smooth. Also, if $k < \infty$, then G cannot be a normalized form of any Reinhardt domain with C^k-smooth, but not C^∞-smooth, boundary; this assertion is proved by the same argument as we used for domains of the form (12) above (see (17)).

Domain of type (15). By transformation (16), the domain G is mapped into a domain of the form (14) corresponding to the case $R = \infty$, so it can be treated as above.

The theorem is proved. ■

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