THE $\overline{\partial}$-NEUMANN PROBLEM IN THE SOBOLEV TOPOLOGY

LUIGI FONTANA, STEVEN G. KRANTZ, AND MARCO M. PELOSO

ABSTRACT. We study the $\overline{\partial}$-Neumann problem using the Sobolev space inner product. We show that the problem can be solved on any smoothly bounded, pseudoconvex domain. We further formulate estimates and the basic results of a Sobolev Hodge theory.

1. INTRODUCTION

Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. We write the coordinates $z_j = x_j + ix_{j+n}$, $j = 1, \ldots, n$, and the standard basis of vector fields $D_k := \partial/\partial x_k$, for $k = 1, \ldots, 2n$. For s a non-negative integer we define the Sobolev inner product $\langle \cdot, \cdot \rangle_s$ to be

$$\langle f, g \rangle_s := \sum_{|\alpha| \le s} \gamma_\alpha \int_\Omega D^\alpha f \overline{D^\alpha g}. \tag{1}$$

Here, and throughout the paper, we use D^α to denote the α-order derivative, where α is a multi-index and we are using standard multi-index notation. Moreover, $\gamma_\alpha := |\alpha|!/\alpha!$ denotes the polynomial coefficient. [The naturality of this choice of the Sobolev inner product will be pointed out and discussed below.]

We define the Sobolev space $W^s(\Omega)$ to be the closure of $C^\infty(\bar{\Omega})$ with respect to the above inner product. We denote by $W^s_{(0,q)}(\Omega)$ the space of $(0, q)$ forms whose coefficients are in $W^s(\Omega)$. If $\phi = \sum_{|J|=q} \phi_J d\bar{z}^J$ and $\psi = \sum_{|J|=q} \psi_J d\bar{z}^J$, then the inner product in $W^s_{(0,q)}(\Omega)$ is defined by

$$\langle \phi, \psi \rangle_s := \sum_{|J|=q} \sum_{|\alpha| \le s} \gamma_\alpha \int_\Omega D^\alpha \phi_J \overline{D^\alpha \psi_J},$$

where we use the standard notation J to denote a q-vector with increasing entries, and α to denote a multi-index. [Note that the inner product of forms of different degrees is defined to be 0.]

1991 *Mathematics Subject Classification.* 32C10 35N15.

Krantz's research was supported in part by Grant DMS-9531967 from the National Science Foundation. Research at MSRI is supported by NSF Grant DMS-9022140.

For a $(0,q)$ form $\phi = \sum_{|J|=q} \phi_J d\bar{z}^J$ with C^∞ coefficients, the operator $\overline{\partial}$ is defined by

$$\overline{\partial}\phi := \sum_{|K|=q+1} \sum_{kJ} \varepsilon^K_{kJ} \frac{\partial \phi_J}{\partial \bar{z}_k} d\bar{z}^K, \tag{2}$$

where ε^K_{kJ} equals the sign of the permutation $kJ \mapsto K$ if $\{k\} \cup J = K$ as sets, and is 0 otherwise. We continue to use $\overline{\partial}$ to denote its closure in the W^s topology. In this way, for each integer $q = 0, 1, \dots, n$, we obtain an unbounded, densely defined, closed operator

$$\overline{\partial} : W^s_{(0,q)}(\Omega) \to W^s_{(0,q+1)}(\Omega).$$

Thus, in particular, $\ker \overline{\partial}$ is a closed subspace in $W^s_{(0,q)}(\Omega)$. Sometimes we shall use the notation $\overline{\partial}_{(0,q)}$ to stress the fact that the operator $\overline{\partial}$ is acting on $(0,q)$ forms.

Consider now the $W^s(\Omega)$-Hilbert space adjoint $\overline{\partial}^*$ of $\overline{\partial}$. We want to study the boundary value problem

$$\begin{cases} (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})u = f & \text{on } \Omega \\ u,\ \overline{\partial}u \in \operatorname{dom} \overline{\partial}^*, \end{cases} \tag{3}$$

where f is a given $(0,q)$ form. When appropriate, we shall refer to this problem as **(3,s)** in order to emphasize that the topology is coming from the W^s inner product. The condition that u and $\overline{\partial}u$ lie in the domain of $\overline{\partial}^*$ leads to the *$\overline{\partial}$-Neumann s-order boundary conditions*. We shall refer below to the $(\overline{\partial}, s)$-Neumann conditions, and the $(\overline{\partial}, s)$-Neumann problem. Notice that if the Hilbert space under consideration is $L^2(\Omega)$ (that is, $s = 0$) with respect to the Lebesgue measure, then the problem **(3,s)** reduces to the classical $\overline{\partial}$-Neumann problem.

J. J. Kohn solved the $\overline{\partial}$-Neumann ($= (\overline{\partial}, 0)$-Neumann) problem in a series of papers in 1963-4 (see [FK] and references therein). This work has proved important in the theory of partial differential equations, in geometry, and in function theory. Recent work of Christ [Ch] has shown that the *canonical solution*—the solution that is minimal in L^2 norm—that arises from Kohn's work in the L^2 topology is not as well behaved as one might have hoped. The program presented in this paper endeavors to seek other canonical solutions that may serve when Kohn's solution will not. This work is also interesting from the point of view of partial differential equations—particularly boundary value problems—and in the study of the energy integral in geometry. We mention that H. Boas [Bo1] and [Bo2] studied properties and regularity of the Hilbert space orthogonal projection of $W^s(\Omega)$ onto the subspace consisting of the holomorphic functions.

The present paper is the first of a series of papers that we devote to the study of the $(\overline{\partial}, s)$-Neumann problem. We begin by showing that problem **(3,s)** can always be solved on any smoothly bounded pseudoconvex domain Ω. This result does not depend on the particular choice of Sobolev inner product. Then we investigate the

$(\overline{\partial}, s)$-Neumann problem more closely by determining a description of the Hilbert space adjoint $\overline{\partial}^*$ of $\overline{\partial}$, and the boundary conditions arising from requiring that u and $\overline{\partial}u$ belong to $\operatorname{dom}\overline{\partial}^*$. While doing this we use the particular choice of the inner product (1) to obtain reasonably clean equations and formulas. We then conclude with some remarks about what lies ahead. In a forthcoming paper we give estimates for the above problem in the special case of a strongly pseudoconvex domain, and with $s = 1$. The foundations for the present work, studied in the real variable context of the de Rham complex, were laid in the papers [FKP1], [FKP2].

We thank H. Boas for making several useful remarks and comments on an earlier version of this paper. We also thank the referee for making helpful suggestions. Work of the second author at MSRI was supported by NSF Grant DMS-9022140.

2. Solvability of the $(\overline{\partial}, s)$-Neumann problem

The aim of the present section is to prove the following theorem.

Theorem 2.1. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Let s, q be positive integers, $0 < q \leq n$. Let $f \in W^s_{(0,q)}(\Omega)$. Then there exists a unique $u \in W^s_{(0,q)}(\Omega)$ that solves the $(\overline{\partial}, s)$-Neumann problem*

$$\begin{cases} (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})u = f & \text{on } \Omega \\ u,\, \overline{\partial}u \in \operatorname{dom}\overline{\partial}^*. \end{cases}$$

Moreover, there exists a constant $c > 0$, independent of f, such that

$$\|u\|_s \leq c\|f\|_s.$$

Proof. The proof is in two steps. In the first, we rely heavily on Kohn's estimates [Ko], to construct and estimate the *canonical solutions* in W^s to the equations $\overline{\partial}u = f$ and $\overline{\partial}^*v = g$. In the second step we prove the solvabilty of the $(\overline{\partial}, s)$-Neumann problem. In the course of the proof, by *orthogonal* we shall always mean orthogonality in the W^s inner product.

By (3.21) in [Ko], since the $\overline{\partial}$-cohomology is trivial on a pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$, we have that $\operatorname{range}\overline{\partial}_{(0,q-1)} = \ker\overline{\partial}_{(0,q)}$. This equality implies that $\operatorname{range}\overline{\partial}_{(0,q-1)}$ is closed in $W^s_{(0,q)}(\Omega)$. Now Lemma 4.1.1 in [Hö1], applied with $F = \operatorname{range}\overline{\partial}_{(0,q-1)}$, gives that

$$\|f\|_s \leq c\|\overline{\partial}^*_{(0,q)}f\|_s$$

for all $f \in \operatorname{range}\overline{\partial}_{(0,q-1)} \cap \operatorname{dom}\overline{\partial}^*_{(0,q)}$. This in turn, by Lemma 4.1.2 in [Hö1], implies that for all v in the orthogonal complement of $\ker\overline{\partial}_{(0,q-1)}$, i.e. in the closure of $\operatorname{range}\overline{\partial}^*_{(0,q)}$, there exists $f \in \operatorname{dom}\overline{\partial}^*_{(0,q)}$ such that $\overline{\partial}^*_{(0,q)}f = v$. Hence, $\operatorname{range}\overline{\partial}^*_{(0,q)}$ is closed as well, and therefore we have the estimate $\|f\|_s \leq C\|\overline{\partial}f\|_s$ for all $f \in$

range $\overline{\partial}^*_{(0,q)} \cap \operatorname{dom} \overline{\partial}_{(0,q-1)}$. Moreover, we have the strong orthogonal decomposition

$$W^s_{(0,q)}(\Omega) = \operatorname{range} \overline{\partial}^*_{(0,q+1)} \oplus \operatorname{range} \overline{\partial}_{(0,q-1)}.$$

Now, given any $g \in W^s_{(0,q)}(\Omega)$, with $\overline{\partial}_{(0,q)} g = 0$, i.e. $g \in \operatorname{range} \overline{\partial}_{(0,q-1)}$, we can find $v \in \operatorname{dom} \overline{\partial}_{(0,q-1)}$, orthogonal to $\ker \overline{\partial}_{(0,q-1)}$, such that $\overline{\partial} v = g$, and we have the estimate

$$\|v\|_s \leq c_s \|g\|_s.$$

We can apply the same argument to the $\overline{\partial}^*$-equation, i.e. given any f with $\overline{\partial}^*_{(0,q)} f = 0$, we can find u orthogonal to $\ker \overline{\partial}^*_{(0,q+1)}$ such that $\overline{\partial}^*_{(0,q+1)} u = f$, with the estimate

$$\|u\|_s \leq c_s \|f\|_s.$$

We shall call such solutions u and v the *s-canonical* solution to the $\overline{\partial}$ and $\overline{\partial}^*$ equation, respectively.

We now establish the solvability of the $(\overline{\partial}, s)$-Neumann problem. We shall suppress the subscripts on the operators $\overline{\partial}$ and $\overline{\partial}^*$ (used to denote the space of forms that is being acted upon), since this will be clear from context. Let $f \in W^s_{(0,q)}(\Omega)$. Then f can be uniquely written as $f = f_1 + f_2$ with $f_1 \in \operatorname{range} \overline{\partial}$ and $f_2 \in \operatorname{range} \overline{\partial}^*$. Let g_1, g_2 be the canonical solution of $\overline{\partial} g_1 = f_1$, and $\overline{\partial}^* g_2 = f_2$, respectively. Since $g_1 \perp \ker \overline{\partial}$ we have that $g_1 \in \operatorname{range} \overline{\partial}^*$, and therefore $\overline{\partial}^* g_1 = 0$. Analogously, $g_2 \in \operatorname{range} \overline{\partial}$ and $\overline{\partial} g_2 = 0$.

Thus we can canonically select u_1, u_2 such that $\overline{\partial}^* u_1 = g_1$ and $\overline{\partial} u_2 = g_2$. Setting $u = u_1 + u_2$ we obtain that

$$(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) u = f,$$

and the desired estimate follows from the corresponding ones for $\overline{\partial}$ and $\overline{\partial}^*$:

$$\begin{aligned} \|u\|_s^2 &= \|u_1\|_s^2 + \|u_2\|_s^2 \\ &\leq c(\|g_1\|_s^2 + \|g_2\|_s^2) \\ &\leq c(\|f_1\|_s^2 + \|f_2\|_s^2) \\ &= c\|f\|_s^2. \end{aligned} \qquad \square$$

We let N_s be the operator on $W^s_{(0,q)}(\Omega)$ defined by

$$(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) N_s f = f \tag{4}$$

for all $f \in W^s_{(0,q)}(\Omega)$. [Notice that the harmonic space for the operator on the left side of (4) is just the zero space—by the preceding arguments. Therefore this last condition uniquely defines N_s.] We call N_s the *Neumann operator* for the $(\overline{\partial}, s)$-Neumann problem. Thus we have proved that N_s is a bounded operator from $W^s_{(0,q)}(\Omega)$ into itself, for $0 < q \leq n$.

Remark 2.2. We want to stress the fact that the results of the present section are *independent* of the particular choice of the Sobolev inner product. In fact, the same arguments work for any $s \geq 0$, not necessarily integral, and any choice of an equivalent norm in $W^s_{(0,q)}(\Omega)$. The form of the inner product (1) will only become relevant in the next section since we seek explicit formulas for $\overline{\partial}^*$ and its domain.

3. The Hilbert space adjoint $\overline{\partial}^*$ of $\overline{\partial}$

In this section we wish to make the $(\overline{\partial}, s)$-Neumann problem more explicit by calculating the domain of $\overline{\partial}^*$, and the operator $\overline{\partial}^*$ itself by showing how it actually operates on the $(0,q)$-forms in its domain. As a result, we shall formulate problem (3) as a boundary value problem in which the equation on the domain is of the form $\Box + G_s$ where $\Box$ is the complex Laplacian and G_s is a so-called *singular Green's operator* (see [Gr]). This result demonstrates a striking difference with the classical case of the $(\overline{\partial}, 0)$-Neumann problem, where the adjoint is taken with respect to the L^2-inner product, and no operator G_s appears. The particular expression of the Hilbert space adjoint $\overline{\partial}$, and of the singular Green's operator G_s arising in the $(\overline{\partial}, s)$-Neumann problem depend on the choice of the inner product in $W^s_{(0,q)}(\Omega)$. We note that with our definition (1) we have

$$\langle f, g\rangle_s = \langle f, g\rangle_0 + \sum_{j=1}^{2n} \langle D_j f, D_j g\rangle_{s-1}. \tag{5}$$

Recall that for a $(0,q)$ form $\phi = \sum_{|J|=q} \phi_J d\bar{z}^J$ with C^∞ coefficients, the operator $\overline{\partial}$ is defined as

$$\overline{\partial}\phi = \sum_{|K|=q+1} \sum_{kJ} \varepsilon^K_{kJ} \frac{\partial \phi_J}{\partial \bar{z}_k} d\bar{z}^K.$$

Then, the *formal adjoint* ϑ of $\overline{\partial}$ is easily calculated to be

$$\vartheta\phi = -\sum_{|I|=q-1} \varepsilon^J_{iI} \frac{\partial \phi_J}{\partial z_i} d\bar{z}^I.$$

We want to compute the Hilbert space adjoint $\overline{\partial}^*$, together with its domain. In carrying out this program, a central role is played by a particular extension of the normal vector field on $b\Omega$ to a suitable tubular neighborhood of $b\Omega$. The differential condition arising in the description of $\operatorname{dom}\overline{\partial}^*$ is most easily expressed if we make the following choices.

Let ϱ be the signed Euclidean distance from $b\Omega$ (negative inside, positive outside). In a suitable tubular neighborhood U of $b\Omega$, this function ϱ is well defined and in $C^\infty(\overline{U})$. We define the vector field N on U by setting $N = \operatorname{grad}\varrho$. Then N is the outward unit vector field, and if we set $N = \sum_{j=1}^{2n} \nu_j D_j$, then $N\nu_j = 0$ on U. [This last one is in fact the property that at several stages makes our computations easier, and the formulas appearing simplier.]

Proposition 3.1. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Let $\overline{\partial}^*$ be the $W^s(\Omega)$-Hilbert space adjoint of $\overline{\partial}$. Then*

$$\operatorname{dom}\overline{\partial}^* \cap C^\infty_{(0,q+1)}(\bar{\Omega}) = \{\psi : N^s\Big(\psi \llcorner \overline{\partial}\varrho\Big)_I = 0 \text{ on } b\Omega,\ \text{for all } I, |I| = q\}.$$

Notice that N^s denotes the s-fold composition of N with itself.

Here the contraction of a $(0, q+1)$ form ψ with a $(0,1)$ form $\omega = \sum_k \omega_k d\bar{z}_k$ is defined by the formula

$$\psi \llcorner \omega := \sum_I \sum_{kK} \varepsilon^K_{kI} \psi_K \bar{\omega}_k d\bar{z}^I.$$

Remark 3.2. Suppose that r is a generic defining function for Ω, C^∞ in a neighborhood of $\overline{\Omega}$, and we wish to express $\operatorname{dom}\overline{\partial}^*$ in terms of r and $\partial/\partial r$. Then we obtain the following description. There exists a differential operator L_s of order s, with $C^\infty(\overline{U})$ coefficients, whose leading term is $(\partial/\partial r)^s$, and such that

$$\operatorname{dom}\overline{\partial}^* \cap C^\infty_{(0,q+1)}(\bar{\Omega}) = \Big\{\psi : L_s\Big(\psi \llcorner \overline{\partial} r\Big)_I = 0 \text{ on } b\Omega,\ \text{for all } I, |I| = q\Big\}.$$

Indeed, on U,

$$\frac{\partial}{\partial r} = |\operatorname{grad} r| N + gX,$$

where $g \in C^\infty(\overline{U})$, $g = 0$ on $b\Omega$, and X is a vector field on U.

We set $\overline{\partial}^* = \vartheta + \mathcal{K}$, and we want to determine $\mathcal{K}$. Our result is the following.

Proposition 3.3. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$, and let s be a positive integer. Let $\overline{\partial}^*$ be the $W^s(\Omega)$-Hilbert adjoint of $\overline{\partial}$, and ϑ be the formal adjoint of $\overline{\partial}$, respectively. Set $\overline{\partial}^* = \vartheta + \mathcal{K}$. Then, for a $(0, q+1)$ form ψ, $\mathcal{K}\psi := \sum_{|I|=q} \omega_I d\bar{z}^I$ is the $(0, q)$ form whose components are solutions of the following boundary value problem:*

$$\begin{cases} \sum_{j=0}^s (-\Delta)^j \omega_I = 0 & \text{on } \Omega \\ \sum_{j=0}^{s+k-1} T_j N^{s+k-1-j} \omega_I = P^{(I)}_{s+k}\psi & \text{on } b\Omega,\ k = 1, \dots, s\,. \end{cases} \tag{6}$$

Here T_k denotes a tangential differential operator of order $\le k$, with C^∞ coefficients, $T_0 = (-1)^{k-1} \cdot id$ on $b\Omega$, and $P^{(I)}_{s+k}$ is a differential operator of order $s+k$ with C^∞ coefficients and acting on the components of ψ.

As a consequence, from the theory of elliptic boundary value problems [LM], we shall obtain the next result.

Corollary 3.4. *Let s a be positive integer. Then, $\mathcal{K}$ is a well defined operator of order 1. More precisely, for all $t > s + 1/2$. there exists a positive constant $C_t > 0$ such that we have the estimate*

$$\|\mathcal{K}\psi\|_{t-1} \le C_t \|\psi\|_t$$

for all $\psi \in C^\infty_{(0,q+1)}(\overline{\Omega})$*. Furthermore, when restricted to purely tangential forms,* $\mathcal{K}$ *is of order* 0*, i.e. for all* $t > s + 1/2$ *there exists* $C_t > 0$ *such that if* $\psi \llcorner \overline{\partial}\varrho = 0$ *in a neighborhood of* $b\Omega$*, then*

$$\|\mathcal{K}\psi\|_{t-1} \leq C_t \|\psi\|_{t-1}.$$

As a consequence of these facts, we obtain the following representation for the $(\overline{\partial}, s)$-Neumann problem. We set $G_s := \overline{\partial}\mathcal{K} + \mathcal{K}\overline{\partial}$. With the notation above, the $(\overline{\partial}, s)$-Neumann problem is equivalent to the boundary value problem

$$\begin{cases} (\Box + G_s)u = f & \text{on } \Omega \\ N^s(u \llcorner \overline{\partial}\varrho) = 0 & \text{on } b\Omega \\ N^s(\overline{\partial}u \llcorner \overline{\partial}\varrho) = 0 & \text{on } b\Omega\,. \end{cases}$$

Here $\Box := \overline{\partial}\vartheta + \vartheta\overline{\partial}$ is the complex Laplacian, and it equals -4Δ on $\Omega \subseteq \mathbb{C}^n$. Notice that G_s is the singular Green's operator we mentioned earlier. The operator G_s is of order 2, so of the same order as the complex Laplacian $\Box$. Moreover notice that $G_s u$ only depends on the boundary values of u and $\overline{\partial}u$ and their derivatives up to order $2s$, and that in general G_s is not diagonal. An analysis of the analogue of the operator G_s in the case of the de Rham complex, appears in [FKP1].

Proof of Theorem 3.1. Let $\phi \in C^\infty_{(0,q)}(\overline{\Omega})$ and $\psi \in C^\infty_{(0,q+1)}(\overline{\Omega})$. Using Green's formula we have

$$\begin{aligned} \langle \overline{\partial}\phi, \psi\rangle_s =& \langle \phi, \overline{\partial}^*\psi\rangle_s = \langle \phi, \vartheta\psi\rangle_s + \langle \phi, \mathcal{K}\psi\rangle_s \\ =& \langle \phi, \vartheta\psi\rangle_s + \sum_{0\leq|\alpha|\leq s} \gamma_\alpha \sum_{KkI} \varepsilon^K_{kI} \int_{b\Omega} D^\alpha\phi_I \overline{D^\alpha\psi_K} \frac{\partial\varrho}{\partial\bar{z}_k}. \end{aligned} \tag{7}$$

Recall that the $(0, q+1)$ form ψ belongs to $\operatorname{dom}\overline{\partial}^*$ if and only if there exists a constant $C_\psi > 0$ such that $|\langle\overline{\partial}\phi, \psi\rangle_s| \leq C_\psi\|\phi\|_s$ for all $\phi \in \operatorname{dom}\overline{\partial}$. Hence $\psi \in \operatorname{dom}\overline{\partial}^*$ if and only if the boundary terms in the calculation (7) above can be bounded by $C_\psi\|\phi\|_s$. By the Sobolev trace theorem we can bound the terms of the form

$$\int_{b\Omega} D^\alpha\phi_I\overline{D^\alpha\psi_K}\frac{\partial\varrho}{\partial\bar{z}_k}$$

when $|\alpha| \leq s - 1$. Thus it suffices to consider the sum

$$\sum_{|\alpha|=s}\sum_{KkI}\int_{b\Omega} D^\alpha\phi_I\overline{D^\alpha\psi_K}\frac{\partial\varrho}{\partial\bar{z}_k}.$$

By integrating by parts we can move tangential derivatives from ϕ to ψ, so only the s normal derivatives on ϕ may cause trouble.

We decompose the standard derivatives in the coordinate directions into their normal and tangential components:

$$D_j = Y_j + \nu_j N,$$

where N is the normal derivative, and Y_j are tangential vector fields. Then

$$D^\alpha = (Y_{\alpha_{p_1}} + \nu_{\alpha_{p_1}} N) \cdots (Y_{\alpha_{p_s}} + \nu_{\alpha_{p_s}} N).$$

Notice that, since $\sum_j \nu_j^2 \equiv 1$ and $N = \sum_j \nu_j D_j$, we have that $\sum_j \nu_j Y_j = 0$. Therefore, when considering s normal derivatives on ϕ_I, we have

$$\begin{aligned}
&\sum_{|\alpha|=s} \gamma_\alpha \sum_{KkI} \\
&\Big[\varepsilon^K_{kI} \int_{b\Omega} (\nu_{\alpha_{p_1}} N) \cdots (\nu_{\alpha_{p_s}} N)\phi_I \overline{(Y_{\alpha_{p_1}} + \nu_{\alpha_{p_1}} N) \cdots (Y_{\alpha_{p_s}} + \nu_{\alpha_{p_s}} N)\psi_K \frac{\partial \varrho}{\partial \bar z_k}}\Big] \\
&= \sum_{|\alpha|=s} \gamma_\alpha \sum_{KkI} \varepsilon^K_{kI} \int_{b\Omega} (\nu_{\alpha_{p_1}})^2 \cdots (\nu_{\alpha_{p_s}})^2 \Big(N^s \phi_I\Big) \overline{\Big(N^s \psi_K\Big)} \frac{\partial \varrho}{\partial \bar z_k} \\
&= \Big(\sum_{|\alpha|=s} \gamma_\alpha (\nu_{\alpha_{p_1}})^2 \cdots (\nu_{\alpha_{p_s}})^2\Big) \sum_I \int_{b\Omega} (N^s \phi_I) \overline{\Big(\sum_{Kk} \varepsilon^K_{kI} N^s (\psi_K \frac{\partial \varrho}{\partial z_k})\Big)}.
\end{aligned}$$

Now, if $\psi \in C^\infty_{(0,q+1)}(\overline{\Omega})$ and

$$0 = \sum_{Kk} \varepsilon^K_{kI} N^s (\psi_K \frac{\partial \varrho}{\partial z_k}) = N^s (\psi \llcorner \overline{\partial} \varrho)_I$$

on $b\Omega$ for all I, then clearly $\psi \in \operatorname{dom} \overline{\partial}^*$.

On the other hand, suppose that $N^s(\psi \llcorner \overline{\partial}\varrho)_I \neq 0$ on $b\Omega$ for a certain I. We may assume that

$$\operatorname{Re}\Big(N^s(\psi \llcorner \overline{\partial}\varrho)_I\Big) \geq 1 \quad \text{on } B(p,\delta) \cap \overline{\Omega},$$

where $B(p,\delta)$ is a small ball center at $p \in \Omega$. For $\varepsilon > 0$, consider the collection of $(0,q)$ forms $\phi^{(\varepsilon)}$,

$$\phi^{(\varepsilon)} := (-\varrho)^{s-1}(-\varrho + \varepsilon)^{3/4} \chi d\bar z^I,$$

where χ is a non-negative C^∞ cut-off function, $\operatorname{supp}\chi \subseteq B(p,\delta)$, and $\chi = 1$ on $B(p,\delta/2)$. Now, an easy calculation shows that

$$\|\phi^{(\varepsilon)}\|_s \leq C_1$$

independently of ε, while

$$\Big| \int_{b\Omega} N^s \phi^{(\varepsilon)}_I \cdot \overline{N^s(\psi \llcorner \overline{\partial}\varrho)_I} \Big| \geq C_2 \varepsilon^{-1/4},$$

which is unbounded, as $\varepsilon \to 0$. This finishes the proof of the proposition. □

Remark 3.5. We observe that $\operatorname{dom} \overline{\partial}^* \cap C^\infty_{(0,q+1)}(\bar\Omega)$ is dense in $W^s_{(0,q+1)}(\Omega)$. Therefore it suffices to show that for any $\varepsilon > 0$ and $\phi \in C^\infty_{(0,q+1)}(\overline{\Omega})$ there exists $\psi \in C^\infty_{(0,q+1)}(\overline{\Omega})$ with $\|\psi\|_s < \varepsilon$ and $\phi - \psi \in \operatorname{dom} \overline{\partial}^*$.

Having fixed ϕ and ε, let $\chi \in C_0^\infty(-1,1)$ and $\chi = 1$ in a neighborhood of the origin. Then the form ψ

$$\psi := (1/s!)(-\varrho)^s \chi(-\varrho/\varepsilon)\Big(N^s(\phi \llcorner \overline{\partial}\varrho)\Big) \wedge \overline{\partial}\varrho$$

satisfies the required conditions.

Proof of Proposition 3.3. We have set $\overline{\partial}^* = \vartheta + \mathcal{K}$, so that for $\psi \in \operatorname{dom} \overline{\partial}^*$ we have

$$\langle \phi, \overline{\partial}^* \psi \rangle_s = \langle \phi, \vartheta\psi \rangle_s + \langle \phi, \mathcal{K}\psi \rangle_s. \tag{8}$$

On the other hand by (7) we see that, for $\psi \in \operatorname{dom} \overline{\partial}^*$ and $\phi \in C^\infty_{(0,q)}(\overline{\Omega})$ we have the equality

$$\langle \overline{\partial}\phi, \psi \rangle_s = \langle \phi, \vartheta\psi \rangle_s + \sum_{0 \le |\alpha| \le s} \gamma_\alpha \sum_{KkJ} \varepsilon^K_{kJ} \int_{b\Omega} D^\alpha \phi_J \overline{D^\alpha \psi_K} \frac{\partial \varrho}{\partial \bar{z}_k};$$

so it follows that

$$\langle \phi, \mathcal{K}\psi \rangle_s = \sum_{0 \le |\alpha| \le s} \gamma_\alpha \sum_{KkJ} \varepsilon^K_{kJ} \int_{b\Omega} D^\alpha \phi_J \overline{D^\alpha \psi_K} \frac{\partial \varrho}{\partial \bar{z}_k}. \tag{9}$$

By choosing ϕ with compact support in Ω we find that $\mathcal{K}\psi$ satisfies

$$\begin{aligned} 0 =& \langle \phi, \mathcal{K}\psi \rangle_s \\ =& \sum_{|J|=q} \sum_{0 \le |\alpha| \le s} \gamma_\alpha \int_\Omega D^\alpha \phi_J \overline{D^\alpha (\mathcal{K}\psi)_J} \\ =& \sum_{|J|=q} \sum_{0 \le |\alpha| \le s} (-1)^{|\alpha|} \gamma_\alpha \int_\Omega \phi_J \overline{D^{2\alpha} (\mathcal{K}\psi)_J}. \end{aligned}$$

Since this holds for all $\phi \in C^\infty_{(0,q)}(\Omega)$ with compact support in Ω, we see that $(\mathcal{K}\psi)_J$ must satisfy the equation

$$0 = \sum_{0 \le |\alpha| \le s} (-1)^{|\alpha|} \gamma_\alpha D^{2\alpha} (\mathcal{K}\psi)_J = \sum_{j=0}^{s} (-\Delta)^j (\mathcal{K}\psi)_J \quad \text{on } \Omega$$

for all J, which is the equation on the interior of Ω that appears in (6).

Now we move on to consider the boundary conditions that $\mathcal{K}\psi$ must satisfy. For $\phi \in C^\infty_{(0,q)}(\overline{\Omega})$, by repeatedly applying Green's theorem to the left hand side of equation

(9), and recalling equation (5), we have

$$\begin{aligned}\langle\phi,\mathcal{K}\psi\rangle_s &= \sum_{|J|=q}\Big(\langle\phi_J,(\mathcal{K}\psi)_J\rangle_0+\sum_{j=1}^{2n}\langle D_j\phi_J,D_j(\mathcal{K}\psi)_J\rangle_{s-1}\Big)\\
&= \sum_{|J|=q}\Big(\int_\Omega \phi_J\overline{(\mathcal{K}\psi)_J}+\sum_{i=1}^{2n}\int_\Omega D_i\phi_J\overline{D_i(\mathcal{K}\psi)_J}\\
&\qquad+\sum_{1\le|\beta|\le s-1}\gamma_\beta\sum_{i=1}^{2n}\int_\Omega D_iD^\beta\phi_J\overline{D_iD^\beta(\mathcal{K}\psi)_J}\Big)\\
&= \sum_{|J|=q}\Big(\int_{b\Omega}\phi_J\overline{N(\mathcal{K}\psi)_J}+\sum_{1le|\beta|\le s-1}\gamma_\beta\int_{b\Omega}D^\beta\phi_J\overline{ND^\beta(\mathcal{K}\psi)_J}\\
&\qquad-\langle\phi_J,\Delta(\mathcal{K}\psi)_J\rangle_{s-1}+\dots\Big),\end{aligned}$$

where the dots stand for terms that do not contribute to any boundary expression.

We iterate this calculation on the last term on the right in the above chain of equalities to obtain that

$$\langle\phi,\mathcal{K}\psi\rangle_s=\sum_{|J|=q}\sum_{i=0}^{s-1}\sum_{|\alpha|\le i}\gamma_\alpha\int_{b\Omega}(D^\alpha\phi_J)\overline{ND^\alpha(-\Delta)^{s-1-i}(\mathcal{K}\psi)_J}+\dots,$$

where the dots have the same meaning as before. ¿From this equation and (9) it follows that, for all J,

$$\begin{aligned}\sum_{i=0}^{s-1}\sum_{|\alpha|\le i}\gamma_\alpha\int_{b\Omega}(D^\alpha\phi_J)\overline{ND^\alpha(-\Delta)^{s-1-i}(\mathcal{K}\psi)_J}&\\
=\sum_{0\le|\alpha|\le s}\gamma_\alpha\sum_{kK}\varepsilon^K_{kJ}\int_{b\Omega}D^\alpha\phi_J\overline{D^\alpha\psi_K}\frac{\partial\varrho}{\partial\bar z_k}.&\qquad(10)\end{aligned}$$

This equation must hold true for all $\phi\in C^\infty_{(0,q)}(\overline{\Omega})$. Thus we need to isolate the terms containing $N^\ell\phi_J$ for $\ell=0,1,\dots,s-1$, and for all J.

Now observe that, if f and g are smooth functions on the boundary, then

$$\begin{aligned}\sum_{j=1}^{2n}\int_{b\Omega}D_jf\overline{D_jg}&=\sum_j\int_{b\Omega}(Y_j+\nu_jN)f\overline{(Y_j+\nu_jN)g}\\
&=\sum_j\int_{b\Omega}Y_jf\overline{Y_jg}+\int_{b\Omega}Nf\overline{Ng},\end{aligned}$$

where we have used the fact that $\sum_j\nu_jY_j=0$. Now

$$D^\alpha=T_{\alpha,|\alpha|}+T_{\alpha,|\alpha|-1}N+\dots+\nu^\alpha N^{|\alpha|},$$

where $T_{\alpha,k}$ is a tangential operator of order $\leq k$, and $\nu := (\nu_1, \ldots, \nu_{2n})$. Therefore the left hand side of (10) equals

$$\sum_{i=0}^{s-1} \sum_{|\alpha| \leq i} \gamma_\alpha \int_{b\Omega} \Big(T_{\alpha,|\alpha|} + T_{\alpha,|\alpha|-1}N + \cdots + \nu^\alpha N^{|\alpha|}\Big)\phi_J \cdot \overline{ND^\alpha(-\Delta)^{s-1-i}(\mathcal{K}\psi)_J} \tag{11}$$

$$= \sum_{\ell=0}^{s-1}\Bigg(\int_{b\Omega} N^\ell \phi_J \cdot \overline{\Big[\sum_{i=\ell}^{s-1} \sum_{\ell \leq |\alpha| \leq i} \gamma_\alpha T^*_{\alpha,|\alpha|-\ell} ND^\alpha(-\Delta)^{s-1-i}(\mathcal{K}\psi)_J\Big]}\Bigg)$$

$$= \sum_{\ell=0}^{s-1}\Bigg(\int_{b\Omega} N^\ell \phi_J \cdot \overline{\sum_{\ell \leq |\alpha| \leq s-1} \Big[\sum_{j=0}^{s-1-|\alpha|} \gamma_\alpha T^*_{\alpha,|\alpha|-\ell} ND^\alpha(-\Delta)^j(\mathcal{K}\psi)_J\Big]}\Bigg). \tag{12}$$

Notice that in the above calculations we have obtained the identity

$$\langle \phi, \mathcal{K}\psi\rangle_s = \sum_J \Bigg(\langle \phi_J, \sum_{j=0}^{s} (-\Delta)^j (\mathcal{K}\psi)_J\rangle_0 + \sum_{\ell=0}^{s-1} \int_{b\Omega} N^\ell \phi_J \cdot \overline{\sum_{\ell \leq |\alpha| \leq s-1} \Big[\sum_{j=0}^{s-1-|\alpha|} \gamma_\alpha T^*_{\alpha,|\alpha|-\ell} ND^\alpha(-\Delta)^j(\mathcal{K}\psi)_J\Big]}\Bigg). \tag{13}$$

In particular, for $\nu := (\nu_1, \ldots, \nu_{2n})$, we have that $T_{\alpha,0} = \nu^\alpha = T^*_{\alpha,0}$ and for ℓ a positive integer we have

$$\sum_{|\alpha|=\ell-1} \gamma_\alpha \nu^\alpha D^\alpha = \sum_{|\beta|=\ell-2} \gamma_\beta \nu^\beta \Big(\sum_{i=1}^{2n} \nu_i D_i\Big) D^\beta = \cdots = N^{\ell-1}. \tag{14}$$

Thus the last summand on the right hand side of (12) (corresponding to $\ell = s-1$) becomes

$$\int_{b\Omega} N^{s-1}\phi_J \cdot \overline{\Bigg(\sum_{|\alpha|=s-1} \gamma_\alpha T^*_{\alpha,0}\Big[ND^\alpha(\mathcal{K}\psi)_J\Big]\Bigg)} = \int_{b\Omega} N^{s-1}\phi_J \cdot \overline{N^s(\mathcal{K}\psi)_J}.$$

The right hand side of (10) can be treated in the same way:

$$\sum_{0 \leq |\alpha| \leq s} \gamma_\alpha \int_{b\Omega} \Bigg(\sum_{\ell=0}^{|\alpha|} T_{\alpha,|\alpha|-\ell} N^\ell \phi_J\Bigg) \overline{\Bigg(\sum_{kK} \varepsilon^K_{kJ} D^\alpha \psi_K \frac{\partial \varrho}{\partial z_k}\Bigg)}$$

$$= \sum_{\ell=0}^{s} \int_{b\Omega} N^\ell \phi_J \cdot \overline{\sum_{\ell \leq |\alpha| \leq s} \gamma_\alpha T^*_{\alpha,|\alpha|-\ell} \Bigg(\sum_{kK} \varepsilon^K_{kJ} D^\alpha \psi_K \frac{\partial \varrho}{\partial z_k}\Bigg)}. \tag{15}$$

Notice that the top order term vanishes since $N^s\phi_J$ is paired with

$$\sum_{|\alpha|=s} \gamma_\alpha T^*_{\alpha,0} \Bigg(\sum_{kK} \varepsilon^K_{kJ} D^\alpha \psi_K \frac{\partial \varrho}{\partial z_k}\Bigg) = \sum_{kK} \varepsilon^K_{kJ} N^s \psi_K \frac{\partial \varrho}{\partial z_k},$$

which equals 0 on $b\Omega$, because $\psi \in \operatorname{dom} \overline{\partial}^*$. ¿From these calculations, and by equating the right hand sides of (12) and (15), we obtain the s boundary equations. Set

$$\sum_{kK} \varepsilon^K_{kJ} D^\alpha \psi_K \frac{\partial \varrho}{\partial z_k} = (L_\alpha \psi)_J .$$

Then, on $b\Omega$, we have

$$N^s (\mathcal{K}\psi)_J = \sum_{s-1\le|\alpha|\le s} \gamma_\alpha T^*_{\alpha,|\alpha|-s+1} (L_\alpha\psi)_J$$

$$\sum_{s-2\le|\alpha|\le s-1} \gamma_\alpha T^*_{\alpha,|\alpha|-s+2} N D^\alpha \Bigg(\sum_{j=0}^{s-1-|\alpha|} (-\Delta)^j (\mathcal{K}\psi)_J \Bigg)$$
$$= \sum_{s-2\le|\alpha|\le s} \gamma_\alpha T^*_{\alpha,|\alpha|-s+2} (L_\alpha\psi)_J$$
$$\cdots \qquad\qquad \cdots$$

$$\sum_{0\le|\alpha|\le s-1} \gamma_\alpha T^*_{\alpha,|\alpha|} N D^\alpha \Bigg(\sum_{j=0}^{s-1-|\alpha|} (-\Delta)^j (\mathcal{K}\psi)_J \Bigg)$$
$$= \sum_{0\le|\alpha|\le s} \gamma_\alpha T^*_{\alpha,|\alpha|} (L_\alpha\psi)_J .$$

Thus we have s boundary equations in $(\mathcal{K}\psi)_J$. Notice that the k^{th} equation has order $s+k-1$ in the normal direction, for $k = 1,\ldots,s$. Since $T^*_{\alpha,0} = \nu^\alpha$ and $-\Delta = -N^2 + T_1 N + T_2$, using formula (14), the operator on the left hand side in the k^{th} equation becomes

$$\begin{aligned} &\sum_{s-k\le|\alpha|\le s-1} \gamma_\alpha T^*_{\alpha,|\alpha|-s+k} N D^\alpha \Bigg(\sum_{j=0}^{s-1-|\alpha|} (-\Delta)^j \Bigg) \\ &= N^{s-k+1} \sum_{j=0}^{k-1} (-\Delta)^j + \cdots + \sum_{|\alpha|=s-1} \gamma_\alpha T_{\alpha,k-1} N D^\alpha \\ &= (-1)^{(k-1)} N^{s+k-1} + T_1 N^{s+k-2} + \cdots + T_{s+k-2} N \end{aligned}$$

as in the statement of the proposition, while the right hand side in the same equation is an operator of order $s+k$ (one order larger than the left hand side), that we denote by $P^{(J)}_{s+k}$. Then we have

$$P^{(J)}_{s+k}(\psi) = \sum_{s-k\le|\alpha|\le s} \gamma_\alpha T^*_{\alpha,|\alpha|-s+k} (L_\alpha\psi)_J . \tag{16}$$

This finishes the proof. $\square$

Before proving Corollary 3.4 we need one more result. Consider the boundary value problem (6) that defines the components of $\mathcal{K}$:

$$\begin{cases} \sum_{j=0}^{s}(-\Delta)^j u = 0 & \text{on } \Omega \\ \sum_{j=0}^{s+\ell} T_j N^{s+\ell-j} u = g_\ell & \text{on } b\Omega,\ \ell = 0,\dots,s-1\,. \end{cases} \tag{17}$$

for given $g_\ell \in C^\infty(b\Omega)$, $\ell = 0,\dots,s-1$. Notice that the operator $\mathcal{K}$ applied to a form ψ gives rise to the composition of a (non-diagonal) differential operator acting on the components of ψ, the restriction to the boundary $b\Omega$, and the solution operator S of the (scalar) boundary value problem (17). Then we have the following.

Lemma 3.6. *The boundary value problem (17) is an elliptic boundary value problem with trivial kernel, that is if $g_\ell = 0$ for $\ell = 0,\dots,s-1$, then $S(g_0,\dots,g_{s-1}) = 0$.*

Proof. In order to prove that the boundary value problem (17) is elliptic, we use the standard definition, see (10.1.1) in [Hö2]. Given any point $p \in b\Omega$ we need to consider a C^∞ change of coordinates that takes p into the origin, flattens the boundary, and such that the transformed vector fields at the origin coincide with the new basis vector fields. We write the new coordinates as $(x_0, x) \in [0,+\infty) \times \mathbb{R}^{2n-1}$. Then, the normal vector field is ∂_{x_0}, and $\partial_1,\dots,\partial_{2n-1}$ are the tangential vector fileds. After taking the Fourier transform in the tangential directions, writing $\xi \in \mathbb{R}^{2n-1}$ for the variable dual to x, we need to show that the ordinary differential equation

$$\begin{cases} (-\partial_{x_0}^2 + |\xi|^2)^s v & = 0 \quad \text{on } [0,+\infty) \\ B_{s,\ell}\, v(0) & = 0 \quad \ell = 0,1,\dots,s-1 \end{cases} \tag{18}$$

admits the trivial solution as the only bounded solution on $[0,+\infty)$. Here $B_{s,\ell}$ denote the top order terms of the boundary operators in (17) in our special chart, after freezing the coefficients and taking the Fourier transform.

We begin by describing the differential operators that give the initial conditions in (18). We then prove that the only bounded solution of (18) is in fact the trivial solution.

The boundary equations in (17) arise from the identity (13). By considering forms of the type $\phi_J d\bar{z}^J$ we may reduce to the case of functions. We set $u = (\mathcal{K}\psi)_J$. Consider the top order terms in (13), change coordinates, and freeze the coefficients. Write $\alpha = (k, \alpha')$ and notice that $\gamma_\alpha = \binom{s-1}{k}\gamma_{\alpha'}$. Then $\partial^\alpha = \partial_{x_0}^k \partial^{\alpha'}$. Notice that the top order term in $T_{\alpha,|\alpha|-\ell}$ equals $\partial^{\alpha'}$, and that $T^*_{\alpha,|\alpha|-\ell} = (-1)^{|\alpha'|}\partial^{\alpha'}$. Then we have that

$$\begin{aligned} B_{s,\ell} &= \sum_{|\alpha'|=0}^{s-1-\ell} \binom{|\alpha'|+\ell}{\ell} \gamma_{\alpha'} (-1)^{|\alpha'|} \partial^{2\alpha'} \partial_{x_0}^{\ell+1} (-\Delta)^{s-1-\ell-|\alpha'|} \\ &= \sum_{j=0}^{s-1-\ell} \binom{j+\ell}{\ell} (-\Delta')^j (-\Delta)^{s-1-\ell-j} \partial_{x_0}^{\ell+1}, \end{aligned}$$

where Δ' is the tangential Laplacian. Now write $\Delta = \partial_{x_0}^2 + \Delta'$. We claim that the following identity holds true

$$\sum_{j=0}^{s-1-k-\ell} \binom{\ell+j}{j}\binom{s-j-\ell-1}{k} = \binom{s}{\ell+k+1}. \tag{19}$$

Assume the claim for now. Then, it turns out that

$$B_{s,\ell} = \sum_{k=0}^{s-1-k} (-1)^k \binom{s}{\ell+k+1} |\xi|^{2(s-1-\ell-k)} \partial_{x_0}^{\ell+2k+1}.$$

Next, let $v = v_\xi$ be a bounded solution of (18) for $\xi \neq 0$. Notice that $v = \left(\sum_{\ell=0}^{s-1} c_\ell x_0^\ell\right) e^{-|\xi| x_0}$. Let $f \in C_0^\infty(\overline{\mathbb{R}_+^{2n-1}})$. Then for any $\xi \neq 0$, by assumption and by integrating by parts we have

$$\begin{aligned}
0 &= -\sum_{\ell=0}^{s-1} \partial_{x_0}^\ell f(0,\xi)\overline{B_{s,\ell} v_\xi(0)} + \int_0^\infty f(x_0,\xi)\overline{(-\partial_{x_0}^2 + |\xi|^2)^s v_\xi}\, dx_0 \\
&= -\sum_{\ell=0}^{s-1} \partial_{x_0}^\ell f(0,\xi)\overline{B_{s,\ell} v_\xi(0)} + \sum_{j=0}^{s} (-1)^j \binom{s}{j} |\xi|^{2(s-j)} \int_0^\infty f(x_0,\xi)\overline{\partial_{x_0}^{2j} v_\xi}\, dx_0 \\
&= -\sum_{\ell=0}^{s-1} \partial_{x_0}^\ell f(0,\xi)\overline{B_{s,\ell} v_\xi(0)} + |\xi|^{2s} \int_0^\infty f(x_0,\xi)\overline{v_\xi}\, dx_0 \\
&\qquad - \sum_{j=1}^{s} (-1)^j \binom{s}{j} |\xi|^{2(s-j)} \Big(|\xi|^{2(s-j)} f(0,\xi) \partial_{x_0}^{2j-1} v_x(0) + \int_0^\infty \partial_{x_0} f(x_0,\xi) \overline{\partial_{x_0}^{2j-1} v_\xi}\, dx_0 \Big) \\
&= -\sum_{\ell=1}^{s-1} \partial_{x_0}^\ell f(0,\xi)\overline{B_{s,\ell} v_\xi(0)} + |\xi|^{2s} \int_0^\infty f(x_0,\xi)\overline{v_\xi}\, dx_0 \\
&\qquad + \sum_{k=0}^{s-1} (-1)^k \binom{s}{k+1} |\xi|^{2(s-k-1)} \int_0^\infty \partial_{x_0} f(x_0,\xi) \overline{\partial_{x_0}^{2k+1} v_\xi}\, dx_0 .
\end{aligned}$$

By applying integration by parts $(s-1)$ more times to the last term in the right hand side above, we obtain that

$$0 = \sum_{j=0}^{s} \binom{s}{j} |\xi|^{2(s-j)} \int_0^\infty \partial_{x_0}^j f(x_0,\xi) \overline{\partial_{x_0}^j v_\xi}\, dx_0 \tag{20}$$

for all $\xi \neq 0$.

Now, for each $\xi \neq 0$ we can pick f so that $f(\cdot,\xi) = v_\xi$. Substituting in (20) we obtain that

$$\sum_{j=0}^{s} \binom{s}{j} |\xi|^{2(s-j)} \int_0^\infty |\partial_{x_0}^j v_\xi(x_0)|^2\, dx_0 = 0,$$

that is, $v_\xi = 0$.

Thus, we only need to prove the claim. If, for $p \geq m$ we set $F_k(p,m) := \sum_{j=0}^m \binom{k+j}{j}\binom{p-j}{m-j}$, we wish to show that

$$F_k(p,m) = \binom{p+k+1}{m}. \tag{21}$$

Observe that (21) holds true for $m = 0,1$ and $p \geq 1$, and for $p = m$, by direct computation and well known properties of binomial coefficients. Assume the statement true for $p-1$ and all $m \leq p-1$. Since

$$F_k(p,m) = F_k(p-1,m) + F_k(p-1,m-1),$$

equality (21) follows by induction and the equality in the case $m = p$. This finishes the proof of the ellipticity of (3.6).

Finally, if all the boundary data g_ℓ in problem (3.6) are identically 0, then the only solution of the boundary value problem is the trivial one. In fact, if u is such a solution, the identity (13) with u in place of $\mathcal{K}\psi$ implies that u is orthogonal in the W^s sense to all $\phi \in C^\infty(\Omega)$, hence $u = 0$. $\square$

Finally, we have:

Proof of Corollary 3.4. Clearly, $\mathcal{K}$ is well defined as composition of differential operators, restriction to the boundary, and the operator S solution of the boundary value problem in the previous Lemma.

Next, we use standard estimates for elliptic boundary value problems, as in [LM] Theorem 5.1, and Lemma 3.6. Recall that $P_{s,k}^{(I)}$ is a differential operator of order $s+k$, containing s at most derivatives in the normal direction. Then we see that for all $t > s + 1/2$

$$\begin{aligned}
\|\mathcal{K}\psi\|_{t-1} &\leq C_t \sum_I \|(\mathcal{K}\psi)_I\|_{t-1} \\
&\leq C_t \sum_I \sum_{k=1}^s \|P_{s,k}^{(I)}\psi\|_{W^{t-1-(s+k-1)-1/2}(b\Omega)} \\
&\leq C_t \sum_I \sum_{k=1}^s \|N^k\psi\|_{W^{t-k-1/2}(b\Omega)} \\
&\leq C_t \sum_I \sum_{k=1}^s \|N^k\psi\|_{t-k} \\
&\leq C_t \|\psi\|_t,
\end{aligned}$$

where we use the assumption $t > s + 1/2$ in order to able be to apply the trace theorem.

Finally notice that $\psi \llcorner \overline{\partial}\varrho = 0$ in a neighborhood of $b\Omega$, $P_{s,k}^{(I)}$ becomes an operator of one degree lower, i.e., of order $s+k-1$. Repeating the argument above, we obtain that, for $t > s + 1/2$

$$\|\mathcal{K}\psi\|_{t-1} \leq C_t \|\psi\|_{t-1}.$$

This concludes the proof of the corollary. $\square$

Final Remarks. The results of Section 3 are obtained under a specific formulation of the Sobolev inner product. If we modify the formulation by choosing other positive coefficients γ_α in the definition of the inner product (1), results analogous to those presented here should still hold. It is also the case that the formulas that arise in these formulations of the norm are probably much less tractable.

The situation seems quite different if we take a generic equivalent norm. Consider, for instance, the weighted theory of the $\overline{\partial}$-Neumann problem, as developed by Kohn in [Ko]. Kohn showed that the regularity properties enjoyed by the canonical solution in the weighted case are in general much stronger than the ones enjoyed by the classical canonical solution (see also the aforementioned work of Christ [Ch]). Therefore, it is clear that much has still to be understood in the general case. We shall provide no details about the treatment of equivalent Sobolev topologies.

In the present paper we have worked with $(0, q)$ forms on a domain Ω in $\mathbb{C}^n$. These results hold true in the case of (p, q) forms, with no change in the proofs. Routine modifications (see [FK]) should allow one to work out the case of a smoothly bounded pseudoconvex domain M' in a complex, or even an almost complex, manifold M.

Of course it is also of interest to work out sharp estimates for the $(\overline{\partial}, s)$ problem, and to calculate the full Hodge and spectral theories; we save that work for a future series of papers.

References

[Ba] D. Barrett, Behavior of the Bergman projection on the Diederich-Fornæss worm, *Acta Math.* **168** (1992), 1-10.

[Bo1] H. P. Boas, Holomorphic reproducing kernels in Reinhardt domains, *Pac. J. Math.* **112** (1984), 273-292.

[Bo2] ———, Sobolev space projections in strictly pseudoconvex domains, *Trans. Amer. Math. Soc.* **288** (1985), 227-240.

[Ca] D. W. Catlin, Global regularity of the $\overline{\partial}$-Neumann problem, *Proc. Symp. Pure Math.* **41** (1984), 39-49.

[Ch] M. Christ, Global C^∞ irregularity of the $\overline{\partial}$-Neumann problem for worms domains, *Journal of the Amer. Math. Soc.*, to appear.

[FK] G. B. Folland and J. J. Kohn, *The Neumann Problem for the Cauchy-Riemann Complex*, Princeton University Press, Princeton, 1972.

[FKP1] L. Fontana, M. M. Peloso, and S. G. Krantz, Hodge theory for the de Rham complex in Sobolev topology, *Memoirs Amer. Math. Soc.*, to appear.

[FKP2] ———, Hodge theory in the Sobolev topology for the de Rham complex on a smoothly bounded domain in Euclidean space, *Electronic Research Announcements* of the American Mathematical Society **1** (1995), 103-107.

[Gr] G. Grubb, *Boundary Value Problems for Pseudo-Differential Operators*, Birkhäuser, Basel 1992 .

[Hö1] L. Hörmander, *An Introduction to Complex Analysis in Several Complex Variables*, North Holland, Amsterdam 1973.

[Hö2] ———, *Linear Partial Differential Operators*, Springer-Verlag, Berlin 1964.

[Ko] J. J. Kohn, Global regularity for $\overline{\partial}$ on weakly pseudo-convex manifolds, *Trans. Amer. Math. Soc.* **181** (1973), 273-292.

[LM] J. L. Lions, E. Magenes, *Problèmes Aux Limites Non Homogènes*, vol. 1, Dunon, Paris 1968.

Luigi Fontana, Dipartimento di Matematica, Via Saldini 50, Università di Milano, 20133 Milano (Italy)
E-mail address: fontana@vmimat.mat.unimi.it

Steven G. Krantz, Mathematical Sciences Research Institute and Department of Mathematics, Washington University, St. Louis, MO 63130 (U.S.A.)
E-mail address: sk@artsci.wustl.edu

Marco M. Peloso, Dipartimento di Matematica, Politecnico di Torino, 10129 Torino (Italy)
E-mail address: peloso@polito.it