SOME DEFINITION OF THE ARTIN EXPONENT OF FINITE GROUPS

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Abstract. The Artin exponent induced from cyclic subgroups of finite groups was studied extensively by T.Y. Lam in [5]. A Burnside ring theoretic version of the results in [5] for p-groups was given in [6]. Here we shall be interested in looking at the Artin exponent induced from the elementary abelian subgroups of finite p-groups using some results of A. Dress in [3].

1. Introduction

The Artin exponent induced from cyclic subgroups of finite groups was studied extensively by T.Y. Lam in [5]. A new setting for the description of these exponents in the sense of [5] using the Burnside ring theoretic methods was given in [6]. In this paper we are interested in defining a similar concept for elementary abelian subgroups of finite groups. More precisely, using basically some results of A. Dress concerning the Burnside rings (see[1], [2], [3]), we shall prove the following results (see chapter three for the explanation of the notation).

Main result Let G be a finite p-group and $e(G)$ the Artin exponent induced from the elementary abelian subgroups of G. One has the following:
(a)If G is abelian, then $e(G) = |G : \overline{U}|$, where $\overline{U}$ is the maximal elementary abelian subgroup of G.
(b) If G is a quaternion or dihedral group then $e(G) = 2$ and $e(G) = 4$ if G is the semidihedral group.
(c) In all other cases $e(G) = |G|/p$.

The above results seem to suggest that the invariant $e(G)$ gives an interesting numerical measure of the deviation of G from being an elementary abelian group. Let me mention where this study fits in. First, the method we are going to adopt here is Burnside ring theoretic and hence affirms the utility of the Burnside ring in the representation theory of finite groups. Second, as is known, the idea of the Artin exponent appeals to number theory – which in themselves have aesthetic values. Finally, exponents of this type have been known to be very useful in some formulation of some induction theorems for various functors on the category of finite groups – see for instance [8].

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This paper is arranged as follows. In section 2, we collect some well known definitions and basic results about the Burnside rings. In section 3, we form our definition and provide some particular examples. In section 4, we prove some useful results from which the main result finally follows in section 5.

2. The Rings $\Omega(G)$ and $\tilde{\Omega}(G)$

Let G be a finite group. A G-set is a finite set on which G acts from the left by permutation. The set of isomorphism classes of finite left G-sets form a commutative semi-ring $\Omega^+(G)$ with addition induced by disjoint union and multiplication induced by cartesian product with diagonal actions. The Grothendieck ring of $\Omega^+(G)$ is denoted by $\Omega(G)$ and called the *Burnside ring* of G. Let $G^{sub/\sim}$ denote the set of conjugate classes of subgroups of G. For any G-set X let $[X]$ be the image in $\Omega(G)$ and write (U) for the set of subgroups conjugate to U in G. Then $\Omega(G)$ is additively a free abelian group with basis $[G/U]$ where (U) is taken over the set $G^{sub/\sim}$. For any G-set X and a subgroup U of G, let $X^U := \{x \in X \mid ux = x \; \forall u \in U\}$, be the set of U-invariant elements of X. Since the map $|X| \longrightarrow |X^U|$ preserves sums and products, it extends to a ring homomorphism

$$\chi_U : \ \Omega(G) \longrightarrow Z : [X] \longrightarrow |X^U|$$

from $\Omega(G)$ to Z (the integers). Now for every $(U) \in G^{sub/\sim}$ let $Z_{(U)}$ be the ring isomorphic to Z. We define the *ghost ring* $\tilde{\Omega}(G)$ as follows:

$$\tilde{\Omega}(G) \ := \prod_{(U)\in G^{sub/\sim}} Z_{(U)}.$$

So $\tilde{\Omega}(G)$ is a ring by pointwise multiplication. It is well known (see [3]) that the product map

$$\chi := \prod_{(U)\in G^{sub/\sim}} \chi_{(U)} : \ \Omega(G) \longrightarrow \tilde{\Omega}(G)$$

is injective. One also has the following important result.

Theorem 2.1. *Identifying $\Omega(G)$ with its image in $\tilde{\Omega}(G)$ with respect to the map $\prod_{(U)\in G^{sub/\sim}} \chi_{(U)}$ one has that $\{n \in Z \mid n \cdot \tilde{\Omega}(G) \subseteq \Omega(G)\} \ = \ |G| \cdot Z$.*

Proof: see [3]

Lemma 2.2. *For every $x \in \Omega(G)$ one has $\sum_{g\in G} \chi_{<g>}(x) \ \equiv \ 0(mod\ |G|)$, where $< g >$ denotes the cyclic group generated by g. (This relation is called the Cauchy-Frobenius-Burnside relation).*

Proof: see [4]

Because for any two subgroups U and V in G one has that $\chi_U = \chi_V$ if and only if $U \overset{G}{\sim} V$ (U and V are conjugate in G) and for x and y in $\Omega(G)$ one has $\chi_U(x) = \chi_U(y)$ for all subgroups U in G if and only if $x = y$ it follows that we can identify each $x \in \Omega(G)$ with the associated map $U \longrightarrow \chi_U(x)$, from the set of all subgroups of G into Z. We shall denote this associated map also by x. So we can consider $\Omega(G)$ in a canonical way as a subring of its ghost ring $\tilde{\Omega}(G)$.

Theorem 2.3. *Let $x \in \tilde{\Omega}(G)$. Then $x \in \Omega(G)$ if and only if for every $U \trianglelefteq V \leq G$ with $(V : U)$ a power of a prime one has*

$$\sum_{vU \in V/U} x(< v \ , \ U >) \ \equiv \ 0 (mod \ (V : U)).$$

Proof: see [2]

For more details on the Burnside ring see [3], [7].

3. The Artin Exponent

We now give the definition of the Artin exponent in terms of the Burnside ring. This arises in the following context.

Definition 3.1. *Let $\mathcal{U}$ denote the family of all elementary abelian subgroups of G. Let $b_{\mathcal{U}}$ be the element of $\tilde{\Omega}(G)$ defined as follows. If (U) is the conjugacy class of the subgroup U of G then*

$$b_{\mathcal{U}}(U) \ := \begin{cases} 1 & if \ U \in \mathcal{U} \\ 0 & if \ U \notin \mathcal{U} \end{cases}.$$

We call the integer $e(G) \ := \ min(n \in N \mid n \cdot b_{\mathcal{U}} \in \Omega(G))$ the Artin exponent of G.

Corollary 3.2. $|G| \cdot b_{\mathcal{U}}(U) \in \Omega(G)$.

Proof: Follows from Theorem 2.1. □

Corollary 3.3. *$e(G)$ divides the order of G.*

Proof: Put $e := e(G)$ and $b := b_{\mathcal{U}}(U)$. Write $|G| = qe + r$ with $0 \leq r < e$. Now $r \cdot b = (|G| - qe) \cdot b = |G| \cdot b - (qe) \cdot b \in \Omega(G)$. Hence $r = 0$ in view of the minimality of e. □

To motivate the general algebraic procedure to be followed in the coming sections we take some particular examples.

Example 3.4. *For a finite group G, let $\mathcal{S}(G)$ denote the set of all subgroups of G. Now let G be a cyclic p-group of order p^γ. That is*
$G =< g >,\ |G| = p^\gamma$ *and* $\mathcal{S}(G) = \{U_0,\ U_1,\ U_2,\ .\ .\ , U_{\gamma-1}\}$, *where*
$U_0 =< g^{p^\gamma} >;\ U_1 =< g^{p^{\gamma-1}} >;\ .\ .\ .\ .\ ;\ U_{\gamma-i}\ = < g^{p^i} >$ *and*
$|U_j| = p^j$. *The family $\mathcal{U}$ of elementary abelian subgroups of G consists of the subgroups U_0 and U_1. From definition 3.1, we have that*
$e(G) \cdot b_{\mathcal{U}} = e(G) \cdot (1,\ 1,\ 0,\ .\ .\ .\ , 0)\ =\ (e(G),\ e(G),\ 0,\ .\ .\ .\ , 0) \in \Omega(G)$. *To finish this example we need the following lemma.*

Lemma 3.5. *Let G denote a cyclic p-group of order p^n. For every $i \in \{0,\ .\ .\ .\ , n\}$ let U_i denote the unique subgroup of order p^i of G. Then given $x(U_i) \in \tilde{\Omega}(G)$ one has that $x(U_i) \in \Omega(G)$ if and only if*

$$p^i \cdot x(U_i)\ +\ \sum_{j=i+1}^{n} (p^j - p^{j-1}) \cdot x(U_j)\ \ \equiv\ \ 0(mod\ p^n),$$

where $|U_j|\ =\ p^j$.

Proof: *For* $x\ \in\ \Omega(G)$ *and* $U\ \trianglelefteq\ V\ \leq\ G$ *we have that*

$$\sum_{vU \in V/U} x(< v\ ,\ U >)\ \equiv\ 0\ (mod\ V : U).$$

Since G is abelian, $U_i\ \trianglelefteq\ G\ \forall i \in \{0,\ .\ .\ .\ , n\}$. Hence by Theorem 2.3

$$\sum_{gU_i \in G/U_i} x_i(< g\ ,\ U_i >)\ \ \equiv\ \ 0(mod\ p^{n-i}).$$

Since $x(< g\ ,\ U_i >)\ =\ x(U_j)$ if and only if $< g\ ,\ U_i > =\ U_j$ we have
$\sum_{gU_i \in G/U_i} x(< g\ ,\ U_i >)\ \ =\ \ \sum_{j=i}^{n} x(U_j) \cdot \#\{gU_i\ \in\ G/U_i\ \mid\ < g\ ,\ U_i > =\ U_j\ \}$. *Since we have that $< g\ ,\ U_i >\ =\ < g >$ if $g\ \notin\ U_i$ then*

$$\begin{aligned} & \sum_{j=i}^{n} x(U)_j\ \cdot \#\{gU_i\ \in\ G/U_i\ \mid\ \ < g\ ,\ U_i > =\ U_j\} \\ =\quad & x(U)_i + \sum_{j=i+1}^{n} x(U_j) \cdot \frac{p^j - p^{j-1}}{p^i}\ \ \equiv\ \ 0(mod\ p^{n-i}) \\ \Longleftrightarrow\quad & x(U_i) \cdot p^i\ +\ \sum_{j=i+1}^{n} x(U_j) \cdot (p^j - p^{j-1})\ \ \equiv\ \ 0(mod\ p^n). \end{aligned}$$

□

Now we finish our example by observing from the above lemma that for $i = 1$ we have $e(G) \cdot p\ \ \equiv\ \ 0(mod\ p^\gamma)$ and for $i = 0$ we have

$$e(G)\ +\ e(G) \cdot (p - 1)\ \ \equiv\ \ 0(mod\ p^\gamma)\ \ \Longleftrightarrow\ \ p^{\gamma-1}\ \mid\ e(G).$$

This implies $e(G) = p^{\gamma-1}$.

Example 3.6. *Consider* $\mathcal{U}$ *a family of elementary abelian subgroups of a finite group* G. *Then, from lemma 3.5, one has that* $e(G) = 1$ *if and only if* $\mathcal{U} = \mathcal{S}(G)$. *In particular, if* G *is elementary abelian then* $e(G) = 1$.

Example 3.7. *Let* G *be the quaternion group of order* 8. *Then it is to see by direct computation that* $e(G) = 2$.

We now generalize the above examples in the following section.

4. Results

We start this section with the following observation.

Lemma 4.1. *For* $\mathcal{U}$ *a family of elementary abelian subgroups of a finite group* G *let* $e_{\mathcal{U}}(G)$ *be the Artin exponent with respect to the family* $\mathcal{U}$. *Assume that* $e_{\mathcal{U}}(G) = 1$, *then for all* $U, V \leq G$ *with* $U \trianglelefteq V$ *and* $(V : U)$ *a prime power, one has that* $U \in \mathcal{U}$ *if and only if* $V \in \mathcal{U}$.

Proof: Assume $(V : U) = p^{\alpha}$ for some $\alpha \geq 0$. Then $x \in \Omega(G)$ implies $x(U) \equiv x(V)(mod\ p)$. Therefore $U \in \mathcal{U}$ implies $x(V) \equiv 1(mod\ p)$, hence $x(V) \neq 0$. We have $x(V) = 1$ and so $V \in \mathcal{U}$. Conversely $U \notin \mathcal{U}$ implies $x(V) \equiv 0(mod\ p)$, hence $x(V) \neq 1$ which implies $x(V) = 0$, that is $V \notin \mathcal{U}$. □

Lemma 4.2. *If* G *is solvable, then* $x \in \Omega(G)$ *implies that* $\mathcal{U} = \mathcal{S}(G)$ *or* $\mathcal{U} = \emptyset$ *(the empty set).*

Proof: If G is solvable then for every U in G there exists a sequence of subgroups $U = U_0, U_1, \ldots, U_k = 1$ such that U_i is normal in U_{i-1} for $i = 1, \ldots, k$ and $(U_{i-1} : U_i)$ is a prime power or 1. Hence if $\mathcal{U} \neq \emptyset$, then by (4.1) one has that $1 \in \mathcal{U}$ and $U \in \mathcal{U}$ for all $U \leq G$. □

Corollary 4.3. *If* G *is a finite* p*-group, then* $e(G) = 1$ *if and only if* $\mathcal{U} = \mathcal{S}(G)$.

Proof: Follows from 4.2. □

Proposition 4.4. *Let* G *be a finite noncyclic abelian* p*-group. Then one has* $e(G) = (G : \overline{U})$, *where* $\overline{U} = \{g \in G \mid g^p = 1\}$ *is the maximal elementary abelian subgroup of* G.

Proof: Let $\mathcal{U}$ be the family of all elementary abelian subgroups of G and let $U \in \mathcal{U}$. Let $e(G)$ and $b_{\mathcal{U}}$ be defined as in 3.1. By definition of $e(G)$ we have $e(G) \cdot b_{\mathcal{U}} \in \Omega(G)$, hence by Theorem 2.3,

$$\sum_{gU \in G/U} e(G) \cdot b_{\mathcal{U}}(< g \ , \ U >) \ \equiv \ 0(mod \ |G \ : \ U|).$$

Since $< g \ , \ U >$ is elementary abelian if and only if $g \in \overline{U}$, we have

$$\begin{aligned} &\sum_{gU \in G/U} b_{\mathcal{U}}(< g \ , \ U >) \\ &= \ |\{gU \in G/U \ \mid \ < g \ , \ U > \text{ is elementary abelian } \}| \\ &= \ |\{gU \in G/U \ \mid \ < g \ , \ U > \leq \ \overline{U}\}| \\ &= \ |\overline{U} \ : \ U|. \end{aligned}$$

Hence it follows that $e(G)$ is the minimal positive integer such that

$$e(G) \cdot |\overline{U} \ : \ U| \ \equiv \ 0(mod \ |G \ : \ U|),$$

that is $e(G) \ = \ (G : \overline{U})$. □

Before we state the next result we need the following (see [9]).

Lemma 4.5. *Let G be a non-abelian p-group of order $p^n (p = 2)$. If there exists a $g \in G$ of order p^{n-1} then G has one of the following presentations for some h in G.*
(A) $g^{2^{n-1}} = 1$, $h^2 = g^{2^{n-2}}$, $hgh^{-1} = g^{-1}$, where $p = 2$ and $n \geq 3$, – the quaternion group.
(B) $g^{2^{n-1}} = 1$, $h^2 = 1$, $hgh^{-1} = g^{-1}$, where $p = 2$ and $n \geq 3$ – the dihedral group.
(C) $g^{2^{n-1}} = 1$, $h^2 = 1$, $hgh^{-1} = g^{1+2^{n-2}}$, where $p = 2$ and $n \geq 4$.
(D) $g^{2^{n-1}} = 1$, $h^2 = 1$, $hgh^{-1} = g^{-1+2^{n-2}}$. where $p = 2$ and $n \geq 4$, – the semi-dihedral group.

Proposition 4.6. *Let G be a finite nonabelian and noncyclic p group. Then one has that*
$e(G) = |G|/p$ unless G is the quaternion, the dihedral or the semi-dihedral group in which cases $e(G) = 2$ for the quaternion or the dihedral groups and $e(G) = 4$ for the semi-dehidral.

Proof: As usual let $\mathcal{U}$ be the family of all elementary abelian subgroups of G and let $U \in \mathcal{U}$. Let $e(G)$ and $b_{\mathcal{U}}$ be defined as before. So $e(G) \cdot b_{\mathcal{U}} \in \Omega(G)$ and $\sum_{gU \in G/U} e(G) \cdot b_{\mathcal{U}}(< g \ , \ U >) \ \equiv \ 0(mod \ |G \ : \ U|)$. To find $e(G)$, it suffices to set

$|U| = p$. Now

$$\begin{aligned}&\sum_{gU\in G/U} b_{\mathcal{U}}(< g\ ,\ U >)\\ &= \ |\{gU \in G/U \ \mid \ < g\ ,\ U > \text{ is elementary abelian }\}|\\ &= \ 1 \ + \ |\{gU \in G/U \ \mid \ < g\ ,\ U > \ \cong \ C_p \times C_p\}|,\end{aligned}$$

where C_p denote a cyclic group of order p. Let

$$\mathcal{N}(g\ ,\ U) \ := \ |\{gU \in G/U \ \mid \ < g\ ,\ U > \cong \ C_p \times C_p\}|.$$

For G a noncyclic p-group, it is easy to see that $\mathcal{N}(g\ ,\ U) \ \equiv \ 0(mod\ p)$ if G is not a quaternion, dihedral or semidihedral group. This implies that $e(G) = |G|/p$ is the desired minimal number such that $e(G) \cdot (1 + \mathcal{N}(g\ ,\ U)) \equiv 0(mod\ (G\ :\ U))$, that is $e(G) = |G|/p$. On the other hand if $\mathcal{N}(g\ ,\ U) \ \not\equiv \ 0(mod\ p)$ then G corresponds to the nonabelian group of order 8, i.e. the quaternion and the dihedral groups and by easy computation we find that in these cases $e(G) = 2$. The semi-dihedral group of order 8 is just the noncyclic abelian group of order 8. So for the semi-dihedral $e(G) = 4$. □

5. Proof of main result

Finally we have;

Theorem 5.1. *Let G be a finite p-group. We have the following.*
(a)If G is abelian, then the Artin exponent $e(G) = |G : U|$, where U is the maximal elementary abelian subgroup of G.
(b) If G is a quaternion or dihedral group, then $e(G) = 2$, and $e(G) = 4$ if G is a semidihedral group.
(c) In all other cases $e(G) = |G|/p$

Proof: Results now follows from proposition 4.4 and proposition 4.6.

References

[1] A. DRESS; A characterization of solvable groups, *Math. Z. 110, (1960), pp. 213-217.*
[2] A. DRESS; Congruence relations characterizing the representation ring of the Symmetric group, *J. Algebra 2, (1986), pp 350 - 363.*
[3] A. DRESS; Notes on the theory of representation of finite groups, *Lecture Notes, Bielefeld, 1971*
[4] A. DRESS, C. SIEBENEICHER, T. YOSHIDA; An application of Burnside rings in elementary finite group theory, *J. Algebra 1, (1992), pp 27 - 44.*
[5] T.Y. LAM; Artin exponent for finite groups, *J. Algebra 9, (1968), pp. 94-119.*
[6] K.K. NWABUEZE; Certain applications of the Burnside ring and its ghost ring in finite group theory I *Bull. Soc. Math. Belgium (to appear).*
[7] T. TOM-DIECK; Transformation groups and representation theory, *Lecture Notes in Math. 444, Springer-Verlag 1975.*
[8] R. SWAN; Induced representation and projective modules, *Ann. Math 71, (1960), pp 552 - 578.*
[9] H. ZASSENHAUS; Theory of Groups, *Chelsea, NewYork, 1958.*

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