APPLICATION OF THE COMPLEX MONGE-AMPÈRE EQUATION TO THE STUDY OF PROPER HOLOMORPHIC MAPPINGS OF STRICTLY PSEUDOCONVEX DOMAINS

STEVEN G. KRANTZ AND SONG-YING LI

ABSTRACT. We construct a special plurisubharmonic defining function for a smoothly bounded strictly pseudoconvex domain so that the determinant of the complex Hessian vanishes to high order on the boundary. This construction, coupled with regularity of solutions of complex Monge-Ampére equation and the reflection principle, enables us to give a new proof of the Fefferman mapping theorem.

1. INTRODUCTION

In classical analysis, an important theorem of Painlevé [Pai] and Kellogg [Kel] states that any conformal mapping between two smoothly bounded domains in the complex plane $\mathbb{C}$ can be extended to be a diffeomorphism on the closures of the domains. This theorem was generalized by C. Fefferman [Fef] in 1974 to strictly pseudoconvex domains in $\mathbb{C}^n$. Fefferman's original proof of this theorem is very technical, relying as it does on deep work on the boundary asymptotics of the Bergman kernel and on the regularity of $\overline{\partial}$-Neumann operator that is due to J. J. Kohn [Ko]. Bell/Ligocka [BeL], and later Bell [Be], gave a simpler proof which deals with more general domains, including pseudoconvex domains of finite type, by using regularity of the Bergman projection and the $\overline{\partial}$-Neumann operator as studied by Kohn [Ko], Catlin [Ca], Boas/Straube [BS], and others.

From the results in [NWY], [Web], [Lem], [PH] and [For], we know that the proof of the Fefferman mapping theorem can be reduced to proving that $\det(\varphi'(z))$ extends continuously across the boundary if $\varphi : \Omega_1 \to \Omega_2$ is a biholomorphic mapping and the $\Omega_j, j = 1, 2$, are smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$; such an argument uses the reflection principle as developed by Nirenberg/Webster/Yang [NWY], Webster [Web], Pinchuk/Hasanov [PH], Coupet [Cou], and more recently F. Forstneric [For].

We know from [Ker] that Painlevé and Kellogg's theorem can be proved by using the regularity of the Dirichlet problem for the Laplacian in a smoothly bounded planar domain, where the property of the Laplacian being conformally invariant

Date: December 2, 1996.
Krantz was partially supported by NSF Grant DMS–9531967, and at MSRI by NSF Grant DMS–9022140. Li was partially supported by NSF Grant DMS–9500758.

plays an important role in the proof. The natural generalization of the Laplacian in one complex variable to several complex variables, with these considerations in mind, is the complex Monge-Ampère equation. In [Ker], Kerzman observed that the proof of the Fefferman mapping theorem would follow from the C^∞ global regularity of the Dirichlet problem of a degenerate complex Monge-Ampère equation. However, counterexamples in Bedford/Fornæss [BF] as well as in Gamelin/Sibony [GS] show that, in general, the degenerate Dirichlet problem for the complex Monge-Ampère equation does not have C^2 boundary regularity. Thus Kerzman's idea does not work in the sense of its original formulation.

The main purpose of the present paper is to give a new method for using the complex Monge-Ampère equation to study the boundary regularity of biholomorphic mappings of strongly pseudoconvex domains; thus, in effect, we validate the program initiated by Kerzman/Kohn/Nirenberg [Ker]. We achieve this goal by proving our Theorem 2.2 and then combining our result with a result of Caffarelli/Kohn/Nirenberg/Spruck [CKNS] on the Dirichlet problem for the complex Monge-Ampère equation. Specifically, we shall prove the following theorem.

THEOREM 1.1. *Let $\Omega_j, j = 1, 2$ be two bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with C^∞ boundary. Let $\varphi : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping. Then for any $\epsilon > 0$ we have*

(i) φ can be extended as a $Lip_1(\overline{\Omega}_1)$ mapping;

(ii) $\det(\varphi') \in Lip_{1/2}(\overline{\Omega}_1)$;

(iii) There is a C^2 defining function ρ for Ω_2 so that, for any $k = 1, \ldots, n$, we have $\sum_{j=1}^n \left[\frac{\partial\rho}{\partial w_j} \circ \varphi\right] \cdot \frac{\partial\varphi_j}{\partial z_k} \in Lip_1(\overline{\Omega}_1)$.

As a corollary of (ii) and of a theorem in [PH] and [Web], we obtain a new proof of the Fefferman mapping theorem.

This paper is organized as follows. In Section 2, our main theorem (Theorem 2.2) is stated and proved. Theorem 1.1 is proved in the second part of Section 2 by combining Theorem 2.2 and Theorem 2.1 (a result in [CKNS]). We shall then include an explanation (in Section 3) of how to combine theorems in [PH] and [Web] together with Theorem 1.1 to obtain a new proof of the Fefferman mapping theorem.

The authors would like to thank Xiaojun Huang and Peter Li for helpful conversations.

2. An Application of the Complex Monge-Ampère Equation

Let us recall a theorem of Caffarelli, Kohn, Nirenberg and Spruck [CKNS].

THEOREM 2.1. *Let Ω be a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$ with C^3 boundary $\partial\Omega$. Let f be a non-negative function on $\overline{\Omega}$ such that $f(z)^{1/n} \in C^{1,1}(\overline{\Omega})$.*

Let $H(u)$ denote the complex Hessian matrix of the function u. Then there is a unique plurisubharmonic function $v \in C^{1,1}(\overline{\Omega})$ satisfying

$$\begin{aligned} \det H(v)(z) &= f(z) \quad \text{for } z \in \Omega; \\ v &= 0 \qquad \text{for } z \in \partial\Omega. \end{aligned} \tag{2.1}$$

Note: The theorem stated in [CKNS] is for $\partial\Omega$ being $C^{3,1}$ but, we may approximate $\partial\Omega$ by a sequence of bounded strictly pseudoconvex domains with C^4 boundaries, and do a careful count of which derivatives are actually used in their proof, to obtain the result stated here.

As we mentioned in the introduction, the following theorem plays an essential role in the proof of Theorem 1.1.

THEOREM 2.2. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary $\partial\Omega$. For any $0 < \epsilon << 1$ and any positive integer q there is a plurisubharmonic defining function $\rho_q \in C^\infty(\overline{\Omega})$ for Ω so that*

$$\det H(\rho_q)(z) \le C dist(z, \partial\Omega)^q, \quad z \in \Omega. \tag{2.3}$$

Proof. Let $\delta(z)$ denote the (signed) distance function from z to $\partial\Omega$. Since Ω is a bounded domain in $\mathbb{C}^n$ with C^∞ boundary $\partial\Omega$, then $\delta(z) \in C^\infty(\overline{\Omega})$ (after modification of the distance function on a compact set in the interior). By a rotation we see that, for any fixed point $z^0 \in \Omega$ near the boundary, we may assume that the z_n direction is the direction $(\frac{\partial\delta(z)}{\partial\overline{z}_1}, \cdots, \frac{\partial\delta}{\partial\overline{z}_n})$ at the point z^0. Let

$$H(\delta)_{n-1}(z) = \left[\frac{\partial^2\delta}{\partial z_\ell \partial\overline{z}_q}(z)\right]_{1\le\ell,q\le n-1}$$

Since Ω is strictly pseudoconvex, there is an $\epsilon > 0$ so that

$$-H(\delta)_{n-1}(z^0) \ge \epsilon I_{n-1}$$

for all $z \in \overline{\Omega}$ with $\delta(z) \le \epsilon$. We may assume that $H(\delta)_{n-1}$ is diagonal at z^0. Note that

$$H(-\delta)(z) = \begin{bmatrix} -H(\delta)_{n-1}(z) & 0 \\ 0 & -\frac{\partial^2\delta}{\partial z_n\partial\overline{z}_n} \end{bmatrix}$$

If it happens that $\sum_{ij} \frac{\partial\delta}{\partial z_i}\frac{\partial\delta}{\partial\overline{z}_j}\frac{\partial^2\delta}{\partial z_i\partial\overline{z}_j} = 0$, then $\rho_q = -\delta(z) + \delta(z)^{2+q}$ is the desired defining function. [In fact it is these terms that distinguish the study of the real Hessian from the more subtle study of the complex Hessian. In particular, we know that $\frac{\partial^2\delta}{\partial x_n^2}$ equals 0, but the term $\partial^2\delta\partial z_n\partial\overline{z}_n$ may not be zero. Therefore estimate (2.3) is easy to check for the determinant of the real Hessian of δ; matters are much trickier for the complex Hessian.]

Now we let

$$\rho(z) = -\delta(z)$$

and

$$r^{[m]}(z) = \rho(z) + \sum_{k=2}^{m} a_k(z)\delta(z)^k \quad z \in \Omega.$$

We will prove inductively that

$$\det(H(r^{[m]})(z)) = b_{m-1}(z) \cdot \rho^{m-1},$$

where b_{m-1} is some smooth function. In particular, $\det(H(r^{[m]}))$ vanishes to order $m-1$ at the boundary.

Now

$$\begin{aligned} \partial_{i\bar{j}} r^{[m]}(z) &= \partial_{i\bar{j}}\rho(z) + \sum_{k=2}^{m} [\partial_{i\bar{j}} a_k\, \rho(z)^2 + k(k-1)a_k \partial_i\rho\partial_{\bar{j}}\rho)]\rho^{k-2} \\ &\quad + \sum_{k=2}^{m} k[\partial_i a_k \partial_{\bar{j}}\rho + \partial_i\rho\partial_{\bar{j}} a_k + a_k \rho_{i\bar{j}}]\rho^{k-1} \end{aligned}$$

Let $H(r^{[m]})_{n-1}(z) = [\frac{\partial^2 r^{[m]}}{\partial z_i \partial \bar{z}_j}]_{(n-1)\times(n-1)}$, and let $B(z)^* = [r_{n\bar{1}}, \cdots, r_{n\overline{n-1}}]$ be a row-vector. Then

$$H(r^{[m]})(z) = \begin{bmatrix} H(r^{[m]})_{n-1}(z) & B(z) \\ B(z)^* & r^{[m]}_{n\bar{n}}(z) \end{bmatrix}$$

Then

$$\det(H(r^{[m]})(z)) = \det(H(r^{[m]})_{n-1}(z))[r^{[m]}_{n\bar{n}} - -B^* H(r^{[m]})_{n-1}(z)''^{-1} B(z)]$$

We know that

$$H(r^{[m]})_{n-1}(z) \geq \epsilon I_{n-1}$$

for all $z \in \Omega_\epsilon$, where ϵ is a positive number depending only on Ω and $\|a_j\|_{C^2(\overline{\Omega})}$. First, we let

$$(d\rho)(z) = \sum_{i,j} \rho_i \rho_{\bar{j}} \rho_{i\bar{j}}\,;$$

then we define

$$a_2(z) = -\frac{(d\rho)(z)}{2((d\rho)(z) + |\partial\rho|^4)}.$$

Thus we have

$$\begin{aligned} r_{n\overline{n}}(z) &= \rho_{n\overline{n}} + \sum_{k=2}^{m}[\partial_{n\overline{n}}a_k\rho^k + k(k-1)a_k|\partial\rho|^2\rho^{k-2}] \\ &\quad + \sum_{k=2}^{m} k[\partial_n a_k\partial_{\overline{n}}\rho + \partial_n\rho\partial_{\overline{n}}a_k + a_k\rho_{n\overline{n}}]\rho^{k-1} \\ &= \sum_{k=2}^{m}[\partial_{n\overline{n}}a_k\rho^k + k(k+1)a_{k+1}|\partial\rho|^2\rho^{k-1}] \\ &\quad + \sum_{k=2}^{m} k[\partial_n a_k\partial_{\overline{n}}\rho + \partial_n\rho\partial_{\overline{n}}a_k + a_k\rho_{n\overline{n}}]\rho^{k-1} \\ &= O(\rho) \end{aligned}$$

Therefore

$$\det(H(r^{[2]})(z)) = b_1(z)\rho(z)$$

Assume that we have constructed $r^{[m]} = \rho(z) + \sum_{k=2}^{m} a_k\rho(z)^k$ such that

$$\det(H(r^{[m]})(z)) = b_{m-1}(z)\rho^{m-1}, \quad z \in \Omega.$$

We consider

$$r^{[m+1]} = r^{[m]} + a_{m+1}\rho(z)^{m+1}.$$

Since

$$\begin{aligned} &H(r^{[m+1]})(z) \\ &= H(r^{[m]}) + \rho(z)^{m+1}H(a_{m+1})(z) + (m+1)a_{m+1}\rho(z)^m H(\rho) \\ &\quad +(m+1)ma_{m+1}\rho^{m-1}\partial\rho(z)\otimes\overline{\partial\rho} \\ &\quad +(m+1)\rho(z)^m[\partial\rho\otimes\overline{\partial a_{m+1}} + \partial a_{m+1}\otimes\overline{\partial\rho}], \end{aligned}$$

it is easy to see that

$$\det(H(r^{[m+1]})(z)) = b_m(z)\rho(z)^m) + \det\Big[H(r^{[m]}) + (m+1)ma_{m+1}\rho^{m-1})\Big].$$

By a rotation, we may let z_n be the complex normal direction of $\partial\Omega_{\delta(z)}$ at z. Thus

$$\begin{aligned} &\det(H(r^{[m]})(z) + (m+1)m\rho(z)^{m-1}\partial\rho(z)\otimes\overline{\partial\rho}) \\ &\qquad = \det M, \end{aligned}$$

where

$$M = \begin{bmatrix} H(r^{[m]})_{n-1}(z) & B(m)(z) \\ B(m)(z)^* & r^{[m]}_{n\overline{n}}(z) + (m+1)m\rho^{m-1}\rho_n\rho_{\overline{n}} \end{bmatrix}.$$

But this

$$\begin{aligned} &= \det(H(r^{[m]})_{n-1}(z)[r^{[m]}_{n\overline{n}} + B(m)^* H(r^{[m]})_{n-1}(z)^{-1} B(m)(z) \\ &\qquad +(m+1)m\rho^{m-1}\rho_n\rho_{\overline{n}}] \\ &= \det(H(r^{[m]})(z)) + \det(H(r^{[m]})_{n-1}(z))(m+1)m\rho_n\rho_{\overline{n}}\rho(z)^{m-1} \\ &= b_{m-1}(z)\rho(z)^{m-1} + \det(H(r^{[m]})_{n-1}(z))(m+1)m\rho_n\rho_{\overline{n}}\rho(z)^{m-1}. \end{aligned}$$

We therefore choose a_{m+1} such that

$$b_{m-1}(z) + (m+1)m\det(H(r^{[m]})_{n-1}(z))\rho_n\rho_{\overline{n}}(z) = 0$$

From this it will follow that

$$\det(H(r^{[m+1]})(z)) = b_m(z)\rho(z)^m.$$

By our construction, we know that $H(r^{[m]})_{n-1}(z)$ is positive definite with least positive eigenvalue ϵ_m for all $z \in \Omega \setminus \Omega_{\epsilon_m}$. Thus if c_m is large enough so that $|\nabla\delta|^2 c_m \geq 2|b_m(z)|$ then, by choosing ϵ_m small enough, we will easily see that the function

$$\rho_m(z) \equiv r^{[m+1]}(z) + c_m\delta(z)^{m+2} \tag{2.4}$$

is positive definite in $\Omega\setminus\Omega_{\epsilon_m}$ and (2.3) holds on $\Omega\setminus\Omega_{\epsilon_m}$. Here $\Omega_t = -z \in \Omega : \delta(z) < t''$. Then we use arguments in [CKNS] and [Li1] to extend ρ_m to be defined on Ω and strictly pseudoconvex on Ω. The proof of the theorem is complete. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1.
Let ρ_{4n} be the plurisubharmonic defining function for Ω_2, with $m = 4n$, that we constructed in Theorem 2.2. We let

$$r(z) = \rho_{4n}(\varphi(z)), \qquad z \in \Omega_1.$$

Then we have

$$\det H(r)(z) = \det(H(\rho_{4n})(\varphi(z)))|\det\varphi'(z)|^2 = f(z), \quad z \in \Omega_1$$

Since $\partial\Omega_2 \in C^\infty$ then $f(z)^{1/n} \in C^2(\overline{\Omega})$ and $f \geq 0$. Theorem 2.1 implies that $r \in C^{1,1}(\overline{\Omega}_1)$. So

$$\|r\|_{C^{1,1}(\overline{\Omega}_1)} \leq C.$$

It is obvious that $\exp(\rho_{4n})$ is strictly plurisubharmonic in Ω_2 and that there is a constant $\epsilon > 0$ so that

$$H(\exp(\rho_{4n})(w) \geq \epsilon I_n, \quad \text{for all} \quad w \in \Omega_2.$$

Thus

$$\begin{aligned}\frac{\partial^2 \exp(r(z))}{\partial z_\ell \partial \overline{z}_\ell} &= \sum_{p,\ell=1}^{n} \frac{\partial^2 \exp(\rho_{4n})}{\partial w_p \partial \overline{w}_\ell}(\varphi(z)) \frac{\partial \varphi_p}{\partial z_\ell} \overline{\frac{\partial \varphi_\ell}{\partial z_\ell}}(z) \\ &\geq \epsilon \sum_{p=1}^{n} \left| \frac{\partial \varphi_p}{\partial z_\ell}(z) \right|^2.\end{aligned}$$

This shows that

$$\|\varphi\|^2_{\mathrm{Lip}(\overline{\Omega}_1)} \leq C\lambda_0^{-1} \| \exp(v)\|_{C^{1,1}(\overline{\Omega}_1)}.$$

Thus we have $\det(\varphi'(z)) \in L^\infty(\Omega_1)$.

If we apply $\frac{\partial^2}{\partial z_\ell \partial z_m}$ to $r(z)$ and use the above result, then we have

$$\left| \sum_{m=1}^{n} \frac{\partial \rho_{4n}}{\partial w_p}(\varphi(z)) \frac{\partial^2 \varphi_p}{\partial z_\ell \partial z_m} \right| \leq C.$$

Let $z^0 \in \Omega_1$ be sufficiently near to $\partial\Omega_1$. Without loss of generality, by applying a rotation, we may assume that $z_1^0, \cdots, z_{n-1}^0$ are complex tangential at z^0 and also that at the point $\varphi(z_0)$ the directions $w_1, \cdots, w_{n-1}$ are complex tangential. Thus

$$\begin{aligned}&\frac{\partial}{\partial z_p} \log(\det(\varphi'(z_0)))(z) \\ &= \sum_{\ell,m=1}^{n} \varphi^{\ell m} \frac{\partial^2 \varphi_m}{\partial z_\ell \partial z_p}(z) \\ &= \sum_{\ell<n} \varphi^{\ell m} \frac{\partial^2 \varphi_m}{\partial z_\ell \partial z_p}(z) + \sum_{m<n} \varphi^{nm} \frac{\partial^2 \varphi_m}{\partial z_n \partial z_p}(z) + \varphi^{nn} \frac{\partial^2 \varphi_n}{\partial z_n \partial z_p}(z),\end{aligned}$$

where $(\varphi^{\ell m})$ is the inverse matrix of $\varphi'(z)$. Since $\varphi \in \mathrm{Lip}_1(\overline{\Omega}_1)$, we have

$$\left| \frac{\partial^2 \varphi_m}{\partial z_\ell \partial z_p}(z_0) \right| \leq C\delta_1(z)^{-1/2}$$

for all $1 \leq \ell \leq n-1$ and $1 \leq p \leq n$. By (3.8), we have

$$\left| \frac{\partial^2 \varphi_n}{\partial z_n \partial z_p}(z_0) \right| \leq C.$$

Now we consider the terms with $1 \leq m \leq n-1$. Since $(0, \cdots, 0, \varphi_n)$ is normal at $\varphi(z_0)$ and $z_1, \cdots, z_{n-1}$ are complex tangential at z_0 to $\partial\Omega_1$, we have

$$\left| \frac{\partial \varphi_n}{\partial z_\ell}(z_0) \right| \leq C\delta_2(\varphi(z_0))^{1/2} \leq C\delta_1(z_0)^{1/2}.$$

Since $\det(\varphi'(z))$ is bounded and

$$|\det(\varphi(z))\varphi^{\ell m}(z)| \leq C,$$

we see that

$$\begin{aligned} & |\det(\varphi'(z)\varphi^{nm}(z_0)\frac{\partial^2\varphi_m}{\partial z_n \partial z_n}(z_0)| \\ \leq \;& C\delta_1(z_0)^{-1}\left|\det(\frac{\partial \varphi_p}{\partial z_\ell})_{1\leq \ell \leq n-1, p\neq m})\right| \\ \leq \;& C\delta_1(z)^{-1/2}. \end{aligned}$$

Combining all the estimates, we have proved that

$$|\nabla \det(\varphi'(z))| = |\det(\varphi'(z))\nabla \det(\varphi'(z)\log\det(\varphi'(z))| \leq C\delta(z)^{-1/2},$$

for all $z \in \Omega_1$. Hence $\det(\varphi'(z)) \in \mathrm{Lip}_{1/2}(\overline{\Omega}_1)$. The proof of Theorem 1.1 is complete.
□

3. Reduction of the Mapping Problem

In this section, we shall state several theorems (from F. Forstneric [For] and Webster [Web]) which show how to connect the mapping problem with the reflection principle.

Let $\Sigma \subset \mathbb{C}^n$ be a maximal totally real submanifold. The "edge-of-the-wedge" domain Σ is locally defined as follows: Near $p \in \Sigma$ we find n smooth real-valued functions $r_1, \cdots, r_n$ so that

$$\Sigma = –z : r_1(z) = \cdots = r_n(z) = 0''$$

and so that the complex gradients

$$\partial r_m = \sum_{p=1}^{n} \frac{\partial r_m}{\partial z_p} dz_p$$

are $\mathbb{C}$-linearly independent on Σ. If U is a neighborhood of p in $\mathbb{C}^n$, and if $\Gamma \subset \mathbb{R}^n$ is an open convex cone with vertex zero, then we define the *wedge with edge* Σ to be

$$W = W(U, \Gamma) = –z \in U : r(z) \in \Gamma''.$$

Here we have used the notation $r(z) = \big(r_1(z), \ldots, r_n(z)\big)$. Then the following theorem is due to Pinchuk/Hasanov [PH] and Coupet [Cou]. Note that, in the statement of the theorem, the phrase "g is asymptotically holomorphic at Σ" has the standard meaning that $\overline{\partial} g$ vanishes to infinite order at Σ.

THEOREM 3.1. *Let $W = W(U, \Gamma)$ be a wedge with a smooth maximal totally real edge $\Sigma \subset \mathbb{C}^n$, let $\Sigma' \subset \mathbb{C}^{n'}$ be another smooth totally real submanifold, and let $F : W \cup \Sigma \to \mathbb{C}^{n'}$ be a continuous mapping that is smooth on W, asymptotically holomorphic at Σ, and such that $F(\Sigma) \subset \Sigma'$. Then the restriction of F to $\Sigma \cap U$ is smooth.*

In order to connect this last theorem to the mapping problem, we need to introduce an important theorem due to Webster [Web]. First we need some notation.

Let M and M' be local strictly convex smooth hypersurfaces containing the origin in $\mathbb{C}^n$ $(n > 1)$. Let D be a domain in $\mathbb{C}^n$, with boundary passing through the origin, that is smoothly bounded and pseudoconvex along $M \subseteq \partial\Omega$, and let D' be a similar such domain bounded in part by M'. We assume that we have a biholomorphic mapping $f : D \to D'$ that extends to a diffeomorphism on $D \cup M$ near the origin and such that this extended map sends M to M'. Since the problem is a local one, we may shrink our sets toward the origin, and we shall freely do so in the sequel.

Let $a \in \mathbb{C}^n \setminus -0''$ and let

$$[a] = -z \in \mathbb{C}^n : \sum_{m=1}^{n} z_m a_m = 0'' \in \mathbb{CP}^{n-1}.$$

If f is a holomorphic mapping, then we let

$$F(z, [a]) = (f(z), [af'(z)^{-1}])$$

where $f'(z)$ is the Jacobian matrix of f at z and $af'(z)^{-1}$ is the matrix product of a row vector a with the matrix $\left[f'(z)^{-1}\right]$.

Recall that $H_z M \subset T_z M$ is the maximal complex subspace of the $(2n-1)$-dimensional real tangent space $T_z M$. We associate to M the smooth submanifold $\tilde{M}$ of $X = \mathbb{C}^n \times \mathbb{C}P^{n-1}$ defined by

$$\tilde{M} = -(z, H_z M) \in X : z \in M'',$$

and

$$\tilde{M}' = -(z, H_z M') \in X : z \in M'''.$$

The following theorem is due to Webster [Web].

THEOREM 3.2. *If $M \subset \mathbb{C}^n$ is a strictly pseudoconvex hypersurface, then the associated manifold $\tilde{M} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ is totally real (whence maximally real).*

If f and $\det(f'(z))$ are continuous up to $D \cup M$, then for each $z \in M$ the derivative $f'(z)$ maps $H_z M$ isomorphically onto $H_{f(z)} M'$. Hence the associated mapping F extends continuously from the domain $\tilde{D}$ to $\tilde{D} \cup \tilde{M}$ and maps $\tilde{M}$ to $\tilde{M}'$. Its restriction to $\tilde{M}$ is given by

$$\tilde{f}(z, H_z M) = (f(z), H_{f(z)} M')$$

When $M = \partial\Omega_1$ and $\partial\Omega_2$ are strictly pseudoconvex and smooth then, by Theorem 1.1, we have that $f(z)$ and $\det(f'(z))$ are continuous on $\overline{\Omega}_1$. Combining this result with Theorem 3.2 and then with and 3.1, we obtain a new proof of the following theorem of C. Fefferman:

THEOREM 3.3 (C. Fefferman)**.** *Let Ω_1, Ω_2 be strongly pseudoconvex domains in $\mathbb{C}^n$ with C^∞ boundaries. Assume that $\Phi : \Omega_1 \to \Omega_2$ is a biholomorphic (proper) mapping. Then Φ extends to a univalent, C^∞ diffeomorphism of $\overline{\Omega}_1$ to $\overline{\Omega}_2$.*

References

[B1] E. Bedford, Survey of Pluri-Potential Theory, *Indiana U. Math. J.*, (1989).11–65.

[BF] E. Bedford and J. E. Fornaess, Counterexamples to regularity for the complex Monge-Ampère equation, *Invent. Math.* 50 (1979), 129–134.

[BR] S. Baouendi and L. Rothschild, A general reflection principle in $\mathbb{C}^2$, *J. of Funct. Anal.* 99 (1991), 409–442.

[BT] E. Bedford and B.A. Taylor, The Dirichlet problem for a complex Monge-Ampère equation, *Invent. Math.* 37(1976), 1–44.

[Be] S. Bell, Biholomorphic mappings and the $\overline{\partial}$-problem, *Ann of Math.* 114(1981), 103–113.

[BeC] S. Bell and D. Catlin, Boundary regularity of proper holomorphic mappings , *Duke Math. J.* 49(1982), 385–396.

[BeL] S. Bell and E. Ligocka, A simplification and extension of Fefferman's theorem on biholomorphic mappings, *Invent. Math.* 57(1980), 283–289.

[BS] H. Boas and E. Straube, Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in $\mathbb{C}^n$ admitting a defining function that is plurisubharmonic on the boundary, *Math. Z.* 206 (1991), 81–88.

[CKNS] L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations, II: complex Monge-Ampère, and uniformly elliptic equations, *Comm. in Pure and Appl. Math.* (1985), 209–252.

[Ca] D. Catlin, Subelliptic estimates for the $\overline{\partial}$-Neumann problem, *Annals of Math.* 126(1987), 131-192.

[CY] S. Y. Cheng and S. T. Yau, On the existence of a complex Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation, *Comm. Pure Appl. Math.* 33 (1980) 507–544.

[Cou] B. Coupet, Regularité d'applications holomorphes sur des varietés totalement reeles, in Thèse, Universite de Provence, 1987.

[DF] K. Diederich and J. Fornaess, Boundary regularity of proper holomorphic mappings, *Invent. Math.* 67(1982), 363–384.

[Fef] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent Math*, 26(1974), 1–65.

[For] F. Forstneric, An elementary proof of Fefferman's theorem, *Exposition Math*, 10(1992), 136–149.

[GS] T. W. Gamelin and N. Sibony, Subharmonicity for uniform algebras, *J. of Funct. Analysis* 35(1980), 64–108.

[He] G. M. Henkin, An analytic polyhedron is not holomorphically equivalent to a strictly pseudoconvex domain, *Dokl. Akad. Nauk SSSR* 210 (1973), 1026–1029.

[HX] X. Huang, Ph.D thesis, Washington University, 1994.

[Hu] Y. V. Hurumov, Boundary smoothness of proper holomorphic mappings of strictly pseudoconvex domains (in Russian), *Mat. Zametki* 48 (1990), no. 6, 149–150.

[Kel] O. D. Kellogg, Harmonic functions and Green's integra l, *Trans. Amer. Math. Soc.* 13(1912), 109–132.

[Ker] N. Kerzman, A Monge-Ampère equation in complex analysis, *Proc. Symp. Pure Math., Amer. Math. Soc.* 30 (1977), 161–167.

[Ko] J. J. Kohn, Harmonic integrals on strongly pseudoconvex manifolds I, *Ann. Math.* 78(1963), 112-148; II, ibid. 70(1964), 450-472.

[Kr] S. G. Krantz, *Partial Differential Equations and Complex Analysis*, CRC Press, Boca Raton, 1992.

[KrP] S. G. Krantz and H. R. Parks, *J. of Diff. Equations*, 40(1981), 116–120.

[Lem] L. Lempert, A precise result on the boundary regularity of biholomorphic mappings, *Math. Z.* 193 (1986), 559-579.

[Li] S-Y. Li, Neumann problems for complex Monge-Ampère equations, *Indiana University J. of Math*, 43(1994), 1099–1122.

[Mr] R. Moriyón, Regularity for the Dirichlet problem for the degenerate complex Monge-Ampère equation, *Comm. Pure Appl. Math.* 35 (1982), 1-27.

[NWY] L. Nirenberg, S. H. Webster and P. Yang, Local Boundary Regularity of Holomorphic Mappings *Comm. Pure Appl. Math.* XXXIII(1980), 305–338.

[Pai] P. Painlevé, Sure les lignes singulares des functions analytiques, Thesis, Gauthier-Villars, 1887.

[PH] S. Pinchuk and S. V. Hasanov, Asymptotically holomorphic functions, (Russian)*Mat. Sb.*, 134(176) (1987), 546–555.

[Web] S. Webster, On the reflection principle in several complex variables, *Proc. Amer. Math. Soc.*, 71(1978), 26–28.

Steven G. Krantz, Department of Mathematics, Box 1146, Washington University in St. Louis, One Brookings Drive, St. Louis, Missouri 63130

E-mail address: sk@math.wustl.edu

Song-Ying Li, Department of Mathematics, University of California, Irvine, CA 92697

E-mail address: sli@math.uci.edu