SCHUBERT POLYNOMIALS, THE BRUHAT ORDER, AND THE GEOMETRY OF FLAG MANIFOLDS

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Abstract. We illuminate the relation between the Bruhat order and structure constants for the polynomial ring in terms of its basis of Schubert polynomials. We use combinatorial, algebraic, and geometric methods, notably a study of intersections of Schubert varieties and maps between flag manifolds. We establish a number of new identities among these structure constants. This leads to formulas for some of these constants and new results on the enumeration of chains in the Bruhat order. A new graded partial order on the symmetric group which contains Young's lattice arises from these investigations. We also derive formulas for certain specializations of Schubert polynomials.

To the memory of Marcel Paul Schützenberger

Introduction

Extending work of Demazure [14] and of Bernstein, Gelfand, and Gelfand [5], in 1982 Lascoux and Schützenberger [33] defined remarkable polynomial representatives for Schubert classes in the cohomology of a flag manifold, which they called Schubert polynomials. For each permutation w in $\mathcal{S}_\infty$, there is a Schubert polynomial $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$. The collection of all Schubert polynomials forms an additive homogeneous basis for this polynomial ring. Thus the identity

$$\mathfrak{S}_u \cdot \mathfrak{S}_v \quad = \quad \sum_w c^w_{u\,v} \mathfrak{S}_w \tag{1}$$

defines integral *structure constants* $c^w_{u\,v}$ for the ring of polynomials with respect to its Schubert basis. Littlewood-Richardson coefficients are a special case of the $c^w_{u\,v}$ as any Schur symmetric polynomial is a Schubert polynomial. The $c^w_{u\,v}$ are positive integers: Evaluating a Schubert polynomial at certain Chern classes gives a Schubert class in the cohomology of the flag manifold. Hence, $c^w_{u\,v}$ enumerates the flags in a suitable

Date: December 4, 1996.

1991 *Mathematics Subject Classification.* 05E15, 14M15, 05E05.

Key words and phrases. Schubert polynomials, Littlewood-Richardson coefficients, Bruhat order, Young's lattice, flag manifold, Grassmannian, Schubert variety.

First author supported in part by an NSERC grant.

Second author supported in part by NSERC grant OGP0170279 and NSF grant DMS-9022140.

triple intersection of Schubert varieties. This evaluation exhibits the cohomology of the flag manifold as the quotient:

$$\mathbb{Z}[x_1, x_2, \ldots]/\langle \mathfrak{S}_w \mid w \notin \mathcal{S}_n \rangle.$$

These constants, $c^w_{u\,v}$, are readily computed: The MAPLE libraries ACE [16] can be used to manipulate Schubert polynomials, Gröbner basis methods are applied in [18], and a new approach, using orbit values of Kostant polynomials, is developed in [6]. However, it remains an open problem to understand these constants combinatorially. By this we mean a combinatorial interpretation or a bijective formula for these constants. We expect such a formula will have the form

$$c^w_{u\,v} \quad = \quad \#\left\{\begin{matrix}\text{(saturated) chains in the Bruhat order on } \mathcal{S}_\infty \text{ from} \\ u \text{ to } w \text{ satisfying } \textit{some} \text{ condition imposed by } v\end{matrix}\right\}. \qquad (2)$$

The Littlewood-Richardson rule [39], which this would generalize, may be expressed in this form (cf. §6.1). A relation between chains in the Bruhat order and the multiplication of Schubert polynomials has previously been noted [27]. A new proof of the classical Pieri's formula for Grassmannians [53] suggests a geometric rationale for such 'chain-theoretic' formulas. Lastly, known formulas for these constants, particularly Monk's formula [42], Pieri-type formulas [33, 52, 57], and other formulas of [52], are all of this form. Recently, Kirillov and Maeno [28] have generalized these chain-theoretic Pieri-type formulas to quantum Schubert polynomials [18]. In §1.1, we give a refinement of (2).

We establish several new identities for the $c^w_{u\,v}$, including Theorems 1.3.1 (*ii*) and 1.3.3 (*ii*). One, Theorem 1.2.1 (*ib*), gives a recursion for $c^w_{u\,v}$ when one of the permutations wu^{-1}, wv^{-1}, or uv^{-1} has a fixed point and a condition on its inversions holds. When $\mathfrak{S}_v$ is a Schur polynomial, we give a chain-theoretic interpretation (Theorem 1.3.2) for some $c^w_{u\,v}$, determine many more (Theorem 6.2.1) in terms of the classical Littlewood-Richardson coefficients, and show how a map that takes certain chains in the Bruhat order to standard Young tableaux and satisfies some additional properties would give a combinatorial interpretation for these $c^w_{u\,v}$ (Theorem 6.3.1).

Most of these identities have an order-theoretic companion which could imply them were a description such as (2) known. The one identity (Theorem 1.3.4) lacking such a companion yields a new result about the enumeration of chains in the Bruhat order (Corollary 1.3.5). In §3, we study a suborder called the k-Bruhat order, which is relevant in (2) when $\mathfrak{S}_v$ is Schur symmetric polynomial in $x_1, \ldots, x_k$. This leads to a new graded partial order on $\mathcal{S}_\infty$ containing every interval in Young's lattice as an induced suborder for which many group homomorphisms are order preserving (Theorem 3.2.3).

Some of these identities require the computation of maps on the cohomology of flag manifolds induced by certain embeddings, including Theorem 4.2.4, Lemma 4.4.1, and Theorem 4.5.4. We use these to determine the effect of some homomorphisms of $\mathbb{Z}[x_1, x_2, \ldots]$ on its Schubert basis. For example, let $P \subset \mathbb{N}$ and list the elements of

P and $\mathbb{N}-P$ in order:

$$P \ :\ p_1 < p_2 < \cdots \qquad\qquad \mathbb{N}-P \ :\ p_1^c < p_2^c < \cdots$$

Define $\Psi_P : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots]$ by:

$$\Psi_P(x_{p_j}) \ = \ y_j \qquad \text{and} \qquad \Psi_P(x_{p_j^c}) \ = \ z_j.$$

Then there exist integers $d_w^{u\,v}$ such that

$$\Psi_P(\mathfrak{S}_w(x)) \ = \ \sum_{u,v} d_w^{u\,v}\mathfrak{S}_u(y)\mathfrak{S}_v(z).$$

We show (Theorem 1.2.2) there exist $\pi \in \mathcal{S}_\infty$ (depending upon w and P) such that $d_w^{u\,v} = c_{\pi\,w}^{(u\times v)\pi}$. In particular, the coefficients $d_w^{u\,v}$ are nonnegative. This generalizes 1.5 of [34], where it is shown that the $d_w^{u\,v}$ are non-negative when $P = \{1, 2, \ldots, n\}$.

Algebraic structures in the cohomology of a flag manifold also yield identities among the $c_{u\,v}^w$ such as $c_{u\,v}^w = c_{v\,u}^w$ (imposed by commutativity) or $c_{u\,v}^w = c_{w_0u\,v}^{w_0w} = c_{\overline{u}\,\overline{v}}^{\overline{w}}$, where $\overline{w} := w_0ww_0$, (imposed by Poincaré duality among the Schubert classes). Such 'algebraic' identities for the classical Littlewood-Richardson coefficients were studied combinatorially in [58, 2, 25, 1, 24]. We expect the identities established here will similarly lead to some beautiful combinatorics, once a combinatorial interpretation for the $c_{u\,v}^w$ is known. These identities impose stringent conditions on the form of any combinatorial interpretation and should be useful in guiding the search for such an interpretation.

Contents

1. Description of results

1.1. **Suborders of the Bruhat order and the $c^w_{u\,v}$.** Suppose the Schubert polynomial $\mathfrak{S}_v$ in (1) is replaced by the Schur polynomial $S_\lambda(x_1, \dots, x_k)$. The resulting identity

$$\mathfrak{S}_u \cdot S_\lambda(x_1, \dots, x_k) \quad = \quad \sum_w c^w_{u\,v(\lambda,k)}\, \mathfrak{S}_w \tag{1.1.1}$$

defines integer constants $c^w_{u\,v(\lambda,k)}$, which we call *Littlewood-Richardson coefficients for Schubert polynomials*, as we show they share many properties with the classical Littlewood-Richardson coefficients. They are related to chains in a suborder of the Bruhat order called the *k-Bruhat order*, $\leq_k$. Its covers coincide with the index of summation in Monk's formula [42]:

$$\mathfrak{S}_u \cdot \mathfrak{S}_{(k,k+1)} \quad = \quad \mathfrak{S}_u \cdot (x_1 + \cdots + x_k) \quad = \quad \sum \mathfrak{S}_{u(a,b)},$$

the sum over all $a \leq k < b$ where $\ell(u(a,b)) = \ell(u)+1$. The set of permutations comparable to the identity in the k-Bruhat order is isomorphic to Young's lattice of partitions with at most k parts. This is the set of Grassmannian permutations with descent k, those permutations whose Schubert polynomials are Schur symmetric polynomials in $x_1, \ldots, x_k$. If f^λ is the number of standard Young tableaux of shape λ, then [41, I.5, Example 2],

$$(x_1 + \cdots + x_k)^m \quad = \quad \sum_\lambda f^\lambda S_\lambda(x_1, \ldots, x_k),$$

the sum over all λ which partition the integer m. Considering the coefficient of $\mathfrak{S}_w$ in the product $\mathfrak{S}_u \cdot (x_1 + \cdots + x_k)^m$ and the definition (1.1.1) of $c^w_{u\,v(\lambda,k)}$, we obtain:

Proposition 1.1.1. *The number of chains in the k-Bruhat order from u to w is*

$$\sum_\lambda f^\lambda c^w_{u\,v(\lambda,k)}.$$

In particular, $c^w_{u\,v(\lambda,k)} = 0$ unless $u \leq_k w$. A chain-theoretic description of the constants $c^w_{u\,v(\lambda,k)}$ should provide a bijective proof of Proposition 1.1.1. By this we mean a function τ from the set of chains in $[u,w]_k$ to the set of standard Young tableaux T whose shape is a partition of $\ell(w) - \ell(u)$ with the further condition that whenever T has shape λ, then $\#\tau^{-1}(T) = c^w_{u\,v(\lambda,k)}$. For the classical Littlewood-Richardson coefficients, Schensted insertion [49] or Schützenberger's *jeu de taquin* [51] provides such a proof. In §6.3 we show (Theorem 6.3.1) that if τ is a function where $\#\tau^{-1}(T)$ depends only upon the shape of T and satisfies a further condition, then $\#\tau^{-1}(T) = c^w_{u\,v(\lambda,k)}$. Such a function τ would be a generalization of Schensted insertion to this setting.

The k-Bruhat order has a more intrinsic formulation, which we establish in §3.1:

Theorem 1.1.2. *Let $u, w \in \mathcal{S}_\infty$. Then $u \leq_k w$ if and only if*

I. *$a \leq k < b$ implies $u(a) \leq w(a)$ and $u(b) \geq w(b)$.*
II. *If $a < b$, $u(a) < u(b)$, and $w(a) > w(b)$, then $a \leq k < b$.*

The k-Bruhat order and its connection to the Littlewood-Richardson coefficients $c^w_{u\,v(\lambda,k)}$ may be generalized, which leads to a refinement of (2). A *parabolic subgroup* P of $\mathcal{S}_\infty$ [11] is a subgroup generated by some adjacent transpositions, $(i, i{+}1)$. Given a parabolic subgroup P of $\mathcal{S}_\infty$, define the *P-Bruhat order* by its covers. A cover $u \lessdot_P w$ in the P-Bruhat order is a cover in the Bruhat order where $u^{-1}w \notin P$. When P is generated by all adjacent transpositions except $(k, k{+}1)$, this is the k-Bruhat order.

Let $I \subset \{1, 2, \ldots, n{-}1\}$ index the adjacent transpositions *not* in P. A *coloured chain* in the P-Bruhat order is a chain together with an element of $I \bigcap \{a, a{+}1, \ldots, b{-}1\}$ for each cover $u \lessdot_P u(a,b)$ in the chain. This notion of colouring the Bruhat order was introduced in [35]. Iterating Monk's rule, we obtain:

$$\left(\sum_{i\in I} \mathfrak{S}_{(i,i+1)}\right)^m \quad = \quad \sum_v f_e^v(P)\,\mathfrak{S}_v, \tag{1.1.2}$$

where $f_e^v(P)$ counts the coloured chains in the P-Bruhat order from e to v. This number, $f_e^v(P)$, is nonzero only for those v which are minimal in their coset vP. More generally, let $f_u^w(P)$ count the coloured chains in the P-Bruhat order from u to w. Multiplying (1.1.2) by $\mathfrak{S}_u$ and equating coefficients of $\mathfrak{S}_w$, gives a generalization of Proposition 1.1.1:

Theorem 1.1.3. *Let $u, w \in \mathcal{S}_\infty$ and P be any parabolic subgroup of $\mathcal{S}_\infty$. Then*

$$f_u^w(P) \quad = \quad \sum_v c^w_{u\,v}\, f_e^v(P).$$

This also shows $c^w_{u\,v} = 0$ unless $u \leq_P w$, whenever v is minimal in vP. Theorem 1.1.3 suggests a refinement of (2): Let $u, v, w \in \mathcal{S}_\infty$, and let P be any parabolic subgroup such that v is minimal in vP. (There always is such a P.) Then, for any coloured chain γ in the P-Bruhat order from e to v, we expect that

$$c^w_{u\,v} \quad = \quad \#\left\{\begin{matrix}\text{coloured chains in the } P\text{-Bruhat order on } \mathcal{S}_\infty \text{ from}\\ u \text{ to } w \text{ which satisfy } \textit{some} \text{ condition imposed by } \gamma\end{matrix}\right\}. \tag{1.1.3}$$

Moreover, this rule should give a bijective proof of Theorem 1.1.3.

This P-Bruhat order may be defined for any parabolic subgroup of any Coxeter group. Likewise, the problem of finding the structure constants for a Schubert basis also generalizes. For Weyl groups, the basis is the Schubert classes in the cohomology of a generalized flag manifold G/B or the analogues of Schubert polynomials in this case [7, 19, 22]. For finite Coxeter groups, the basis is the 'Schubert classes' of Hiller [26] in the coinvariant algebra. Likewise, Theorem 1.1.3 and the expectation (1.1.3) have analogues in this more general setting. Of the known formulas in this setting [12, 9, 43, 55, 45, 46, 47] (see also the survey [44]), few [12, 9, 43, 55] have been expressed in such a chain-theoretic manner.

1.2. **Substitutions and the Schubert basis.**

In §§4.3 and 4.4, we study the $c^w_{u\,v}$ when $w(p) = u(p)$ for some p. This leads to a formula for the substitution of 0 for x_p in terms of the Schubert basis, a recursion for some $c^w_{u\,v}$, and new identities. For $w \in \mathcal{S}_{n+1}$ and $1 \leq p \leq n+1$, let $w/_p \in \mathcal{S}_n$ be defined by deleting the pth row and $w(p)$th column from the permutation matrix of w. If $y \in \mathcal{S}_n$ and $1 \leq q \leq n+1$, then $\varepsilon_{p,q}(y) \in \mathcal{S}_{n+1}$ is the permutation such that $\varepsilon_{p,q}(y)/_p = y$ and $\varepsilon_{p,q}(y)(p) = q$. The index of summation in a particular case of the Pieri-type formula [33, 52, 57],

$$\mathfrak{S}_v \cdot (x_1 \cdots x_{p-1}) \quad = \quad \sum_{v \xrightarrow{c_p} w} \mathfrak{S}_w,$$

defines the relation $v \xrightarrow{c_p} w$, which is described in more detail before Theorem 4.2.4. Define $\Psi_p : \mathbb{Z}[x_1, x_2, \dots] \to \mathbb{Z}[x_1, x_2, \dots]$ by

$$\Psi_p(x_j) \;=\; \begin{cases} x_j & \text{if } j < p \\ 0 & \text{if } j = p \\ x_{j-1} & \text{if } j > p \end{cases} .$$

Theorem 1.2.1. *Let $u, w \in \mathcal{S}_\infty$ and $p \in \mathbb{N}$.*

(*i*) *Suppose $w(p) = u(p)$ and $\ell(w) - \ell(u) = \ell(w/_p) - \ell(u/_p)$. Then*

(*a*) $\varepsilon_{p,u(p)} : [u/_p, w/_p] \xrightarrow{\sim} [u, w]$.

(*b*) *For any $v \in \mathcal{S}_\infty$,*

$$c^w_{u\,v} \;=\; \sum_{\substack{y \,\in\, \mathcal{S}_\infty \\ v \xrightarrow{c_p} \varepsilon_{p,1}(y)}} c^{w/_p}_{u/_p\, y}.$$

(*ii*) *For any $v \in \mathcal{S}_\infty$,*

$$\Psi_p(\mathfrak{S}_v) = \sum_{\substack{y \,\in\, \mathcal{S}_\infty \\ v \xrightarrow{c_p} \varepsilon_{p,1}(y)}} \mathfrak{S}_y.$$

The first statement (Lemma 4.1.1 (*ii*)) is proven using combinatorial arguments, while the second (Theorem 4.4.2) and third (Theorem 4.3.2) are proven by computing certain maps on cohomology. Since $c^w_{u\,v} = c^w_{v\,u} = c^{w_0 u}_{v\, w_0 w}$, Theorem 1.2.1 (*ib*) gives a recursion for $c^w_{u\,v}$ when one of wu^{-1}, wv^{-1}, or uv^{-1} has a fixed point and the condition on lengths is satisfied.

We also compute the effect of other substitutions of the variables in terms of the Schubert basis: Let $P \subset \mathbb{N}$ and list the elements of P and $\mathbb{N} - P$ in order:

$$P \;:\; p_1 < p_2 < \cdots \qquad\qquad \mathbb{N} - P \;:\; p^c_1 < p^c_2 < \cdots$$

Define $\Psi_P : \mathbb{Z}[x_1, x_2, \dots] \to \mathbb{Z}[y_1, y_2, \dots, z_1, z_2, \dots]$ by:

$$\Psi_P(x_{p_j}) \;=\; y_j \qquad \text{and} \qquad \Psi_P(x_{p^c_j}) \;=\; z_j.$$

In Remark 4.6.2, we define an infinite set I_P of permutations with the following property:

Theorem 1.2.2. *For any $w \in \mathcal{S}_\infty$, there exists an integer N such that if $\pi \in I_P$ and $\pi \notin \mathcal{S}_N$, then*

$$\Psi_P(\mathfrak{S}_w) \;=\; \sum_{u,\,v} c^{(u\times v)\cdot\pi}_{\pi\; w}\; \mathfrak{S}_u(y)\; \mathfrak{S}_v(z).$$

A precise version of Theorem 1.2.2 (Theorem 4.6.1) is proven in §4.6. Theorem 1.2.2 gives infinitely many identities of the form $c^{(u\times v)\cdot\pi}_{\pi\; w} = c^{(u\times v)\cdot\sigma}_{\sigma\; w}$ for $\pi, \sigma \in I_P$. Moreover, for these u, v, and π with $c_{\pi\, w}^{(u\times v)\pi} \neq 0$, we have $[\pi,\ (u\times v)\cdot\pi] \simeq [e, u]\times[e,\ v]$, which is suggestive of a chain-theoretic basis for these identities.

Theorem 1.2.2 extends 1.5 of [34], where it is shown that the $d_w^{u\,v}([n])$ are non-negative. A combinatorial proof of the non-negativity of these coefficients $d_w^{u\,v}(P)$ and of Theorem 1.2.1 (*ii*) using, perhaps, one of the combinatorial constructions of Schubert polynomials [30, 3, 8, 20, 21, 56] may provide insight into the problem of determining the $c_{u\,v}^w$.

Theorems 1.2.1 (*ii*) and 1.2.2 enable the computation of rather general substitutions: Let $P_\bullet := (P_0, P_1, \ldots)$ be any (finite or infinite) partition of $\mathbb{N}$. For $i > 0$, let $\underline{x}^{(i)} := x_1^{(i)}, x_2^{(i)}, \ldots$ be a set of variables in bijection with P_i. Define $\Psi_{P_\bullet} : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots]$ by

$$\Psi_{P_\bullet}(x_j) \;=\; \begin{cases} 0 & \text{if } j \in P_0 \\ x_l^{(i)} & \text{if } j \text{ is the } l\text{th element of } P_i \end{cases}.$$

Corollary 1.2.3. *For any partition $P_\bullet$ of $\mathbb{N}$ and $w \in \mathcal{S}_\infty$,*

$$\Psi_{P_\bullet}(\mathfrak{S}_w(x)) \;=\; \sum_{u_1,u_2,\ldots} d_w^{u_1,u_2,\cdots}(P_\bullet)\, \mathfrak{S}_{u_1}(\underline{x}^{(1)})\mathfrak{S}_{u_1}(\underline{x}^{(2)})\cdots,$$

where each $d_w^{u_1,u_2,\cdots}(P_\bullet)$ is a(n explicit) sum of products of the $c_{v\,y}^z$, hence non-negative.

A *ballot sequence* [48, §4.9] $A = (a_1, a_2, \ldots)$ is a sequence of non-negative integers where, for each $i, j \geq 1$,

$$\#\{k \leq j \mid a_k = i\} \quad \geq \quad \#\{k \leq j \mid a_k = i+1\}.$$

(Traditionally, the $a_i > 0$. One should consider $a_i = 0$ as a vote for 'none of the above'.) Given a ballot sequence A, define $\Psi_A : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[x_1, x_2, \ldots]$ by

$$\Psi_A(x_i) \quad = \quad \begin{cases} 0 & a_i = 0 \\ x_{a_i} & a_i \neq 0 \end{cases}.$$

Corollary 1.2.4. *For any ballot sequence A and $w \in \mathcal{S}_n$, there exist non-negative integers $d_w^u(A)$ for $u, w \in \mathcal{S}_\infty$ such that*

$$\Psi_A(\mathfrak{S}_w(x)) \quad = \quad \sum_u d_w^u(A)\, \mathfrak{S}_u(x).$$

Moreover, each $d_w^u(A)$ is a(n explicit) sum of products of the $c_{v\,y}^z$.

Proof. If $P_0 := \{i \mid a_i = 0\}$ and for $j > 0$

$$P_j \;:=\; \{i \mid a_i \text{ is the } j\text{th occurrence of some integer in } A\},$$

then $\Psi_A = \Delta \circ \Psi_{(P_0, P_1, \ldots)}$, where Δ is the diagonal map, $\Delta(x_j^{(i)}) = x_j$. ∎

1.3. **Identities when $\mathfrak{S}_v$ is a Schur polynomial.** If λ, μ, and ν are partitions with at most k parts then the classical Littlewood-Richardson coefficients $c^{\nu}_{\mu\,\lambda}$ are defined by the identity

$$S_\mu(x_1,\ldots,x_k)\cdot S_\lambda(x_1,\ldots,x_k) \quad = \quad \sum_\nu c^{\nu}_{\mu\,\lambda} S_\nu(x_1,\ldots,x_k).$$

The $c^{\nu}_{\mu\,\lambda}$ depend only upon λ and the skew partition ν/μ. That is, if κ and ρ are partitions with at most l parts, and $\kappa/\rho = \nu/\mu$, then for any partition λ,

$$c^{\nu}_{\mu\,\lambda} \quad = \quad c^{\kappa}_{\rho\,\lambda}.$$

Moreover, $c^{\kappa}_{\rho\,\lambda}$ is the coefficient of $S_\kappa(x_1,\ldots,x_l)$ when $S_\rho(x_1,\ldots,x_l)\cdot S_\lambda(x_1,\ldots,x_l)$ is expressed as a sum of Schur polynomials. The order type of the interval in Young's lattice from μ to ν is determined by ν/μ. These facts generalize to the Littlewood-Richardson coefficients $c^{w}_{u\,v(\lambda,k)}$.

If $u \leq_k w$, let $[u,w]_k$ be the interval between u and w in the k-Bruhat order, a graded poset. Permutations ζ and η are *shape equivalent* if there exist sets of integers $P = \{p_1 < \cdots < p_n\}$ and $Q = \{q_1 < \cdots < q_n\}$, where ζ (respectively η) acts as the identity on $\mathbb{N} - P$ (respectively $\mathbb{N} - Q$), and

$$\zeta(p_i) \ = \ p_j \quad \Longleftrightarrow \quad \eta(q_i) \ = \ q_j.$$

Theorem 1.3.1. *Suppose $u \leq_k w$ and $x \leq_l z$ where wu^{-1} is shape equivalent to zx^{-1}. Then*

(*i*) $[u,w]_k \simeq [x,z]_l$. *When $wu^{-1} = zx^{-1}$, this isomorphism is given by $v \mapsto vu^{-1}x$.*
(*ii*) *For any partition λ, $c^{w}_{u\,v(\lambda,k)} = c^{z}_{x\,v(\lambda,l)}$.*

Theorem 1.3.1 (*i*) is a consequence of Theorems 3.1.3 and 3.2.3, which are proven using combinatorial arguments. Theorem 1.3.1 (*ii*) is proven in §5.1 using geometric arguments. By Theorem 1.3.1, we may define the constant c^{ζ}_{λ} for $\zeta \in \mathcal{S}_\infty$ and λ a partition by $c^{\zeta}_{\lambda} := c^{\zeta u}_{u\,v(\lambda,k)}$ and also define $|\zeta| := \ell(\zeta u) - \ell(u)$ for any $u \in \mathcal{S}_\infty$ with $u \leq_k \zeta u$. In §3.2, this analysis leads to a graded partial order $\preceq$ on $\mathcal{S}_\infty$ with rank function $|\zeta|$ which has the defining property: Let $[e,\zeta]_\preceq$ be the interval in the $\preceq$-order from the identity to ζ. If $u \leq_k \zeta u$, then the map $[e,\zeta]_\preceq \to [u,\zeta u]_k$ defined by

$$\eta \quad \longmapsto \quad \eta\, u$$

is an order isomorphism. Any interval in Young's lattice is an induced suborder in $(\mathcal{S}_\infty, \preceq)$, as is the lattice of partitions with at most l parts (these are embedded differently for different l). Proposition 1.1.1 may be stated in term of this order: $\sum_\lambda f^\lambda c^{\zeta}_{\lambda}$ counts the chains in $[e,\zeta]_\preceq$. This order is studied further in [4].

For $\eta \in \mathcal{S}_n$, the map $\eta \mapsto \overline{\eta}$ induces an order isomorphism $[e,\zeta]_\preceq \xrightarrow{\sim} [e,\overline{\zeta}]_\preceq$. The involution $\mathfrak{S}_w \mapsto \mathfrak{S}_{\overline{w}}$ shows $c^{\zeta}_{\lambda} = c^{\overline{\zeta}}_{\lambda^t}$, where λ^t is the conjugate or transpose of λ. Thus $[e,\zeta]_\preceq \simeq [e,\eta]_\preceq$ is not sufficient to guarantee $c^{\zeta}_{\lambda} = c^{\eta}_{\lambda}$. We express some of the Littlewood-Richardson coefficients in terms of chains in the Bruhat order. If $u \lessdot_k u(a,b)$ is a cover in the k-Bruhat order, label that edge of the Hasse diagram with the integer $u(b)$. The *word* of a chain in the k-Bruhat order is the sequence of labels of edges in the chain.

Theorem 1.3.2. *Suppose $u \leq_k w$ and wu^{-1} is shape equivalent to $v(\mu,l)\cdot v(\nu,l)^{-1}$, for some l and partitions μ,ν. Then, for any partition λ and standard Young tableau T of shape λ,*

$$c^w_{u\,v(\lambda,k)} \;=\; \#\left\{\begin{array}{c}\text{Chains in } k\text{-Bruhat order from } u \text{ to } w \text{ whose word}\\ \text{has recording tableau } T \text{ for Schensted insertion}\end{array}\right\}.$$

Theorem 1.3.2 gives a combinatorial proof of Proposition 1.1.1 for some u, w. We prove this in §6.1 using Theorem 1.3.1 and combinatorial arguments.

If a skew partition $\theta = \rho\coprod\sigma$ is the union of incomparable skew partitions ρ and σ, then

$$\rho\coprod\sigma \;\simeq\; \rho\times\sigma,$$

as graded posets. The skew Schur function S_θ is defined [41, I.5] to be $\sum_\lambda c^\theta_\lambda S_\lambda$ and $S_{\rho\coprod\sigma} = S_\rho\cdot S_\sigma$ [41, I.5.7]. Thus

$$c^{\rho\coprod\sigma}_\lambda \quad=\quad \sum_{\mu,\,\nu} c^\lambda_{\mu\,\nu}\, c^\rho_\mu\, c^\sigma_\nu. \tag{1.3.1}$$

Permutations ζ and η are *disjoint* if ζ and η have disjoint supports and $|\zeta\eta| = |\zeta|+|\eta|$.

Theorem 1.3.3. *Let ζ and η be disjoint permutations. Then*

(*i*) *The map $(\xi,\chi)\mapsto\xi\chi$ induces an isomorphism of graded posets*

$$[e,\zeta]_\preceq \times [e,\eta]_\preceq \xrightarrow{\sim} [e,\zeta\eta]_\preceq.$$

(*ii*) *For any partition λ, $c^{\zeta\eta}_\lambda \;=\; \sum_{\mu,\,\nu} c^\lambda_{\mu\,\nu}\, c^\zeta_\mu\, c^\eta_\nu$.*

The first statement is proven in §3.3 (Theorem 3.3.4), using a characterization of disjointness related to non-crossing partitions [31], and the second in §5.2 using geometry.

Our last identity has no analogy with the classical Littlewood-Richardson coefficients. Let $(1\,2\,\ldots\,n)$ be the permutation which cyclicly permutes $[n]$.

Theorem 1.3.4 (Cyclic Shift)**.** *Suppose $\zeta\in S_n$ and $\eta = \zeta^{(1\,2\,\ldots\,n)}$. Then, for any partition λ, $c^\zeta_\lambda = c^\eta_\lambda$.*

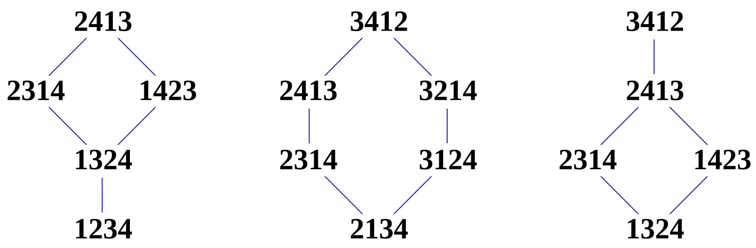
This is proven in §5.3 using geometry. Combined with Proposition 1.1.1, we obtain:

Corollary 1.3.5. *If $u\leq_k w$ and $x\leq_k z$ with $wu^{-1}, zx^{-1}\in\mathcal{S}_n$ and $(wu^{-1})^{(1\,2\,\ldots\,n)} = zx^{-1}$, then the two intervals $[u,w]_k$ and $[x,z]_k$ each have the same number of chains.*

The two intervals $[u,w]_k$ and $[x,z]_k$ of Corollary 1.3.5 are typically non-isomorphic: For example, in $\mathcal{S}_4$ let $u = 1234$, $x = 2134$, and $v = 1324$. If $\zeta = (1243)$, $\eta = (1423) = \zeta^{(1234)}$, and $\xi = (1342) = \eta^{(1234)}$, then

$$u \ \leq_2 \ \zeta u, \quad x \ \leq_2 \ \eta x, \quad \text{and} \quad v \ \leq_2 \ \xi v.$$

Here are the intervals $[u,\zeta u]_2$, $[x,\eta x]_2$, and $[v,\xi v]_2$.



The Theorems of this section, together with the 'algebraic' identities $c^w_{u\,v} = c^{w_0 w}_{w_0 u\,v} = c^{\overline{w}}_{\overline{u}\,\overline{v}}$, greatly reduce the number of distinct Littlewood-Richardson coefficients $c^w_{u\,v(\lambda,k)}$ which need to be determined. We make this precise. For λ a partition, let λ^t be its conjugate, or transpose. Note $\overline{v(\lambda,k)} = v(\lambda^t, n-k)$.

Theorem 1.3.6 (Symmetries of the $c^w_{u\,v(\lambda,k)}$). *Let λ be a partition and $\zeta \in \mathcal{S}_n$. Then*

$$c^{\zeta}_{\lambda} \ = \ c^{\zeta^{-1}}_{\lambda^t} \ = \ c^{\overline{\zeta}}_{\lambda^t} \ = \ c^{\zeta^{(1\,2\,\ldots\,n)}}_{\lambda}.$$

Let D_n be the dihedral group with $2n$ elements. These identities show that the action of $\mathbb{Z}/2\mathbb{Z} \times D_n$ on these coefficients leaves their values invariant.

2. Preliminaries

2.1. **Permutations.** Let $\mathcal{S}_n$ be the group of permutations of $[n] := \{1, 2, \ldots, n\}$. Let (a,b) be the transposition interchanging $a < b$. The *length* $\ell(w)$ of a permutation $w \in \mathcal{S}_n$ counts the *inversions*, $\{i < j \mid w(i) > w(j)\}$, of w. The Bruhat order $\leq$ on $\mathcal{S}_n$ is the partial order whose cover relation is $w \lessdot w(a,b)$ if $w(a) < w(b)$ and whenever $a < c < b$, either $w(c) < w(a)$, or $w(b) < w(c)$. Thus $\ell(w) + 1 = \ell(w(a,b))$, so the Bruhat order is graded by length with minimal element the identity e. If $u \leq w$, let $[u,w] := \{v \mid u \leq v \leq w\}$ be the interval between u and w in $\mathcal{S}_n$, a poset graded by $\ell(v) - \ell(u)$. Let $w_0^{(n)} \in \mathcal{S}_n$ (or simply w_0) be defined by $w_0(j) = n + 1 - j$.

A permutation $w \in \mathcal{S}_n$ acts on $[n{+}1]$, fixing $n+1$. Thus $\mathcal{S}_n \subset \mathcal{S}_{n+1}$. Define $\mathcal{S}_\infty := \bigcup_n \mathcal{S}_n$, the permutations of the positive integers $\mathbb{N}$ fixing all but finitely many integers. For $P = \{p_1 < p_2 < \cdots\} \subset \mathbb{N}$, define $\phi_P : \mathcal{S}_{\#P} \to \mathcal{S}_\infty$ by requiring that ϕ_P act as the identity on $\mathbb{N} - P$ and $\phi_P(\zeta)(p_i) = p_{\zeta(i)}$. This injective homomorphism is a map of posets, but not of graded posets, as ϕ_P typically does not preserve length. If $P = \{n{+}1, n{+}2, \ldots\}$, then ϕ_P does preserve length. For this P, set $1^n \times w := \phi_P(w)$.

If there exist permutations ξ, ζ, and η and sets of positive integers P, Q such that $\phi_P(\xi) = \zeta$ and $\phi_Q(\xi) = \eta$, then ζ and η are *shape equivalent*.

2.2. **Schubert polynomials.** Lascoux and Schützenberger developed the elementary theory of Schubert polynomials in a series of papers [33, 34, 35, 36, 37, 38]. A self-contained exposition of some of this elegant theory is found in [40]. For an interesting historical survey, see [32].

$\mathcal{S}_n$ acts on polynomials in $x_1, \ldots, x_n$ by permuting the variables. For a polynomial f, $f - (i, i{+}1)f$ is antisymmetric in x_i and x_{i+1}, hence divisible by $x_i - x_{i+1}$. Define the *divided difference* operator

$$\partial_i = (x_i - x_{i+1})^{-1}(e - (i, i{+}1)).$$

If $w = (a_1, a_1{+}1) \cdots (a_p, a_p{+}1)$ is a factorization of w into adjacent transpositions with minimal length ($p = \ell(w)$), then $\partial_{a_1} \circ \cdots \circ \partial_{a_p}$ depends only upon w, defining an operator ∂_w for each $w \in \mathcal{S}_n$. For $w \in \mathcal{S}_n$, Lascoux and Schützenberger [33] defined the *Schubert polynomial* $\mathfrak{S}_w$ by

$$\mathfrak{S}_w \; := \; \partial_{w^{-1}w_0}\left(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}\right).$$

The degree of ∂_i is -1, so $\mathfrak{S}_w$ is homogeneous of degree $\binom{n}{2} - \ell(w^{-1}w_0) = \ell(w)$. Since $w_0^{(n)} = (n, n{+}1) \cdots (2,3)(1,2)w_0^{(n+1)}$ and $x_1^{n-1} \cdots x_{n-1} = \partial_n \circ \cdots \circ \partial_1(x_1^n \cdots x_{n-1}^2 x_n)$, $\mathfrak{S}_w$ is independent of n (when $w \in \mathcal{S}_n$). This defines polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$ for $w \in \mathcal{S}_\infty$.

A *partition* λ is a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ of integers. Each λ_j is a *part* of λ. For partitions λ and μ, write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all i. *Young's lattice* is the set of partitions ordered by $\subset$. The partition with n parts each equal to m is written m^n. For a partition λ with $\lambda_{k+1} = 0$, the Schur polynomial $S_\lambda(x_1, \ldots, x_k)$ is

$$S_\lambda(x_1, \ldots, x_k) \; := \; \frac{\det \left| x_j^{k-i+\lambda_i} \right|_{i,j=1}^{k}}{\det \left| x_j^{k-i} \right|_{i,j=1}^{k}}.$$

$S_\lambda(x_1, \ldots, x_k)$ is symmetric in $x_1, \ldots, x_k$ and homogeneous of degree $|\lambda| := \lambda_1 + \cdots + \lambda_k$.

A permutation w is *Grassmannian of descent* k if $j \neq k \Rightarrow w(j) < w(j+1)$. A Grassmannian permutation w with descent k defines, and is defined by, a partition λ with $\lambda_{k+1} = 0$:

$$\lambda_{k+1-j} \; = \; w(j) - j \qquad j = 1, \ldots, k.$$

(The condition $w(k{+}1) < w(k{+}2) < \cdots$ determines the remaining values of w.) In this case, write $w = v(\lambda, k)$. The *raison d'etre* for this definition is that $\mathfrak{S}_{v(\lambda,k)} = S_\lambda(x_1, \ldots, x_k)$. Thus the Schubert polynomials form a basis for $\mathbb{Z}[x_1, x_2, \ldots]$ which contains all Schur symmetric polynomials $S_\lambda(x_1, \ldots, x_k)$ for all λ and k.

2.3. **The flag manifold.** Let $V \simeq \mathbb{C}^n$. A *flag* $F_\bullet$ in V is a sequence

$$\{0\} \;=\; F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n \;=\; V,$$

of subspaces with $\dim_{\mathbb{C}} F_i = i$. Flags $F_\bullet$ and $F_\bullet'$ are *opposite* if $F_{n-j} \cap F_j' = \{0\}$ for all j. The set of all flags is an $\binom{n}{2}$-dimensional complex manifold, $\mathbb{F}\ell V$ (or $\mathbb{F}\ell_n$), called the *flag manifold*. There is a tautological flag $mathcalF_\bullet$ of bundles over $\mathbb{F}\ell V$ whose fibre at $F_\bullet$ is $F_\bullet$. Let $-x_i$ be the first Chern class of the line bundle $\mathcal{F}_i/\mathcal{F}_{i-1}$. Borel [10] showed the cohomology ring of $\mathbb{F}\ell V$ to be

$$\mathbb{Z}[x_1, \ldots, x_n]/\langle e_i(x_1, \ldots, x_n) \mid i = 1, \ldots, n\rangle,$$

where $e_i(x_1, \ldots, x_n)$ is the ith elementary symmetric polynomial in $x_1, \ldots, x_n$.

Given a subset $S \subset V$, let $\langle S\rangle$ be its linear span. For subspaces $W \subset U$, let $U - W$ be their set-theoretic difference. An ordered basis $f_1, f_2, \ldots, f_n$ for V determines a flag $E_\bullet := \langle\langle f_1, \ldots, f_n\rangle\rangle$, where $E_i = \langle f_1, \ldots, f_i\rangle$ for $1 \leq i \leq n$. A fixed flag $F_\bullet$ gives a decomposition due to Ehresmann [17] of $\mathbb{F}\ell V$ into affine cells indexed by permutations w of S_n. The cell determined by w has two equivalent descriptions:

$$X_w^\circ F_\bullet \quad := \quad \left\{ \begin{array}{l} \{E_\bullet \in \mathbb{F}\ell V \mid \dim E_i \bigcap F_j = \#\{p \leq i \mid w(p) > n - j\}\}, \\ \{E_\bullet = \langle\langle f_1, \ldots, f_n\rangle\rangle \mid f_i \in F_{n+1-w(i)} - F_{n-w(i)},\ 1 \leq i \leq n\}. \end{array} \right.$$

Its closure is the Schubert subvariety $X_w F_\bullet$, which has complex codimension $\ell(w)$. Also, $u \leq w \Leftrightarrow X_u F_\bullet \supset X_w F_\bullet$. The *Schubert class* $[X_w F_\bullet]$ is the cohomology class Poincaré dual to the fundamental cycle of $X_w F_\bullet$. These Schubert classes form a basis for the cohomology. Schubert polynomials were defined so that $\mathfrak{S}_w(x_1, \ldots, x_n) = [X_w F_\bullet]$. We write $\mathfrak{S}_w$ for $[X_w F_\bullet]$.

If $F_\bullet$ and $F_\bullet'$ are opposite flags, then $X_u F_\bullet \bigcap X_v F_\bullet'$ is an irreducible, generically transverse intersection, a consequence of [15] (cf. [52, §5]). Thus its codimension is $\ell(u) + \ell(v)$, and the fundamental cycle of $X_u F_\bullet \bigcap X_v F_\bullet'$ is Poincaré dual to $\mathfrak{S}_u \cdot \mathfrak{S}_v$. Since

$$\mathbb{Z}[x_1, \ldots, x_n] \;\longrightarrow\; \mathbb{Z}[x_1, \ldots, x_{n+m}]/\langle e_i(x_1, \ldots, x_{n+m}) \mid i = 1, \ldots, n+m\rangle,$$

is an isomorphism on $\mathbb{Z}\langle x_1^{a_1} \cdots x_n^{a_n} \mid a_i < m\rangle$, identities of Schubert polynomials follow from product formulas for Schubert classes. The Schubert basis is self-dual for the intersection pairing: If $\ell(w) + \ell(v) = \binom{n}{2}$, then

$$\mathfrak{S}_w \cdot \mathfrak{S}_v \quad = \quad \left\{ \begin{array}{ll} \mathfrak{S}_{w_0} & v = w_0 w \\ 0 & \text{otherwise} \end{array} \right. .$$

Let $Grass_k V$ be the Grassmannian of k-dimensional subspaces of V, a $k(n-k)$-dimensional manifold. A flag $F_\bullet$ induces a cellular decomposition indexed by partitions $\lambda \subset (n-k)^k$. The closure of the cell indexed by λ is the Schubert variety $\Omega_\lambda F_\bullet$:

$$\Omega_\lambda F_\bullet \quad := \quad \{H \in Grass_k V \mid \dim H \bigcap F_{n+j-k-\lambda_j} \geq j,\ j = 1, \ldots, k\}.$$

The cohomology class Poincaré dual to the fundamental cycle of $\Omega_\lambda F_\bullet$ is $S_\lambda(x_1,\ldots,x_k)$, where $x_1,\ldots,x_k$ are negative Chern roots of the tautological k-plane bundle on $Grass_k V$. Write S_λ for $S_\lambda(x_1,\ldots,x_k)$, if k is understood. These *Schubert classes* form a basis for cohomology, $\mu\subset\lambda \Leftrightarrow \Omega_\mu F_\bullet \supset \Omega_\lambda F_\bullet$, and if $F_\bullet, F_\bullet'$ are opposite flags, then

$$[\Omega_\mu F_\bullet \bigcap \Omega_\nu F_\bullet'] \;=\; [\Omega_\mu F_\bullet]\;\cdot\;[\Omega_\nu F_\bullet'] \;=\; \sum_{\lambda\subset(n-k)^k} c^\lambda_{\mu\,\nu}\, S_\lambda,$$

where the $c^\lambda_{\mu\,\nu}$ are the Littlewood-Richardson coefficients [24].

This Schubert basis is self-dual: If $\lambda\subset(n-k)^k$, then let λ^c, the *complement* of λ, be the partition $(n-k-\lambda_k,\ldots,n-k-\lambda_1)$. Suppose $|\lambda|+|\mu|=k(n-k)$, then

$$S_\lambda(x_1,\ldots,x_k)\cdot S_\mu(x_1,\ldots,x_k) \quad = \quad \begin{cases} S_{(n-k)^k} & \text{if } \mu=\lambda^c \\ 0 & \text{otherwise}\end{cases}.$$

We suppress the dependence of λ^c on n and k, which may be determined by context.

A map $f: X\to Y$ between manifolds induces a homomorphism $f_*: H^*X\to H^*Y$ of abelian groups via the functorial map on homology and the Poincaré duality isomorphism between homology and cohomology. While f_* is not a map of graded rings, it does satisfy the projection formula (cf. [23, 8.1.7]): Let $\alpha\in H^*X$ and $\beta\in H^*Y$, then

$$f_*(f^*\alpha\cap\beta) \quad = \quad \alpha\cap f_*\beta. \tag{2.3.1}$$

For a(n oriented) manifold X of dimension d, $H^dX=\mathbb{Z}\cdot[\mathrm{pt}]$ is generated by the class of a point. Let $\deg: H^*X\to\mathbb{Z}$ be the map which selects the coefficient of [pt] Then $\deg(f_*\beta)=\deg(\beta)$.

Let $\pi_k:\mathbb{F}\ell V \twoheadrightarrow Grass_k V$ be defined by $\pi_k(E_\bullet)=E_k$. Then $\pi_k^{-1}\Omega_\lambda F_\bullet = X_{v(\lambda,k)}F_\bullet$ and $\pi_k: X_{w_0 v(\lambda^c,k)}F_\bullet \twoheadrightarrow \Omega_\lambda F_\bullet$ is generically one-to-one. Thus on cohomology,

$$\begin{aligned}\pi^* S_\lambda \quad &= \quad \mathfrak{S}_{v(\lambda,k)}\\ (\pi_k)_*\mathfrak{S}_w \quad &= \quad \begin{cases} S_\lambda & \text{if } w=w_0v(\lambda^c,k)\\ 0 & \text{otherwise}\end{cases}.\end{aligned}$$

By the Künneth formula, the cohomology of $\mathbb{F}\ell V\times\mathbb{F}\ell W$ ($\dim W=m$) has an integral basis of classes $\mathfrak{S}_u\otimes\mathfrak{S}_x$ for $u\in\mathcal{S}_n$ and $x\in\mathcal{S}_m$. Likewise the cohomology of $Grass_k V\times Grass_l W$ has a basis $S_\lambda\otimes S_\mu$ for $\lambda\subset(n-k)^k$ and $\mu\subset(m-l)^l$.

While we use the cohomology rings of complex varieties, our results and methods are valid for the Chow rings [23] and l-adic (étale) cohomology [13] of these same varieties over any field.

3. Orders on $\mathcal{S}_\infty$

3.1. **The k-Bruhat order.** The k-Bruhat order, $\leq_k$, is a suborder of the Bruhat order on $\mathcal{S}_\infty$ which is linked to the Littlewood-Richardson coefficients $c^w_{u\,v(\lambda,k)}$. It appeared in [35], where it was called the k-coloured Ehresmanoëdre. Its covers are given by the index of summation in Monk's formula [42]:

$$\mathfrak{S}_u \cdot (x_1 + \cdots + x_k) \quad = \quad \sum_{\substack{u \leq_k w \\ \ell(w)=\ell(u)+1}} \mathfrak{S}_w$$

Thus w covers u in the k-Bruhat order if w covers u in the Bruhat order, so that $w = u(a,b)$ and $\ell(w) = \ell(u) + 1$, with the additional requirement that $a \leq k < b$. The k-Bruhat order has the following non-recursive characterization.

Theorem 1.1.2. *Let $u, w \in \mathcal{S}_\infty$. Then $u \leq_k w$ if and only if*

I. *$a \leq k < b$ implies $u(a) \leq w(a)$ and $u(b) \geq w(b)$.*
II. *If $a < b$, $u(a) < u(b)$, and $w(a) > w(b)$, then $a \leq k < b$.*

Proof. The idea is to show that the transitive relation $u \trianglelefteq_k w$ defined by the conditions I and II coincides with the k-Bruhat order. If $u \lessdot_k u(a,b)$ is a cover, then $u \trianglelefteq_k u(a,b)$. Thus $u \leq_k w$ implies $u \trianglelefteq_k w$. Algorithm 3.1.1, which, given $u \trianglelefteq_k w$, produces a chain in the k-Bruhat order from u to w, completes the proof. ▛

Algorithm 3.1.1 (Produces a chain in the k-Bruhat order).
`input:` *Permutations $u, w \in \mathcal{S}_\infty$ with $u \trianglelefteq_k w$.*
`output:` *A chain in the k-Bruhat order from w to u.*
Output w. While $u \neq w$, do
1 *Choose $a \leq k$ with $u(a)$ minimal subject to $u(a) < w(a)$.*
2 *Choose $k < b$ with $u(b)$ maximal subject to $w(b) < w(a) \leq u(b)$.*
3 *$w := w(a,b)$, output w.*

At every iteration of 1, *$u \trianglelefteq_k w$. Moreover, this algorithm terminates in $\ell(w) - \ell(u)$ iterations and the sequence of permutations produced is a chain in the k-Bruhat order from w to u.*

Proof. It suffices to consider a single iteration. We first show it is possible to choose a and b, then $u \trianglelefteq_k w(a,b)$, and lastly $w(a,b) \lessdot_k w$ is a cover in the k-Bruhat order.

In 1, $u \neq w$, so one may always choose a. Suppose $u \trianglelefteq_k w \in \mathcal{S}_n$ and it is not possible to choose b. In that case, if $j > k$ and $w(j) < w(a)$, then also $u(j) < w(a)$. Similarly, if $j \leq k$ and $w(j) < w(a)$, then $u(j) \leq w(j) < w(a)$. Thus $\alpha < w(a) \Leftrightarrow uw^{-1}(\alpha) < w(a)$, which contradicts $uw^{-1}(w(a)) = u(a) < w(a)$.

Let $w' := w(a,b)$. Note that $w(b) \geq u(a)$ implies Condition I for (u, w'). Suppose $w(b) < u(a)$. Set $b_1 := u^{-1}w(b)$. Then $w(b_1) \neq u(b_1)$ and the minimality of $u(a)$ shows that $b_1 > k$ and $w(b_1) < u(b_1)$. Similarly, if $b_2 := u^{-1}w(b_1)$, then $b_2 > k$ and $w(b_2) < u(b_2)$. Continuing, we obtain a sequence $b_1, b_2, \ldots$ with $u(a) > u(b_1) > u(b_2) > \cdots$, a contradiction.

We show (u, w') satisfies Condition II. Suppose $i < j$ and $u(i) < u(j)$. If $j \leq k$, then $w(i) < w(j)$. To show $w'(i) < w'(j)$, it suffices to consider the case $j = a$. But then $u(i) < u(a)$, and thus $u(i) = w(i) = w'(i)$, by the minimality of $u(a)$. Then $w'(i) < u(a) \leq w(b) = w'(a)$. Similarly, if $k < i$, then $w'(i) < w'(j)$.

Finally, suppose w does not cover w' in the k-Bruhat order. Since $w(a) > w(b)$, there exists a c with $a < c < b$ and $w(a) > w(c) > w(b)$. If $k < c$, then Condition II implies $u(c) > u(b)$ and then the maximality of $u(b)$ implies $w(a) < w(c)$, a contradiction. The case $c \leq k$ similarly leads to a contradiction. ◾

Remark 3.1.2. Algorithm 3.1.1 depends only upon $\zeta = wu^{-1}$.
`input:` *A permutation $\zeta \in \mathcal{S}_\infty$.*
`output:` *Permutations $\zeta, \zeta_1, \ldots, \zeta_m = e$ such that if $u \leq_k \zeta u$, then*

$$u \lessdot_k \zeta_{m-1}u \lessdot_k \cdots \lessdot_k \zeta_1 u \lessdot_k \zeta u$$

is a saturated chain in the k-Bruhat order.

Output ζ. While $\zeta \neq e$, do

1 *Choose α minimal subject to $\alpha < \zeta(\alpha)$.*
2 *Choose β maximal subject to $\zeta(\beta) < \zeta(\alpha) \leq \beta$.*
3 *$\zeta := \zeta(\alpha, \beta)$, output ζ.*

To see this is equivalent to Algorithm 3.1.1, set $\alpha = u(a)$ and $\beta = u(b)$ so that $w(a) = \zeta(\alpha)$ and $w(b) = \zeta(\beta)$. Thus $w(a, b) = \zeta u(a, b) = \zeta(\alpha, \beta)u$.

More is true, the full interval $[u, w]_k$ depends only upon wu^{-1}:

Theorem 3.1.3. *If $u \leq_k w$ and $x \leq_k y$ with $wu^{-1} = zx^{-1}$, then the map $v \mapsto vu^{-1}x$ induces an isomorphism of graded posets $[u, w]_k \xrightarrow{\sim} [x, z]_k$.*

This is a consequence of the following lemma.

Lemma 3.1.4. *Let $u \leq_k w$ and $x \leq_k z$ with $wu^{-1} = zx^{-1}$. If $u \lessdot_k (\alpha, \beta)u$ is a cover with $(\alpha, \beta)u \leq_k w$, then $x \lessdot_k (\alpha, \beta)x$ is a cover with $(\alpha, \beta)x \leq_k z$*

Proof. Let $\zeta = wu^{-1} = zx^{-1}$. By the position of γ in u, we mean $u^{-1}(\gamma)$.

Suppose $(\alpha, \beta)x$ does not cover x in the k-Bruhat order, so there is a γ with $\alpha < \gamma < \beta$ and $x^{-1}(\alpha) < x^{-1}(\gamma) < x^{-1}(\beta)$. Then, in one line notation, x and z are as illustrated:

$$\begin{array}{ll} x: & \ldots\ \alpha\ \ldots\ \gamma\ \ldots\ \beta\ \ldots \\ z: & \ldots\zeta(\alpha)\ldots\zeta(\gamma)\ldots\zeta(\beta)\ldots \end{array}$$

Since $u \lessdot_k (\alpha,\beta)u$ is a cover in the k-Bruhat order, either $k < u^{-1}(\beta) < u^{-1}(\gamma)$ or else $u^{-1}(\gamma) < u^{-1}(\alpha) \leq k$. We illustrate u, $(\alpha,\beta)u$, and w for each possibility:

$$\begin{array}{rcc}
 & k < u^{-1}(\beta) < u^{-1}(\gamma) & u^{-1}(\gamma) < u^{-1}(\alpha) \leq k \\
u: & \ldots\ \alpha\ \ldots\ \beta\ \ldots\ \gamma\ \ldots & \ldots\ \gamma\ \ldots\ \alpha\ \ldots\ \beta\ \ldots \\
(\alpha,\beta)u: & \ldots\ \beta\ \ldots\ \alpha\ \ldots\ \gamma\ \ldots & \ldots\ \gamma\ \ldots\ \beta\ \ldots\ \alpha\ \ldots \\
w: & \ldots\zeta(\alpha)\ldots\zeta(\beta)\ldots\zeta(\gamma)\ldots & \ldots\zeta(\gamma)\ldots\zeta(\alpha)\ldots\zeta(\beta)\ldots
\end{array}$$

Assume $k < u^{-1}(\beta) < u^{-1}(\gamma)$. Then Theorem 1.1.2 and $(\alpha,\beta)u \leq_k w$ imply $\gamma \geq \zeta(\gamma)$ and $\zeta(\beta) < \zeta(\gamma)$, since $\alpha < \gamma$ and both have positions greater than k in $(\alpha\,\beta)u$. Let $c := x^{-1}(\gamma)$. If $c \leq k$, then $x \leq_k z$ implies $\gamma \leq \zeta(\gamma)$ so $\gamma = \zeta(\gamma)$. Also, $\alpha < \gamma$ implies $\zeta(\alpha) < \zeta(\gamma)$ and thus $\zeta(\gamma) = \gamma < \beta \leq \zeta(\alpha)$, a contradiction. Similarly, if $c > k$, then $\gamma < \beta$ implies $\zeta(\gamma) < \zeta(\beta)$, another contradiction. The other possibility, $u^{-1}(\gamma) < u^{-1}(\alpha)$, leads to a similar contradiction. Thus $x \lessdot_k (\alpha,\beta)x$ is a cover in the k-Bruhat order.

To show $y := (\alpha,\beta)x \leq_k z$, first note that the pair (y,z) satisfy condition I of Theorem 1.1.2, because $(\alpha,\beta)u \leq_k w$. For condition II, we need only show:

a) If $\alpha < \gamma < \beta$ and $x^{-1}(\gamma) < x^{-1}(\alpha)$, so that $\gamma = yx^{-1}(\gamma) < yx^{-1}(\alpha) = \beta$, then $zx^{-1}(\gamma) = \zeta(\gamma) < \zeta(\beta) = zx^{-1}(\alpha)$, and

b) If $\alpha < \gamma < \beta$ and $x^{-1}(\beta) < x^{-1}(\gamma)$, so that $\alpha = yx^{-1}(\beta) < yx^{-1}(\gamma) = \gamma$, then $\zeta(\alpha) < \zeta(\gamma)$.

If $\alpha < \gamma < \beta$, then one of these two possibilities does occur, as $x \lessdot_k (\alpha,\beta)x = y$ is a cover in the k-Bruhat order. Suppose $x^{-1}(\gamma) < x^{-1}(\alpha)$, as the other case is similar.

Since $x^{-1}(\gamma) < k$ and $x \leq_k z$, we have $\gamma \leq \zeta(\gamma)$, by condition I. If $u^{-1}(\gamma) < u^{-1}(\alpha)$, then $(\alpha\,\beta)u \leq_k w \Rightarrow \zeta(\gamma) < \zeta(\alpha)$. If $u^{-1}(\beta) < u^{-1}(\gamma)$, then $\gamma = \zeta(\gamma)$, and so $\zeta(\gamma) = \gamma < \beta \leq \zeta(\alpha)$. Since $u \leq_k (\alpha\,\beta)u$, we cannot have $u^{-1}(\alpha) < u^{-1}(\gamma) < u^{-1}(\beta)$. ▛

Define $\mathrm{up}_\zeta := \{\alpha \mid \alpha < \zeta(\alpha)\}$ and $\mathrm{down}_\zeta := \{\beta \mid \beta > \zeta(\beta)\}$.

Theorem 3.1.5. *Let $\zeta \in \mathcal{S}_\infty$.*

(i) *For any $u \in \mathcal{S}_\infty$, $u \leq_k \zeta u$ if and only if the following conditions are satisfied.*

(a) $u^{-1}\mathrm{up}_\zeta \subset \{1,\ldots,\mathrm{k}\}$,

(b) $u^{-1}\mathrm{down}_\zeta \subset \{\mathrm{k}+1,\mathrm{k}+2,\ldots\}$, *and*

(c) *For all $\alpha,\beta \in \mathrm{up}_\zeta$ (respectively $\alpha,\beta \in \mathrm{down}_\zeta$), $\alpha < \beta$ and $u^{-1}(\alpha) < u^{-1}(\beta)$ together imply $\zeta(\alpha) > \zeta(\beta)$.*

(ii) *If $\#\mathrm{up}_\zeta \leq \mathrm{k}$, then there is a Grassmannian permutation u such that $u \leq_k \zeta u$.*

Proof. Statement (i) is a consequence of Theorem 1.1.2. For (ii), let $\{a_1,\ldots,a_k\} \subset \mathbb{N}$ contain up_ζ and possibly some fixed points of ζ, and let $\{a_{k+1},a_{k+2},\ldots\}$ be the complementary set in $\mathbb{N}$. Suppose these sets are indexed so that $\zeta(a_i) < \zeta(a_{i+1})$ for $i \neq k$. Define $u \in \mathcal{S}_\infty$ by $u(i) = a_i$ Then ζu is Grassmannian with descent k, and Theorem 1.1.2 implies $u \leq \zeta u$. ▛

3.2. **A new partial order on $\mathcal{S}_\infty$.** For $\zeta \in \mathcal{S}_\infty$, define $|\zeta|$ to be the quantity:

$$\#\{(\alpha,\beta)\in\zeta(\mathrm{up}_\zeta)\times\zeta(\mathrm{down}_\zeta)\,|\,\alpha>\beta\}\;-\;\#\{\mathrm{a},\mathrm{b}\in\mathrm{up}_\zeta\,|\,\mathrm{a}>\mathrm{b}\text{ and }\zeta(\mathrm{a})<\zeta(\mathrm{b})\}$$
$$-\;\#\{a,b\in\mathrm{down}\zeta\,|\,\mathrm{a}>\mathrm{b}\text{ and }\zeta(\mathrm{a})<\zeta(\mathrm{b})\}\;-\;\#\{(\mathrm{a},\mathrm{b})\in\mathrm{up}_\zeta\times\mathrm{down}_\zeta\,|\,\mathrm{a}>\mathrm{b}\}.$$

Lemma 3.2.1. *If $u \leq_k \zeta u$, then $\ell(u) + |\zeta| = \ell(\zeta u)$.*

Proof. By Theorem 3.1.3, $\ell(\zeta u) - \ell(u)$ depends only upon ζ. Computing this for the permutation u defined in the proof of Theorem 3.1.5, shows it equals $|\zeta|$: If $c = \zeta(c)$, then the number of inversions involving c in u equals the number involving c in ζu. The first term of the expression for $|\zeta|$ counts the remaining inversions in ζu and the last three terms the remaining inversions in u. ∎

By Theorem 3.1.3, the interval $[u, \zeta u]_k$ depends only upon ζ if $u \leq_k \zeta u$. A closer examination of our arguments shows it is independent of k, as well. That is, if $x \leq_l \zeta x$, then the map $v \mapsto xu^{-1}v$ defines an isomorphism $[u,\, \zeta u]_k \xrightarrow{\sim} [x,\, \zeta x]_l$. This motivates the following definition.

Definition 3.2.2. For $\zeta, \eta \in \mathcal{S}_\infty$, let $\eta \preceq \zeta$ if there exists $u \in \mathcal{S}_\infty$ and a positive integer k such that $u \leq_k \eta u \leq_k \zeta u$. If u is chosen as in the proof of Theorem 3.1.5, then we see that $\eta \preceq \zeta$ if

1. if $\alpha < \eta(\alpha)$, then $\eta(\alpha) \leq \zeta(\alpha)$,
2. if $\alpha > \eta(\alpha)$, then $\eta(\alpha) \geq \zeta(\alpha)$, and
3. if $\alpha, \beta \in \mathrm{up}_\zeta$ (respectively, $\alpha, \beta \in \mathrm{down}_\zeta$) with $\alpha < \beta$ and $\zeta(\alpha) < \zeta(\beta)$, then $\eta(\alpha) < \eta(\beta)$.

We illustrate $\preceq$ on $\mathcal{S}_4$:

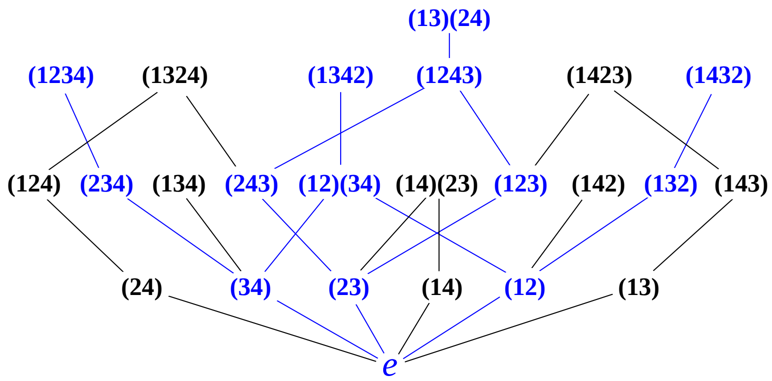


FIGURE 1. $\preceq$ on $\mathcal{S}_4$

Theorem 3.2.3. *Suppose $u, \zeta, \eta, \xi \in \mathcal{S}_\infty$.*

(i) $(\mathcal{S}_\infty, \preceq)$ is a graded poset with rank function $|\zeta|$.

(*ii*) *The map $\lambda \mapsto v(\lambda, k)$ exhibits Young's lattice of partitions with at most k parts as an induced suborder of $(\mathcal{S}_\infty, \preceq)$.*

(*iii*) *If $u \leq_k \zeta u$, then the map $\eta \mapsto \eta u$ induces an isomorphism $[e,\, \zeta]_\preceq \xrightarrow{\sim} [u,\, \zeta u]_k$.*

(*iv*) *If $\eta \preceq \zeta$, then the map $\xi \mapsto \xi\eta^{-1}$ induces an isomorphism $[\eta, \zeta]_\preceq \xrightarrow{\sim} [e, \zeta\eta^{-1}]_\preceq$.*

(*v*) *For any infinite set $P \subset \mathbb{N}$, $\phi_P : \mathcal{S}_\infty \to \mathcal{S}_\infty$ is an injection of graded posets. Thus, if $\zeta, \eta \in \mathcal{S}_\infty$ are shape equivalent, then $[e,\, \zeta]_\preceq \simeq [e,\, \eta]_\preceq$.*

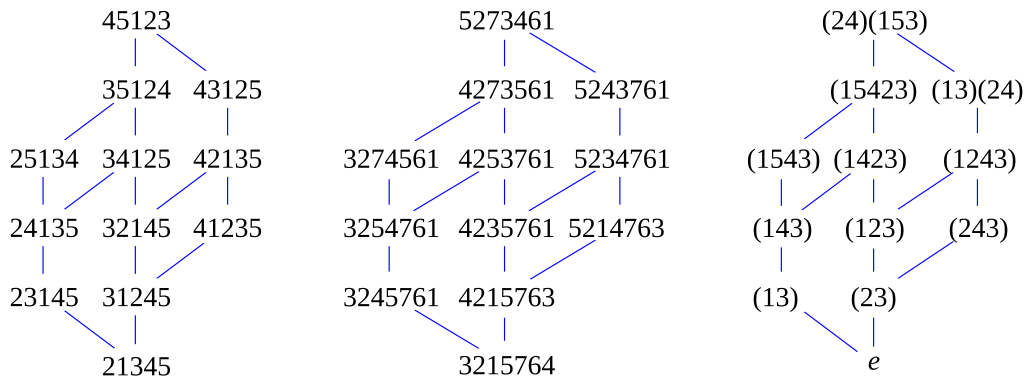
(*vi*) *The map $\eta \mapsto \eta\zeta^{-1}$ induces an order reversing isomorphism between $[e, \zeta]_\preceq$ and $[e, \zeta^{-1}]_\preceq$.*

(*vii*) *The homomorphism $\zeta \mapsto \overline{\zeta}$ on $\mathcal{S}_n$ induces an automorphism of $(\mathcal{S}_n, \preceq)$.*

Theorem 1.3.1 (i) is an immediate consequence of the definition of $\preceq$ and (v).

Proof. Statements (i)–(v) follow from the definitions. Suppose $u \leq_k \eta u \leq_k \zeta u$ with $u, \eta u, \zeta u \in \mathcal{S}_n$. If $w := \zeta u$, then $ww_0 \leq_{n-k} \eta\zeta^{-1}ww_0 \leq_{n-k} \zeta^{-1}ww_0$, which proves (vi). Similarly, $u \leq_k w \Leftrightarrow \overline{u} \leq_{n-k} \overline{w}$ implies (vii). ∎

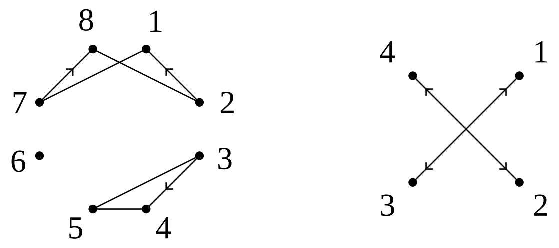
Example 3.2.4. Let $\zeta = (24)(153)$ and $\eta = (35)(174) = \phi_{\{1,3,4,5,7\}}(\zeta)$. Then $21345 \leq_2 45123 = \zeta \cdot 21345$ and $3215764 \leq_3 5273461 = \eta \cdot 3215764$. Below are the intervals $[21342,\, 45123]_2$, $[3215764,\, 5273461]_3$, and $[e, \zeta]_\preceq$.



3.3. **Disjoint permutations.** Let $\zeta \in \mathcal{S}_n$ and $1, \ldots, n$ be the vertices of a convex planar n-gon numbered consecutively. Define Γ_ζ, a directed geometric graph to be the union of directed chords $\langle \alpha, \zeta(\alpha) \rangle$ for α in the support, supp_ζ, of ζ.

Permutations ζ and η are *disjoint* if the edge sets of their geometric graphs Γ_ζ and Γ_η (drawn on the same n-gon) are disjoint as *subsets of the plane*. This implies (but is not equivalent to) $\mathrm{supp}_\zeta \bigcap \mathrm{supp}_\eta = \emptyset$. Disjointness may be rephrased in terms of partitions [54] of $[n]$. Suppose for simplicity, that supp_ζ and supp_η partition $[n]$. Then ζ and η are disjoint if and only if there is a non-crossing partition [31] π of $[n]$ refining the partition $(\mathrm{supp}_\zeta,\, \mathrm{supp}_\eta)$, and which is refined by the partition given by the cycles of ζ and η.

We compare the graphs of the pairs of permutations $(1782), (345)$ and $(13), (24)$.

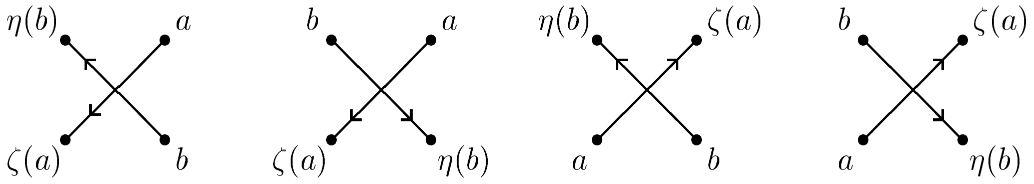


The first pair is disjoint and the second is not. We relate this definition to that given in §1.3.

Lemma 3.3.1. *Let $\zeta, \eta \in \mathcal{S}_\infty$. Then the edges of Γ_ζ are disjoint from the edges of Γ_η if and only if ζ and η have disjoint support and $|\zeta| + |\eta| = |\zeta\eta|$.*

Proof. Suppose ζ and η have disjoint support and let $\langle a, \zeta(a)\rangle$ be an edge of Γ_ζ and $\langle b, \eta(b)\rangle$ be an edge of Γ_η. Consider the contribution of the endpoints of these edges to $|\zeta\eta| - |\zeta| - |\eta|$. This contribution is zero if the edges do not cross, which proves the forward implication.

For the reverse, suppose these edges cross. Then the contribution is 1 if $a < \zeta(a)$ and $b > \eta(b)$ (or vice-versa), and 0 otherwise. Because each edge is part of a directed cycle in the graph, if one edge of Γ_ζ crosses an edge of Γ_η, then there are at least four crossings, one of each type illustrated.



Here, the numbers increase in a clockwise direction, with the least number in the northeast ($\nearrow$). Thus $|\zeta\eta| > |\zeta| + |\eta|$. ∎

Lemma 3.3.2. *Let $\alpha < \beta$ and $\zeta \in \mathcal{S}_\infty$ and suppose $\zeta \lessdot (\alpha, \beta)\zeta$ is a cover. Then*

(i) α and β are connected in the geometric graph $\Gamma_{(\alpha,\beta)\zeta}$.
(ii) If $\langle c, d\rangle$ is any chord of the n-gon meeting Γ_ζ, then $\langle c, d\rangle$ meets $\Gamma_{(\alpha,\beta)\zeta}$.
(iii) If p and q are connnected in Γ_ζ, then they are connected in $\Gamma_{(\alpha,\beta)\zeta}$.
(iv) If ζ and η are disjoint permutations and $\zeta' \preceq \zeta$, then ζ' and η are disjoint.

Proof. Suppose $u \in \mathcal{S}_\infty$ with $u \leq_k \zeta u \lessdot_k (\alpha, \beta)\zeta u$. Define i and j by $\zeta u(i) = \alpha$ and $\zeta u(j) = \beta$, and set $a = u(i)$ and $b = u(j)$. Since $\zeta u \lessdot_k (\alpha, \beta)\zeta u$ is a cover, $i \leq k < j$, and thus $a \leq \alpha < \beta \leq b$, since $u \leq_k \zeta u$. Thus the edges $\langle a, \beta\rangle$ and $\langle b, \alpha\rangle$ of $\Gamma_{(\alpha,\beta)\zeta}$ meet, proving (i).

For (ii), note that $\Gamma_{(\alpha,\beta)\zeta}$ differs from Γ_ζ only by the (possible) deletion of edges $\langle a,\alpha\rangle$ and $\langle b,\beta\rangle$ and the addition of the edges $\langle a,\beta\rangle$ and $\langle b,\alpha\rangle$. Checking all possibilities for the chords $\langle c,d\rangle$, $\langle a,\alpha\rangle$, and $\langle b,\beta\rangle$ shows (ii).

Statement (iii) follows from (ii) by considering edges of $\Gamma_\zeta - \langle a,\alpha\rangle - \langle b,\beta\rangle$.

The contrapositive of (iv) is also a consequence of (ii); If ζ' and η are not disjoint and $\zeta' \preceq \zeta$, then ζ and η are not disjoint. ▛

Lemma 3.3.3. *Suppose ζ and η are disjoint permutations. For any $u \in \mathcal{S}_\infty$,*

$$u \ \leq_k \ \zeta\eta u \quad \Longleftrightarrow \quad u \ \leq_k \ \zeta u \quad \textit{and} \quad u \ \leq_k \ \eta u.$$

Proof. Suppose $u \leq_k \zeta\eta u$. Let $i \leq k$ so that $u(i) \leq \zeta\eta u(i)$. Since ζ and η have disjoint supports, $u(i) \leq \zeta u(i)$. Similarly, if $k < j$, then $u(j) \geq \zeta u(j)$, showing Condition I of Theorem 1.1.2 holds for the pair $(u, \zeta u)$.

For Condition II, suppose $i < j$, with $u(i) < u(j)$ and $\zeta u(i) > \zeta u(j)$. If $j \leq k$, this implies $u(i) \in \mathrm{supp}_\zeta$. Since $u \leq_k \zeta\eta u$, and ζ, η have disjoint supports, we have $\zeta u(i) = \eta\zeta u(i) < \eta\zeta u(j)$, which implies $u(j) \in \mathrm{supp}_\eta$ and so

$$u(i) \ < \ u(j) \ < \ \zeta u(i) \ < \ \eta u(j).$$

But then the edge $\langle u(i), \zeta u(i)\rangle$ of Γ_ζ meets the edge $\langle u(j), \eta u(j)\rangle$ of Γ_η, a contradiction. The assumption that $k < i$ leads similarly to a contradiction. Thus $u \leq_k \zeta u$ and similarly, $u \leq_k \eta u$.

Suppose now that $u \leq_k \zeta u$ and $u \leq_k \eta u$. Condition I of Theorem 1.1.2 for $(u, \zeta\eta u)$ holds as ζ and η have disjoint support. For condition II, let $i < j$ with $u(i) < u(j)$ and suppose that $j \leq k$. If the set $\{u(i), u(j)\}$ meets at most one of supp_ζ or supp_η , say supp_ζ, then $u \leq_k \zeta u$ implies $\zeta\eta u(i) < \zeta\eta u(j)$. Suppose now that $u(i)$ is in the support of ζ and $u(j)$ is in the support of η. Since $u \leq_k \zeta u$, we have $\zeta u(i) < \zeta u(j) = u(j)$. But $u \leq_k \eta u$ implies $u(j) \leq \eta u(j)$. Thus $\eta\zeta u(i) = \zeta u(i) < u(j) \leq \eta u(j) = \eta\zeta u(j)$. Similar arguments suffice when $k < i$. ▛

Theorem 3.3.4. *Suppose ζ and η are disjoint. Then the map $[e,\zeta]_\preceq \times [e,\eta]_\preceq \to [e,\zeta\eta]_\preceq$ defined by $(\zeta',\eta') \mapsto \zeta'\eta'$ is an isomorphism of graded posets.*

Proof. By Lemmas 3.3.2 and 3.3.3, this map is an injection of graded posets. For surjectivity, let $\xi \preceq \zeta\eta$. By Lemma 3.3.2 (iii) and downward induction from $\zeta\eta$ to ξ, Γ_ξ has no edges connecting supp_ζ to supp_η. Set $\xi' := \xi|_{\mathrm{supp}_\zeta}$, and $\xi'' := \xi|_{\mathrm{supp}_\eta}$. Then $\xi = \xi'\xi''$ and ξ' and ξ'' are disjoint. Surjectivity will follow by showing $\xi' \preceq \zeta$ and $\xi'' \preceq \eta$.

It suffices to consider the case $\xi \prec\!\!\cdot\, (\alpha,\beta)\xi = \zeta\eta$ is a cover. By Lemma 3.3.2 (i), α and β are connected in $\Gamma_{\zeta\eta}$, so we may assume that α, β are both in the support of ζ. Then $\xi'' = \eta$ and $(\alpha,\beta)\xi' = \zeta$. We show that $\xi' \preceq (\alpha,\beta)\xi' = \zeta$ is a cover, which will complete the proof.

Choose $u \in \mathcal{S}_\infty$ with $u \leq_k \xi u \leq_k \zeta\eta u$. Let $a := (\xi' u)^{-1}(\alpha)$ and $b := (\xi' u)^{-1}(\beta)$. Since ξ' and η are disjoint, $\alpha, \beta \notin \mathrm{supp}_\eta$ and so $a, b \notin \mathrm{supp}_\eta$. Thus $(\alpha,\beta)\xi'\eta u = \xi'\eta u(a,b)$, showing $a \leq k < b$, as $\xi'\eta u \lessdot_k (\alpha,\beta)\xi'\eta u$.

Since ξ' and η are disjoint and $\xi = \xi'\eta$, Lemma 3.3.3 implies $u \leq_k \xi' u$. Thus $|\xi'| + \ell(u) = \ell(\xi' u)$. But since ξ' and η are disjoint and $\xi'\eta \lessdot \zeta\eta$ is a cover, we have

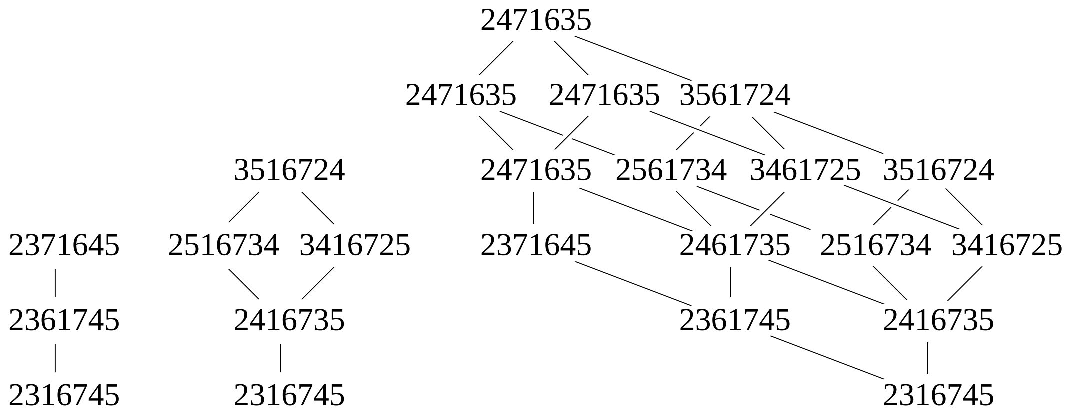
$$|\zeta| + |\eta| \;=\; |\zeta\eta| \;=\; 1 + |\xi'\eta| \;=\; 1 + |\xi'| + |\eta|,$$

so $\ell(\xi' u) + 1 = \ell(\xi' u(a,b))$. Since $a \leq k < b$ and $\zeta u = \xi' u(a,b)$, this implies $\xi' u \lessdot_k \zeta u$.

Example 3.3.5. Let $\zeta = (2354)$ and $\eta = (176)$, which are disjoint. Let $u = 2316745$. Then

$$u \leq_3 \zeta\eta u = 3571624, \quad u \leq_3 \zeta u = 3516724, \quad \text{and } u \leq_3 \eta u = 2371645.$$

The intervals $[u, \zeta u]_3$, $[u, \eta u]_3$, and $[u,\ \zeta\eta u]_3$ are illustrated below.



4. Cohomological formulas and identities for the $c^w_{u\,v}$

4.1. **Two maps on $\mathcal{S}_\infty$.** For positive integers p, q and $w \in \mathcal{S}_\infty$, define $\varepsilon_{p,q}(w) \in \mathcal{S}_\infty$ by

$$\varepsilon_{p,q}(w)(j) \quad = \quad \begin{cases} w(j) & j < p \text{ and } w(j) < q \\ w(j)+1 & j < p \text{ and } w(j) \geq q \\ q & j = p \\ w(j-1) & j > p \text{ and } w(j) < q \\ w(j-1)+1 & j > p \text{ and } w(j) \geq q \end{cases} .$$

Note that $\varepsilon_{p,p} = \phi_{\mathbb{N}-\{p\}}$. However, if $p \neq q$, then this injection, $\varepsilon_{p,q} : \mathcal{S}_\infty \hookrightarrow \mathcal{S}_\infty$, is not a group homomorphism. The map $\varepsilon_{p,q}$ has a left inverse $/_p : \mathcal{S}_\infty \to \mathcal{S}_\infty$: For

$x \in \mathcal{S}_\infty$, define $x/_p$ by

$$x/_p(j) \quad = \quad \begin{cases} x(j) & j < p \text{ and } x(j) < x(p) \\ x(j)-1 & j < p \text{ and } x(j) > x(p) \\ x(j+1) & j \geq p \text{ and } x(j) < x(p) \\ x(j+1)-1 & j \geq p \text{ and } x(j) > x(p) \end{cases}.$$

Representing permutations as matrices, the effect of $/_p$ on x is to erase the pth row and $x(p)$th column. The effect of $\varepsilon_{p,q}$ is to expand the matrix by adding a new pth row and qth column consisting mostly of zeroes, but with a 1 in the (p,q)th position. For example,

$$\varepsilon_{3,3}(23154) \;=\; 243165 \quad \text{and} \quad 264351/_3 \;=\; 25341$$

These maps have some order-theoretic properties.

Lemma 4.1.1. *Suppose $x \leq z$ and p, q are positive integers. Then*

(i) $\varepsilon_{p,q}(x) \leq \varepsilon_{p,q}(z)$.

(ii) If $\ell(z) - \ell(x) = \ell(\varepsilon_{p,q}(z)) - \ell(\varepsilon_{p,q}(x))$, *then*

$$\varepsilon_{p,q} \;:\; [x,z] \;\xrightarrow{\sim}\; [\varepsilon_{p,q}(x), \varepsilon_{p,q}(z)].$$

(iii) If $x, z \in \mathcal{S}_n$ and either of p or q is equal to either 1 or $n+1$, then $\ell(z) - \ell(x) = \ell(\varepsilon_{p,q}(z)) - \ell(\varepsilon_{p,q}(x))$.

(iv) If $x \leq_k z$ and $x(p) = z(p)$, then $x/_p \leq_{k'} z/_p$ and $[x,z]_k \simeq [x/_p, z/_p]_{k'}$, where k' is equal to k if $k < p$ and $k-1$ otherwise. Furthermore, $zx^{-1} = \varepsilon_{x(p),x(p)}(z/_p(x/_p)^{-1})$.

Proof. Suppose $x \lessdot x(a,b)$ is a cover. Then $\varepsilon_{p,q}(x) < \varepsilon_{p,q}(x(a,b))$ is a cover if either $p \leq a$ or $b < p$, or else $a < p \leq b$ and either $q \leq x(a)$ or $x(b) < q$. If however, $a < p \leq b$ and $x(a) < q \leq x(b)$, then there is a chain of length 3 from $\varepsilon_{p,q}(x)$ to $\varepsilon_{p,q}(x(a,b)) = \varepsilon_{p,q}(x)(a, b{+}1)$:

$$\varepsilon_{p,q}(x) \;\lessdot\; \varepsilon_{p,q}(x)(a,p) \;\lessdot\; \varepsilon_{p,q}(x)(a, b{+}1, p) \;\lessdot\; \varepsilon_{p,q}(x)(a, b{+}1).$$

The lemma follows from this observation. For example, under the hypothesis of (ii), the number of inversions in $\varepsilon_{p,q}(z)$ involving q equals the number of inversion in $\varepsilon_{p,q}(x)$ involving q. Thus, if $\varepsilon_{p,q}(x) \leq u \leq \varepsilon_{p,q}(z)$, then $u(p) = q$. ∎

4.2. **An embedding of flag manifolds.** Let $W \subset V$ with $W \simeq \mathbb{C}^n$ and $V \simeq \mathbb{C}^{n+1}$. Suppose $f \in V - W$ so that $V = \langle W, f\rangle$. For $p \in [n{+}1]$ define the injection $\psi_p : \mathbb{F}\ell W \hookrightarrow \mathbb{F}\ell V$ by

$$(\psi_p E_\bullet)_j \quad = \quad \begin{cases} E_j & \text{if } j < p \\ \langle E_{j-1}, f\rangle & \text{if } j \geq p \end{cases}$$

Proposition 4.2.1 ([52], Lemma 12). *Let $E_\bullet \in \mathbb{F}\ell W$ and $w \in \mathcal{S}_n$. Then, for any $p, q \in [n{+}1]$,*

$$\psi_p X_w E_\bullet \quad \subset \quad X_{\varepsilon_{p,q}(w)} \psi_{n+2-q} E_\bullet.$$

Recall that e is the identity permutation.

Corollary 4.2.2. *Let $w \in \mathcal{S}_n$ and $E_\bullet, E'_\bullet \in \mathbb{F}\ell W$ be opposite flags. Then $\psi_1 E_\bullet$ and $\psi_{n+1}E'_\bullet$ are opposite flags in $\mathbb{F}\ell V$ and*

$$\psi_p X_w E_\bullet \;=\; X_{\varepsilon_{p,1}(w)}\psi_{n+1}E_\bullet \bigcap X_{\varepsilon_{p,n+1}(e)}\psi_1 E'_\bullet \;=\; X_{\varepsilon_{p,1}(e)}\psi_{n+1}E'_\bullet \bigcap X_{\varepsilon_{p,n+1}(w)}\psi_1 E_\bullet\,.$$

Proof. Since $X_e E'_\bullet = \mathbb{F}\ell W$, Proposition 4.2.1 with $q = 1$ or $n+1$ implies $\psi_p X_w E_\bullet$ is a subset of either intersection:

$$X_{\varepsilon_{p,1}(w)}\psi_{n+1}E_\bullet \bigcap X_{\varepsilon_{p,n+1}(e)}\psi_1 E'_\bullet \qquad \text{or} \qquad X_{\varepsilon_{p,1}(e)}\psi_{n+1}E'_\bullet \bigcap X_{\varepsilon_{p,n+1}(w)}\psi_1 E_\bullet\,.$$

Since $E_\bullet$ and $E'_\bullet$ are opposite flags, $\psi_{n+1}E_\bullet$ and $\psi_1 E'_\bullet$ are opposite flags, so both intersections are generically transverse and irreducible. Since

$$\ell(\varepsilon_{p,1}(w)) \;=\; \ell(w) + p - 1 \qquad \text{and} \qquad \ell(\varepsilon_{p,n+1}(w)) \;=\; \ell(w) + n + 1 - p,$$

both intersections have the same dimension as $\psi_p X_w E_\bullet$, proving equality. ∎

Since $\varepsilon_{p,n+1}(e) = v(n{+}1{-}p,\, p)$, where $n+1-p$ is the partition of $n+1-p$ into a single part, we see that $\mathfrak{S}_{\varepsilon_{p,n+1}(e)} = h_{n+1-p}(x_1, \ldots, x_p)$, the complete symmetric polynomial of degree $n{+}1{-}p$ in $x_1, \ldots, x_p$. Similarly, $\mathfrak{S}_{\varepsilon_{p,1}(e)} = e_{p-1}(x_1, \ldots, x_{p-1}) = x_1 \cdots x_{p-1}$, as $\varepsilon_{p,1} = v(1^{p-1}, p-1)$, where 1^{p-1} is the partition of $p{-}1$ into $p{-}1$ equal parts, each of size 1.

Corollary 4.2.3. *Let $w \in \mathcal{S}_n$. In $H^*\mathbb{F}\ell V$,*

$$\mathfrak{S}_{\varepsilon_{p,1}(w)} \cdot h_{n+1-p}(x_1, \ldots, x_p) \quad = \quad \mathfrak{S}_{\varepsilon_{p,n+1}(w)} \cdot x_1 \cdots x_{p-1}$$

and these products are equal to $(\psi_p)_* \mathfrak{S}_w$.

We use this to compute ψ_p^*. The Pieri-type formulas of [52] show that if $u \in \mathcal{S}_n$ and $k, m \leq n$ positive integers, then

$$\mathfrak{S}_u \cdot \mathfrak{S}_{w_0 w} \cdot e_m(x_1 \cdots x_k) \;=\; \begin{cases} 1 & u \xrightarrow{c_{k,m}} w \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \tag{4.2.1}$$

$$\mathfrak{S}_u \cdot \mathfrak{S}_{w_0 w} \cdot h_{n+1-m}(x_1, \ldots, x_k) \;=\; \begin{cases} 1 & u \xrightarrow{r_{k,m}} w \\ 0 & \text{otherwise} \end{cases}, \tag{4.2.2}$$

where $u \xrightarrow{c_{k,m}} w$ if there is a (saturated) chain in the k-Bruhat order from u to w:

$$u \;\lessdot_k\; (\alpha_1, \beta_1)u \;\lessdot_k\; \cdots \;\lessdot_k\; (\alpha_m, \beta_m)\cdots(\alpha_1, \beta_1)u \;=\; w$$

such that $\beta_1 > \cdots > \beta_m$. When $k = m$, it follows that $\{\alpha_1, \ldots, \alpha_k\} = \{u(1), \ldots, u(k)\}$. When $k = m = p-1$, we write $\xrightarrow{c_p}$ for this relation. Similarly, $u \xrightarrow{r_{k,m}} w$ if there is a chain in the k-Bruhat order:

$$u \;\lessdot_k\; (\alpha_1, \beta_1)u \;\lessdot_k\; \cdots \;\lessdot_k\; (\alpha_{n+1-m}, \beta_{n+1-m})\cdots(\alpha_1, \beta_1)u \;=\; w$$

such that $\beta_1 < \beta_2 < \cdots < \beta_{n+1-m}$.

Theorem 4.2.4. *Let $x \in \mathcal{S}_{n+1}$. In $H^*\mathbb{F}\ell_n$,*

$$(i)\ \ \psi_p^* \mathfrak{S}_x \;=\; \sum_{\substack{w \in \mathcal{S}_n \\ x \xrightarrow{c_p} \varepsilon_{p,1}(w)}} \mathfrak{S}_w \;=\; \sum_{\substack{w \in \mathcal{S}_n \\ x \xrightarrow{r_{p,n+1-p}} \varepsilon_{p,n+1}(w)}} \mathfrak{S}_w .$$

$$(ii)\ \ \psi_p^*(x_i) \;=\; \begin{cases} x_i & i < p \\ 0 & i = p \\ x_{i-1} & i > p \end{cases} .$$

Proof. In $H^*\mathbb{F}\ell_n$,

$$\psi_p^* \mathfrak{S}_x \;=\; \sum_{w \in \mathcal{S}_n} \deg(\mathfrak{S}_{w_0 w} \cdot \psi_p^* \mathfrak{S}_x)\, \mathfrak{S}_w .$$

By the projection formula (2.3.1) and Corollary 4.2.3, we have

$$\deg(\mathfrak{S}_{w_0 w} \cdot \psi_p^* \mathfrak{S}_x) \;=\; \deg(\mathfrak{S}_x \cdot (\psi_p)_* \mathfrak{S}_{w_0 w}) \;=\; \deg(\mathfrak{S}_x \cdot \mathfrak{S}_{\varepsilon_{p,n+1}(w_0 w)} \cdot x_1 \cdots x_{p-1}).$$

Note that $\varepsilon_{p,n+1}(w_0 w) = w_0^{(n+1)} \varepsilon_{p,1}(w)$. By (4.2.1), the triple product

$$\mathfrak{S}_x \cdot \mathfrak{S}_{\varepsilon_{p,n+1}(w_0 w)} \cdot x_1 \cdots x_{p-1}$$

is zero unless $x \xrightarrow{c_p} \varepsilon_{p,1}(w)$, and in this case it equals $\mathfrak{S}_{w_0^{(n+1)}}$. This establishes the first equality of (i). For the second, use the other formula for $(\psi_p)_* \mathfrak{S}_w$ from Corollary 4.2.3 and (4.2.2).

For (ii), let $\mathcal{F}_\bullet$ be the tautological flag on $\mathbb{F}\ell_{n+1}$, $\mathcal{E}_\bullet$ the tautological flag on $\mathbb{F}\ell_n$, and 1 the trivial line bundle. Then

$$\psi_p^*(\mathcal{F}_i/\mathcal{F}_{i-1}) \;=\; \begin{cases} \mathcal{E}_i/\mathcal{E}_{i-1} & \text{if } i < p \\ 1 & \text{if } i = p \\ \mathcal{E}_{i-1}/\mathcal{E}_{i-2} & \text{if } i > p \end{cases} ,$$

Since $-x_i$ is the Chern class of both $\mathcal{F}_i/\mathcal{F}_{i-1}$ and $\mathcal{E}_i/\mathcal{E}_{i-1}$, we are done. ∎

4.3. **The endomorphism $x_p \mapsto 0$.** For $p \in \mathbb{N}$ and $x \in \mathcal{S}_\infty$, define

$$A_p(x) \;:=\; \{u \in \mathcal{S}_\infty \mid x \xrightarrow{c_p} \varepsilon_{p,1}(u)\}.$$

Lemma 4.3.1. *If $x \in \mathcal{S}_n$ and $p \leq n$, then $A_p(x) \;=\; \{u \in \mathcal{S}_n \mid x \xrightarrow{r_{p,n+1-p}} \varepsilon_{p,n+1}(u)\}$.*

Proof. If $x \in \mathcal{S}_n$, $p \leq n$, and $x \xrightarrow{c_p} w$, then $w \in \mathcal{S}_{n+1}$, so $A_p(x) \subset \mathcal{S}_n$. But then $A_p(x)$ and $\{u \in \mathcal{S}_n \mid x \xrightarrow{r_{p,n+1-p}} \varepsilon_{p,n+1}(u)\}$ index the two equal sums in Theorem 4.2.4(i). ∎

Let $\Psi_p : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[x_1, x_2, \ldots]$ be defined by

$$\Psi_p(x_i) \;=\; \begin{cases} x_i & \text{if } i < p \\ 0 & \text{if } i = p \\ x_{i-1} & \text{if } i > p \end{cases} .$$

Theorem 4.3.2. *For $x \in \mathcal{S}_\infty$, and $p \in \mathbb{N}$, $\Psi_p \mathfrak{S}_x = \displaystyle\sum_{u \in A_p(x)} \mathfrak{S}_u$.*

Proof. For $p \leq n+1$, the homomorphism Ψ_p induces the map $\psi_p^* : H^*\mathbb{F}\ell_{n+1} \to H^*\mathbb{F}\ell_n$, by Theorem 4.2.4 (ii). Choosing n large enough completes the proof. ∎

Corollary 4.3.3. *For $w, x, y \in \mathcal{S}_\infty$ and $p \in \mathbb{N}$,*

$$\sum_{u \in A_p(x)} \sum_{v \in A_p(y)} c^w_{u\,v} \quad = \quad \sum_{\substack{z \\ w \in A_p(z)}} c^z_{x\,y}.$$

Proof. Apply Ψ_p to the identity $\mathfrak{S}_x \cdot \mathfrak{S}_y = \sum_z c^z_{x\,y} \mathfrak{S}_z$ to obtain:

$$\sum_{u \in A_p(x)} \sum_{v \in A_p(y)} \mathfrak{S}_u \cdot \mathfrak{S}_v \quad = \quad \sum_z c^z_{x\,y} \sum_{w \in A_p(z)} c^z_{x\,y}\, \mathfrak{S}_w.$$

Expanding the product $\mathfrak{S}_u \cdot \mathfrak{S}_v$ and equating the coefficients of $\mathfrak{S}_w$ proves the identity. ∎

Example 4.3.4. We illustrate the effect of Ψ_3 with an example. Since

$$\begin{aligned} \mathfrak{S}_{413652} \;=\;& x_1^4x_2x_4x_5 + x_1^3x_2^2x_4x_5 + x_1^3x_2x_4^2x_5 + \\ & x_1^4x_2x_3x_4 + x_1^4x_2x_3x_5 + x_1^4x_3x_4x_5 + x_1^3x_2^2x_3x_4 + x_1^3x_2^2x_3x_5 + x_1^3x_2x_3^2x_4 + \\ & x_1^3x_2x_3^2x_5 + x_1^3x_2x_3x_4^2 + x_1^3x_3^2x_4x_5 + x_1^3x_3x_4^2x_5 + 2 \cdot x_1^3x_2x_3x_4x_5, \end{aligned}$$

we have

$$\Psi_3(\mathfrak{S}_{413652}) \;=\; x_1^4x_2x_3x_4 + x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4.$$

However,

$$\begin{aligned} \mathfrak{S}_{52341} &= x_1^4x_2x_3x_4 \\ \mathfrak{S}_{42531} &= x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4, \end{aligned}$$

which shows

$$\Psi_3(\mathfrak{S}_{413652}) \;=\; \mathfrak{S}_{52341} + \mathfrak{S}_{42531}.$$

To see this agrees with Theorem 4.3.2, compute the permutations w such that $x \xrightarrow{c_3} w$:

$$\begin{tikzcd}[column sep=small]
623451 & & \underline{631452} & & \underline{531642} & & 523641 \\
& 613452 \arrow[ul, dash] \arrow[ur, dash] & & & & 513462 \arrow[ul, dash] \arrow[ur, dash] & \\
& & & 413652 \arrow[ull, dash] \arrow[urr, dash] & & &
\end{tikzcd}$$

Of these, only the two underlined permutations are of the form $\varepsilon_{3,1}(u)$:

$$631452 \;=\; \varepsilon_{3,1}(52341) \quad \text{and} \quad 531642 \;=\; \varepsilon_{3,1}(42531).$$

Lemma 4.3.5. *Let λ be a partition and p, k positive integers. Then $A_p(v(\lambda,k)) = \{(v(\lambda,k')\}$, where $k' = k-1$ if $p \leq k$ and k otherwise.*

Proof. By the combinatorial definition of Schur functions [48, §4.4], $\Psi_p(\mathfrak{S}_{v(\lambda,k)}) = \mathfrak{S}_{v(\lambda,k')}$. ∎

Lemma 4.3.5 implies that $v(\lambda,k')$ is the only solution x to the equation $v(\lambda,k) \xrightarrow{c_p} \varepsilon_{p,1}(x)$, a statement about chains in the Bruhat order.

4.4. Identities for $c^z_{x\,y}$ when $x(p) = z(p)$.

Lemma 4.4.1. *Let $x, z \in \mathcal{S}_{n+1}$ with $x(p) = z(p)$ for some $p \in [m+1]$ and suppose $\ell(z) - \ell(x) = \ell(z/_p) - \ell(x/_p)$. In $H^*\mathbb{F}\ell_{n+1}$,*

$$(\psi_p)_* \left(\mathfrak{S}_{x/_p} \cdot \mathfrak{S}_{w_0^{(n)}(z/_p)}\right) \quad = \quad \mathfrak{S}_x \cdot \mathfrak{S}_{w_0^{(n+1)}z}.$$

Proof. Let $E_\bullet, E'_\bullet$ be opposite flags in W. By Proposition 4.2.1,

$$\psi_p\left(X_{w_0^{(n)}(z/_p)}E_\bullet \bigcap X_{x/_p}E'_\bullet\right) \quad = \quad X_{w_0^{(n+1)}z}\psi_{z(p)}E_\bullet \bigcap X_x \psi_{n+2-x(p)}E'_\bullet \tag{4.4.1}$$

Note that $w_0^{(n)}(z/_p) = (w_0^{(n+1)}z)/_p$. Since $x(p) = z(p)$, the flags $\psi_{z(p)}E_\bullet$ and $\psi_{n+2-x(p)}E'_\bullet$ are opposite in V. Moreover, as $\ell(z) - \ell(x) = \ell(z/_p) - \ell(x/_p)$, both sides of (4.4.1) have the same dimension, so they are equal, proving the lemma. ∎

Theorem 4.4.2. *Let $x, z \in \mathcal{S}_\infty$ with $x(p) = z(p)$ and suppose that $\ell(z) - \ell(x) = \ell(z/_p) - \ell(x/_p)$. Then, for any $y \in \mathcal{S}_\infty$ and positive integer p,*

$$c^z_{x\,y} \quad = \quad \sum_{v \in A_p(y)} c^{z/_p}_{x/_p\,v}$$

Proof. It suffices to compute this in $H^*\mathbb{F}\ell_{n+1}$, for n such that $p \leq n$, $y \in \mathcal{S}_{n+1}$ and $A_p(y) \subset \mathcal{S}_n$. By Lemma 4.4.1,

$$\mathfrak{S}_x \cdot \mathfrak{S}_{w_0^{(n+1)}z} \;=\; (\psi_p)_*\left(\mathfrak{S}_{x/_p} \cdot \mathfrak{S}_{w_0^{(n)}(z/_p)}\right) \;=\; (\psi_p)_*\left(\sum_{v \in \mathcal{S}_n} c^{w_0^{(n)}v}_{x/_p\ w_0^{(n)}(z/_p)}\ \mathfrak{S}_{w_0^{(n)}v}\right).$$

Since $c^{w_0^{(n)}v}_{u\,w_0^{(n)}w} = c^w_{u\,v}$ for $u, v, w \in \mathcal{S}_n$ and $\varepsilon_{p,1}(w_0^{(n)}v) = w_0^{(n+1)}\varepsilon_{p,1}(v)$,

$$\begin{aligned}\mathfrak{S}_x \cdot \mathfrak{S}_{w_0^{(n+1)}z} &= \sum_{v\in\mathcal{S}_n} c^{z/p}_{x/p\,v}(\psi_p)_*\left(\mathfrak{S}_{w_0^{(n)}v}\right)\\ &= \sum_{v\in\mathcal{S}_n} c^{z/p}_{x/p\,v}\mathfrak{S}_{w_0^{(n+1)}\varepsilon_{p,1}(v)} \cdot x_1\cdots x_{p-1},\end{aligned}$$

by Corollary 4.2.3. Thus

$$\begin{aligned}c^z_{x\,y} &= \deg\left(\mathfrak{S}_x \cdot \mathfrak{S}_{w_0^{(n+1)}z} \cdot \mathfrak{S}_y\right)\\ &= \sum_{v\in\mathcal{S}_n} c^{z/p}_{x/p\,v} \cdot \deg\left(\mathfrak{S}_{w_0^{(n+1)}\varepsilon_{p,1}(v)} \cdot (x_1\cdots x_{p-1}) \cdot \mathfrak{S}_y\right)\\ &= \sum_{v\in A_p(y)} c^{z/p}_{x/p\,v}. \qquad \blacksquare\end{aligned}$$

When $p = 1$, this has the following consequence:

Corollary 4.4.3. *If $x(1) = z(1)$, then $c^z_{x\,y} = 0$ unless $y = 1 \times v$. In that case, $c^z_{x\ 1\times v} = c^{z/1}_{x/1\ v}$.*

4.5. **Products of flag manifolds.** Let $P, Q \in \binom{[n+m]}{n}$, that is, $P, Q \subset [n+m]$ and each has order n. Index the sets P, Q and their complements P^c, Q^c as follows:

$$\begin{array}{rclcrclcl} P &=& p_1 < \cdots < p_n & \qquad & P^c &:=& [n+m] - P &=& p^c_1 < \cdots < p^c_m\\ Q &=& q_1 < \cdots < q_n & & Q^c &:=& [n+m] - Q &=& q^c_1 < \cdots < q^c_m\end{array}$$

Define a function $\varepsilon_{P,Q} : \mathcal{S}_n \times \mathcal{S}_m \hookrightarrow \mathcal{S}_{m+n}$ by:

$$\begin{array}{rcll}\varepsilon_{P,Q}(v,w)(p_i) &=& q_{v(i)} & \qquad i = 1,\ldots,n\\ \varepsilon_{P,Q}(v,w)(p^c_j) &=& q^c_{w(j)} & \qquad i = 1,\ldots,m.\end{array}$$

As permutation matrices, $\varepsilon_{P,Q}(v,w)$ is obtained from v and w by placing the entries of v in the blocks $P \times Q$ and those of w in the blocks $P^c \times Q^c$. If $P = [n+1] - \{p\}$ and $Q = [n+1] - \{q\}$, then $\varepsilon_{P,Q}(v,e) = \varepsilon_{p,q}(v)$.

Suppose $V \simeq \mathbb{C}^n$, $W \simeq \mathbb{C}^m$, and $P \in \binom{[n+m]}{n}$. Define a map

$$\psi_P \ : \ \mathbb{F}\ell V \times \mathbb{F}\ell W \quad\hookrightarrow\quad \mathbb{F}\ell(V \oplus W)$$

by $\psi_P(E_\bullet, F_\bullet)_j = \langle E_i, F_{i'} \,|\, p_i, p^c_{i'} \leq j\rangle$. Equivalently, if $e_1, \ldots, e_n$ is a basis for V and $f_1, \ldots, f_m$ a basis for W, then $\psi_P(\langle\langle e_1,\ldots,e_n\rangle\rangle, \langle\langle f_1,\ldots,f_m\rangle\rangle) = \langle\langle g_1,\ldots,g_{n+m}\rangle\rangle$, where $g_{p_i} = e_i$ and $g_{p^c_i} = f_i$. From this, it follows that if $E_\bullet, E'_\bullet \in \mathbb{F}\ell V$ and $F_\bullet, F'_\bullet \in \mathbb{F}\ell W$ are pairs of opposite flags, then $\psi_P(E_\bullet, F_\bullet)$ and $\psi_{w_0^{(n+m)}P}(E'_\bullet, F'_\bullet)$ are opposite flags in $V \oplus W$.

Lemma 4.5.1. *Let $P, Q \in \binom{[n+m]}{n}$, $v \in \mathcal{S}_n$, and $w \in \mathcal{S}_m$. Then, for $E_\bullet \in \mathbb{F}\ell V$ and $F_\bullet \in \mathbb{F}\ell W$,*

$$\begin{aligned}\psi_P\left(X_{w_0^{(n)}v}E_\bullet \times X_{w_0^{(m)}w}F_\bullet\right) &\subset X_{w_0^{(n+m)}\varepsilon_{P,Q}(v,w)}\psi_Q(E_\bullet, F_\bullet)\\ \psi_P\left(X_vE_\bullet \times X_wF_\bullet\right) &\subset X_{\varepsilon_{P,Q}(v,w)}\psi_{w_0^{(n+m)}Q}(E_\bullet, F_\bullet).\end{aligned}$$

Proof. For a flag $G_\bullet$, define $G_j^\circ := G_j - G_{j-1}$. By the definition of ψ_Q, we have $E_i^\circ \subset \psi_Q(E_\bullet, F_\bullet)^\circ_{q_i}$ and $F_i^\circ \subset \psi_Q(E_\bullet, F_\bullet)^\circ_{q_i^c}$. Since

$$w_0^{(n+m)}Q \quad = \quad n+m+1-q_n \;<\; \cdots \;<\; n+m+1-q_1,$$

$E_{n+1-j} \subset \psi_{w_0^{(n+m)}Q}(E_\bullet, F_\bullet)_{n+m+1-q_j}$, and $F_{n+1-j} \subset \psi_{w_0^{(n+m)}Q}(E_\bullet, F_\bullet)_{n+m+1-q'_j}$, the lemma is a consequence of the definitions of Schubert varieties and ψ_P. ∎

Corollary 4.5.2. *Let $E_\bullet, E'_\bullet \in \mathbb{F}\ell V$ and $F_\bullet, F'_\bullet \in \mathbb{F}\ell W$ be pairs of opposite flags and let $P \in \binom{[n+m]}{n}$. Set $Q = \{m+1, \ldots, m+n\}$. Then for any $v \in \mathcal{S}_n$ and $w \in \mathcal{S}_m$,*

$$\begin{aligned}\psi_P\left(X_vE_\bullet \times X_wF_\bullet\right) &= X_{\varepsilon_{P,[n]}(v,w)}\psi_Q(E_\bullet, F_\bullet) \bigcap X_{\varepsilon_{P,Q}(e,e)}\psi_{[n]}(E'_\bullet, F'_\bullet)\\ &= X_{\varepsilon_{P,[n]}(v,e)}\psi_Q(E_\bullet, F'_\bullet) \bigcap X_{\varepsilon_{P,Q}(e,w)}\psi_{[n]}(E'_\bullet, F_\bullet)\\ &= X_{\varepsilon_{P,[n]}(e,w)}\psi_Q(E'_\bullet, F_\bullet) \bigcap X_{\varepsilon_{P,Q}(v,e)}\psi_{[n]}(E_\bullet, F'_\bullet)\\ &= X_{\varepsilon_{P,[n]}(e,e)}\psi_Q(E'_\bullet, F'_\bullet) \bigcap X_{\varepsilon_{P,Q}(v,w)}\psi_{[n]}(E_\bullet, F_\bullet).\end{aligned}$$

Proof. Since $w_0^{(n+m)}[n] = Q$, $X_eE_\bullet = \mathbb{F}\ell V$, and $X_eF_\bullet = \mathbb{F}\ell W$, Lemma 4.5.1 shows that $\psi_P\left(X_vE_\bullet \times X_wF_\bullet\right)$ is a subset of any of the four intersections. Equality follows as they have the same dimension. Indeed, for $x, z \in \mathcal{S}_n$ and $y, u \in \mathcal{S}_m$,

$$\begin{aligned}\ell(\varepsilon_{P,[n]}(x,y)) &= \ell(x) + \ell(y) + \#\{i \in [n], j \in [m] \mid p_i > p_j^c\}\\ \ell(\varepsilon_{P,Q}(z,u)) &= \ell(z) + \ell(u) + \#\{i \in [n], j \in [m] \mid p_j^c > p_i\}.\end{aligned}$$

Thus $\ell(\varepsilon_{P,[n]}(x,y)) + \ell(\varepsilon_{P,Q}(z,u)) = \ell(x) + \ell(y) + \ell(z) + \ell(u) + n \cdot m$ and so

$$\binom{n+m}{2} - \ell(\varepsilon_{P,[n]}(x,y)) - \ell(\varepsilon_{P,Q}(z,u)) \;=\; \binom{n}{2} + \binom{m}{2} - \ell(x) - \ell(y) - \ell(z) - \ell(u).$$

If (x,y,z,u) is one of $(v,w,e,e), (v,e,e,w), (e,w,v,e), (e,e,v,w)$, then these are, respectively, the the dimension of one of the intersections and the dimension of $X_vE_\bullet \times X_wF_\bullet$. ∎

Corollary 4.5.3. *Let $Q = \{m+1, \ldots, m+n\} = w_0^{(n+m)}[n]$. For any $v \in \mathcal{S}_n$, $w \in \mathcal{S}_m$, and $P \in \binom{[n+m]}{n}$, the following identities hold in $H^*\mathbb{F}\ell_{n+m}$:*

$$\begin{aligned}\mathfrak{S}_{\varepsilon_{P,[n]}(v,w)}\cdot\mathfrak{S}_{\varepsilon_{P,Q}(e,e)} &= \mathfrak{S}_{\varepsilon_{P,[n]}(v,e)}\cdot\mathfrak{S}_{\varepsilon_{P,Q}(e,w)} = \\ &\mathfrak{S}_{\varepsilon_{P,[n]}(e,w)}\cdot\mathfrak{S}_{\varepsilon_{P,Q}(v,e)} = \mathfrak{S}_{\varepsilon_{P,[n]}(e,e)}\cdot\mathfrak{S}_{\varepsilon_{P,Q}(v,w)},\end{aligned}$$

and this common cohomology class is $(\psi_P)_*(\mathfrak{S}_v\otimes\mathfrak{S}_w)$.

Theorem 4.5.4. *Let* $x\in\mathcal{S}_{n+m}$ *and* $P\in\binom{[n+m]}{n}$. *Then*

(i)
$$\begin{aligned}\psi_P^*\mathfrak{S}_x &= \sum_{v\in\mathcal{S}_n,\ w\in\mathcal{S}_m} c^{\varepsilon_{P,[n]}(v,w)}_{\varepsilon_{P,[n]}(e,e)\ x}\ \mathfrak{S}_v\otimes\mathfrak{S}_w\\ &= \sum_{v\in\mathcal{S}_n,\ w\in\mathcal{S}_m} c^{\varepsilon_{P,[n]}(v,w_0^{(m)})}_{\varepsilon_{P,[n]}(e,w_0^{(m)}w)\ x}\ \mathfrak{S}_v\otimes\mathfrak{S}_w\\ &= \sum_{v\in\mathcal{S}_n,\ w\in\mathcal{S}_m} c^{\varepsilon_{P,[n]}(w_0^{(n)},w)}_{\varepsilon_{P,[n]}(w_0^{(n)}v,e)\ x}\ \mathfrak{S}_v\otimes\mathfrak{S}_w\\ &= \sum_{v\in\mathcal{S}_n,\ w\in\mathcal{S}_m} c^{\varepsilon_{P,[n]}(w_0^{(n)},w_0^{(m)})}_{\varepsilon_{P,[n]}(w_0^{(n)}v,w_0^{(m)}w)\ x}\ \mathfrak{S}_v\otimes\mathfrak{S}_w\end{aligned}$$

(ii) *Let* $Q=\{m+1,\ldots,m+n\}$. *For any* $v\in\mathcal{S}_n$ *and* $w\in\mathcal{S}_m$, *we have*

$$\begin{array}{ccccccc} c^{\varepsilon_{P,[n]}(v,w)}_{\varepsilon_{P,[n]}(e,e)\ x} &=& c^{\varepsilon_{P,[n]}(v,w_0^{(m)})}_{\varepsilon_{P,[n]}(e,w_0^{(m)}w)\ x} &=& c^{\varepsilon_{P,[n]}(w_0^{(n)},w)}_{\varepsilon_{P,[n]}(w_0^{(n)}v,e)\ x} &=& c^{\varepsilon_{P,[n]}(w_0^{(n)},w_0^{(m)})}_{\varepsilon_{P,[n]}(w_0^{(n)}v,w_0^{(m)}w)\ x}\\ \| && \| && \| && \|\\ c^{\varepsilon_{P,Q}(v,w)}_{\varepsilon_{P,Q}(e,e)\ x} &=& c^{\varepsilon_{P,Q}(v,w_0^{(m)})}_{\varepsilon_{P,Q}(e,w_0^{(m)}w)\ x} &=& c^{\varepsilon_{P,Q}(w_0^{(n)},w)}_{\varepsilon_{P,Q}(w_0^{(n)}v,e)\ x} &=& c^{\varepsilon_{P,Q}(w_0^{(n)},w_0^{(m)})}_{\varepsilon_{P,Q}(w_0^{(n)}v,w_0^{(m)}w)\ x}\end{array}$$

Remark 4.5.5. Each structure constant in (ii) is of the form $c^{\zeta y}_{y\,x}$, where ζ is, respectively, $v\times w, v\times\overline{w}^{-1},\overline{v}^{-1}\times w$, and $\overline{v}^{-1}\times\overline{w}^{-1}$. Each interval $[y,\zeta y]$ is isomorphic to $[e,v]\times[e,w]$. This is consistent with the expectation that the $c^z_{y\,x}$ should only depend upon $[y,z]$ and x.

Proof. In (ii), the second row is a consequence of the first as $c^z_{y\,x}=c^{w_0^{(n+m)}y}_{w_0^{(n+m)}z\ x}$, for $x,y,z\in\mathcal{S}_{n+m}$. The first row of equalities is a consequence of the identities in (i). For (i), there exist integral constants $d^{v\,w}_x$ defined by the identity

$$\psi_P^*\mathfrak{S}_x = \sum d^{v\,w}_x\ \mathfrak{S}_v\otimes\mathfrak{S}_w.$$

Since the Schubert basis is self-dual with respect to the intersection pairing, we have

$$\begin{aligned}d^{v\,w}_x &= \deg\left(\psi_P^*\mathfrak{S}_x\cdot(\mathfrak{S}_{w_0^{(n)}v}\otimes\mathfrak{S}_{w_0^{(m)}w})\right)\\ &= \deg\left(\mathfrak{S}_x\cdot(\psi_P)_*(\mathfrak{S}_{w_0^{(n)}v}\otimes\mathfrak{S}_{w_0^{(m)}w})\right).\end{aligned}$$

Each expression for $(\psi_P)_*(\mathfrak{S}_{w_0^{(n)}v} \otimes \mathfrak{S}_{w_0^{(m)}w})$ of Corollary 4.5.3 yields one of the sums in (i). For example, the last expression in Corollary 4.5.3 yields

$$\begin{aligned} d_x^{v\,w} &= \deg\Big(\mathfrak{S}_x \cdot \mathfrak{S}_{\varepsilon_{P,[n]}(e,e)} \cdot \mathfrak{S}_{w_0^{(n+m)}\varepsilon_{P,[n]}(v,w)}\Big) \\ &= c^{\varepsilon_{P,[n]}(v,w)}_{\varepsilon_{P,[n]}(e,e)\ x}, \end{aligned}$$

since $w_0^{(n+m)}\varepsilon_{P,[n]}(v,w) = \varepsilon_{P,w_0^{(n+m)}[n]}\Big(w_0^{(n)}v, w_0^{(m)}w\Big)$. ∎

Corollary 4.5.6. *Let $u, v, w \in \mathcal{S}_n$ and $x, y, z \in \mathcal{S}_\infty$. Then $c^{w\times z}_{u\times x\ \ v\times y} = c^w_{u\,v} \cdot c^z_{x\,y}$.*

Proof. Choose m so that $x, y, z \in \mathcal{S}_m$. Since $\varepsilon_{[n],[n]}(u, x) = u \times x$, the first identity of Theorem 4.5.4 (i) implies $\psi^*_{[n]}\mathfrak{S}_{u\times x} = \mathfrak{S}_u \otimes \mathfrak{S}_x$. Then

$$\begin{aligned} c^w_{u\,v} \cdot c^z_{x\,y} &= \deg\Big((\mathfrak{S}_u \otimes \mathfrak{S}_x) \cdot (\mathfrak{S}_v \otimes \mathfrak{S}_y) \cdot (\mathfrak{S}_{w_0^{(n)}w} \otimes \mathfrak{S}_{w_0^{(m)}z})\Big) \\ &= \deg\Big(\psi^*_{[n]}(\mathfrak{S}_{u\times x} \cdot \mathfrak{S}_{v\times y}) \cdot (\mathfrak{S}_{w_0^{(n)}w} \otimes \mathfrak{S}_{w_0^{(m)}z})\Big) \\ &= \deg\Big(\mathfrak{S}_{u\times x} \cdot \mathfrak{S}_{v\times y} \cdot \mathfrak{S}_{w_0^{(n+m)}(w\times z)}\Big) \\ &= c^{w\times z}_{u\times x\ \ v\times y}, \end{aligned}$$

as $(\psi_{[n]})_*\mathfrak{S}_{w_0^{(n+m)}(w\times z)} = \mathfrak{S}_{w_0^{(n)}w} \otimes \mathfrak{S}_{w_0^{(m)}z}$, by Corollary 4.5.3. ∎

4.6. **Maps $\mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots]$.** Let $P \subset \mathbb{N}$, define $P^c := \mathbb{N} - P$, and suppose P^c is infinite. Enumerate P and P^c as follows:

$$P\ :\ p_1\ <\ p_2\ <\ \begin{cases} \cdots < p_s & \text{if } \#P = s \\ \cdots & \text{otherwise} \end{cases} \qquad\qquad P^c\ :\ p_1^c\ <\ p_2^c\ <\ \cdots$$

Define $\Psi_P : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots]$ by

$$x_{p_i} \longmapsto y_i \qquad x_{p_i^c} \longmapsto z_i.$$

Then there exist integer constants $d^{u\,v}_w(P)$ for $u, v, w \in \mathcal{S}_\infty$ defined by the identity:

$$\Psi_P(\mathfrak{S}_w(x)) \;=\; \sum_{u,v} d^{u\,v}_w(P)\ \mathfrak{S}_u(y)\ \mathfrak{S}_v(z).$$

For $l, d \in \mathbb{N}$ and $R \subset \{d+1, \ldots, d+2l\}$ with $\#R = l$, define $\overline{P}(l, d, R) := (P \bigcap [d]) \bigcup R$.

Theorem 4.6.1. *Let $P \subset \mathbb{N}$ and $w \in \mathcal{S}_\infty$. For any integers $l > \ell(w)$ and d exceeding the last descent of w and any subset R of $\{d+1, \ldots, d+2l\}$ of cardinality l, set*

$n := \#\overline{P}(l,d,R)$, $m := d+2l-n$, and $\pi := \varepsilon_{\overline{P}(l,d,R),\,[n]}(e,e)$. Then $d^{u\,v}_{w}(P) = 0$ unless $u \in \mathcal{S}_n$ and $\in \mathcal{S}_m$, and in that case,

$$d^{u\,v}_{w}(P) \;=\; c^{(u\times v)\pi}_{\pi\; w}.$$

Moreover, $d^{u\,v}_{w}(P) \neq 0$ implies that $a := \#P\bigcap[d]$ exceeds the last descent of u and $b := d-a$ exceeds the last descent of v.

Remark 4.6.2. Theorem 4.6.1 generalizes [34, 1.5] (see also [40, 4.19]) where it is shown that $d^{u\,v}_{w}([a]) \geq 0$. Define I_P to be

$$\{\varepsilon_{\overline{P}(l,l,R),[n]}(e,e) \mid l \in \mathbb{N},\;\; n = l + \#(P\bigcap[l]),\;\text{ and } R \subset \{l+1,\ldots,3l\}, \#R = l\}.$$

For $w \in \mathcal{S}_n$, let N be an integer such that $N/3$ exceeds both the last descent the length of w. If $\pi \in I_P$ with $\pi \notin \mathcal{S}_N$, then $\pi = \varepsilon_{\overline{P}(l,d,R),[n]}(e,e)$ for l,d,R satisfying the conditions of Theorem 4.6.1 and so $d^{u\,v}_{w}(P) \;=\; c^{(u\times v)\pi}_{\pi\, w}$ for any $\pi \in I_P - \mathcal{S}_N$, which establishes Theorem 1.2.2.

Apply the ring homomorphism Ψ_P to both sides of the product:

$$\mathfrak{S}_w(x)\;\mathfrak{S}_\gamma(x) \quad=\quad \sum_\zeta c^{\zeta}_{w\,\gamma}\,\mathfrak{S}_\zeta(x).$$

If we expand this in terms of $\mathfrak{S}_\eta(y)\;\mathfrak{S}_\xi(z)$ and equate the coefficients, we get a corollary.

Corollary 4.6.3. *Let $w,\gamma,\eta,\xi \in \mathcal{S}_\infty$, and $P \subset \mathbb{N}$. Then there exists an integer $N \in \mathbb{N}$ such that if $\pi \in I_P - \mathcal{S}_N$, then*

$$\sum_\zeta c^{(\eta\times\xi)\pi}_{\pi\;\;\zeta}\;c^{\zeta}_{w\;\gamma} \quad=\quad \sum_{u,v,\alpha,\beta} c^{(u\times v)\pi}_{\pi\;\;w}\;c^{(\alpha\times\beta)\pi}_{\pi\;\;\gamma}\;c^{\eta}_{u\;\alpha}\;c^{\xi}_{v\;\beta}.$$

Proof of Theorem 4.6.1. First, a Schubert polynomial $\mathfrak{S}_\pi(x) \in \mathbb{Z}[x_1,\ldots,x_s]$ if and only if s exceeds the last descent of π [33] (see also [40, 4.13]). Thus, $\mathfrak{S}_w(x) \in \mathbb{Z}[x_1,\ldots,x_d]$, and if $d^{u\,v}_{w}(P) \neq 0$, then $\mathfrak{S}_u(y) \in \mathbb{Z}[y_1,\ldots,y_a]$ and $\mathfrak{S}_v(z) \in \mathbb{Z}[z_1,\ldots,z_b]$, hence a, respectively, b, exceeds the last descent of u, respectively v. Since $\deg\mathfrak{S}_w(x) \leq l$, both $\deg\mathfrak{S}_u(y)$ and $\deg\mathfrak{S}_v(z)$ are at most l. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[x_1,\ldots,x_d] \arrow[r, hook, "\iota"] & \mathbb{Z}[x_1,\ldots,x_{n+m}] \arrow[r, "\overline{\Psi_P}"] \arrow[d] & \mathbb{Z}[y_1,\ldots,y_n,z_1,\ldots,z_m] \arrow[d] \\
& H^*\mathbb{F}\ell_{n+m} \arrow[r, "\psi^*_{\overline{P}}"] & H^*\mathbb{F}\ell_n \otimes H^*\mathbb{F}\ell_m
\end{tikzcd}$$

Here, $\overline{\Psi_P}$ is the restriction of $\Psi_{\overline{P}}$ to $\mathbb{Z}[x_1,\ldots,x_{n+m}]$. The vertical arrows are injective on the module $\mathbb{Z}\langle x_1^{\alpha_1}\cdots x_d^{\alpha_d} \mid \alpha_i \leq l\rangle$ and its image

$$\mathbb{Z}\langle y_1^{\beta_1}\cdots y_a^{\beta_a} z_1^{\gamma_1}\cdots z_b^{\gamma_b} \mid \beta_i,\gamma_j \leq l\rangle \quad\subset\quad \mathbb{Z}[y_1,\ldots,y_n,z_1,\ldots,z_m].$$

Moreover, since $P\bigcap[d] = \overline{P}\bigcap[d]$, the composition, $\Psi_{\overline{P}}\circ\iota$, of the top row coincides with $\Psi_P\circ\iota$. Since $\mathfrak{S}_w(x)\in\mathbb{Z}\langle x_1^{\alpha_1}\cdots x_d^{\alpha_d}\mid \alpha_i\leq l\rangle$, the cohomological formula for $\psi^*_{\overline{P}}(\mathfrak{S}_w)$ in Theorem 4.5.4 computes $\Psi_P(\mathfrak{S}_w(x))$. ▛

In the statement of the Theorem 4.6.1, $\ell(w)$ could be replaced by $\max_i\{\deg_{x_i}(\mathfrak{S}_w(x))\}$.

4.7. **Products of Grassmannians.** Let $k\leq n$ and $l\leq m$ be integers and $V\simeq\mathbb{C}^n$ and $W\simeq\mathbb{C}^m$. Define $\varphi_{k,l}: \mathit{Grass}_kV\times \mathit{Grass}_lW\hookrightarrow \mathit{Grass}_{k+l}(V\oplus W)$ by

$$\varphi_{k,l}\ :\ (H,K)\quad\longmapsto\quad H\oplus K.$$

Theorem 4.7.1.

(*i*) *For any Schubert class $S_\lambda\in H^*\mathit{Grass}_{k+l}V\oplus W$,*

$$\varphi^*_{k,l}(S_\lambda)\quad=\quad\sum_{\mu,\nu}c^\lambda_{\mu\,\nu}S_\mu\otimes S_\nu.$$

(*ii*) *If $S_{\mu^c}\otimes S_{\nu^c}\in H^*\mathit{Grass}_kV\otimes H^*\mathit{Grass}_lW$, then*

$$(\varphi_{k,l})_*(S_{\mu^c}\otimes S_{\nu^c})\quad=\quad\sum_\lambda c^\lambda_{\mu\,\nu}S_{\lambda^c},$$

where λ^c,μ^c, and ν^c are defined by $\mu^c_i=n-k-\mu_{k+1-i}$, $\nu^c_i=m-l-\nu_{l+1-i}$, and $\lambda^c_i=m+n-k-l-\lambda_{k+l+1-i}$.

Remark 4.7.2. If $-x_1,\ldots,-x_k$ are the Chern roots of the tautological k-plane bundle over Grass_kV, and $-y_1,\ldots,-y_l$ those of the tautological l-plane bundle over Grass_lW, and $f\in H^*\mathit{Grass}_{k+l}V\oplus W$ (which is a symmetric polynomial in the negative Chern roots of the tautological bundle over $\mathit{Grass}_{k+l}V\oplus W$). Then

$$\varphi^*_{k,l}f\ =\ f(x_1,\ldots,x_k,y_1,\ldots,y_l).$$

Let $\Lambda=\Lambda(z)$ be the ring of symmetric functions, the inverse limit (over n) in the category of graded rings of the rings of symmetric polynomials in the variables $z_1,\ldots,z_n$. Fixing λ and choosing k,l,n, and m large enough gives a new proof of [41, I.5.9]:

Proposition 4.7.3 ([41, I.5.9])**.** *Let λ be a partition and x, y be infinite sets of variables. Then*

$$S_\lambda(x,y)\ =\ \sum_{\mu,\nu}c^\lambda_{\mu\,\nu}\,S_\mu(x)\cdot S_\nu(y),$$

where S_μ denotes the Schur function basis of the ring Λ of symmetric functions.

If we define a linear map $\Delta:\Lambda(z)\to\Lambda(x)\otimes_{\mathbb{Z}}\Lambda(y)$ by $\Delta(f(z))=f(x,y)$, then Δ is induced by the maps $\varphi^*_{k,l}$. Moreover, the obvious commutative diagrams of spaces give a new proof of [41, I.5.24], that Λ is a cocommutative Hopf algebra with comultiplication Δ.

Proof of Theorem 4.7.1. The first statement is a consequence of the second: Schubert classes form a basis for the cohomology ring, so there exist integral constants $d_\lambda^{\mu\,\nu}$ such that

$$\varphi_{k,l}^*(S_\lambda) \quad = \quad \sum_{\mu,\nu} d_\lambda^{\mu\,\nu} S_\mu \otimes S_\nu.$$

Since the Schubert basis diagonalizes the intersection pairing,

$$d_\lambda^{\mu\,\nu} \quad = \quad \deg(\varphi_{k,l}^*(S_\lambda)\cdot(S_{\mu^c}\otimes S_{\nu^c})).$$

Apply $(\varphi_{k,l})_*$ and use the second assertion to obtain

$$\begin{aligned} d_\lambda^{\mu\,\nu} &= \deg(S_\lambda\cdot(\varphi_{k,l})_*(S_{\mu^c}\otimes S_{\nu^c}))\\ &= S_\lambda\cdot\sum_\kappa c_{\mu\,\nu}^\kappa S_{\kappa^c}\\ &= c_{\mu\,\nu}^\lambda. \end{aligned}$$

The second assertion is a consequence of the following lemma.

Lemma 4.7.4. *Suppose μ,ν are partitions with $\mu\subset(n-k)^k$ and $\nu\subset(m-l)^l$. Let $E_\bullet\in\mathbb{F}\ell V$ and $F_\bullet\in\mathbb{F}\ell W$ and let $G_\bullet'$ be any flag opposite to $\psi_{[n]}(E_\bullet,F_\bullet)$ with $G_m'=W$. Then*

$$\varphi_{k,l}\left(\Omega_{\mu^c}E_\bullet\times\Omega_{\nu^c}F_\bullet\right) \quad = \quad \Omega_{\rho^c}\psi_{[n]}(E_\bullet,F_\bullet)\bigcap\Omega_{(m-l)^k}G_\bullet{}', \tag{4.7.1}$$

where ρ is the partition

$$\nu_1+(n-k)\ \geq\ \cdots\ \geq\ \nu_l+(n-k)\ \geq\ \mu_1\ \geq\cdots\ \geq\mu_k.$$

We finish the proof of Theorem 4.7.1. Lemma 4.7.4 implies

$$\begin{aligned} \left(\varphi_{k,l}\right)_*\left(S_{\mu^c}\otimes S_{\nu^c}\right) &= \left[\Omega_{\rho^c}\psi_{[n]}(E_\bullet,F_\bullet)\bigcap\Omega_{(n-k)^l}G_\bullet{}'\right]\\ &= \sum_\lambda c_{\rho^c\,(n-k)^l}^{\lambda^c}\ S_{\lambda^c}. \end{aligned}$$

Since $\deg(S_\alpha\cdot S_\beta\cdot S_\gamma)=c_{\beta\,\gamma}^{\alpha^c}$, we see that

$$c_{\rho^c\,(n-k)^l}^{\lambda^c} \ = \ c_{(n-k)^l\,\lambda}^{\rho} \ = \ c_\lambda^{\rho/(n-k)^l} \ = \ c_\lambda^{\mu\coprod\nu} \ = \ c_{\mu\,\nu}^\lambda.$$

Here, $\mu\coprod\nu$ is a skew partition with two components μ and ν and the last equality is a special case of (1.3.1) in §1.3. ∎

Proof of Lemma 4.7.4. Since $\Omega_{(n-k)^l}G_\bullet{}'=\{M\in Grass_{k+l}V\oplus W\mid \dim M\bigcap G_m'\geq l\}$ and $G_m'=W$, we see that $\varphi_{k,l}(Grass_kV\times Grass_lW)\subset\Omega_{(n-k)^l}G_\bullet{}'$. It is an exercise in the definition of the Schubert varieties involved and of $\psi_{[n]}(E_\bullet,F_\bullet)$ to see that

$$\varphi_{k,l}(\Omega_{\mu^c}E_\bullet\times\Omega_{\nu^c}F_\bullet)\ \subset\ \Omega_{\rho^c}\psi_{[n]}(E_\bullet,F_\bullet),$$

which shows the inclusion $\subset$ in (4.7.1). Equality follows as they have the same dimension: The intersection has dimension $|\rho|-|(n-k)^l| \;=\; |\mu|+|\nu|$, the dimension of $\Omega_{\mu^c}E_\bullet \times \Omega_{\nu^c}F_\bullet$. ∎

5. Symmetries of the Littlewood-Richardson coefficients

5.1. **Proof of Theorem 1.3.1 (*ii*).** Combining Lemma 4.3.5 with Theorem 4.4.2, we deduce:

Lemma 5.1.1. *Suppose $x \leq_k z$ and $x(p) = z(p)$. Let $k' = k-1$ if $p < k$ and $k' = k$ otherwise. Then for any partition λ, we have*

$$c^{z}_{x\, v(\lambda,k)} \quad = \quad c^{z/_p}_{x/_p\, v(\lambda,k')}.$$

Note that zx^{-1} and $z/_p(x/_p)^{-1}$ are shape-equivalent, by Lemma 4.1.1 (*iv*).

Lemma 5.1.2. *Let $x, z, u, w \in \mathcal{S}_n$. Suppose $x \leq_k z$, $u \leq_k w$, and $zx^{-1} = wu^{-1}$. Further suppose that w is Grassmannian with descent k, the permutation wu^{-1} has no fixed points, and for $k < i \leq n$, $u(i) = x(i)$. Then, for any partition λ with at most k parts,*

$$c^{w}_{u\, v(\lambda,k)} \quad = \quad c^{z}_{x\, v(\lambda,k)}.$$

Proof of Theorem 1.3.1 (*ii*) using Lemma 5.1.2. We reduce Theorem 1.3.1 (*ii*) to the special case of Lemma 5.1.2. First, by Lemma 5.1.1, it suffices to prove Theorem 1.3.1 (*ii*) when $x, z, u, w \in \mathcal{S}_n$, $k = l$, with $wu^{-1} = zx^{-1}$ and the permutation wu^{-1} has no fixed points.

Define $s \in \mathcal{S}_n$ by

$$s(i) \quad := \quad \begin{cases} u(i) & 1 \leq i \leq k \\ x(i) & k < i \leq n \end{cases}$$

and set $t := wu^{-1}s$. Then $s \leq_k t$ and

$$t(i) \quad = \quad \begin{cases} w(i) & 1 \leq i \leq k \\ z(i) & k < i \leq n \end{cases}.$$

It suffices to show separately that $c^{w}_{u\, v(\lambda,k)}$ and $c^{t}_{s\, v(\lambda,k)}$ each equal $c^{z}_{t\, v(\lambda,k)}$. Thus we may further assume $u(i) = x(i)$ for $1 \leq i \leq k$ or $u(i) = x(i)$ for $k < i \leq n$.

Suppose that $u(i) = x(i)$ for $1 \leq i \leq k$. If for $v \in \mathcal{S}_n$, $\overline{v} := w_0 v w_0$,

$$c^{z}_{x\, v(\lambda,k)} \quad = \quad c^{w}_{u\, v(\lambda,k)} \quad \Longleftrightarrow \quad c^{\overline{z}}_{\overline{x}\, \overline{v(\lambda,k)}} \quad = \quad c^{\overline{w}}_{\overline{u}\, \overline{v(\lambda,k)}}.$$

Set $l = n-k$ and λ^t the partition conjugate to λ. Then $\overline{x} \leq_{k'} \overline{z}$, $\overline{u} \leq_{k'} \overline{w}$, $\overline{z}(\overline{x}^{-1}) = \overline{wu}^{-1}$, $\overline{v(\lambda,k)} = v(\lambda^t, l)$, and $\overline{x}(i) = \overline{u}(i)$ for $l < i \leq n$. Thus we may assume $x(i) = u(i)$ for $1 \leq i \leq k$.

Finally, there is a Grassmannian permutation $t \in \mathcal{S}_n$ with descent k and a permutation $s \in \mathcal{S}_n$ such that $t = wu^{-1}s$. Thus it suffices to further assume that w is Grassmannian with descent k, the situation of Lemma 5.1.2. ◼

We prove Lemma 5.1.2 by studying two intersections of Schubert varieties and their image under the projection $\mathbb{F}\ell V \twoheadrightarrow Grass_k V$. Let $e_1, \ldots, e_n$ be a basis for V and set $F_\bullet = \langle\langle e_1, \ldots, e_n\rangle\rangle$. Let $M(w) \subset M_{n\times n}\mathbb{C}$ be the set of matrices satisfying the conditions:

(a) $M(w)_{i,w(i)} = 1$
(b) $M(w)_{i,j} = 0$ if either $w(i) < j$ or else $w^{-1}(j) < i$.

Then $M(w) \simeq \mathbb{C}^{\ell(w)}$ as the only unconstrained entries of $M(w)$ are $M(w)_{i,j}$ when $j < w(i)$ and $i < w^{-1}(j)$, and there are $\ell(w)$ such entries.

Example 5.1.3. Let $w = 25134 \in \mathcal{S}_5$, a Grassmannian permutation with descent 2. Then $M(w)$ is the set of matrices

$$\left\{ \left[\begin{array}{ccccc} a & 1 & 0 & 0 & 0 \\ b & 0 & c & d & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \;\middle|\; \text{for any } (a,b,c,d) \in \mathbb{C}^4 \right\}$$

Fix a basis $e_1, \ldots, e_n$ for V. For $\alpha \in M(w)$, and $1 \leq i \leq n$, define the vector $f_i(\alpha) := \sum_j \alpha_{i,j} e_j$. Then $f_1(\alpha), \ldots, f_n(\alpha)$ are the 'row vectors' of the matrix α and they form a basis for V as α has determinant $(-1)^{\ell(w)}$. Set $E_\bullet(\alpha) = \langle\langle f_1(\alpha), \ldots, f_n(\alpha)\rangle\rangle$. Since $f_i(\alpha) \in F_{w(i)} - F_{w(i)-1}$, we see that $E_\bullet(\alpha) \in X^\circ_{w_0 w} F_\bullet$. Moreover, given $E_\bullet \in X^\circ_{w_0 w} F_\bullet$, restricted row reduction on a basis for $E_\bullet$ shows there is a unique $\alpha \in M(w)$ with $E_\bullet = E_\bullet(\alpha)$. In the case that w is Grassmannian with descent k, matrices in $M(w)$ have a simple form: if $k < i$, then $f_i(\alpha) = e_{w(i)}$.

For opposite flags $F_\bullet, F_\bullet'$, $\mathfrak{S}_{w_0 w} \cdot \mathfrak{S}_u$ is the class Poincaré dual to the fundamental cycle of $X_{w_0 w} F_\bullet \bigcap X_u F_\bullet'$. We use the projection formula (2.3.1) to compute the coefficient $c^w_{u\,v(\lambda,k)}$:

$$\begin{aligned} c^w_{u\,v(\lambda,k)} &= \deg(S_\lambda(x_1, \ldots, x_k) \cdot \mathfrak{S}_{w_0 w} \cdot \mathfrak{S}_u) \\ &= \deg(\pi_k)_*(S_\lambda(x_1, \ldots, x_k) \cdot \mathfrak{S}_{w_0 w} \cdot \mathfrak{S}_u) \\ &= \deg(S_\lambda \cdot (\pi_k)_*(\mathfrak{S}_{w_0 w} \cdot \mathfrak{S}_u)) \end{aligned}$$

Thus Lemma 5.1.2 is a consequence of the following calculation:

Lemma 5.1.4. *Let u, w, x, z satisfy the hypotheses of Lemma 5.1.2. Then, if $F_\bullet$ and $F_\bullet'$ are opposite flags in V,*

$$\pi_k\left(X_{w_0 w} F_\bullet \bigcap X_u F_\bullet'\right) \quad = \quad \pi_k\left(X_{w_0 z} F_\bullet \bigcap X_x F_\bullet'\right).$$

Proof. Let $e_1,\ldots,e_n$ be a basis for V such that $F_\bullet = \langle\langle e_1,\ldots,e_n\rangle\rangle$ and $F_\bullet' = \langle\langle e_n,\ldots,e_1\rangle\rangle$, and define $M(w)$ as before. Let $A \subset M(w)$ consist of those matrices α such that $E_\bullet(\alpha) \in X_u^\circ F_\bullet'$. If $j > k$, set $g_j(\alpha) = f_j(\alpha) = e_{w(j)}$. For $j \leq k$ construct $g_j(\alpha)$ inductively, setting $g_j(\alpha)$ to be the intersection of $F'_{n+1-u(j)}$ and the affine space $f_j(\alpha) + \langle g_i(\alpha) \mid i < j \text{ and } u(i) < u(j)\rangle$. Since $E_\bullet(\alpha) \in X_u F_\bullet$ and $\dim E_j(\alpha) \bigcap F'_{n+1-u(j)} = \#\{i \leq j \mid u(i) > u(j)\}$, this intersection consists of a single, non-zero vector, $g_j(\alpha)$.

Then the algebraic map $A \ni \alpha \mapsto (g_1(\alpha),\ldots,g_n(\alpha)) \in V^n$ gives a parameterized basis of V. Moreover, $E_\bullet(\alpha) = \langle\langle g_1(\alpha),\ldots,g_n(\alpha)\rangle\rangle$ for all $\alpha \in A$, and if $1 \leq j \leq k$, then $g_j(\alpha) \in F'_{n+1-u(j)} \bigcap F_{w(j)}$. Note that for $\alpha \in A$,

$$G_\bullet(\alpha) \;:=\; \langle\langle g_{u^{-1}x(1)}(\alpha),\ldots,g_{u^{-1}x(n)}(\alpha)\rangle\rangle \;\in\; X_{w_0 z}F_\bullet \bigcap X_x F_\bullet'. \qquad (5.1.1)$$

and thus A also parameterizes a subset of $X_{w_0 z}F_\bullet \bigcap X_x F_\bullet'$. Indeed, for $1 \leq j \leq k$, $g_{u^{-1}x(j)} \in F'_{n+1-x(j)} \bigcap F_{y(j)}$. Also for $j > k$, we have $u^{-1}x(j) = j = w^{-1}y(j)$, thus $g_j(\alpha) = f_j(\alpha) = e_{y(j)}$ and $G_j(\alpha) = E_j(\alpha)$. The definition of Schubert cells for the flag manifold in §2.3 then implies (5.1.1).

Both cycles $X_{w_0 w}F_\bullet \bigcap X_u F_\bullet'$ and $X_{w_0 z}F_\bullet \bigcap X_x F_\bullet'$ are irreducible and have the same dimension, $\ell(w) - \ell(u) = |wu^{-1}|$. Since $G_\bullet(\alpha) = G_\bullet(\beta)$ if and only if $\alpha = \beta$, the loci of flags $\{G_\bullet(\alpha) \mid \alpha \in A\}$ is dense in $X_x F_\bullet' \bigcap X_{w_0 y}F_\bullet$. Finally, for $\alpha \in A$, we have $G_k(\alpha) = E_k(\alpha)$, as $u^{-1}x$ permutes $\{1,\ldots,k\}$, which completes the proof. ▪

5.2. **Proof of Theorem 1.3.3 (*ii*).** We show that if ζ and η are disjoint permutations and λ any partition, then

$$c_\lambda^{\zeta\eta} \;=\; \sum_{\mu,\nu} c^\lambda_{\mu\,\nu}\; c^\zeta_\mu\; c^\eta_\nu.$$

Lemma 5.2.1. *Let $\zeta,\eta \in \mathcal{S}_{n+m}$ be disjoint permutations. Suppose $k \geq \#up_\zeta$, $l \geq \#up_\eta$, $n \geq \#supp_\zeta$, and $m \geq \#supp_\eta$. Let $u \in \mathcal{S}_{n+m}$ be a permutation such that $u \leq_{k+l} \zeta\eta u$. Let Q be any element of $\binom{[n+m]-supp_\eta}{n}$ which contains $supp_\zeta$ for which $k = \#u^{-1}(Q)\bigcap[k+l]$. Set $Q^c := [n+m] - Q$.*

Define $\zeta' \in \mathcal{S}_n$ and $\eta' \in \mathcal{S}_m$ by $\phi_Q(\zeta') = \zeta$ and $\phi_{Q^c}(\eta') = \eta$. Set $P = u^{-1}(Q)$, $P^c = u^{-1}(Q^c)$, and define $v \in \mathcal{S}_n$ and $w \in \mathcal{S}_m$ by $u(p_i) = q_{v(i)}$ and $u(p_i^c) = q^c_{w(i)}$, where

$$\begin{array}{lllllllllll} P &=& p_1 &<& p_2 &<& \cdots &<& p_n \\ Q &=& q_1 &<& q_2 &<& \cdots &<& q_n \end{array} \qquad \begin{array}{lllllllllll} P^c &=& p_1^c &<& p_2^c &<& \cdots &<& p_m^c \\ Q^c &=& q_1^c &<& q_2^c &<& \cdots &<& q_m^c \end{array}$$

Then

(*i*) *$v \leq_k \zeta' v$ and $w \leq_l \eta' w$,*
(*ii*) *$u = \varepsilon_{P,Q}(v,w)$ and $\zeta\eta u = \varepsilon_{P,Q}(\zeta' v, \eta' w)$, and*

(iii) For any pairs of opposite flags $E_\bullet, E'_\bullet \in \mathbb{F}\ell_n$ and $F_\bullet, F'_\bullet \in \mathbb{F}\ell_m$,

$$\psi_P\left[\left(X_{w_0^{(n)}\zeta' v}E_\bullet \bigcap X_v E'_\bullet\right) \times \left(X_{w_0^{(m)}\eta' w}F_\bullet \bigcap X_w F'_\bullet\right)\right] = X_{w_0^{(n+m)}\zeta\eta u}\psi_Q(E_\bullet, F_\bullet) \bigcap X_u \psi_{w_0^{(m+n)}Q}(E'_\bullet, F'_\bullet).$$

Proof. Since $u \leq_{k+l} \zeta\eta u$, (i) follows from Theorem 1.1.2 and the definitions. The second statement is also immediate. For (iii), Lemma 4.5.1 shows the inclusion $\subset$. Since ζ' is shape equivalent to ζ, η' to η, and ζ and η are disjoint, $|\zeta\eta| = |\zeta'| + |\eta'|$, showing both cycles have the same dimension, and hence are equal, as $\psi_Q(E_\bullet, F_\bullet)$ and $\psi_{w_0^{(m+n)}Q}(E'_\bullet, F'_\bullet)$ are opposite flags. ∎

Note that if $u \leq_k \zeta u$, then

$$c_\lambda^\zeta \quad = \quad \deg(S_\lambda \cdot (\pi_k)_*(\mathfrak{S}_{w_0\zeta u} \cdot \mathfrak{S}_u)).$$

So the skew Littlewood-Richardson coefficients c_λ^ζ are defined by the identity in $H^* Grass_k V$:

$$(\pi_k)_*(\mathfrak{S}_{w_0\zeta u}\cdot\mathfrak{S}_u) \quad = \quad \sum_{\lambda\subset(n-k)^k} c_\lambda^\zeta\, S_{\lambda^c}. \tag{5.2.1}$$

Proof of Theorem 1.3.3 (ii). We use the notation of Lemma 5.2.1. The following diagram commutes since $[k+l] = \{p_1, \ldots, p_k, p_1^c, \ldots, p_l^c\}$.

$$\begin{array}{ccc}
\mathbb{F}\ell_n \times \mathbb{F}\ell_m & \xrightarrow{\quad\psi_P\quad} & \mathbb{F}\ell_{n+m} \\
{\scriptstyle \pi_k \times \pi_l}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{k+l}} \\
Grass_k\mathbb{C}^n \times Grass_l\mathbb{C}^m & \xrightarrow{\varphi_{k,l}} & Grass_{k+l}\mathbb{C}^{n+m}
\end{array}$$

From this and Lemma 5.2.1, we see that

$$\pi_{k+l}\left(X_{w_0^{(n+m)}\zeta\eta u}\psi_Q(E_\bullet, F_\bullet) \bigcap X_u \psi_{w_0^{(m+n)}Q}(E'_\bullet, F'_\bullet)\right)$$

is equal to

$$\varphi_{k,l}\left(\pi_k\left(X_{w_0^{(n)}\zeta' v}E_\bullet \bigcap X_v E'_\bullet\right) \times \pi_l\left(X_{w_0^{(m)}\eta' w}F_\bullet \bigcap X_w F'_\bullet\right)\right).$$

Thus $(\pi_{k+l})_*\left(\mathfrak{S}_{w_0^{(n+m)}\zeta\eta u} \cdot \mathfrak{S}_u\right)$ is equal to

$$(\varphi_{k,l})_*\left((\pi_k)_*\left(\mathfrak{S}_{w_0^{(n)}\zeta' v} \cdot \mathfrak{S}_v\right) \otimes (\pi_l)_*\left(\mathfrak{S}_{w_0^{(m)}\eta' w} \cdot \mathfrak{S}_w\right)\right).$$

This, together with (5.2.1), gives

$$\begin{aligned}
\sum_\lambda c_\lambda^{\zeta\eta} S_{\lambda^c} &= (\pi_{k+l})_* \left(\mathfrak{S}_{w_0^{(n+m)}\zeta\eta u} \cdot \mathfrak{S}_u\right) \\
&= (\varphi_{k,l})_* \left(\sum_\mu c_\mu^{\zeta'} S_{\mu^c} \otimes \sum_\nu c_\nu^{\eta'} S_{\nu^c}\right) \\
&= \sum_{\mu,\nu} c_\mu^{\zeta'} c_\nu^{\eta'} (\varphi_{k,l})_* \left(S_{\mu^c} \otimes\ S_{\nu^c}\right) \\
&= \sum_{\mu,\nu} c_\mu^{\zeta'} c_\nu^{\eta'} \sum_\lambda c_{\mu\,\nu}^\lambda S_{\lambda^c}.
\end{aligned}$$

This completes the proof, as ζ', ζ and η', η are shape equivalent pairs. ∎

5.3. **Theorem 1.3.4′ (Cyclic Shift).** *Let $u, w, x, z \in \mathcal{S}_\infty$ with $u \leq_k w$ and $x \leq_l z$. Suppose $wu^{-1} \in \mathcal{S}_n$ and zx^{-1} is shape equivalent to $(wu^{-1})^{(1\,2\,\ldots\,n)^t}$, for some t. For any partition λ,*

$$c^w_{u\,v(\lambda,k)} \quad = \quad c^z_{x\,v(\lambda,l)}.$$

Proof. By Theorem 1.3.1 (*ii*), it suffices to prove a restricted case. Suppose $u, w \in \mathcal{S}_n$, $u \leq_k w$, and w is Grassmannian with descent k. The idea is to construct permutations $x, z \in \mathcal{S}_n$ with $x \leq_k z$ and $zx^{-1} = (wu^{-1})^{(1\,2\,\ldots\,n)}$ for which

$$\pi_k \left(X_{w_0 w} F_{\bullet} \bigcap X_u F_{\bullet}'\right) \quad = \quad \pi_k \left(X_{w_0 z} G_{\bullet} \bigcap X_x G_{\bullet}'\right), \tag{5.3.1}$$

where $e_1, \ldots, e_n$ be a basis for V and the flags $F_{\bullet}, F_{\bullet}', G_{\bullet}$, and $G_{\bullet}'$ are

$$\begin{aligned}
F_{\bullet} &= \langle\langle e_1, \ldots, e_n\rangle\rangle & F_{\bullet}' &= \langle\langle e_n, \ldots, e_1\rangle\rangle \\
G_{\bullet} &= \langle\langle e_n, e_1, \ldots, e_{n-1}\rangle\rangle & G_{\bullet}' &= \langle\langle e_{n-1}, \ldots, e_1, e_n\rangle\rangle.
\end{aligned}$$

Then (5.3.1) implies the identity $c^w_{u\,v(\lambda,k)} = c^z_{x\,v(\lambda,k)}$, which completes the proof.

If $wu^{-1}(n) = n$, then $zx^{-1} = 1 \times wu^{-1}$, which is shape equivalent to wu^{-1}, and the result follows by Theorem 1.3.1 (*ii*). Assume $wu^{-1}(n) \neq n$. Then $w(k) = n$ and $u(k) < n$, as w is Grassmannian with descent k. Set $m := u(k)$, $p := u^{-1}(n)(> k)$, and $l := w(p)$. Define $x \in \mathcal{S}_n$ by:

$$x(j) \quad = \quad \begin{cases} u(j) + 1 & 1 \leq j < k \text{ or } p < j \\ 1 & j = k \\ m + 1 & j = k + 1 \\ u(j-1) + 1 & k + 1 < j \leq p \end{cases}.$$

Then $x \leq_k z := (wu^{-1})^{(1\,2\,\ldots\,n)}x$ where

$$z(j) \quad = \quad \begin{cases} w(j)+1 & 1 \leq j < k \text{ or } p < j \\ l+1 & j = k \\ 1 & j = k+1 \\ w(j-1)+1 & k+1 < j \leq p \end{cases} .$$

To show (5.3.1), let $g_1(\alpha), \ldots, g_n(\alpha)$ for $\alpha \in A$ be the parameterized basis for flags $E_\bullet(\alpha) \in X_u^\circ F_\bullet' \bigcap X_{w_0 w}^\circ F_\bullet$ constructed in the proof of Lemma 5.1.4. Since $g_k(\alpha) \in F'_{n+1-u(k)} \bigcap F_{w(k)}$, $u(k) = m$, and $w(k) = n$, there exist regular functions $\beta_j(\alpha)$ on A such that

$$g_k(\alpha) = e_n + \sum_{j=m}^{n-1} \beta_j(\alpha) e_j.$$

Since $F_1' = \langle e_n \rangle \subset E_p(\alpha) - E_{p-1}(\alpha)$ and $g_p(\alpha) = e_l$, there exist regular functions $\delta_j(\alpha)$ on A with $\delta_p(\alpha)$ nowhere vanishing such that

$$\begin{aligned} e_n \quad &= \quad \sum_{j=1}^{p} \delta_j(\alpha) g_j(\alpha) \\ &= \quad g_k(\alpha) + \sum_{j=1}^{k-1} \delta_j(\alpha) g_j(\alpha) + \sum_{j=k+1}^{p} \delta_j(\alpha) e_{w(j)}, \end{aligned}$$

as $g_k(\alpha)$ is the only vector among the $g_j(\alpha)$ in which e_n has a non-zero coefficient. Thus

$$e_n - \sum_{j=k+1}^{p} \delta_j(\alpha) e_{w_j} \quad = \quad g_k(\alpha) + \sum_{j=1}^{k-1} \delta(\alpha) g_j(\alpha)$$

is a vector in $E_k(\alpha) - E_{k-1}(\alpha)$.

Define a basis $h_1(\alpha), \ldots, h_n(\alpha)$ for V by

$$h_j(\alpha) \quad = \quad \begin{cases} g_j(\alpha) & 1 \leq j < k \text{ or } p < j \\ e_n - \left(\sum_{j=k+1}^{p} \delta_j(\alpha) e_{w(j)} \right) & j = k \\ e_n & j = k+1 \\ g_{j-1}(\alpha) & k+1 < j \leq p \end{cases} .$$

We claim $E_\bullet'(\alpha) := \langle\langle h_1(\alpha), \ldots, h_n(\alpha) \rangle\rangle$ is a flag in $X_{w_0 z} G_\bullet \bigcap X_x G_\bullet{}'$, which implies (5.3.1): Since $h_k(\alpha) \in E_k(\alpha) - E_{k-1}(\alpha)$ and $h_j(\alpha) = g_j(\alpha)$ for $j < k$, we have

$$E_k'(\alpha) \quad = \quad \langle h_1(\alpha), \ldots, h_k(\alpha) \rangle \quad = \quad E_k(\alpha).$$

Thus if $\alpha \neq \alpha'$, then $E_\bullet'(\alpha) \neq E_\bullet'(\alpha')$ and so $\{E_\bullet'(\alpha) \mid \alpha \in A\}$ is a subset of the intersection $X_{w_0 z} G_\bullet \bigcap X_x G_\bullet{}'$ of dimension equal to $\dim A = \ell(w) - \ell(u) = \ell(z) - \ell(x)$, the dimension of $X_{w_0 z} G_\bullet \bigcap X_x G_\bullet{}'$. Thus $\{E_\bullet'(\alpha) \mid \alpha \in A\}$ is dense, and so $E_k'(\alpha) = E_k(\alpha)$ implies (5.3.1).

For notational convenience, set $G^\circ_j := G_j - G_{j-1}$, and similarly for F°_j. To establish this claim, we first show that $h_j(\alpha) \in G^\circ_{z(j)}$ for $j = 1, \ldots, n$, which shows $h_1(\alpha), \ldots, h_n(\alpha)$ is a parameterized basis for V and $E'_\bullet(\alpha) \in X_{w_0 z} G_\bullet$. Then, for a fixed $\alpha \in A$, we construct $h'_1, \ldots, h'_n$ which satisfy $E'_\bullet(\alpha) = \langle\langle h'_1, \ldots, h'_n \rangle\rangle$ and $h'_j \in G'_{n+1-x(j)}$ for $j = 1, \ldots, n$, showing $E'_\bullet(\alpha) \in X_x G_\bullet{}'$.

Note that if $i < n$, then $G_{i+1} = \langle e_n, F_i \rangle$. Thus $h_j(\alpha) \in F^\circ_{w(j)} \subset G^\circ_{z(j)}$ for $1 \leq j < k$ and $p < j$, and if $k+1 < j \leq p$, then $h_j(\alpha) \in F^\circ_{w(j-1)} \subset G^\circ_{z(j)}$. Then, since $G_1 = \langle e_n \rangle$, we see that $h_{k+1}(\alpha) = e_n \in G'_1 = G'_{n+1-x(k+1)}$. Finally, since w is Grassmannian of descent k, if $k+1 \leq i \leq p$, then $w(i) \leq w(p) = l$, which shows $h_k(\alpha) \in G^\circ_{l+1} = G^\circ_{z(k)}$. Thus $E'_\bullet(\alpha) \in X^\circ_{w_0 z} G_\bullet$.

We now show that $E'_\bullet(\alpha) \in X_x G_\bullet{}'$. Note that if $a \leq b < n$, then $F'_{n+1-a} \bigcap F_b \subset G'_{n-a} \cap G_{b+1}$. Thus if $1 \leq j < k$, $h_j(\alpha) = g_j(\alpha) \in F'_{n+1-u(j)} \bigcap F_{w(j)} \subset G'_{n+1-x(j)}$. Since $x(k) = 1$, we see that $h_k(\alpha) \in G'_{n+1-x(k)} = V$. Fix $\alpha \in A$ and set $h'_j = h_j(\alpha)$ for $1 \leq j \leq k$. Define

$$h'_{k+1} \ := \ g_k(\alpha) - e_n \ = \ \sum_{j=m}^{n-1} \beta_j(\alpha) e_j \ \in \ G'_{n+1-(m+1)} \ = \ G'_{n+1-x(k+1)}.$$

Since $h'_{k+1} + h_{k+1}(\alpha) = g_k(\alpha)$, we see that $E'_{k+1}(\alpha) = \langle E'_k(\alpha), H'_{k+1} \rangle$.

Finally, since $E_\bullet(\alpha) \in X_u F_\bullet{}'$, if $k < j$ there exists a vector

$$g'_j \ := \ \sum_{i \leq j} \gamma_{i,j} g_i(\alpha) \ \in \ F'_{n+1-u(j)}$$

such that $\langle E_{j-1}(\alpha), g'_j \rangle = E_j(\alpha)$. For $k+1 < j \leq p$, set

$$h'_j \ = \ g'_{j-1} - \gamma_{k,j-1} e_n \ \in \ \langle e_{n-1} \ldots, e_{n+1-u(j-1)} \rangle \ = \ G'_{n+1-x(j)},$$

as as $g_k(\alpha)$ is the only vector among $\{g_1(\alpha), \ldots, g_n(\alpha)\}$ which is not in the span of $e_1, \ldots, e_{n-1}$. If $p < j$, set $h'_j = g'_j - \gamma_{k,j} e_n \in G'_{n+1-x(j)}$. Then $\langle\langle h'_1, \ldots, h'_n \rangle\rangle = E'_\bullet(\alpha)$, completing the proof. ∎

6. Formulas for some Littlewood-Richardson coefficients

6.1. **A chain-theoretic interpretation.** We give a chain-theoretic interpretation for some Littlewood-Richardson coefficients c^ζ_λ in terms chains in either the k-Bruhat order or the $\preceq$-order, similar to the main results of [52]. If either $u \lessdot_k (\alpha, \beta) u$ or $\zeta \prec\!\!\cdot\, (\alpha, \beta)\zeta$ is a cover, label that edge in the Hasse diagram with the integer $\beta = \max\{\alpha, \beta\}$. Given a saturated chain in the k-Bruhat order from u to ζu, equivalently, a saturated $\preceq$-chain from e to ζ, the *word* of that chain is its sequence of edge labels. Given a word $\omega = a_1.a_2 \ldots a_m$, Schensted insertion [49] or [48, §3.3] of ω into the empty tableau gives a pair (S,T) of Young tableaux, where S is the *insertion tableau* and T the *recording tableau* of ω.

Let $\mu \subset \lambda$ be partitions. A permutation ζ is *shape-equivalent* to a skew Young diagram λ/μ if there is a k such that ζ is shape-equivalent to $v(\lambda,k)\cdot v(\mu,k)^{-1}$. It follows that ζ is shape equivalent to some skew partition λ/μ if and only if whenever $\alpha,\beta \in \mathrm{up}_\zeta$ or $\alpha,\beta\in \mathrm{down}_\zeta$,

$$\alpha \;<\; \beta \quad \Longleftrightarrow \zeta(\alpha)\;<\;\zeta(\beta).$$

We prove a stronger version of Theorem 1.3.2:

Theorem 6.1.1. *Let $\mu \subset \lambda$ be partitions and suppose $\zeta \in \mathcal{S}_\infty$ is shape equivalent to λ/μ. Then, for any partition ν*

(*i*) $c^\zeta_\nu = c^{\lambda/\mu}_\nu$, *and*

(*ii*) *For any standard Young tableau T of shape ν,*

$$c^\zeta_\nu \;=\; \#\left\{\begin{array}{c}\preceq\text{-chains from } e \text{ to } \zeta \text{ whose}\\ \text{word has recording tableau } T\end{array}\right\}$$

Equivalently, if $u \leq_k w$ and $wu^{-1}=\zeta$, then

$$c^w_{u\,v(\nu,k)} \;=\; \#\left\{\begin{array}{c}\text{Chains in } k\text{-Bruhat order from } u \text{ to}\\ w \text{ whose word has recording tableau } T\end{array}\right\}$$

Remark 6.1.2. Theorem 6.1.1 (*ii*) gives a combinatorial proof of Proposition 1.1.1, when wu^{-1} is shape equivalent to a skew partition. Theorem 6.1.1 (*ii*) is similar in form to Theorem 8 of [52]:

Theorem 8 [52]. *Suppose $\nu=(p,1^{q-1})$, a partition of 'hook' shape. Then for any $u,w\in\mathcal{S}_\infty$ and $k\in\mathbb{N}$, the constant $c^w_{u\,v(\nu,k)}$ counts either set*

$$(i)\ \left\{\begin{array}{l}\text{Chains in } k\text{-Bruhat order from } u \text{ to } w \text{ with}\\ \text{word } \ a_1<\cdots<a_p>a_{p+1}>\cdots>a_{p+q-1}.\end{array}\right\}.$$

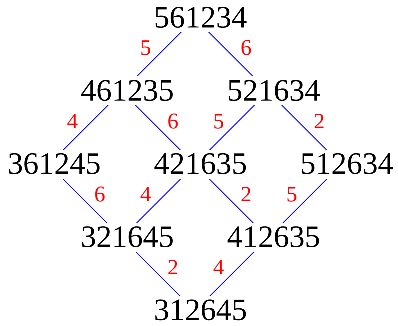
$$(ii)\ \left\{\begin{array}{l}\text{Chains in } k\text{-Bruhat order from } u \text{ to } w \text{ with}\\ \text{word } \ a_1>\cdots>a_q<a_{q+1}<\cdots<a_{p+q-1}.\end{array}\right\}.$$

The recording tableaux of words in (*i*) all have the integers $1,2,\ldots,p$ in the first row and $1,p+1,\ldots,p{+}q{-}1$ in the first column. Furthermore, these are the only words with this recording tableaux. Similarly, the recording tableaux of words in (*ii*) all have the integers $1,2,\ldots,q$ in the first column and $1,q{+}1,\ldots,p{+}q{-}1$ in the first row. However, Theorem 1.3.2 is *not* a generalization of this result: The permutation $\zeta := (143652)$ is not shape equivalent to any skew partition as $4,5\in \mathrm{up}_\zeta$ but $\zeta(4)>\zeta(5)$. Nevertheless, $c^\zeta_{(4,1)}=1$. Interestingly, ζ satisfies the conclusions of Theorem 1.3.2.

While the hypothesis of Theorem 1.3.2 is not necessary for the conclusion to hold, some hypotheses are necessary: Let $\zeta = (162)(354)$, a product of two disjoint 3-cycles. Then $\zeta^{(1\cdots 6)} = (132)(465) = v(\ydiagram{2,4}, 2)\cdot v(\ydiagram{2}, 2)^{-1}$. Hence, by Theorem 1.3.4, we have:

$$c^{\zeta}_{\ydiagram{4}} \quad = \quad c^{\zeta}_{\ydiagram{1,3}} \quad = \quad c^{\zeta}_{\ydiagram{2,2}} \quad = \quad 1.$$

(This may also be seen as a consequence of Theorem 1.3.3 and the form of the Pieri-type formula in [33], or of the main result, Theorem 5, of [52].) If $u = 312645$, then $\zeta u = 561234$ and the labeled Hasse diagram of $[u, \zeta u]_2$ is:



The labels of the six chains are:

$$2456,\ 2465,\ 2645,\ 4526,\ 4256,\ 4265$$

and these have (respective) recording tableaux:

$$\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 4 \\ 1 & 2 & 3 \end{ytableau} \qquad \begin{ytableau} 3 \\ 1 & 2 & 4 \end{ytableau} \qquad \begin{ytableau} 3 \\ 1 & 2 & 4 \end{ytableau} \qquad \begin{ytableau} 2 \\ 1 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 2 & 4 \\ 1 & 3 \end{ytableau}\ .$$

This list omits the tableau $\begin{ytableau} 3 & 4 \\ 1 & 2 \end{ytableau}$, and the third and fourth tableaux are identical.

Proof of Theorem 1.3.2. Suppose first that $\zeta = v(\lambda, k)\cdot v(\mu, k)^{-1}$. Then

$$[e, \zeta]_{\preceq} \quad \simeq \quad [v(\mu, k),\ v(\lambda, k)]_k \quad \simeq \quad [\mu, \lambda]_{\subset}.$$

The first isomorphism preserves the edge labeling of the Hasse diagrams, and in the second the labels of the k-Bruhat order correspond to diagonals in a Young diagram: If $\nu \subset\!\!\!\cdot\ \nu'$ is a cover in Young's lattice, there is a unique i such that $\nu_i \neq \nu'_i$, and in that case, $\nu_i + 1 = \nu'_i$, and the label of the corresponding edge in the k-Bruhat order is $k - i + \nu'_i$, the diagonal on which the new box in ν' lies.

A chain in Young's lattice from μ to λ is a standard skew tableau R of shape λ/μ. Consider the word, $a_1 \ldots a_m$, of that chain as a two-rowed array:

$$w \quad = \quad \left(\begin{array}{cccc} 1 & 2 & \cdots & m \\ a_1 & a_2 & \cdots & a_m \end{array} \right).$$

Then the entry i of R is in the a_ith diagonal.

Let S and T be, respectively, the insertion and recording tableaux for that two-rowed array. Consider the two-rowed array consisting of the columns $\binom{a_i}{i}$ arranged in lexicographic order: that is, $\binom{a_i}{i}$ is to the left of $\binom{a_j}{j}$ if either $a_i < a_j$ or $a_i = a_j$ and $i < j$. Then the insertion and recording tableaux of this new array are T and S, respectively [50, 29].

The second row of this new array, the word inserted to obtain T, is the 'diagonal' word of the skew tableau R. That is, the entries of R read lexicographically by diagonal. By Lemma 6.1.3 (proven below), the diagonal word is Knuth-equivalent to the original word. Thus T is the unique tableau of partition shape Knuth-equivalent to R. This gives a combinatorial bijection

$$\left\{\begin{array}{c}\preceq\text{-chains from } e \text{ to } \zeta \text{ whose}\\ \text{word has recording tableau } T\end{array}\right\} \quad\Longleftrightarrow\quad \left\{\begin{array}{c}\text{Skew tableaux } R \text{ of shape}\\ \lambda/\mu \text{ Knuth-equivalent to } T\end{array}\right\},$$

proving the theorem in this case, as it is well-known that (see, for example [48, §4.9]),

$$c_\nu^{\lambda/\mu} \quad = \quad \#\left\{\begin{array}{c}\text{Skew tableaux } R \text{ of shape}\\ \lambda/\mu \text{ Knuth-equivalent to } T\end{array}\right\}.$$

Now suppose ζ is shape-equivalent to $v(\lambda,k)\cdot v(\mu,k)^{-1}$. By Theorem 1.3.1 (ii), $c_\nu^\zeta = c_\nu^{\lambda/\mu}$, proving (i). Assume λ, μ, and k have been chosen so that $\zeta = \phi_P\left(v(\lambda,k)\cdot v(\mu,k)^{-1}\right)$, for some P. By Theorem 3.2.3 (iii), ϕ_P induces an isomorphism

$$\phi_P \;:\; [e,\, v(\lambda,k)\cdot v(\mu,k)^{-1}]_\preceq \;\xrightarrow{\sim}\; [e,\,\zeta]_\preceq.$$

Moreover, if $\eta \lessdot (\alpha,\beta)\eta$ is a cover in $[e,\, v(\lambda,k)\cdot v(\mu,k)^{-1}]_\preceq$, then $\phi_P\eta \lessdot (\phi_P(\alpha\,\beta))\phi_P\eta$ is a cover in $[e,\zeta]_\preceq$ which has label p_β, where $P = p_1 < p_2 < \cdots$. Thus, if γ is a chain in $[e,\, v(\lambda,k)\cdot v(\mu,k)^{-1}]_\preceq$ whose word $a_1,\ldots,a_m$ has recording tableaux T, then $\phi_P(\gamma)$ is a chain in $[e,\zeta]_\preceq$ with word $p_{a_1},\ldots,p_{a_m}$, which also has recording tableaux T. $\blacksquare$

Order the diagonals of a skew Young tableau R beginning with the diagonal incident to the the end of the first column of R. The *diagonal word* of R is the entries of R listed in lexicographic order by diagonal, with magnitude breaking ties. The tableau on the left below has diagonal word 7 58 379 148 26 26 5 8. If we apply Schensted insertion to the initial segment 7 58 379 148, (those diagonals incident upon the first column), we obtain the tableau on the right, whose row word equals this initial segment.

$$\begin{array}{|c|c|c|cc}\hline 7&8&9\\\hline 5&7&8\\\cline{1-3} 3&4&6&\multicolumn{1}{c|}{6}\\\cline{1-4} 1&2&2&\multicolumn{1}{c|}{5}&\multicolumn{1}{c|}{8}\\\hline\end{array}\qquad\qquad\begin{array}{|c|c|c|}\cline{1-1} 7\\\cline{1-2} 5&\multicolumn{1}{c|}{8}\\\hline 3&7&9\\\hline 1&4&8\\\hline\end{array}$$

This observation is the key to the proof of the following lemma.

Lemma 6.1.3. *The diagonal word of a skew tableau is Knuth-equivalent to its column word.*

Proof. For a skew tableau R, let $d(R)$ be its diagonal word. We show that $d(R)$ is Knuth equivalent to the word $c.d(R')$, the concatenation of the first column c of R and $d(R')$, where R' is obtained from R by the removal of its first column. An induction completes the proof.

Suppose the first column of R has length b and R has r diagonals. For $1 \leq j \leq b$ let $w_j := a_1^j \ldots a_{s_j}^j$ be the subword of $d(R)$ consisting of the jth diagonal. Then $a_1^j < \cdots < a_{s_j}^j$, $s_1 \leq s_2 \leq \cdots \leq s_b$, and if $t \leq s_j$, then $a_t^j > a_t^{j+1} > \cdots > a_t^b$, as these are consecutive entries in the tth column of R.

Consider the insertion tableaux T_l of the word $w_1.w_2.\ldots.w_l$ for $1 \leq l \leq b$. Then the tth column of T_l is $a_t^j > \cdots > a_t^l$, where $s_{j-1} < t \leq s_j$. Hence $c.d(R') = c.row(T').w_{b+1} \ldots w_r$, where $row(T')$ is the row word of the tableau obtained from T_b by removing its first column, which is c. Since the column word of a tableau is Knuth-equivalent to its row word, we are done. ▪

6.2. **Skew permutations.** Define the set of *skew permutations* to be the smallest set of permutations containing all permutations $v(\lambda, k) \cdot v(\mu, k)^{-1}$ which is closed under:

1. Shape equivalence. If η is shape equivalent to a skew permutation ζ, then η is skew.
2. Cyclic shift. If $\zeta \in \mathcal{S}_n$ is skew, then so is $\zeta^{(1\,2\,\ldots\,n)}$.
3. Products of disjoint permutations. If ζ, η are disjoint and skew, then $\zeta\eta$ is skew.

A *shape* of a skew permutation ζ is a (non-unique!) skew partition θ which is defined inductively. If ζ is shape equivalent to λ/μ, then ζ has shape λ/μ. If $\zeta \in \mathcal{S}_n$ is a skew permutation with shape θ, then $\zeta^{(1\,2\,\ldots\,n)}$ has shape θ. If ζ and η are disjoint skew permutations with respective shapes ρ and σ, then $\zeta\eta$ has skew shape $\rho \coprod \sigma$.

Theorem 6.2.1. *Let ζ be a skew permutation with shape θ, then*

(*i*) *For all partitions ν,*

$$c_\nu^\zeta \quad = \quad c_\nu^\theta.$$

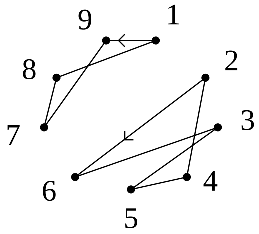
(*ii*) *The number of chains in the interval $[e, \zeta]_\preceq$ is equal to the number of standard Young tableaux of shape θ.*

Proof. The number of standard skew tableaux of shape θ is $\sum_\lambda f^\lambda c_\lambda^\theta$, hence (*ii*) is consequence of (*i*) and Proposition 1.1.1. To show (*i*), we need only consider the last part (3.) of the recursive definition of skew permutations, by Theorems 1.3.1 (*ii*) and 1.3.4. Suppose ζ and η are disjoint skew permutations with respective shapes ρ

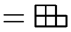
and σ, and for all partitions ν, $c^{\zeta}_{\nu} = c^{\rho}_{\nu}$ and $c^{\eta}_{\nu} = c^{\sigma}_{\nu}$. Then by Theorem 1.3.3 (*ii*),

$$\begin{aligned} c^{\zeta\eta}_{\nu} &= \sum_{\lambda,\mu} c^{\nu}_{\lambda\,\mu}\, c^{\zeta}_{\lambda}\, c^{\eta}_{\mu} \\ &= \sum_{\lambda,\mu} c^{\nu}_{\lambda\,\mu}\, c^{\rho}_{\lambda}\, c^{\sigma}_{\mu} \\ &= c^{\rho \amalg \sigma}_{\nu}. \end{aligned}$$

Example 6.2.2. Consider the geometric graph of the permutation (1978)(26354):



Thus the two cycles $\zeta = (1978)$ and $\eta = (26354)$ are disjoint.

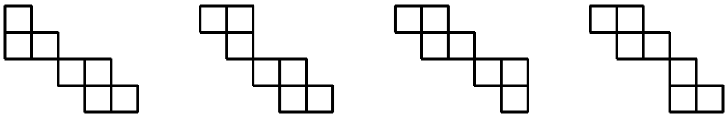
Note that ζ is shape equivalent to (1423) and $(1423)^{(1234)} = (1342)$. Similarly, η is shape equivalent to (15243) and $(15243)^{(12345)} = (13542)$. Both of these cycles, (1423) and (15243), are skew partitions: Let $\lambda =$, $\mu =$ , $\nu =$ . Then

$$v(\lambda, 2) \;=\; 13245, \qquad v(\mu, 2) \;=\; 34125, \qquad v(\nu, 2) \;=\; 35124$$

and

$$\begin{aligned} v(\lambda, 2) &\leq_2 (1342) \cdot v(\lambda, 2) &= v(\mu, 2), \\ v(\lambda, 2) &\leq_2 (13542) \cdot v(\lambda, 2) &= v(\nu, 2). \end{aligned}$$

Hence, for any partition κ, $c^{\zeta}_{\kappa} = c^{\mu/\lambda}_{\kappa}$ and $c^{\eta}_{\kappa} = c^{\nu/\lambda}_{\kappa}$. Thus it follows that $c^{\zeta\eta}_{\kappa} = c^{\rho}_{\kappa}$, where ρ is any of the four skew partitions:



6.3. **Further remarks.** For small symmetric groups, it is instructive to examine all permutations and determine to which class they belong.

	skew partitions	shape equivalent to a skew partition	skew permutation
$\mathcal{S}_4$	14	21	24
$\mathcal{S}_5$	42	79	120
$\mathcal{S}_6$	132	311	678

If ζ is one of the 42 permutations in $\mathcal{S}_6$ which are not skew permutations, and ζ is not among

$$(125634),\ (145236),\ (143652),\ (163254),\ (153)(246),\ \text{or}\ (135)(264),$$

then there is a skew partition θ such that $c^\zeta_\nu = c^\theta_\nu$ for all partitions ν. It would be interesting to understand why this occurs for all but these 6 permutations. Is there a wider class of permutations ζ such that there exists a skew partition θ with $c^\zeta_\nu = c^\theta_\nu$ for all partitions ν?

For the six 'exceptional' permutations ζ, there is a skew partition θ for which $c^\zeta_\nu = c^\theta_\nu$ for all $\nu \subset a^b$, where $a = \#\mathrm{up}_\zeta$ and $b = \#\mathrm{down}_\zeta$. For example, let $\zeta = (153)(246)$. If $u = 214365$, then $u \leq_3 \zeta u$ and there are 42 chains in $[u,\, \zeta u]_3$. Also

$$c^\zeta_{(3,3)} \;=\; 1, \qquad c^\zeta_{(3,2,1)} \;=\; 2, \qquad \text{and} \qquad c^\zeta_{(2,2,2)} \;=\; 1,$$

which verifies Proposition 1.1.1 as $f^{(3,3)} = 5$, $f^{(3,2,1)} = 16$, and $f^{(3,3)} = 5$. In this case, $\theta = (4,3,2,1)/(2,2)$.

A combinatorial interpretation of the Littlewood-Richardson coefficients $c^w_{u\,v(\lambda,k)}$ should also give a bijective proof of Proposition 1.1.1. We show a partial converse to this, that a function τ from chains to standard Young tableaux satisfying some extra conditions will provide a combinatorial interpretation of the Littlewood-Richardson coefficients $c^w_{u\,v(\lambda,k)}$.

Let $\mathrm{ch}[u,w]_k$ denote the set of (saturated) chains in the interval $[u,w]_k$. For a partition μ and integer m, let $\mu * m$ be the set of partitions λ with $\lambda - \mu$ a horizontal strip of length m. These partitions arise in the classical Pieri's formula:

$$S_\mu(x_1,\ldots,x_k)\cdot h_m(x_1,\ldots,x_k) \;=\; \sum_{\lambda\in\mu*m} S_\lambda(x_1,\ldots,x_k).$$

If T is a standard tableau of shape μ and m and integer, let $T * m$ be the set of tableau U which contain T as an initial segment such that $U - T$ is a horizontal strip whose entries increase from left to right.

Theorem 6.3.1. *Suppose that for any $u \leq_k w$, there is a map*

$$\begin{array}{rcl} ch[u,w]_k & \longrightarrow & \left\{\begin{array}{c} \text{Standard Young tableau } T \text{ whose} \\ \text{shape is a partition of } \ell(w)-\ell(u) \end{array}\right\} \\ \gamma & \longmapsto & \tau(\gamma) \end{array}$$

such that

1. $d^w_{u\,v(\lambda,k)} := \#\{\gamma \in ch[u,w]_k \mid \tau(\gamma) = T\}$ *depends only upon the shape λ of the standard tableau T.*
2. *If $\gamma = \delta.\varepsilon$ is the concatenation of two chains δ and ε, then $\tau(\delta)$ is a subtableau of $\tau(\gamma)$. (This means that $\tau(\gamma)$ is a recording tableau.)*
3. *Suppose $\gamma = \delta.\varepsilon$ with $\delta \in ch[u,x]_k$, and hence $\varepsilon \in ch[x,w]_k$. Then $\tau(\delta.\varepsilon) \in \tau(\delta) * m$ only if $x \xrightarrow{r_{k,m}} w$, and $\varepsilon(\delta) := \varepsilon \in ch[x,w]_k$ is unique for this to occur.*

Then, for any standard tableau T of shape λ and $u \leq_k w$,

$$c^w_{u\;v(\lambda,k)} \;=\; d^w_{u\;v(\lambda,k)}.$$

Such a map τ is a generalization of Schensted insertion. In that respect, the existence of such a map would generalize Theorem 6.1.1.

Proof. We induct on λ. Assume the theorem holds for all u, w, and partitions π either with fewer rows than λ, or if λ and π have the same number of rows, then the last row of π is shorter than the last row of λ.

The form of the Pieri-type formulas expressed in [52, 57] (also §4.2) and condition (3) prove the theorem when λ consists of a single row. Assume that λ has more than one row and set μ to be λ with its last row removed. Let m be the length of the last row of λ and T be any tableau of shape μ. Recall that $U \mapsto \mathrm{shape}(U)$ is a one-to-one correspondence between tableaux $T * m$ and $\mu * m$.

By the definition of $c^x_{u\, v(\mu,k)}$, we have

$$\mathfrak{S}_u \cdot S_\mu(x_1, \ldots, x_k) \;=\; \sum_{u \leq_k x} c^x_{u\, v(\mu,k)}\, \mathfrak{S}_x.$$

So

$$\mathfrak{S}_u \cdot S_\mu(x_1, \ldots, x_k) \cdot h_m(x_1, \ldots, x_k) \;=\; \sum_w \left(\sum_{\substack{u \leq_k x \\ x \xrightarrow{r_{k,m}} w}} c^x_{u\, v(\mu,k)} \right) \mathfrak{S}_w.$$

But this equals

$$\mathfrak{S}_u \cdot \sum_{\pi \in \mu * m} S_\pi(x_1, \ldots, x_k) \;=\; \sum_w \left(\sum_{\pi \in \mu * m} c^w_{u\, v(\pi,k)} \right) \mathfrak{S}_w.$$

Hence

$$\sum_{\pi \in \mu * m} c^w_{u\, v(\pi,k)} \;=\; \sum_{\substack{u \leq_k x \\ x \xrightarrow{r_{k,m}} w}} c^x_{u\, v(\mu,k)}.$$

We exhibit a bijection between the two sets

$$M_{T,k,m} \;:=\; \coprod_{\substack{u \leq_k x \\ x \xrightarrow{r_{k,m}} w}} \{\delta \in \mathrm{ch}[u, x]_k \mid \tau(\delta) = T\}$$

and $\coprod_{\pi \in \mu * m} L_\pi$, where

$$L_\pi \;:=\; \{\gamma \in \mathrm{ch}[u, w]_k \mid \tau(\gamma) \in T * m \text{ and } \tau(\gamma) \text{ has shape } \pi\}.$$

This will complete the proof. Indeed, by the induction hypothesis

$$\# M_{T,k,m} \;=\; \sum_{\substack{u \leq_k x \\ x \xrightarrow{r_{k,m}} w}} c^x_{u\, v(\mu,k)}$$

and for $\pi \in \mu * m$ with $\pi \neq \lambda$,

$$\# L_\pi \;=\; c^w_{u\,v(\pi,k)}.$$

Thus the bijection shows

$$c^w_{u\,v(\lambda,k)} \;=\; \sum_{\substack{u\,\leq_k\,x \\ x \xrightarrow{r_{k,m}} w}} c^x_{u\,v(\mu,k)} - \sum_{\pi\in\mu * m,\ \pi\neq\lambda} c^w_{u\,v(\pi,k)} \;=\; \# L_\lambda,$$

which is $\#\tau^{-1}(U)$, for any U of shape λ.

To construct the desired bijection, consider first the map

$$M_{T,k,m} \;\longrightarrow\; \coprod_{\pi\in\mu * m} L_\pi$$

defined by $\delta \in \mathrm{ch}[u,x]_k \mapsto \delta.\varepsilon(\delta)$. By property 3, $\tau(\delta.\varepsilon(\delta)) \in T * m$, so this injective map has the stated range. To see it is surjective, let $\pi \in \mu * m$ and $\gamma \in L_\pi$. Let δ be the first $|\mu|$ steps in the chain γ, so that $\gamma = \delta.\varepsilon$ and suppose $\delta \in \mathrm{ch}[u,x]_k$. Then $\tau(\delta) = T$ so $\tau(\delta.\varepsilon) \in \tau(\delta) * m$. By 3, this implies $x \xrightarrow{r_{k,m}} w$, and hence $\delta \in M_{T,k,m}$.

Appendix A. Illustrating the geometric theorems

These appendices are intended for informal distribution with this manuscript and will not appear in the published version. They contain no results, only examples which we hope may illustrate some of the main results of this manuscript. This appendix is intended to illustrate the geometric results in the previous sections, particularly of Section 5. We hope this may help others think about Schubert varieties and intersections of Schubert varieties.

Throughout, let $e_1, \ldots, e_n$ be a fixed, ordered basis for the vector space $\mathbb{C}^n$. We use this basis to obtain a parameterization for Schubert cells and their intersections. Flags are represented by $n \times n$ matrices M: Let $(g_1, \ldots, g_n) := M \cdot e^T$ be the ordered basis given by the 'change of basis' matrix M. The ith row of M gives the components of g_i. Then M represents the flag $\langle\langle g_1, \ldots, g_n\rangle\rangle$. We adopt some conventions for the entries of M: a dot $(\cdot)$ will denote an entry of zero and an asterix $(*)$ an entry which may assume any value in $\mathbb{C}$.

One last convention is that the flags $E_\bullet, F_\bullet$, etc. will always be defined to be $E_\bullet := \langle\langle e_1, \ldots, e_n\rangle\rangle$ and the 'primed' flags $E'_\bullet, F'_\bullet$, etc. which are opposite to their unprimed cousins will be defined by $E'_\bullet := \langle\langle e_n, e_{n-1}, \ldots, e_2, e_1\rangle\rangle$. We refer to these as the *standard flags*.

A.1. **Theorem 1.3.1 (*ii*).** In Theorem 1.3.1 (*ii*), we studied $X_{w_0w}E_\bullet \bigcap X_uE'_\bullet$ and $X_{w_0z}E_\bullet \bigcap X_xE'_\bullet$ where $u \leq_k w$, $x \leq_k z$, and $wu^{-1} = zx^{-1}$. The main geometric result was that, in $Grass_k\mathbb{C}^n$,

$$\pi_k\left(X_{w_0w}E_\bullet \bigcap X_uE'_\bullet\right) \;=\; \pi_k\left(X_{w_0z}E_\bullet \bigcap X_xE'_\bullet\right).$$

The general case of Theorem 1.3.1 (*ii*) was reduced to Lemma 5.1.2, where w was Grassmannian of descent k, and $k < i \Longrightarrow u(i) = x(i)$ (and hence also $w(i) = z(i)$). The first example illustrates this case.

Let $n = 7$, $k = 4$, and

$$\begin{array}{rclcrcl} u &=& 1436257 &\qquad& x &=& 4631257\\ w &=& 4567123 && z &=& 5764123 \end{array}$$

The following matrices respectively represent general flags in the Schubert cells $X^\circ_{w_0w}E_\bullet$, $X^\circ_uE'_\bullet$, $X^\circ_{w_0z}E_\bullet$, and $X^\circ_xE'_\bullet$:

$$\begin{matrix} * & * & * & 1 & \cdot & \cdot & \cdot\\ * & * & * & \cdot & 1 & \cdot & \cdot\\ * & * & * & \cdot & \cdot & 1 & \cdot\\ * & * & * & \cdot & \cdot & \cdot & 1\\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{matrix} \qquad \begin{matrix} 1 & * & * & * & * & * & *\\ \cdot & \cdot & \cdot & 1 & * & * & *\\ \cdot & \cdot & 1 & \cdot & * & * & *\\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & *\\ \cdot & 1 & \cdot & \cdot & * & \cdot & *\\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & *\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{matrix} \qquad \begin{matrix} * & * & * & * & 1 & \cdot & \cdot\\ * & * & * & * & \cdot & * & 1\\ * & * & * & * & \cdot & 1 & \cdot\\ * & * & * & 1 & \cdot & \cdot & \cdot\\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{matrix} \qquad \begin{matrix} \cdot & \cdot & \cdot & 1 & * & * & *\\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & *\\ \cdot & \cdot & 1 & \cdot & * & \cdot & *\\ 1 & * & \cdot & \cdot & * & \cdot & *\\ \cdot & 1 & \cdot & \cdot & * & \cdot & *\\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & *\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{matrix}$$

w is chosen to be Grassmannian so that the cell $X^\circ_{w_0w}E_\bullet$ has a particularly simple form, which gave an easy parameterization for the intersection of the two cells, $X^\circ_{w_0w}E_\bullet \bigcap X^\circ_uE'_\bullet$. In the proof of Lemma 5.1.4 we describe how to find bases parameterized by $A := \{M \in M(w) \mid M \in X^\circ_uE_\bullet\}$. In practice, this method may be used to determine the subvariety A of $M(w)$.

First, let $g_1, \ldots, g_7$ be the rows of the following matrix, where $\alpha, \beta, \gamma, \delta, x, \rho, \sigma$, and τ are arbitrary elements of $\mathbb{C}$ with $\alpha\delta x\tau \neq 0$:

$$\begin{matrix} \alpha & \beta & \gamma & 1 & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot\\ \cdot & \cdot & x & \rho & \sigma & 1 & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1\\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

These parameters were chosen so that for $j = 1, 2, 3, 4$, g_j is a general element of $E_{w(j)} \bigcap E'_{n+1-u(j)}$ which does not lie in either of $E_{w(j)-1}$ or $E'_{n-u(j)}$, hence the 1's, the condition on α, δ, x, τ, and the 0's ($\cdot$) in their initial columns.

This matrix determines a flag $G_\bullet := \langle\langle g_1, \ldots, g_n \rangle\rangle \in X^\circ_{w_0 w} E_\bullet$, since it is in $M(w)$. Also, since $g_j \in E'_{n+1-u(j)} - E'_{n-u(j)}$ for $j \leq k$, at least $G_1, \ldots, G_k$ satisfy the conditions for the flag $G_\bullet$ to be in $X^\circ_u E'_\bullet$. The remaining conditions for $G_\bullet \in X^\circ_u E'_\bullet$,

$$G_{j-1} \bigcap E'_{n+1-u(j)} \subsetneq G_j \bigcap E'_{n+1-u(j)} \qquad \text{for} \qquad k < j,$$

impose additional restrictions on the parameters. In practice this means we seek conditions to ensure that $(\mathbb{C}^\times g_j + G_{j-1}) \bigcap E'_{n+1-u(j)}$ is non-empty. For instance, for $j = 6$, since $\langle g_5, g_5 \rangle = (*, *, 0, 0, 0, 0, 0)$ and $E_{n+1-u(6)} = (0, 0, 0, 0, *, *, *)$, some cancellation must occur. Indeed, since

$$\begin{aligned} -\alpha g_5 - \beta g_6 + g + 1 &= (0, 0, \gamma, 1, 0, 0, 0) \\ g_3 &= (0, 0, x, \rho, \sigma, 1, 0), \end{aligned}$$

we must have $\gamma\rho - x = 0$ in order that $(\mathbb{C}^\times g_6 + G_5) \bigcap E'_{n+1-u(6)} \neq \emptyset$. In the general situation, more complicated determinantal conditions may arise.

From these considerations, we arrive at a parameterization for $X_{w_0 w} E_\bullet \bigcap X_u E'_\bullet$.

$$\begin{array}{ccccccc} \alpha & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\ \cdot & \cdot & \gamma\rho & \rho & \sigma & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccc} \alpha & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta & 1 & \cdot & \cdot \\ \cdot & \cdot & \gamma\rho & \rho & \sigma & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau & 1 \\ \cdot & \beta & \gamma & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sigma & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

For $\alpha, \ldots, \tau \in \mathbb{C}^\times$, both matrices represent the same flag in the intersection. To see this, let $g_1, \ldots, g_7$ be the basis determined by the first matrix, and $g'_1, \ldots, g'_7$ the basis determined by the second matrix. Then, by the definition (2.3) of Schubert cells in §2.3, the flags $\langle\langle g_1, \ldots, g_7 \rangle\rangle \in X^\circ_{w_0 w} E_\bullet$ and $\langle\langle g'_1, \ldots, g'_7 \rangle\rangle \in X^\circ_u E'_\bullet$. Since $g_i = g'_i$ for $i = 1, 2, 3, 4$,

$$\begin{aligned} g'_5 &= g_1 - \alpha g_5 \\ g'_6 &= g_3 - \rho(g_1 - \alpha g_5 - \beta g_6) \\ g'_7 &= g_4 - \tau \left[g_3 - \rho(g_1 - \alpha g_5 - \beta g_6) - \sigma \left(g_2 - \delta(g_1 - \alpha g_5 - \beta g_6 - \gamma g_7) \right) \right], \end{aligned}$$

showing $\langle\langle g_1, \ldots, g_7 \rangle\rangle = \langle\langle g'_1, \ldots, g'_7 \rangle\rangle$. Lastly, since $\ell(w) - \ell(u) = 12 - 5 = 7$, and $E_\bullet, E'_\bullet$ are opposite flags, we see that $X^\circ_u E'_\bullet \bigcap X^\circ_{w_0 w} E_\bullet$ is irreducible of dimension 7. This identifies a 7-parameter family of flags in this intersection, which must be dense.

Similarly, (with the same restrictions on parameters), the two matrices below both represent the same flag in $X^\circ_{w_0 z}E_\bullet \bigcap X^\circ_x E'_\bullet$:

$$
\begin{array}{ccccccc}
\cdot&\cdot&\cdot&\delta&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\tau&1\\
\cdot&\cdot&\gamma\rho&\rho&\sigma&1&\cdot\\
\alpha&\beta&\gamma&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{array}
\qquad
\begin{array}{ccccccc}
\cdot&\cdot&\cdot&\delta&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\tau&1\\
\cdot&\cdot&\gamma\rho&\rho&\sigma&1&\cdot\\
\alpha&\beta&\gamma&1&\cdot&\cdot&\cdot\\
\cdot&\beta&\gamma&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\sigma&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{array}
$$

If $h_1, \ldots, h_7$ is the basis determined by the first matrix, then $h_1 = g_2$, $h_2 = g_4$, $h_3 = g_3$, and $h_4 = g_1$. Thus $\langle g_1, g_2, g_3, g_4\rangle = \langle h_1, h_2, h_3, h_4\rangle$, which proves

$$\pi_k\left(X_u E'_\bullet \bigcap X_{w_0 w} E_\bullet\right) \;=\; \pi_k\left(X_x E'_\bullet \bigcap X_{w_0 y} E_\bullet\right).$$

This is true even when u, w, x, z do not satisfy the extra hypotheses of Lemma 5.1.2 (one may construct a proof using the geometric analogs of the arguments that reduce Theorem 1.3.1 (ii) to Lemma 5.1.2). We illustrate this on another example.

Here, let $n = 7$, $k = 3$, and

$$
\begin{array}{rclcrcl}
u &=& 2134765 &\qquad& x &=& 2316475\\
w &=& 3571624 && z &=& 3752164
\end{array}
$$

Then the following four matrices represent, respectively, the Schubert cells $X^\circ_{w_0 w}E_\bullet$, $X^\circ_u E'_\bullet$, $X^\circ_{w_0 z}E_\bullet$, and $X^\circ_x E'_\bullet$:

$$
\begin{array}{ccccccc}
&&1&\cdot&\cdot&\cdot&\cdot\\
&&\cdot&*&1&\cdot&\cdot\\
&&\cdot&*&\cdot&*&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&*&\cdot&*&\cdot&1&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot
\end{array}
\quad
\begin{array}{ccccccc}
\cdot&1&*&*&*&*&*\\
1&\cdot&*&*&*&*&*\\
\cdot&\cdot&1&*&*&*&*\\
\cdot&\cdot&\cdot&1&*&*&*\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot
\end{array}
\qquad
\begin{array}{ccccccc}
&&1&\cdot&\cdot&\cdot&\cdot\\
&&\cdot&*&*&*&1\\
&&\cdot&*&1&\cdot&\cdot\\
*&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&*&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot
\end{array}
\quad
\begin{array}{ccccccc}
\cdot&1&*&*&*&*&*\\
\cdot&\cdot&1&*&*&*&*\\
1&\cdot&\cdot&*&*&*&*\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&*\\
\cdot&\cdot&\cdot&1&*&\cdot&*\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot
\end{array}
$$

Consider the (equivalent pairs of) parameterizations for flags in the intersections of the cells, $X^\circ_{w_0 w}E_\bullet \bigcap X^\circ_u E'_\bullet$ (the left-hand pair), and $X^\circ_{w_0 z}E_\bullet \bigcap X^\circ_x E'_\bullet$ (the right-hand pair):

$$
\begin{array}{ccccccc}
\cdot&\alpha&1&\cdot&\cdot&\cdot&\cdot\\
\beta&\alpha\gamma&\gamma&\delta&1&\cdot&\cdot\\
\cdot&\cdot&\rho&\delta\sigma&\sigma&\tau&1\\
\beta&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\rho&\delta\sigma&\sigma&\tau&\cdot\\
\cdot&\alpha&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\rho&\delta\sigma&\cdot&\cdot&\cdot
\end{array}
\quad
\begin{array}{ccccccc}
\cdot&\alpha&1&\cdot&\cdot&\cdot&\cdot\\
\beta&\alpha\gamma&\gamma&\delta&1&\cdot&\cdot\\
\cdot&\cdot&\rho&\delta\sigma&\sigma&\tau&1\\
\cdot&\cdot&\cdot&\delta&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\tau&1\\
\cdot&\cdot&\cdot&\cdot&\sigma&\tau&1
\end{array}
\qquad
\begin{array}{ccccccc}
\cdot&\alpha&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\rho&\delta\sigma&\sigma&\tau&1\\
\beta&\alpha\gamma&\gamma&\delta&1&\cdot&\cdot\\
\beta\sigma&\alpha\rho&\cdot&\cdot&\cdot&\cdot&\cdot\\
\beta&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\rho&\delta\sigma&\sigma&\tau&\cdot\\
\beta&\alpha\gamma&\gamma&\delta&\cdot&\cdot&\cdot
\end{array}
\quad
\begin{array}{ccccccc}
\cdot&\alpha&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\rho&\delta\sigma&\sigma&\tau&1\\
\beta&\alpha\gamma&\gamma&\delta&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\tau&1\\
\cdot&\cdot&\cdot&\delta&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot
\end{array}
$$

To see that each pair of matrices does indeed give the same flag, let $g_1, \ldots, g_7$, $g'_1, \ldots, g'_7$, $h_1, \ldots, h_7$, and $h'_1, \ldots, h'_7$ be the bases given by the four matrices (read left-to-right). Then, for $i = 1, 2, 3$, $g_i = g'_i$ and $h_i = h'_i$. Also,

$$\begin{aligned} g'_4 &= -\alpha g_1 + g_2 - g_4 \\ g'_5 &= g_3 - g_5 \\ g'_6 &= g_3 - \sigma g'_4 - \rho(g_1 - g_6) \\ g'_7 &= g_3 - g_7 \end{aligned}$$

and

$$\begin{aligned} h'_4 &= h_2 - \sigma h_3 + h_4 + (\gamma - \rho) h_1 \\ h'_5 &= h_3 - \gamma h_1 - h_5 \\ h'_6 &= h_2 - h_6 \\ h'_7 &= h_3 - h_7, \end{aligned}$$

thus, $\langle\langle g_1, \ldots, g_7 \rangle\rangle = \langle\langle g'_1, \ldots, g'_7 \rangle\rangle$ and $\langle\langle h_1, \ldots, h_7 \rangle\rangle = \langle\langle h'_1, \ldots, h'_7 \rangle\rangle$. As before, these parameterized bases give dense subsets of flags in each of $X_{w_0 w} E_\bullet \bigcap X_u E'_\bullet$ and $X_{w_0 z} E_\bullet \bigcap X_x E'_\bullet$. Moreover, since $\langle g_1, g_2, g_3 \rangle = \langle h_1, h_2, h_3 \rangle$, we see that

$$\pi_k \left(X_{w_0 w} E_\bullet \bigcap X_u E'_\bullet \right) \; = \; \pi_k \left(X_{w_0 z} E_\bullet \bigcap X_x E'_\bullet \right).$$

A.2. **Theorem 1.3.3** (ii)**.** We complete Example 6.2.2, giving the geometric side of the story. The permutation (1978)(26354) is the disjoint product of $\zeta = (1978)$ and $\eta = (26354)$. Note that $u = 372186945 \leq_4 586913724 = (\zeta\eta)u =: w$. Let $G_\bullet$ and $G_\bullet{}'$ be the standard flags in $\mathbb{C}^9$. The following matrices parameterize the Schubert cells $X^\circ_{w_0 586913724} G_\bullet$ and $X^\circ_{372186945} G_\bullet{}'$:

$$\begin{array}{ccccccccc} \cdot & \cdot & 1 & * & * & * & * & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * & * \\ \cdot & 1 & \cdot & * & * & * & \cdot & * & * \\ 1 & \cdot & \cdot & * & * & * & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & * & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccccc} * & * & * & * & 1 & \cdot & \cdot & \cdot & \cdot \\ * & * & * & * & \cdot & * & * & 1 & \cdot \\ * & * & * & * & \cdot & 1 & \cdot & \cdot & \cdot \\ * & * & * & * & \cdot & \cdot & * & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & * & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

As before, here are two parameterized matrices, each of which give bases defining the same flag in the intersection of the two cells:

$$
\begin{array}{ccccccccc}
\cdot & \cdot & a & b & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & 1 & \cdot \\
\cdot & c & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -c & as & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & a & b & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\qquad\qquad
\begin{array}{ccccccccc}
\cdot & \cdot & a & b & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & 1 & \cdot \\
\cdot & c & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

It is clearer to display two matrices 'on top of each other':

$$
\begin{array}{ccccccccc}
\cdot & \cdot & a & b & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & 1 & \cdot \\
\cdot & c & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \\
1\,\vert & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\,\vert & \gamma & 1 \\
\cdot & -c & as\,\vert & \cdot & \cdot\,\vert & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1\,\vert & \cdot\,\vert & 1 \\
\cdot & 1\,\vert & \cdot\,\vert & bs & s & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & a & b\,\vert & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

The vertical lines in the last 5 rows illustrate that the left- and right-sides of those rows come from different (equivalent) bases. The shading accentuates its 'block form': Let $Q = \{1,7,8,9\}$ and $P = \{2,4,5,7\} = u^{-1}(Q)$. The $P^c = \{1,3,6,8,9\} = u^{-1}(Q^c)$, where $Q^c = \{2,3,4,5,6\}$. Then the shaded regions are $(P \times Q) \bigcup (P^c \times Q^c)$. We see that $\zeta' := (1423)$ and $\eta' := (15243)$ are uniquely defined by $\phi_Q \zeta' = \zeta$ and $\phi_{Q^c}\eta' = \eta$. Moreover, we may define permutations v and w as in Lemma 5.2.1; let $v = 2134$ and $w = 21534$. Then

1. $v \leq_2 \zeta' v = 3412$ and $w \leq_2 \eta' w = 45213$.
2. $u = \varepsilon_{P,Q}(v,w)$ and $(\zeta\eta)u = \varepsilon_{P,Q}(\zeta' v, \eta' w)$.

For the last part of Lemma 5.2.1, let $F_\bullet, F'_\bullet$ be the standard flags in $\mathbb{C}^4$, and $E_\bullet, E'_\bullet$ the standard flags in $\mathbb{C}^5$. Then the following four matrices parameterize the Schubert cells $X^\circ_{w_0\zeta' v}F_\bullet$, $X^\circ_v F'_\bullet$, $X^\circ_{w_0\eta' w}E_\bullet$, and $X^\circ_w E'_\bullet$:

$$
\begin{array}{cccc}
\ast & \ast & 1 & \cdot \\
\ast & \ast & \cdot & 1 \\
1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot
\end{array}
\qquad
\begin{array}{cccc}
\cdot & 1 & \ast & \ast \\
1 & \cdot & \ast & \ast \\
\cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & \cdot
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\ast & \ast & \ast & 1 & \cdot \\
\ast & \ast & \ast & \cdot & 1 \\
\ast & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot
\end{array}
\qquad
\begin{array}{ccccc}
\cdot & 1 & \ast & \ast & \ast \\
1 & \cdot & \ast & \ast & \ast \\
\cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & \ast & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot
\end{array}
$$

Then the following two matrices parameterize the two intersections. Again, we have drawn two matrices on top of each other.

$$\begin{array}{cccc} \cdot & \alpha & 1 & \cdot \\ \beta & \cdot & \gamma & 1 \\ 1 & \cdot & \gamma & 1 \\ \cdot & 1 & \cdot & 1 \end{array} \qquad \begin{array}{ccccc} \cdot & a & b & 1 & \cdot \\ c & \cdot & bs & s & 1 \\ -c & as & \cdot & \cdot & 1 \\ 1 & \cdot & bs & s & 1 \\ \cdot & a & b & 1 & \cdot \end{array}$$

Next, note that $G_\bullet = \psi_Q(E_\bullet, F_\bullet)$ and $G_\bullet{}' = \psi_{w_0^{(9)}Q}(E_\bullet', F_\bullet')$. Finally, verifying that

$$\psi_P\left[\left(X_{w_0^{(4)}\zeta' v}E_\bullet \bigcap X_v E_\bullet'\right) \times \left(X_{w_0^{(5)}\eta' w}F_\bullet \bigcap X_w F_\bullet'\right)\right]$$

is equal to

$$X_{w_0^{(9)}(\zeta\eta)u}G_\bullet \bigcap X_u G_\bullet{}'.$$

may be done by comparing these parameterized matrices.

The final part of the proof of Theorem 1.3.3 (ii) involves comparing images of these intersections under projections to Grassmannians. The row spans of the next three parameterized matrices represent $\pi_2\left(X^\circ_{w_0^{(4)}\zeta' v}E_\bullet \cap X^\circ_v E_\bullet'\right)$, $\pi_2\left(X^\circ_{w_0^{(5)}\eta' w}F_\bullet \cap X^\circ_w F_\bullet'\right)$, and $\pi_4\left(X^\circ_{w_0^{(9)}(\zeta\eta)u}G_\bullet \cap X^\circ_u G_\bullet{}'\right)$ in each of $Grass_2\mathbb{C}^4$, $Grass_2\mathbb{C}^5$, and $Grass_4\mathbb{C}^9$, respectively.

$$\begin{array}{cccc} \cdot & \alpha & 1 & \cdot \\ \beta & \cdot & \gamma & 1 \end{array} \qquad \begin{array}{ccccc} \cdot & a & b & 1 & \cdot \\ c & \cdot & bs & s & 1 \end{array} \qquad \begin{array}{ccccccccc} \cdot & \cdot & a & b & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & 1 & \cdot \\ \cdot & c & \cdot & bs & s & 1 & \cdot & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & 1 \end{array}$$

Thus

$$\pi_4\left(X_{w_0^{(9)}(\zeta\eta)u}\psi_Q(E_\bullet, F_\bullet) \bigcap X_u \psi_{w_0^{(9)}Q}(E_\bullet', F_\bullet')\right)$$

is equal to

$$\varphi_{2,2}\left(\pi_2\left(X_{w_0^{(4)}\zeta' v}E_\bullet \bigcap X_v E_\bullet'\right) \times \pi_2\left(X_{w_0^{(5)}\eta' w}F_\bullet \bigcap X_w F_\bullet'\right)\right).$$

Consider the images of $X_{w_0^{(4)}\zeta' v}E_\bullet \times X_{w_0^{(5)}\eta' w}F_\bullet$ and $X_v E_\bullet' \times X_w F_\bullet'$ under ψ_P in $\mathbb{F}\ell_9$:

$$\begin{array}{ccccccccc} \circ & * & * & * & 1 & \cdot & \cdot & \cdot & \cdot \\ * & \circ & \circ & \circ & \cdot & \circ & * & 1 & \cdot \\ \circ & * & * & * & \cdot & 1 & \cdot & \cdot & \cdot \\ * & \circ & \circ & \circ & \cdot & \cdot & * & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & 1 & \cdot & \cdot & \cdot & \circ & \cdot & \cdot \\ \cdot & \circ & \cdot & \circ & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccccc} \cdot & \cdot & 1 & * & * & * & \circ & \circ & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * & * \\ \cdot & 1 & \cdot & * & * & * & \cdot & \circ & \circ \\ 1 & \cdot & \cdot & \circ & \circ & \circ & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \circ & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & * & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{array}$$

This should be compared with the first figure of this section, which shows the cells $X^\circ_{w_0 586913724} G_\bullet$ and $X^\circ_{372186945} G_\bullet{}'$. Here, the circles ($\circ$) indicate the 'surprise' entries of 0; those which are not zero in that first figure. This illustrates the two inclusions, and serves to illustrate Lemma 4.5.1:

$$\begin{aligned} \psi_P \left(X_{w_0^{(4)} \zeta' v} E_\bullet \times X_{w_0^{(5)} \eta' w} F_\bullet \right) \;&\subset\; X^\circ_{w_0 586913724} G_\bullet \\ \psi_P \left(X_v E'_\bullet \times X_w F'_\bullet \right) \;&\subset\; X^\circ_{372186945} G_\bullet{}' \end{aligned}$$

A.3. **Theorem 1.3.4.** We illustrate 'cyclic shift'. Let $u = 21354$ and $w = 45123$ so that $wu^{-1} = \zeta = (15243)$. Define $x, z \in \mathcal{S}_5$ as in the proof of Theorem 1.3.4′ (§ 5.3) to be $x = 31245$ and $z = 53124$. Then $zx^{-1} = (13542) = \zeta^{(12345)}$. Here, $p = 4$, $m = 1$, and $l = 2$. Let $F_\bullet, F_\bullet'$ be the standard flags for $\mathbb{C}^5$. We define $G_\bullet = \langle\langle e_5, e_1, e_2, e_3, e_4 \rangle\rangle$ and $G_\bullet{}' = \langle\langle e_4, e_3, e_2, e_1, e_5 \rangle\rangle$. Then, with respect to these flags, the Schubert cells $X^\circ_{w_0 w} F_\bullet$, $X^\circ_u F_\bullet'$, $X^\circ_{w_0 z} G_\bullet$, and $X^\circ_x G_\bullet{}'$ are:

$$\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ * & * & * & 1 & \cdot \\ * & * & * & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{array} \qquad \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \cdot & 1 & * & * & * \\ 1 & \cdot & * & * & * \\ \cdot & \cdot & 1 & * & * \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{array} \qquad\qquad \begin{array}{ccccc} e_5 & e_1 & e_2 & e_3 & e_4 \\ * & * & * & * & 1 \\ * & * & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{array} \qquad \begin{array}{ccccc} e_5 & e_1 & e_2 & e_3 & e_4 \\ \cdot & \cdot & 1 & * & * \\ 1 & * & \cdot & * & * \\ \cdot & 1 & \cdot & * & * \\ \cdot & \cdot & \cdot & 1 & * \\ \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

Here, since the flags are different, the columns of the matrices on the right correspond to different elements of our fixed basis, e_1, e_2, e_3, e_4, e_5, as indicated.

Here are two matrices giving (equivalent) parameterized bases for flags in the intersection of the cells $X^\circ_{w_0 w} F_\bullet \bigcap X^\circ_u F_\bullet'$:

$$\begin{array}{ccccc} \cdot & a & b & 1 & \cdot \\ c & \cdot & bd & d & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{array} \qquad\qquad \begin{array}{ccccc} \cdot & a & b & 1 & \cdot \\ c & \cdot & bd & d & 1 \\ \cdot & \cdot & bd & d & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{array}$$

Here $a, b, c, d \in \mathbb{C}^\times$, showing that $(\mathbb{C}^\times)^4$ parameterizes the set A of the intersection of cells. Let $g_1, \ldots, g_5$ be the basis given by the left matrix and $g_1', \ldots, g_5'$ the basis given by the right matrix. Since

$$g_2(a,b,c,d) \;=\; e_5 \; + c\, e_1 + bd\, e_3 + d\, e_4,$$

$(\beta_1, \beta_2, \beta_3, \beta_4) = (c, 0, bd, d)$ are regular functions on A. Also, since

$$e_5 = -d\, g_1 + g_2 - c\, g_3 + da\, g_4,$$

$\delta_1 = -d$, $\delta_3 = -c$, and $\delta_4 = da$ are regular functions on A with δ_4 nowhere vanishing. The bases $h_1, \ldots, h_5$ and $h_1', \ldots, h_5'$ defined in the proof of Theorem 1.3.4′, are parameterized by the following two matrices:

$$\begin{array}{ccccc} \cdot & \cdot & a & b & 1 \\ 1 & -c & da & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{array} \qquad \begin{array}{ccccc} \cdot & \cdot & a & b & 1 \\ 1 & -c & da & \cdot & \cdot \\ \cdot & c & \cdot & bd & d \\ \cdot & \cdot & \cdot & bd & d \\ \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

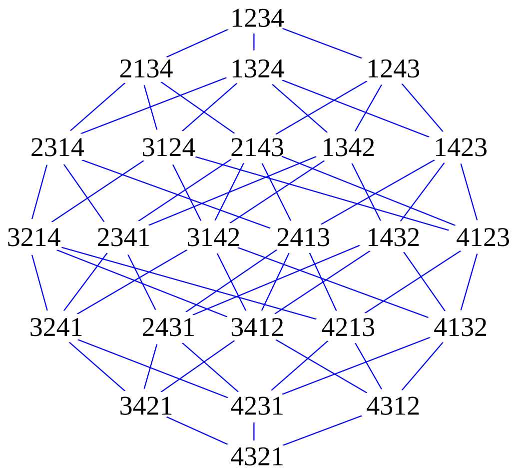
These matrices give equivalent bases, and hence define the same flag. Also, comparing them to the rightmost two matrices in the first figure of this subsection, shows this flag is in the intersection $X^\circ_{w_0z}G_\bullet \bigcap X^\circ_x G_\bullet{}'$. Since the first two rows of each matrix have the same span,

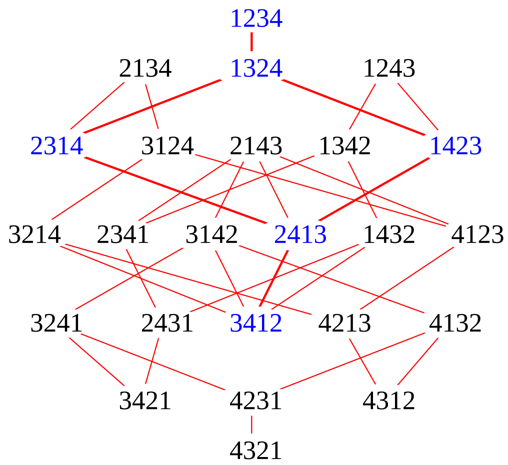
$$\pi_2\left(X^\circ_{w_0w}F_\bullet \bigcap X^\circ_u F_\bullet{}'\right) \;=\; \pi_2\left(X^\circ_{w_0z}G_\bullet \bigcap X^\circ_x G_\bullet{}'\right),$$

which is the main geometric result needed to deduce Theorem 1.3.4′.

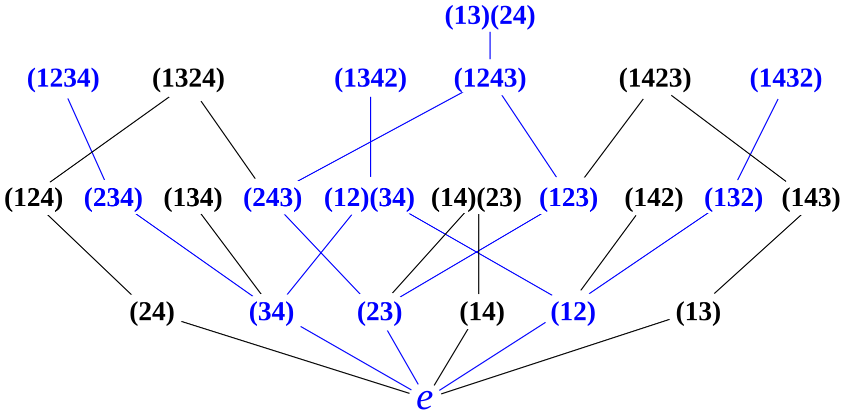
Appendix B. Combinatorial and algebraic examples

B.1. **Suborders of $\mathcal{S}_4$.** The Bruhat order is one of our main objects of study in this paper. Here is a picture of the (full) Bruhat order and the 2-Bruhat order on $\mathcal{S}_4$.

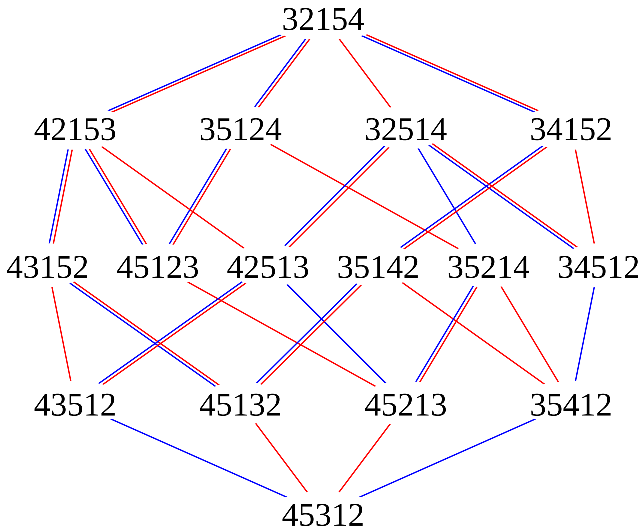




For comparison, here is the $\preceq$-order on $\mathcal{S}_4$ (reproduced from §3.2).



B.2. **Chains in the P-Bruhat order.** Theorem 1.1.3 describes the relation between chains in the P-Bruhat order and the structure constants $c^w_{u\,v}$, when v is a minimal coset representative in vP. We consider an instance of this. Let $P := \langle(1,2),(4,5)\rangle \subset \mathcal{S}_5$. Then $32154 \leq_P 45312$ and this is the interval $[32154,\ 45312]_P$:



The multiple edges are those with two possible colourings. One may verify that $f^{45312}_{32154}(P) = 57$. To check Theorem 1.1.3, we first compute $c^{45312}_{32154\ v}$ for those $v \in \mathcal{S}_5$ of length 4 which are minimal in their P-coset.

$$25134, \quad 34125, \quad 24315, \quad 15324, \quad 14523, \quad \text{and} \quad 23514.$$

The first two are Grassmannian of descent 2, and the last two are Grassmannian of descent 3. Since $32154 \not\leq_2 45312$, we have

$$c^{45312}_{32154\ 25134} \quad = \quad c^{45312}_{32154\ 34125} \quad = \quad 0,$$

Let $\zeta = (13425)$. Then $45312 = \zeta \cdot 32145$ and $(13425)^{(12345)} = (12435)$. Since $(12435) = v(⊞, 3) \cdot v(▫, 3)^{-1}$ and $32154 \leq_3 45312$, Theorem 1.3.4 implies

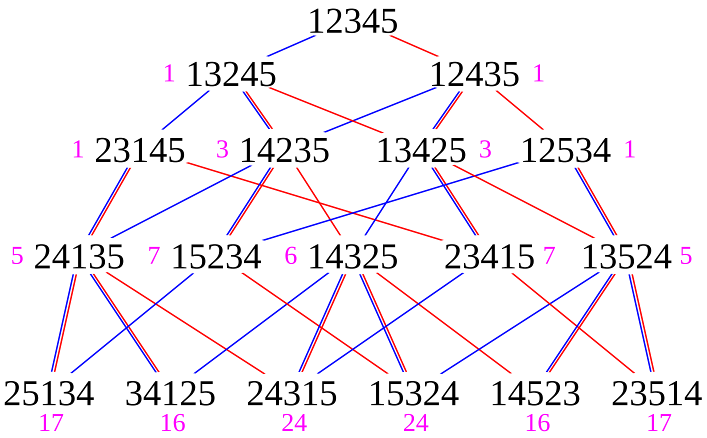
$$c^{45312}_{32154\ 14523} \quad = \quad c^{⊞}_{⊞} \quad = \quad 1 \qquad \text{and} \qquad c^{45312}_{32154\ 23514} \quad = \quad c^{⊞}_{⊞} \quad = \quad 1.$$

Next, let $F_\bullet, F_\bullet', F_\bullet''$ be in general position. If $E_\bullet \in X_{15324}F_\bullet \bigcap X_{32154}F_\bullet'$, then $E_2 \subset F_4'$ and $E_2 \supset F_1$, contradicting $F_\bullet$ and $F_\bullet'$ in general position. Thus

$$c^{45312}_{32154\ 15324} \quad = \quad \#\left(X_{15324}F_\bullet \bigcap X_{32154}F_\bullet' \bigcap X_{w_0 45312}F_\bullet''\right) \quad = \quad 0.$$

To compute $c^{45312}_{32154\ 24315} = \deg\left(\mathfrak{S}_{w_0 45312} \cdot \mathfrak{S}_{32154} \cdot \mathfrak{S}_{24315}\right)$, note that $\mathfrak{S}_{w_0\ 45312} = \mathfrak{S}_{21354} = \mathfrak{S}_{(1,2)} \cdot \mathfrak{S}_{(4,5)}$. Two applications of Monk's formula show $c^{45312}_{32154\ 24315} = 1$. (The other computations could also have proceeded via Monk's formula.)

To compute $f_e^v(P)$ for these minimal coset representatives, consider the part of the P-Bruhat order rooted at e and restricted to permutations of length at most 4:



The small numbers adjacent to each permutation v are $f_e^v(P)$. Thus

$$\sum_v f_e^v(P)\ c^{45312}_{32154\ v} \quad = \quad 17 \cdot 0 \ + \ 16 \cdot 0 \ + \ 24 \cdot 1 \ + \ 24 \cdot 0 \ + \ 16 \cdot 1 \ + \ 17 \cdot 1 \quad = \quad 57,$$

which equals $f^{45312}_{32154}(P)$.

B.3. **Instance of Theorem 1.2.2.** We consider $\Psi_{\{1,3,5,\ldots\}}(\mathfrak{S}_{516432})$.

$$\begin{aligned}\mathfrak{S}_{516432} \quad = \quad & x_1^4x_2^2x_3^3x_5 + x_1^4x_2x_3^3x_4x_5 + x_1^4x_3^3x_4^2x_5 \\ & + x_1^4x_2^3x_3^2x_5 + x_1^4x_2^2x_3^3x_4 + x_1^4x_2^2x_3^2x_4x_5 + x_1^4x_2x_3^3x_4^2 + x_1^4x_2x_3^2x_4^2x_5 \\ & + x_1^4x_2^3x_3^2x_4 + x_1^4x_2^3x_3x_4x_5 + x_1^4x_2^2x_3^2x_4^2 + x_1^4x_2^2x_3x_4^2x_5 \\ & + x_1^4x_2^3x_3x_4^2 + x_1^4x_2^3x_4^2x_5.\end{aligned}$$

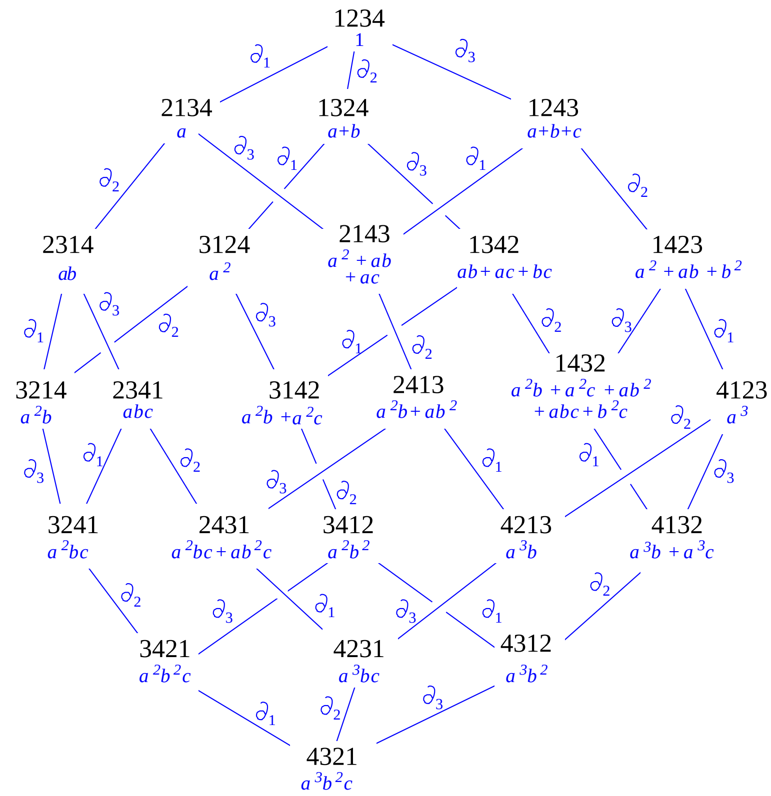
$\Psi_{\{1,3,5,\ldots\}}(\mathfrak{S}_{516432}) = \mathfrak{S}_{516432}(y_1, z_1, y_2, z_2, y_3, z_3, \ldots)$, which is

$$\begin{aligned} & y_1^4y_2^3y_3(z_1^2 + z_1z_2 + z_2^2) \ + \ y_1^4y_2^2y_3(z_1^3 + z_1z_2^2 + z_1^2z_2) \ + \ y_1^4y_2^3(z_1^2z_2 + z_1z_2^2) \\ & \quad + \ (y_1^4y_2^2 + y_1^4y_2y_3)(z_1^3z_2 + z_1^2z_2^2) \ + \ (y_1^4y_2 + y_1^4y_3)z_1^3z_2^2.\end{aligned}$$

Using the definition of Schubert polynomials in §2.2, one may check

$$\begin{array}{rclcrcl}
\mathfrak{S}_{54213} &=& x_1^4x_2^3x_3 & \qquad & \mathfrak{S}_{53214} &=& x_1^4x_2^2x_3\\
\mathfrak{S}_{54213} &=& x_1^4x_2^3 & & \mathfrak{S}_{53124} &=& x_1^4x_2^2\\
\mathfrak{S}_{52314} &=& x_1^4x_2x_3 & & \mathfrak{S}_{51324} &=& x_1^4x_2+x_1^4x_3
\end{array}$$

The Schubert polynomials $\mathfrak{S}_w$ for $w\in\mathcal{S}_4$ are indicated below. The Schubert polynomial $\mathfrak{S}_w$ is written below the permutation w, and these data are displayed at the vertices of the permutahedron (or at vertices of the Cayley graph of $\mathcal{S}_4$). The divided difference operators are also displayed on the edges of this figure.



We see that $\Psi_{\{1,3,5,\ldots\}}\mathfrak{S}_{516432}=\mathfrak{S}_{516432}(y_1,z_1,y_2,z_2,y_3,z_3,\ldots)$ is equal to

$$\mathfrak{S}_{54213}(y)\mathfrak{S}_{1423}(z)\ +\ \mathfrak{S}_{53214}(y)\left[\mathfrak{S}_{4123}(z)+\mathfrak{S}_{2413}(z)\right]\ +\ \mathfrak{S}_{54123}(y)\mathfrak{S}_{2413}(z)$$

$$+\ \left[\mathfrak{S}_{53124}(y)+\mathfrak{S}_{52314}(y)\right]\left[\mathfrak{S}_{4213}(z)+\mathfrak{S}_{3412}(z)\right]\ +\ \mathfrak{S}_{51324}(y)\mathfrak{S}_{4312}(z).$$

B.4. **Automorphisms of $(\mathcal{S}_\infty,\preceq)$.** The definition of the k-Bruhat orders imply that if $u,w\in\mathcal{S}_n$, and $k<n$, then the following are equivalent:

$$u\leq_k w \qquad w_0w\leq_k w_0u \qquad ww_0\leq_{n-k} uw_0 \qquad w_0ww_0\leq_{n-k} w_0uw_0.$$

These induce the following isomorphisms (which were stated in Theorem 3.2.3) of intervals in the $\preceq$-order on $\mathcal{S}_\infty$. Suppose $\zeta \in \mathcal{S}_n$ and $\overline{\zeta} = w_0\zeta w_0$. Then

$$[e,\zeta]_\preceq \quad\simeq\quad [e,\overline{\zeta}]_\preceq \quad\simeq\quad [e,\zeta^{-1}]^{\mathrm{op}}_\preceq \quad\simeq\quad [e,\overline{\zeta}^{-1}]^{\mathrm{op}}_\preceq .$$

These are illustrated in the posets displayed in §B.6.

B.5. **Canonical algorithms?** Besides Algorithm 3.1.1, there are three other 'canonical' algorithms for finding a chain between u and w when $u \leq_k w$, each induced from Algorithm 3.1.1 by one of the automorphisms of the previous section. For example, here is one.

Algorithm B.5.1 (Produces a chain in the k-Bruhat order)**.**
`input:` *Permutations $u, w \in \mathcal{S}_\infty$ with $u \leq_k w$.*
`output:` *A chain in the k-Bruhat order from w to u.*
Output w. While $u \neq w$, do

1 *Choose $a \leq k$ with $w(a)$ maximal subject to $u(a) < w(a)$.*
2 *Choose $k < b$ with $w(b)$ minimal subject to $w(b) \leq u(a) < u(b)$.*
3 *$u := u(a,b)$, output u.*

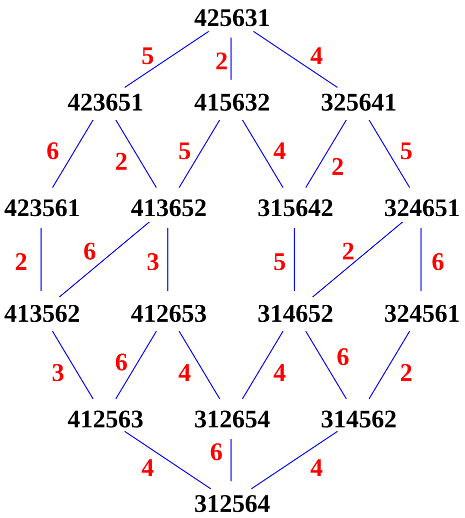
In general, these algorithms produce different chains. In S_7, consider the two permutations $2317546 <_3 4671235$. Here are chains produced by the four algorithms:

2317546	2317546	2317546	2317546
2417536	2417536	2371546	2371546
2517436	2517436	2571346	2571346
2617435	4517236	2671345	3571246
4617235	4617235	3671245	4571236
4671235	4671235	4671235	4671235

B.6. **Schensted insertion and the $c^w_{u\,v(\lambda,k)}$.** In §6.3, we discussed how the conclusion of Theorem 6.1.1 holds for many permutations in $\mathcal{S}_6$, even most which are not skew permutations. We illustrate that here.

Let $\zeta = (143562)$. Then $214365 \leq_4 \zeta \cdot 214365 = 345612$. Below, we display the labeled Hasse diagram of $[214365,\ 345612]_4$ and beside it a table of the words of the

14 chains in this interval, each displayed above its insertion and recording tableau.

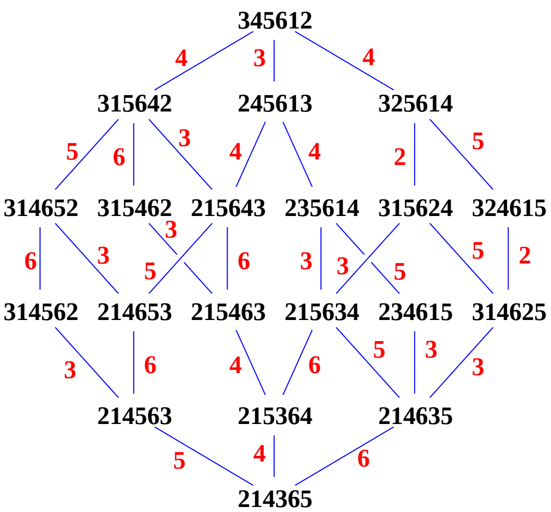


46542	64325	64542	64352
6 5	6 4	6 5	6 5
5 4	4 3	4 4	4 3
3 3	3 2	3 2	3 2
24 12	25 15	25 13	25 14

46254	46524	42654	64524	64254
6 5	6 4	6 5	6 4	6 3
45 34	45 35	45 24	45 25	45 25
24 12	24 12	24 13	24 13	24 14

46352	46325	43652	43625	43265
4 5	4 4	4 5	4 4	4 3
36 34	36 35	36 24	36 25	36 25
25 12	25 12	25 13	25 13	25 14

Note that $\eta := (125634) = \zeta^{(123456)}$ and $312564 \leq_4 \eta \cdot 312564 = 425635$. We continue this example, and illustrate Theorem 1.3.4. Below are the labeled Hasse diagram of $[312564, 425635]_4$, and the insertion and recording tableaux for all 14 chains in this interval.

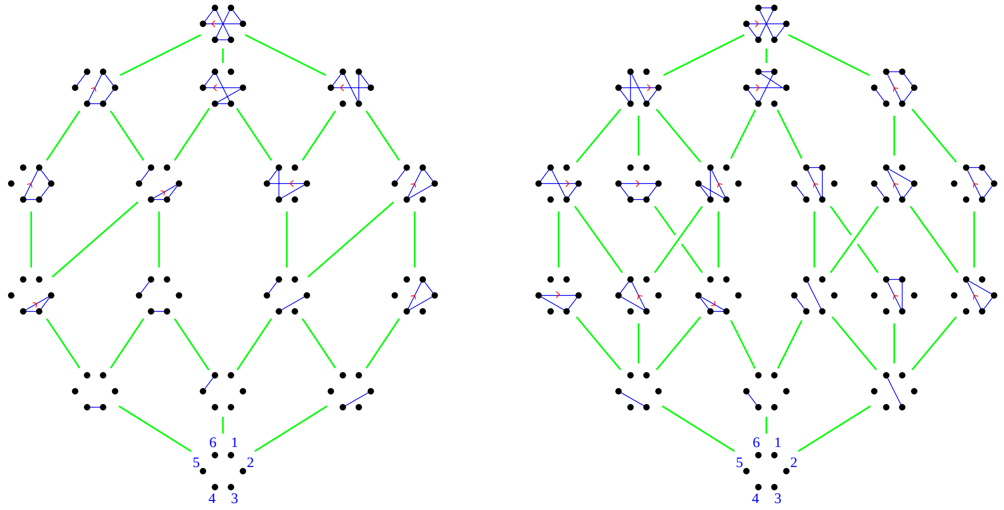


56543	65324	63543	65343
6 5	6 4	6 5	6 5
5 4	5 3	5 4	5 3
4 3	3 2	4 2	4 2
35 12	24 15	33 13	33 14

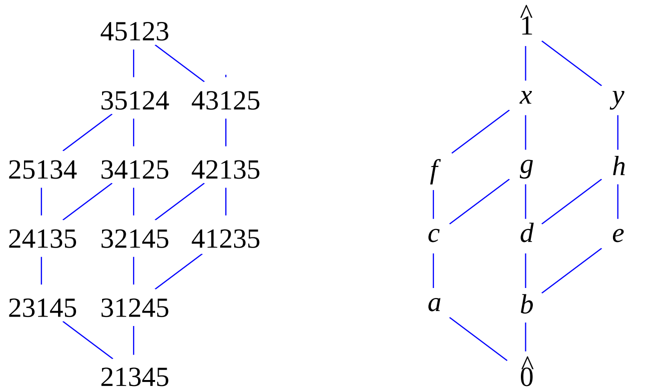
56343	56324	54643	54634	54364
5 5	5 4	5 5	5 4	5 3
46 34	36 35	46 24	46 25	46 25
33 12	24 12	34 13	34 13	34 14

56354	56534	53654	63524	63254
6 5	6 4	6 5	6 4	6 3
55 34	55 35	55 24	35 25	35 25
34 12	34 12	34 13	24 13	24 14

For these last two intervals, it is interesting to view them with the permutation $v \in [u, w]_k$ replaced by the geometric graph of vu^{-1}. This gives an idea of the effect of a 'cyclic shift' on the $\preceq$-order.



B.7. **Simplicial complexes and $\leq_k$.** In the theory of partially ordered sets, one often constructs a simplicial complex $\Delta(P)$ from a poset, P. We compute one such for an interval in the k-Bruhat order, which shows these intervals are not in general shellable. We illustrate this with one example drawn from this paper. In Example 3.2.4, we considered the interval $[21342,\ 45123]_2$. We display that interval below, together with the Hasse diagram of an isomorphic poset:

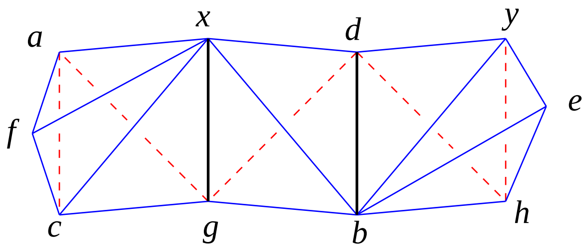


The simplicial complex $\Delta(P)$ associated to a poset P has as simplices all chains, including the non-maximal ones. In our case above, the maximal simplices are

$$\{a, c, f, x\},\ \{a, c, g, x\},\ \{b, d, g, x\},\ \{b, d, h, y\},\ \{b, e, h, y\}.$$

While the pairs $(\{a, c, f, x\}, \{a, c, g, x\})$ and $(\{b, d, h, y\}, \{b, e, h, y\})$ are attached along facets ($\{a, c, x\}$ and $\{b, h, y\}$, respectively), the pairs $(\{a, c, g, x\}, \{b, d, g, x\})$ and $(\{b, d, g, x\}, \{b, d, h, y\})$ are not. They are attached along codimension 2 faces, $\{g, x\}$

and $\{b, d\}$, respectively. Thus this simplicial complex is not shelable. Below, we display a geometric realization of this simplicial complex:



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