ON k-ORDERED HAMILTONIAN GRAPHS

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Abstract. A Hamiltonian graph G of order n is k-ordered, $2 \leq k \leq n$, if for every sequence $v_1, v_2, \dots, v_k$ of k distinct vertices of G, there exists a Hamiltonian cycle that encounters $v_1, v_2, \dots, v_k$ in this order. In this paper, answering a question of Ng and Schultz, we give a sharp bound for the minimum degree guaranteeing that a graph is a k-ordered Hamiltonian graph under some mild restrictions. More precisely, we show that there are $\varepsilon, n_0 > 0$ such that if G is a graph of order $n \geq n_0$ with minimum degree at least $\lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1$ and $2 \leq k \leq \varepsilon n$, then G is a k-ordered Hamiltonian graph. It is also shown that this bound is sharp for every $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

1. Introduction

1.1. **Notations and definitions.** For basic graph concepts see the monograph of Bollobás [1].

$+$ will sometimes be used for disjoint union of sets. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $N(v)$ is the set of neighbours of $v \in V$. Hence the size of $N(v)$ is $|N(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. $\nu(G)$ is the size of a maximum matching in G. For $A \subset V(G)$ we write $N(A) = \cap_{v \in A} N(v)$, the set of common neighbours. $N(x, y, z, \ldots)$ is shorthand for $N(\{x, y, z, \ldots\})$. For a vertex $v \in V$ and set $U \subset V - \{v\}$, we write $deg(v, U)$ for the number of edges from v to U. When A, B are disjoint subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. For non-empty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the **density** of the graph between A and B.

Definition 1. *The bipartite graph $G = (A, B, E)$ is ε-***regular** *if*

$$X \subset A,\ Y \subset B,\ |X| > \varepsilon |A|,\ |Y| > \varepsilon |B| \quad \textit{imply} \quad |d(X, Y) - d(A, B)| < \varepsilon,$$

otherwise it is ε-irregular.

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We will often say simply that "the pair (A, B) is ε-regular" with the graph G implicit. We will also need a stronger version.

Definition 2. *(A, B) is (ε, δ)* **super-regular** *if it is ε-regular and*

$$deg(a) > \delta|B| \ \forall a \in A, \quad deg(b) > \delta|A| \ \forall b \in B.$$

1.2. **k-ordered Hamiltonian graphs.** Let G be a graph on $n \geq 3$ vertices. A **Hamiltonian cycle** (**path**) of G is a cycle (path) containing every vertex of G. A **Hamiltonian graph** is a graph containing a Hamiltonian cycle. A classical result of Dirac [3] asserts that if $\delta(G) \geq n/2$, then G is Hamiltonian. A **Hamiltonian-connected** graph is a graph in which every pair of vertices can be connected with a Hamiltonian path.

The following interesting concept was created by Chartrand: For a positive integer $2 \leq k \leq n$, and for a sequence $S = v_1, v_2, \ldots, v_k$ of k distinct vertices, a cycle C in G is called a $v_1 - v_2 - \ldots - v_k$-cycle, or shortly an S-cycle, if the vertices of S are encountered on C in the specified order. For a Hamiltonian graph G we say that G is **k-ordered** if for every sequence $S = v_1, v_2, \ldots, v_k$ of k distinct vertices, there exists a Hamiltonian S-cycle. It is not hard to see that every Hamiltonian graph is both 2-ordered and 3-ordered. Furthermore, a Hamiltonian graph G of order n is n-ordered if and only if $G = K_n$. Also, if G is k-ordered, then G is l-ordered for every $2 \leq l \leq k$ (see [9]).

A natural question is whether we can obtain a Dirac type condition on the minimum degree for guaranteeing that the graph is a k-ordered Hamiltonian graph. Indeed, the first result of this type was obtained in [9]. In this paper it was shown (among other results) that if $3 \leq k \leq n$ and $\delta(G) \geq \frac{n}{2} + k - 3$, then G is a k-ordered Hamiltonian graph. The authors raised the question whether this can be improved. In this paper our goal is to determine the best possible bound under the restrictions that n is sufficiently large and k does not grow too quickly with n. More precisely, our main result is the following.

Theorem 1. *There exist $\kappa, n_0 > 0$ such that if a graph G has order $n \geq n_0$ with*

$$\delta(G) \geq \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1, \tag{1}$$

and $2 \leq k \leq \kappa n$, then G is a k-ordered Hamiltonian graph.

It is not hard to see that this bound is best possible for most k-s.

Theorem 2. *For any $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$ there exists a graph G of order n with*

$$\delta(G) = \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 2, \tag{2}$$

such that G is not a k-ordered Hamiltonian graph.

2. The main tools

In the proof the following lemma of Szemerédi plays a central role.

Lemma 1 (Regularity Lemma [11]). *For every positive ε and positive integer m there are positive integers M and n_1 with the following property: for every graph G with $n \geq n_1$ vertices there is a partition of the vertex set into $l+1$ classes (clusters)*

$$V = V_0 + V_1 + V_2 + ... + V_l$$

such that

- $m \leq l \leq M$
- $|V_1| = |V_2| = ... = |V_l|$
- $|V_0| < \varepsilon n$
- *at most εl^2 of the pairs $\{V_i, V_j\}$ are ε-irregular.*

We will use the following simple consequence of Lemma 1.

Lemma 2 (Degree form). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex-set V into $l+1$ clusters $V_0, V_1, ..., V_l$, and there is a subgraph $G' = (V, E')$ with the following properties:*

- $l \leq M$,
- $|V_0| \leq \varepsilon |V|$,
- *all clusters V_i, $i \geq 1$, are of the same size $L \leq \lceil \varepsilon |V| \rceil$.*
- $deg_{G'}(v) > deg_G(v) - (d + \varepsilon)|V|$ *for all* $v \in V$,
- $G'|_{V_i} = \emptyset$ *(V_i are independent in G'),*
- *all pairs $G'|_{V_i \times V_j}$, $1 \leq i < j \leq l$, are ε-regular, each with a density either 0 or exceeding d.*

The other main tool asserts that if (A, B) is a super-regular pair with $|A| = |B|$ and $x \in A, y \in B$, then there is a Hamiltonian path starting with x and ending with y. This is a very special case of the Blow-up Lemma [5], and it has already appeared in [4] and in [10]. More precisely.

Lemma 3. *For every $\delta > 0$ there are $\varepsilon_0, n_2 > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $n \geq n_2$, $G = (A, B)$ is an (ε, δ) super-regular pair with $|A| = |B| = n$ and $x \in A$, $y \in B$, then there is a Hamiltonian path in G starting with x and ending with y.*

We will also use two simple Pósa-type lemmas on Hamiltonian-connectedness. The second one is the bipartite version of the first one.

Lemma 4 (see [2]). *Let G be a graph on $n \geq 3$ vertices with degrees $d_1 \leq d_2 \leq \ldots \leq d_n$ such that for every $2 \leq k \leq \frac{n}{2}$ we have $d_{k-1} > k$. Then G is Hamiltonian-connected.*

Lemma 5 (see [2]). *Let $G = (A, B)$ be a bipartite graph with $|A| = |B| = n \geq 2$ with degrees $d_1 \leq d_2 \leq \ldots \leq d_n$ from A and with degrees $d'_1 \leq d'_2 \leq \ldots \leq d'_n$ from B. Suppose that for every $2 \leq j \leq \frac{n+1}{2}$ we have $d_{j-1} > j$ and that for every $2 \leq k \leq \frac{n+1}{2}$ we have $d'_{k-1} > k$. Then G is Hamiltonian-connected.*

Finally we will use the following simple fact.

Lemma 6 (Erdős, Pósa, see [1]). *Let G be a graph on n vertices. Then*

$$\nu(G) \geq \min\{\delta(G), \frac{n-1}{2}\}.$$

In case we have a good upper bound on the maximum degree of G, we can strengthen this lemma in the following way.

Lemma 7. *In a graph G of order n*

$$\nu(G) \geq \delta(G)\frac{n}{2(\delta(G) + \Delta(G))} \geq \delta(G)\frac{n}{4\Delta(G)}.$$

In fact, let us take a maximal matching M with m edges. Then for the number of edges E between M and $V(G) \setminus M$ we get $\delta(G)(n - 2m) \leq E \leq 2m\Delta(G)$, which proves the lemma.

3. Proof of Theorem 1

3.1. **Outline of the proof.** During the past couple of years the first author, together with J. Komlós and E. Szemerédi, developed a new method in graph theory based on the Regularity Lemma and the Blow-up Lemma. The method is usually applied to find certain spanning subgraphs in dense graphs. Typical examples are spanning trees (Bollobás-conjecture, see [4]), Hamiltonian cycles or powers of Hamiltonian cycles (Pósa-Seymour conjecture, see [6, 7]) or H-factors for a fixed graph H (Alon-Yuster conjecture, see [8]). In this paper we apply this method again.

We will use the following main parameters

$$0 < \kappa \ll \varepsilon \ll d \ll \beta \ll \alpha \ll 1, \tag{3}$$

where $a \ll b$ means that a is sufficiently small compared to b. For simplicity we do not compute the actual dependencies, although it could be done. Throughout the rest of the proof we assume that

$$2 \leq k \leq \kappa n. \tag{4}$$

We apply Lemma 2 for G, with ε and d as in (3). We get a partition of $V = \cup_{0 \leq i \leq l} V_i$. We define the following so-called **reduced graph** G_r: The vertices of G_r are the clusters V_i, $1 \leq i \leq l$, in the partition and there is an edge between two clusters if they form an ε-regular pair in G' with density exceeding d. Since in G'

$$\delta(G') > \delta(G) - (d + \varepsilon)n \geq \left(\lceil\frac{n}{2}\rceil + \lfloor\frac{k}{2}\rfloor - 1\right) - (d + \varepsilon)n \geq \left(\frac{1}{2} - (d + \varepsilon)\right) n,$$

an easy calculation shows that in G_r we have

$$\delta(G_r) \geq \left(\frac{1}{2} - 3d\right) l. \tag{5}$$

Then Lemma 6 implies that we can find a matching M in G_r of size at least $\left(\frac{1}{2} - 3d\right) l$. Put $|M| = m$. Let us put the vertices of the clusters not covered by M into the exceptional set V_0. For simplicity V_0 still denotes the resulting set. Then

$$|V_0| \leq 6dlL + \varepsilon n \leq 7dn. \tag{6}$$

Denote the i-th pair in M by (V_1^i, V_2^i) for $1 \leq i \leq m$. Let $S = v_1, v_2, \ldots, v_k$ be any sequence drawn from $V(G)$.

The rest of the paper is organized as follows. In the next section we show that if certain extremal conditions hold then we can find directly the desired Hamiltonian cycle without using the above method. Then assuming that the extremal conditions do not hold we do the following. In Section 3.3 first we find a short S-path P. Then we find short connecting paths between the consecutive edges in the matching M (for $i = m$ the next edge is $i = 1$). The first connecting path between (V_1^1, V_2^1) and (V_1^2, V_2^2) will also contain P, the others have length exactly 3. In Section 3.4 we will take care of the exceptional vertices and make some adjustments by extending some of the connecting paths so that the distribution of the remaining vertices inside each edge in M is perfect, i.e. there are the same number of vertices left in both clusters of the edge. Finally applying Lemma 3 we close the Hamiltonian cycle in each edge. We give the simple proof of Theorem 2 in Section 4. We finish by some open problems in Section 5.

3.2. **Extremal cases.** In this section we show that if G satisfies certain extremal conditions, then we can find the desired Hamiltonian cycle directly. The first extremal case is the following.

Lemma 8. *There exists an $n_3 > 0$ such that the following holds. Assume that G is a graph on $n \geq n_3$ vertices satisfying (1), k satisfies (4), and there are $A, B \subset V(G)$ such that*

- $A \cap B = \emptyset, |A|, |B| \geq (1 - \alpha)\frac{n}{2}$,
- $d(A, B) < \alpha$.

Then G is a k-ordered Hamiltonian graph.

Proof: First we find the set (denoted by $Exc(A)$) of exceptional vertices $x \in A$ for which $deg(x, B) \geq \sqrt{\alpha}|B|$. The density assumption implies that the number of these exceptional vertices is at most $\sqrt{\alpha}|A|$. Similarly we find the set (denoted by $Exc(B)$) of exceptional vertices $y \in B$ for which $deg(y, A) \geq \sqrt{\alpha}|A|$. Again we have $|Exc(B)| \leq \sqrt{\alpha}|B|$. We remove the vertices of $Exc(A)$ from A and the vertices of

$Exc(B)$ from B and we form

$$E = Exc(A) \cup Exc(B) \cup (V(G) \setminus (A \cup B)).$$

For each vertex $z \in E$, if $deg(z, A) \geq deg(z, B)$, then we add z to A, and we add z to B in the opposite case. For simplicity we still denote the resulting sets by A and B. It is not hard to see that in $G|_A$ and in $G|_B$ apart from at most $3\sqrt{\alpha}n$ exceptional vertices all the degrees are at least $(1 - \alpha^{1/4})\frac{n}{2}$, and the degrees of the exceptional vertices are at least $\frac{n}{5}$.

Let $S = v_1, v_2, \ldots, v_k$ be any sequence drawn from $V(G)$, and let

$$A^* = \{v_1, v_2, \ldots, v_k\} \cap A \quad \text{and} \quad B^* = \{v_1, v_2, \ldots, v_k\} \cap B.$$

The graph H is the spanning subgraph of G with edges set

$$E(H) = E(G) \cap ((B \times (A \setminus A^*)) \cup (A \times (B \setminus B^*))),$$

that is, the graph obtained from G by removing all edges whose endpoints are each in A or each in B or each in $\{v_1, v_2, \ldots, v_k\}$. We now give a lower bound on the size of an arbitrary vertex cover C of $E(H)$. By König's Theorem (see [1]) this gives us the same lower bound on the size of a maximum matching. If $(A \setminus A^*) \subset C$ or $(B \setminus B^*) \subset C$, then $|C| \geq k$. Otherwise there exist $x \in (A \setminus A^*) \setminus C$ and $y \in (B \setminus B^*) \setminus C$ and $N(x) \cup N(y) \subset C$. We have

$$|C| \geq deg_H(x) + deg_H(y) \geq \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1 - (|A| - 1) + \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1 - (|B| - 1) =$$

$$= 2\lfloor \frac{k}{2} \rfloor + 2\lceil \frac{n}{2} \rceil - n.$$

Here the last expression is $k - 1$ if k is odd and n is even, and at least k in all other cases. Therefore, we can always find a matching M with size $k - 1$ if k is odd and with size k if k is even. We are going to use the edges in M as "bridges" on the transitions between A and B.

We are going to use the following fact.

Fact 1. *For any k pairs $(u_1, w_1), \ldots, (u_k, w_k)$ of vertices in A (analogously in B), there exist k internally disjoint paths $P(u_1, w_1), \ldots, P(u_k, w_k)$ of lengths at most 4 such that $P(u_i, w_i)$ has u_i and w_i as endpoints.*

Indeed, this follows from (4) and the fact that the number of exceptional vertices in A is much smaller than the minimum degree in $G|_A$.

Given our matching M and the sequence $S = v_1, v_2, \ldots, v_k$, first we construct an S-cycle C_{k+1} (not necessarily Hamiltonian). Let $v_{k+1} = v_1$. Initially $C_1 = (v_1)$, and the following cases are applied iteratively for adding v_{i+1} to C_i until $v_{k+1} = v_1$ is added again, in which case C_{k+1} is an S-cycle (note that the intermediate C_i-s, $i \leq k$ are only paths):

- If $\{v_i, v_{i+1}\} \subset A$, then replace v_i by $P(v_i, v_{i+1})$ in C_i, yielding C_{i+1}.

- If $v_i \in A$ and $v_{i+1} \in B$ and v_i is incident with an edge (v_i, w) of M, then add $P(w, v_{i+1})$ to C_i, yielding C_{i+1}.
- If $v_i \in A$ and $v_{i+1} \in B$ and v_i is not incident with an edge of M, but v_{i+1} is incident with an edge (w, v_{i+1}) of M, then replace v_i by $P(v_i, w)$ and v_{i+1}.
- If $v_i \in A$ and $v_{i+1} \in B$ and v_i and v_{i+1} are not incident with edges of M, then there must be an edge $(u, w) \in M$ such that $u \in A \setminus A^*$ and $w \in B \setminus B^*$. Replace v_i with $P(v_i, u)$ and $P(w, v_{i+1})$ in C_i, yielding C_{i+1}.
- The cases for $v_i \in B$ are handled analogously.

Here we used the fact that the number of edges in M is at least the number of transitions we must make between A and B, thus we never run out of bridges in M. If $C = C_{k+1}$ contains two consecutive vertices u_j and u_{j+1} of A but not all of A belongs to C (an analogous argument holds for B), then let T contain u_j, u_{j+1} and all the vertices of A not in C. It is not hard to see that the degree conditions in Lemma 4 are satisfied in $G|_T$ (with much room to spare) and thus $G|_T$ is Hamiltonian-connected, i.e. there is a Hamiltonian path in $G|_T$ from u_j to u_{j+1}. We add the intermediate vertices of this path to C between u_j and u_{j+1}.

Let T' be the set of vertices not yet in C. If $T' = \emptyset$, then we are finished. Otherwise either $\{v_1, v_2, \ldots, v_k\} \subset A$ and $T' = B$ or $\{v_1, v_2, \ldots, v_k\} \subset B$ and $T' = A$. Without loss of generality, assume $T' = A$ and let x_i and x_j two consecutive vertices in C. Choose distinct $y_i, y_j \in B$ such that $(x_i, y_i), (x_j, y_j) \in E(G)$. Again Lemma 4 implies that $G|_{T'}$ is Hamiltonian- connected, thus there is a Hamiltonian path in $G|_{T'}$ from y_i to y_j. We add this path between x_i and x_j in C to get the desired Hamiltonian cycle.

The second extremal case is the following.

Lemma 9. *There exists an $n_4 > 0$ such that the following holds. Assume that G is a graph on $n \geq n_4$ vertices satisfying (1), k satisfies (4), and there are $A, B \subset V(G)$ such that*

- $A \cap B = \emptyset, |A|, |B| \geq (1-\alpha)\frac{n}{2}$,
- $d(A, B) > (1-\alpha)$.

Then G is a k-ordered Hamiltonian graph.

Proof: We start similarly to the proof of Lemma 8. First we find the set (denoted by $Exc(A)$) of exceptional vertices $x \in A$ for which $deg(x, B) < (1-\sqrt{\alpha})|B|$. The density assumption implies that the number of these exceptional vertices is at most $\sqrt{\alpha}|A|$. Similarly we find the set (denoted by $Exc(B)$) of exceptional vertices $y \in B$ for which $deg(y, A) < (1-\sqrt{\alpha})|A|$. Again we have $|Exc(B)| \leq \sqrt{\alpha}|B|$. We remove the vertices of $Exc(A)$ from A and the vertices of $Exc(B)$ from B and we form

$$E = Exc(A) \cup Exc(B) \cup (V(G) \setminus (A \cup B)).$$

For each vertex $z \in E$, if $deg(z, A) \geq deg(z, B)$, then we add z to B, and we add z to A in the opposite case. For simplicity we still denote the resulting sets by A and

B. It is not hard to see that in $G|_{A\times B}$ apart from at most $3\sqrt{\alpha}n$ exceptional vertices all the degrees are at least $(1-\alpha^{1/4})\frac{n}{2}$, and the degrees of the exceptional vertices are at least $\frac{n}{5}$.

Without loss of generality, assume that $|A|-|B| = r \geq 0$. We also know that $r \leq 3\sqrt{\alpha}n$. Our goal is to achieve $r = 0$ since then we can apply Lemma 5. If there is a vertex $x \in A$ for which

$$deg(x, A) \geq \alpha^{1/4}|A|, \tag{7}$$

then we remove x from A and add it to B. We iterate this procedure until either there is no more vertex satisfying (7) or $|A| = |B|$. Let $S = v_1, v_2, \ldots, v_k$ be any sequence drawn from $V(G)$ and assume that the first case is true. Since we have $\Delta(G|_A) < \alpha^{1/4}|A|$, (1) and Lemma 7 imply that $G|_A$ has a r-matching M denoted by $(u_1, w_1), \ldots, (u_r, w_r)$. In case $|A| = |B|$ we have $r = 0$ and $M = \emptyset$. This time the matching M will be used to balance the discrepancy between $|A|$ and $|B|$.

First we construct a short path P that is an S-path and contains all the edges of M. For this purpose we use the following fact (similar to Fact 1).

Fact 2. *For any $r+k-1$ pairs $(u'_1, w'_1), \ldots, (u'_{r+k-1}, w'_{r+k-1})$ of vertices, there exist $r+k-1$ internally disjoint paths $P(u'_1, w'_1), \ldots, P(u'_{r+k-1}, w'_{r+k-1})$ in $G|_{A\times B}$ of lengths at most 5 such that $P(u'_i, w'_i)$ has u'_i and w'_i as endpoints.*

Then in case $r > 0$ the path P is the following (as a sequence of vertices):

$$P = u_1, P(w_1, u_2), P(w_2, u_3), \ldots, P(w_{r-1}, u_r),$$

$$P(w_r, v_1), (P(v_1, v_2) \setminus v_1), \ldots, (P(v_{k-1}, v_k) \setminus v_{k-1}).$$

Furthermore, if both u_1 and v_k fall into A or into B, then we add one more vertex v from the other set to the end of P such that $(v_k, v) \in E(G)$. In case $r = 0$ P is the following:

$$P = P(v_1, v_2), \ldots, (P(v_{k-1}, v_k) \setminus v_{k-1}).$$

Again, if both v_1 and v_k fall into the same set we add one more vertex from the other set to the end of P. Lemma 5 finds the remaining part of the Hamiltonian S-cycle in both cases. Note that for this purpose we could also use Lemma 3, the remaining bipartite graph is super-regular with the appropriate choice of parameters, but here the much simpler Lemma 5 also suffices.

An easy consequence of Lemmas 8 and 9 is our main extremal case.

Lemma 10. *There exists an $n_5 > 0$ such that the following holds. Assume that G is a graph on $n \geq n_5$ vertices satisfying (1), k satisfies (4), and there are $A, B \subset V(G)$ (not necessarily disjoint) such that*

- $(1-\beta)\frac{n}{2} \leq |A|, |B| \leq \frac{n}{2}$,
- $d(A, B) < \beta$.

Then G is a k-ordered Hamiltonian graph.

Proof: We have three cases:
Case 1: $|A \cap B| \geq (1 - \sqrt{\beta})\frac{n}{2}$. In this case the statement follows from Lemma 9.
Case 2: $\sqrt{\beta}\frac{n}{2} \leq |A \cap B| < (1 - \sqrt{\beta})\frac{n}{2}$. This case is not possible under the given assumptions.
Case 3: $|A \cap B| < \sqrt{\beta}\frac{n}{2}$. The statement follows from Lemma 8.

3.3. **Connecting paths.** In the remainder of the proof of Theorem 1 we may assume that the extremal conditions in Lemma 10 do not hold, since otherwise we can find the desired Hamiltonian cycle directly. For constructing the connecting paths we are going to use the following fact several times.

Fact 3. *If $x, y \in V(G)$ then there are at least dn internally disjoint paths of length 3 connecting x and y.*

Indeed, we apply Lemma 10 with A and B chosen as follows: A is an arbitrary subset of $N_G(x)$ with $|A| = \lfloor \frac{n}{2} \rfloor$ and B is an arbitrary subset of $N_G(y)$ with $|B| = \lfloor \frac{n}{2} \rfloor$.

Let $S = v_1, v_2, \ldots, v_k$ be any sequence drawn from $V(G)$. Again first we construct a short S-path P. Applying Fact 3, we first connect v_1 and v_2 with a path of length 3, then we connect v_2 and v_3 with a path of length 3 that is internally disjoint from the first connecting path between v_1 and v_2, etc. finally we connect v_{k-1} and v_k with a path of length 3 that is internally disjoint from all the connecting paths constructed so far.

For the first connecting path P_1 between (V_1^1, V_2^1) and (V_1^2, V_2^2), first we connect a typical vertex u of V_2^1 (more precisely a vertex u with $deg(u, V_1^1) \geq (d - \varepsilon)|V_1^1|$, most vertices in V_2^1 satisfy this) and v_1 with a path of length 3, and then we connect v_k and a typical vertex w of V_1^2 (so $deg(w, V_2^2) \geq (d - \varepsilon)|V_2^2|$) with a path of length 3. To construct the second connecting path P_2 between (V_1^2, V_2^2) and (V_1^3, V_2^3) we just connect a typical vertex of V_2^2 and a typical vertex V_1^3 with a path of length 3. Continuing in this fashion, finally we connect a typical vertex of V_2^m with a typical vertex of V_1^1 with a path of length 3. Thus P_1 has length $3(k+1)$, all other P_i-s have length 3.

We remove the vertices on these connecting paths from the clusters, but for simplicity we keep the notation for the resulting clusters. These connecting paths will be parts of the final Hamiltonian cycle. If the number of remaining vertices (in the clusters and in V_0) is odd, then we take another typical vertex w of V_1^2 and we extend P_1 by a path of length 3 that ends with w. So we may always assume that the number of remaining vertices is even.

3.4. **Adjustments and the handling of the exceptional vertices.** We already have an exceptional set V_0 of vertices in G. We add some more vertices to V_0 to achieve super-regularity. From V_1^i (and similarly from V_2^i) we remove all vertices u for which $deg(u, V_2^i) < (d - \varepsilon)|V_2^i|$. ε-regularity guarantees that at most $\varepsilon|V_1^i| \leq \varepsilon L$ such vertices exist in each cluster V_1^i.

Thus using (6), we still have

$$|V_0| \leq 7dn + 2\varepsilon n \leq 9dn.$$

Since we are looking for a Hamiltonian cycle, we have to include the vertices of V_0 on the Hamiltonian cycle as well. We are going to extend some of the connecting paths P_i, so now they are going to contain the vertices of V_0. Let us consider the first vertex (in an arbitrary ordering of the vertices in V_0) w in V_0. We find a pair (V_1^i, V_2^i) such that either

$$deg(w, V_1^i) \geq d|V_1^i|, \tag{8}$$

or

$$deg(w, V_2^i) \geq d|V_2^i|. \tag{9}$$

We assign w to the pair (V_1^i, V_2^i). We extend P_{i-1} (for $i = 1$, P_m) in (V_1^i, V_2^i) by a path of length 3 in case (8) holds, and by a path of length 4 in case (9) holds, so that now the path ends with w. To finish the procedure for w, in case (8) holds we add one more vertex w' to P_{i-1} after w such that $(w, w') \in E(G)$ and w' is a typical vertex of V_1^i, so $deg(w', V_2^i) \geq (d - \varepsilon)|V_2^i|$. In case (9) holds we add two more vertices w', w'' to P_{i-1} after w such that $(w, w'), (w', w'') \in E(G)$, w' is a typical vertex of V_2^i and w'' is a typical vertex of V_1^i.

After handling w, we repeat the same procedure for the other vertices in V_0. However, we have to pay attention to several technical details. First, of course in repeating this procedure we always consider the remaining free vertices in each cluster; the vertices on the connecting paths are always removed. Second, we make sure that we never assign too many vertices of V_0 to one pair (V_1^i, V_2^i). It is not hard to see (using $d \ll 1$) that we can guarantee that we always assign at most $\sqrt{d}|V_1^i|$ vertices of V_0 to a pair (V_1^i, V_2^i). Finally, since we are removing vertices from a pair (V_1^i, V_2^i), we might violate the super-regularity. Note that we never violate the ε-regularity. Therefore, we do the following. After handling (say) $\lfloor d^2 n \rfloor$ vertices from V_0, we update V_0 as follows. In a pair (V_1^i, V_2^i) we remove all vertices u from V_1^i (and similarly from V_2^i) for which $deg(u, V_2^i) < (d - \varepsilon)|V_2^i|$ (again, we consider only the remaining vertices). Again, we added at most $2\varepsilon n$ vertices to V_0. In V_0 we handle these vertices first and then we move on to the other vertices in V_0.

After we are done with all the vertices in V_0, we might have a small discrepancy ($\leq 2\sqrt{d}|V_1^i|$) among the remaining vertices in V_1^i and in V_2^i in a pair. Therefore, we have to make some adjustments. Let us take a pair (V_1^i, V_2^i) with a discrepancy ≥ 2 (if one such pair exists), say $|V_1^i| \geq |V_2^i| + 2$ (only remaining vertices are considered). Using Lemma 10 we find an **alternating path** (with respect to M) in G_r of length 6 starting with V_1^i and ending with V_2^i. Let us denote this path by

$$V_1^i, V_2^{i_1}, V_1^{i_1}, V_1^{i_2}, V_2^{i_2}, V_1^i, V_2^i$$

(the construction is similar if the clusters in $(V_1^{i_1}, V_2^{i_1})$ or in $(V_1^{i_2}, V_2^{i_2})$ are visited in different order). We remove a typical vertex from V_1^i and we add it to $V_1^{i_1}$, then we remove a typical vertex from $V_1^{i_1}$ and we add it to $V_2^{i_2}$, finally we remove a typical vertex from $V_2^{i_2}$ and we add it to V_2^i. When we add a new vertex to a pair (V_1^j, V_2^j), we extend the connecting path P_{j-1} by a path of length 4 in the pair so that it now includes the new vertex.

Now we are one step closer to the perfect distribution, and by iterating this procedure we can assure that the discrepancy in every pair is ≤ 1. We consider only those pairs for which the discrepancy is exactly 1, so in particular the number of remaining vertices in one such a pair is odd. From the construction it follows that we have an even number of such pairs. We pair up these pairs arbitrarily. If (V_1^i, V_2^i) and (V_1^j, V_2^j) is one such pair with $|V_1^i| = |V_2^i| + 1$ and $|V_1^j| = |V_2^j| + 1$ (otherwise similar), then similar to the construction above, we find an alternating path in G_r of length 6 between V_1^i and V_2^j, and we move a typical vertex of V_1^i through the intermediate clusters to V_2^j.

Thus we may assume that the distribution is perfect, in every pair (V_1^i, V_2^i) we have the same number of vertices left. In this case Lemma 3 closes the Hamiltonian cycle in every pair.

4. Proof of Theorem 2

We consider the graph G with vertices

$$\left\{u_1, \dots, u_{\lfloor \frac{n}{2} \rfloor}, w_1, \dots, w_{\lceil \frac{n}{2} \rceil}\right\}$$

such that $U = \left\{u_1, \dots, u_{\lfloor \frac{n}{2} \rfloor}\right\}$ and $W = \left\{w_1, \dots, w_{\lceil \frac{n}{2} \rceil}\right\}$ induce complete subgraphs of G. The edges of G between U and W are

$$\left(U \times \left\{w_1, \dots, w_{\lfloor \frac{k}{2} \rfloor}\right\}\right) \cup \left(W \times \left\{u_1, \dots, u_{\lfloor \frac{k}{2} \rfloor - 1}\right\}\right).$$

It is easily seen that

$$\delta(G) = \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 2,$$

as required. Furthermore G does not contain a Hamiltonian cycle which encounters

$$u_{\lfloor \frac{k}{2} \rfloor} - w_{\lfloor \frac{k}{2} \rfloor + 1} - u_{\lfloor \frac{k}{2} \rfloor + 1} - w_{\lfloor \frac{k}{2} \rfloor + 2} - \dots - u_{2\lfloor \frac{k}{2} \rfloor - 1} - w_{2\lfloor \frac{k}{2} \rfloor} \left(-u_{2\lfloor \frac{k}{2} \rfloor} \quad \text{if } k \text{ is odd} \quad\right)$$

in this order. This follows from the fact, that every $u_i - w_{i+1}$ and $w_i - u_i$ transition uses at least one vertex from the vertices

$$\left\{w_1, \dots, w_{\lfloor \frac{k}{2} \rfloor}\right\} \cup \left\{u_1, \dots, u_{\lfloor \frac{k}{2} \rfloor - 1}\right\}.$$

However, the number of transitions is always more than the number of vertices in this set. Here we also used the fact that we have enough vertices in U and W since $k \leq \lfloor \frac{n}{2} \rfloor$. Thus G is not a k-ordered Hamiltonian graph, finishing the proof of Theorem 2.

5. Open Problems

The obvious open problem is to determine the best possible minimum degree requirement for *every* k, n pair satisfying $2 \leq k \leq n$ and guaranteeing that the graph is a k-ordered Hamiltonian graph. Theorem 1 is valid only for $n \geq n_0$ and $2 \leq k \leq \kappa n$.

Another open problem is to determine the best possible Ore-type condition. In [9] it is shown that if $n \geq 3$, $3 \leq k \leq n$ and $deg(u) + deg(v) \geq n + 2k - 6$ for every pair u, v of nonadjacent vertices of G, then G is a k-ordered Hamiltonian graph.

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