$\mathbf{R}^2$-IRREDUCIBLE UNIVERSAL COVERING SPACES OF $\mathbf{P}^2$-IRREDUCIBLE OPEN 3-MANIFOLDS

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Abstract. An irreducible open 3-manifold W is $\mathbf{R}^2$-irreducible if it contains no non-trivial planes, i.e. given any proper embedded plane Π in W some component of $W - \Pi$ must have closure an embedded halfspace $\mathbf{R}^2 \times [0, \infty)$. In this paper it is shown that if M is a connected, $\mathbf{P}^2$-irreducible, open 3-manifold such that $\pi_1(M)$ is finitely generated and the universal covering space $\widetilde{M}$ of M is $\mathbf{R}^2$-irreducible, then either $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$ or $\pi_1(M)$ is a free product of infinite cyclic groups and fundamental groups of closed, connected surfaces other than S^2 or $\mathbf{P}^2$. Given any finitely generated group G of this form, uncountably many $\mathbf{P}^2$-irreducible, open 3-manifolds M are constructed with $\pi_1(M) \cong G$ such that the universal covering space $\widetilde{M}$ is $\mathbf{R}^2$-irreducible and not homeomorphic to $\mathbf{R}^3$; the $\widetilde{M}$ are pairwise non-homeomorphic. Relations are established between these results and the conjecture that the universal covering space of any irreducible, orientable, closed 3-manifold with infinite fundamental group must be homeomorphic to $\mathbf{R}^3$.

1. Introduction

Suppose M is a connected, $\mathbf{P}^2$-irreducible, open 3-manifold with $\pi_1(M)$ finitely generated and non-trivial. It is easy to construct examples of such M for which the universal covering space $\widetilde{M}$ is not homeomorphic to $\mathbf{R}^3$. Start with any 3-manifold N satisfying the given conditions. Let U be a **Whitehead manifold**, i.e. an irreducible, contractible, open 3-manifold which is not homeomorphic to $\mathbf{R}^3$(see e.g. [16], [4]). Choose end-proper embeddings of $[0, \infty)$ in each of N and U. (A map between manifolds is **end-proper** if pre-images of compact sets are compact; it is **∂-proper** if the pre-image of the boundary is the boundary; it is **proper** if it has both these properties. These terms are applied to a submanifold if its inclusion map has the corresponding property.) Let X and Y be the exteriors of these rays. (The **exterior** of a submanifold is the closure of the complement of a regular neighborhood of it.) ∂X and ∂Y are each planes. We identify them to obtain a $\mathbf{P}^2$-irreducible open 3-manifold M with $\pi_1(M) \cong \pi_1(N)$. Let $p : \widetilde{M} \to M$ be the universal covering map. Then $\widetilde{M}$, $p^{-1}(X)$, and $p^{-1}(Y)$ are $\mathbf{P}^2$-irreducible [5]. Each component $\widetilde{Y}$ of $p^{-1}(Y)$

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has interior $\widetilde{U}$ homeomorphic to U and so contains a compact, connected subset J which does not lie in a 3-ball in $\widetilde{U}$. If $\widetilde{M}$ were homeomorphic to $\mathbf{R}^3$ then J would lie in a 3-ball B in $\widetilde{M}$. Standard general position and minimality arguments applied to ∂B and $\partial\widetilde{Y}$ would then yield a 3-ball B' in $\widetilde{U}$ containing J, a contradiction. Alternatively, one could use the Tucker Compactification Theorem [14] to obtain a compact polyhedron K in $\widetilde{U}$ such that some component V of $\widetilde{U} - K$ has non-finitely generated fundamental group. But this is impossible since the union of V and $\widetilde{M} - \widetilde{U}$ is a component of $\widetilde{M} - K$ whose fundamental group is isomorphic to $\pi_1(V)$.

In this example $\partial\widetilde{Y}$ is a **non-trivial plane** in $\widetilde{M}$, i.e. a proper plane Π such that no component of $\widetilde{M} - \Pi$ has closure homeomorphic to $\mathbf{R}^2 \times [0, \infty)$ with $\Pi = \mathbf{R}^2 \times \{0\}$. This paper shows that it is harder to find examples if one rules out this behavior by requiring that $\widetilde{M}$ be **$\mathbf{R}^2$-irreducible** in the sense that, in addition to being irreducible, it contains no non-trivial planes.

Define a **closed surface group** to be the fundamental group of a closed, connected 2-manifold.

Theorem 1. *Let M be a connected, $\mathbf{P}^2$-irreducible, open 3-manifold with $\pi_1(M)$ finitely generated. If the universal covering space $\widetilde{M}$ of M is $\mathbf{R}^2$-irreducible, then either*

(1) *$\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$ or*
(2) *$\pi_1(M)$ is a free product of infinite cyclic groups and infinite closed surface groups.*

The second possibility can be disjoint from the first.

Theorem 2. *Suppose G is a free product of finitely many infinite cyclic groups and infinite closed surface groups. Then there is a $\mathbf{P}^2$-irreducible open 3-manifold M such that $\pi_1(M) \cong G$ and $\widetilde{M}$ is an $\mathbf{R}^2$-irreducible Whitehead manifold. Moreover, for each given G there are uncountably many such M for which the $\widetilde{M}$ are pairwise non-homeomorphic.*

This generalizes an example of Scott and Tucker [12] for which G is infinite cyclic.

These results have a bearing on the following well-known problem.

Conjecture 1 (Universal Covering Conjecture)**.** *Let X be a closed, connected, irreducible, orientable 3-manifold with $\pi_1(X)$ infinite. Then the universal covering space $\widetilde{X}$ of X is homeomorphic to $\mathbf{R}^3$.*

Since there are only countably many homeomorphism types of closed 3-manifolds Theorem 2 implies that there must exist uncountably many $\mathbf{R}^2$-irreducible Whitehead manifolds $\widetilde{M}$ which cover open 3-manifolds M with $\pi_1(M) \cong G$ but cannot cover a closed 3-manifold. This generalizes a result of Tinsley and Wright [13] which shows that there must exist uncountably many non-$\mathbf{R}^2$-irreducible Whitehead manifolds $\widetilde{M}$ which cover open 3-manifolds M with $\pi_1(M)$ infinite cyclic but cannot cover a closed 3-manifold. Unfortunately this argument does not provide any *specific* such examples. Specific examples of non-$\mathbf{R}^2$-irreducible Whitehead manifolds $\widetilde{M}$ which cover open 3-manifolds M with $\pi_1(M)$ infinite cyclic or, more generally, a countable free group, but cannot cover a closed 3-manifold are given in [9] and [10], respectively. At the time of this writing the problem of providing specific examples of $\mathbf{R}^2$-irreducible Whitehead manifolds which non-trivially cover other open 3-manifolds but cannot cover a closed 3-manifold is still open.

One can make several conjectures related to Conjecture 1. We consider the selection below. In all of them G is assumed to be a finitely generated group of covering translations acting on a Whitehead manifold W with quotient a 3-manifold M.

Conjecture 2. *G is a free product of infinite cyclic groups and fundamental groups of ∂-irreducible Haken manifolds.*

Conjecture 3. *G is a free group or contains an infinite closed surface group.*

Conjecture 4. *If W is $\mathbf{R}^2$-irreducible, then G is a free product of infinite cyclic groups and infinite closed surface groups.*

A proper plane Π in W is **equivariant** if for each $g \in G$ either $g(\Pi) = \Pi$ or $\Pi \cap g(\Pi) = \emptyset$.

Conjecture 5 (Special Equivariant Plane Conjecture)**.** *If G is not a free product of infinite cyclic groups and infinite closed surface groups, then W contains a non-trivial equivariant plane.*

Conjecture 6 (Equivariant Plane Conjecture)**.** *If W contains a non-trivial plane, then it contains a non-trivial equivariant plane.*

These conjectures are related as follows.

Theorem 3. $(4) \Leftarrow (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (5) \Leftarrow (4+6)$

Theorems 1 and 3 are proven in section 2. Theorem 2 is proven in sections 3–7. Section 3 presents a modified version of the criterion used by Scott and Tucker [12] for showing that a 3-manifold is $\mathbf{R}^2$-irreducible. Sections 4 and 5 treat, respectively, the special cases in which G is an infinite cyclic group and an infinite closed surface group. The constructions and notation of these special cases are used in section 6, which treats the general case. Section 7 shows how to get uncountably many M with non-homeomorphic $\widetilde{M}$ for each group G.

2. The proofs of Theorems 1 and 3

Lemma 2.1. *Let M be a connected, $\mathbf{P}^2$-irreducible, open 3-manifold. Let Q be a compact, connected, 3-dimensional submanifold of M such that ∂Q is incompressible* *in M and $\pi_1(Q)$ is not an infinite closed surface group. Let $p : \widetilde{M} \to M$ be the universal covering map and G the group of covering translations. Let $\widetilde{Q}$ be a component of $p^{-1}(Q)$. Then*

1. *Each component of $p^{-1}(\partial Q)$ is a plane.*
2. *There is no component Π of $\partial\widetilde{Q}$ which is invariant under the subgroup G_0 of G consisting of those covering translations which leave $\widetilde{Q}$ invariant.*
3. *If each component of $\partial\widetilde{Q}$ is a trivial plane, then $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$.*

Proof. (1) follows from the incompressibility of ∂Q in M.

Suppose S is a component of ∂Q and Π is a component of $p^{-1}(S)$ which is invariant under G_0. Since the restriction of p to $\widetilde{Q}$ is the universal covering space of Q and the restriction of G_0 to $\widetilde{Q}$ is the group of covering translations we have that $\pi_1(S) \to \pi_1(Q)$ is an isomorphism, contradicting our assumption on $\pi_1(Q)$. This establishes (2).

We now prove (3). Suppose that each component Π of $\partial\widetilde{Q}$ bounds an end-proper halfspace H_Π in $\widetilde{M}$. Let K_Π be the closure of the component of $\widetilde{M} - \Pi$ which does not contain $int\,\widetilde{Q}$.

Assume that for all such Π we have $H_\Pi = K_\Pi$. Then $\widetilde{M}$ is the union of $\widetilde{Q}$ and an open collar attached to $\partial\widetilde{Q}$, hence $\widetilde{M}$ is homeomorphic to $int\,\widetilde{Q}$. Since Q is Haken, the Waldhausen Compactification Theorem [15] implies that $\widetilde{Q}$ is homeomorphic to a closed 3-ball minus a closed subset of its boundary, hence $int\,\widetilde{Q}$ is homeomorphic to $\mathbf{R}^3$, and we are done.

Thus we may assume that for some Π we have $H_\Pi \neq K_\Pi$. Then $H_\Pi \cap K_\Pi = \Pi$ and $H_\Pi \cup K_\Pi = \widetilde{M}$. Now G_0 has an element g such that $g(\Pi) \neq \Pi$. Since $\widetilde{Q} \subseteq H_\Pi$ and $g(\widetilde{Q}) = \widetilde{Q}$ we must have $g(K_\Pi) \subseteq H_\Pi$. Since $\mathbf{R}^2 \times [0,\infty)$ is $\mathbf{R}^2$-irreducible (see e.g. [8]) it follows that K_Π is homeomorphic to $\mathbf{R}^2 \times [0,\infty)$. Thus $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$. □

Proof of Theorem 1. By passing to a covering space of M, if necessary, we may assume that $\pi_1(M)$ is indecomposable with respect to free products and is neither an infinite cyclic group nor an infinite closed surface group. Let C be the Scott compact core [11] of M, i.e. C is a compact, connected, 3-dimensional submanifold of M such that $\pi_1(C) \to \pi_1(M)$ is an isomorphism. The conditions on $\pi_1(M)$ imply that ∂C is incompressible in M. We thus can apply Lemma 2.1 with $Q = C$ to finish the proof. □

Proof of Theorem 3. We first show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). If (1) is true, then M must be non-compact; this follows from the fact that if M were closed and non-orientable, then it would be Haken and so have universal covering space homeomorphic to $\mathbf{R}^3$. Let C be the Scott compact core for M. Since M is irreducible we may assume that no component of ∂C is a 2-sphere; it follows that C is irreducible. If C is ∂-irreducible, then we are done. If C is not ∂-irreducible, then there is a finite set of compressing disks for ∂C in C which express C as a ∂-connected sum of 3-balls and ∂-irreducible Haken manifolds, thus yielding (2). Clearly (2) $\Rightarrow$ (3). Suppose (3) is true and M is closed. If G is free, then M is by [2, Theorem 5.2] a connected sum of 2-sphere bundles over S^1, hence is not aspherical, hence W is not contractible. If G contains an infinite closed surface group, then by a result of Hass, Rubinstein, and Scott [1] W is homeomorphic to $\mathbf{R}^3$.

Clearly Theorem 1 and the fact that M cannot be closed and non-orientable show that (1) $\Rightarrow$ (4).

We now show that (1) $\Rightarrow$ (5). Let C be the Scott compact core of M. Then the assumptions on G imply that there is a set of compressing disks for ∂C in C such that some component Q of C split along this collection of disks satisfies the hypotheses of Lemma 2.1. Thus any component of the pre-image of ∂Q is an equivariant non-trivial plane.

We next show that (5) $\Rightarrow$ (1). Assume M is closed. If $\pi_1(M)$ is a free product of infinite cyclic groups and infinite closed surface groups, then we apply (3) to obtain (1). If $\pi_1(M)$ is not such a group, then the existence of an equivariant plane, together with the compactness of M, implies that M is Haken, and so (1) follows by Waldhausen [15].

Finally we show that (4 + 6) $\Rightarrow$ (1). If W is $\mathbf{R}^2$-irreducible, then (4) implies the hypothesis of (2), hence implies (1). If W is not $\mathbf{R}^2$-irreducible, then (6) implies as before that M is Haken, thus (1) holds. $\square$

3. Nice quasi-exhaustions and $\mathbf{R}^2$-irreducibility

We shall reformulate a criterion due to Scott and Tucker [12] for a $\mathbf{P}^2$-irreducible open 3-manifold to be $\mathbf{R}^2$-irreducible. A proper plane Π in an open 3-manifold W is **homotopically trivial** if for any compact subset C of W the inclusion map of Π is end-properly homotopic to a map whose image is disjoint from C.

Lemma 3.1. *Let W be an irreducible, open 3-manifold, and let Π be a proper plane in W. If Π is homotopically trivial, then Π is trivial.*

Proof. This is Lemma 4.1 of [12]. $\square$

Lemma 3.2. *Let W be a connected, irreducible, open 3-manifold, and let $\{C_n\}_{n\geq 1}$, be a sequence of compact 3-dimensional submanifolds of W such that $C_n \subseteq int\, C_{n+1}$ and*

1. *each C_n is irreducible,*
2. *each ∂C_n is incompressible in $W - int\, C_n$,*
3. *if D is a proper disk in C_{n+1} which is in general position with respect to ∂C_n such that ∂D is not null-homotopic in ∂C_{n+1}, then $D \cap \partial C_n$ has at least two*

components which are not null-homotopic in ∂C_n and bound disjoint disks in D.

Then any proper plane in W can be end-properly homotoped off C_n for any n.

Proof. This is Lemma 4.2 of [12]. □

The precise criterion we shall use is as follows.

Lemma 3.3. *Let W be a connected, irreducible, open 3-manifold. Suppose that for each compact subset K of W there is a sequence $\{C_n\}_{n\geq 1}$ of compact 3-dimensional submanifolds such that $C_n \subseteq int\, C_{n+1}$ and*

(1) *each C_n is irreducible,*
(2) *each ∂C_n is incompressible in $W - int\, C_n$ and has positive genus,*
(3) *each $C_{n+1} - int\, C_n$ is irreducible, ∂-irreducible, and anannular,*
(4) *$K \subseteq C_1$.*

Then W is $\mathbf{R}^2$-irreducible.

Proof. Let D be a disk as in part (iii) of Lemma 3.2. If every component of $D \cap \partial C_n$ is null-homotopic in ∂C_n, then one can isotop D so that $D \cap C_n = \emptyset$ and hence ∂C_{n+1} is compressible in $C_{n+1} - int\, C_n$. If only one component α of $D \cap \partial C_n$ is not null-homotopic in ∂C_n, then $\partial D \cup \alpha$ bounds an annulus A which can be isotoped so that $A \cap \partial C_n = \alpha$, hence $C_{n+1} - int\, C_n$ is not anannular. If no two of the components of $D \cap \partial C_n$ which are not null-homotopic in ∂C_n bound disjoint disks in D, then these components must be nested on D. We can isotop D to remove null-homotopic components and then intermediate annuli to again get an incompressible annulus joining ∂C_{n+1} to ∂C_n. Now apply Lemma 3.2 and then Lemma 3.1. □

Let $\{C_n\}$ be a sequence of compact, connected 3-dimensional submanifolds of an irreducible, open 3-manifold W such that $C_n \subseteq int\, C_{n+1}$ such that $W - int\, C_n$ has no compact components. This will be called a **quasi-exhaustion** for W. A quasi-exhaustion for W whose union is W is an **exhaustion** for W. A quasi-exhaustion is **nice** if it satisfies conditions (1)–(3) of Lemma 3.3. Thus that lemma can be rephrased by saying that if every compact subset of W is contained in the first term of a nice quasi-exhaustion, then W is $\mathbf{R}^2$-irreducible.

We shall need some tools for constructing Whitehead manifolds with nice quasi-exhaustions. Define a compact, connected 3-manifold Y to be **nice** it is is $\mathbf{P}^2$-irreducible, ∂-irreducible, and anannular, it contains a two-sided proper incompressible surface, and it is not a 3-ball; define it to be **excellent** if, in addition, every connected, proper, incompressible surface of zero Euler characteristic in Y is ∂-parallel. So in particular an excellent 3-manifold is anannular and atoroidal while a nice 3-manifold is anannular but may contain a non-∂-parallel incompressible torus.

A proper 1-manifold in a compact 3-manifold is **excellent** if its exterior is excellent; it is **poly-excellent** if the union of each non-empty subset of the set of its components is excellent.

Lemma 3.4. *Every proper 1-manifold in a compact, connected 3-manifold whose boundary contains no 2-spheres or projective planes is homotopic rel ∂ to an excellent proper 1-manifold.*

Proof. This is a special case of Theorem 1.1 of [6]. □

Define a k-**tangle** to be a disjoint union of k proper arcs in a 3-ball.

Lemma 3.5. *For all $k \geq 1$ poly-excellent k-tangles exist.*

Proof. This is Theorem 6.3 of [7]. □

We shall also need the following criterion for gluing together excellent 3-manifolds to get an excellent 3-manifold.

Lemma 3.6. *Let Y be a compact, connected 3-manifold. Let S be a compact, proper, two-sided surface in Y. Let Y' be the 3-manifold obtained by splitting Y along S. Let S' and S'' be the two copies of S which are identified to obtain Y. If each component of Y' is excellent, S', S'', and $(\partial Y') - int\,(S' \cup S'')$ are incompressible in Y', and each component of S has negative Euler characteristic, then Y is excellent.*

Proof. This is Lemma 2.1 of [6]. □

4. The infinite cyclic case

In [12] Scott and Tucker described an $\mathbf{R}^2$-irreducible Whitehead manifold which is an infinite cyclic covering space. This section gives a general procedure for constructing such examples. The construction introduced here will be incorporated into that for the general case in section 6.

Let $P_n = D_n \times [0,1]$, where D_n is the disk of radius n. We call P_n a **pillbox**. Identify $D_n \times \{0\}$ with $D_n \times \{1\}$ to obtain a solid torus Q_n. Let R_n be a solid torus and H_n a 1-handle $D \times [0,1]$ joining $\partial D_n \times (0,1)$ to ∂R_n. Let $V_n = P_n \cup H_n \cup R_n$ and $M_n = Q_n \cup H_n \cup R_n$. We call V_n an **eyebolt**. We embed M_n in the interior of M_{n+1} as follows.

We choose a collection of arcs θ_0, θ_1, α_0, α_1, α_2, β_1, β_2, γ_1, γ_2, δ_1, δ_2, ε in P_{n+1}which satisfy certain conditions described below. θ_0, θ_1, and α_0 meet in a common endpoint in $int\,P_{n+1}$ but are otherwise disjoint. The other endpoints of θ_0 and α_0 lie in $(int\,D_{n+1}) \times \{0\}$; that of θ_1 lies in $(int\,D_{n+1}) \times \{1\}$. We let $\theta = \theta_0 \cup \theta_1$. All the other arcs are proper arcs in P_{n+1}which are disjoint from each other and from $\theta \cup \alpha_0$. γ_1, β_2, and δ_2 run from $(int\,D_{n+1}) \times \{0\}$ to itself. γ_2, β_1, and $\delta_1$run from $(int\,D_{n+1}) \times \{1\}$ to itself. α_1 runs from $(int\,D_{n+1}) \times \{0\}$ to $(int\,D_{n+1}) \times \{1\}$. α_2 runs from $(int\,D_{n+1}) \times \{1\}$ to $int\,(P_{n+1} \cap H_{n+1})$. ε runs from $int\,(P_{n+1} \cap H_{n+1})$ to itself. We denote the image in Q_{n+1} of an arc by the same symbol, relying on the context to distinguish an arc in P_{n+1} from its image in Q_{n+1}. We require that θ be a simple closed curve in Q_{n+1} and that $\alpha_0 \cup \beta_1 \cup \gamma_1 \cup \delta_1 \cup \alpha_1 \cup \beta_2 \cup \gamma_2 \cup \delta_2$ is an arc consisting of subarcs which occur in the given order. We require that any union of these arcs which contains α_0 and at least one other arc has excellent exterior in P_{n+1}, and that the same is true for any union of these arcs which contains neither θ_0, θ_1, nor α_0. This can be achieved as follows. Note that the exterior of α_0 in P_{n+1} is a 3-ball B. Choose a poly-excellent 11-tangle in B and then slide its endpoints so that exactly two of the arcs meet a regular neighborhood of α_0. Extend them to meet α_0 in the desired configuration.

Next let κ_1, κ_2, and κ_3 be product arcs in H_{n+1} joining $(int\, D)\times\{0\}$ to $(int\, D)\times\{1\}$. Let $R_n \subseteq int\, R_{n+1}$ be any null-homotopic embedding. Let λ_1 and λ_2 be disjoint proper arcs in $R_{n+1} - int\, R_n$ with λ_1 joining $int\,(H_{n+1} \cap R_{n+1})$ to itself and λ_2 joining $int\,(H_{n+1} \cap R_{n+1})$ to ∂R_n. We require $\lambda_1 \cup \lambda_2$ to be excellent in $R_{n+1} - int\, R_n$. We also require that these arcs, together with ε, fit into an arc whose subarcs form the sequence κ_1, λ_1, κ_2, ε, κ_3, λ_2 and that κ_1 meets α_2 in a common endpoint.

Now we embed P_n in P_{n+1} as a regular neighborhood of the arc θ so that the two disks of $P_n \cap (D_{n+1} \times \{0,1\})$ are identified to give an embedding of Q_n in Q_{n+1}. Note that these embeddings are not consistent with the product structures. From the discussion above we have an arc ω in $M_{n+1} - int\,(Q_n \cup R_n)$ running from ∂Q_n to ∂R_n. We embed H_n as a regular neighborhood of ω. We change notation slightly by now letting α_0 be the old α_0 minus its intersection with the interior of Q_n.

We let M be the direct limit of the M_n and let $p : \widetilde{M} \to M$ be the universal covering map. $p^{-1}(Q_n) = p^{-1}(P_n)$ is the union of pillboxes $P_{n,j} = D_n \times [j, j+1]$ meeting along the $D_n \times \{j\}$ to form $D_n \times \mathbf{R}$. Note that this embedding is not the product embedding. $p^{-1}(R_n)$ is a disjoint union of solid tori $R_{n,j}$. $p^{-1}(H_n)$ is a disjoint union of 1-handles $H_{n,j}$ joining $\partial D_n \times (j, j+1)$ to $\partial R_{n,j}$; these are regular neighborhoods of lifts ω_j of ω. $p^{-1}(M_n) = p^{-1}(V_n)$ is the union of $p^{-1}(P_n)$, $p^{-1}(H_n)$, and $p^{-1}(R_n)$. It is the union of eyebolts $V_{n,j} = P_{n,j} \cup H_{n,j} \cup R_{n,j}$ meeting along the $D_n \times \{j\}$. $\widetilde{M}$ is the nested union of the $p^{-1}(M_n)$.

Let $\Sigma_n^m = \cup_{j=-m}^m V_{n,j}$ and $\Lambda_n^m = P_{n,-(m+1)} \cup P_{n,m+1}$. Let $\Phi_1^m = \emptyset$, and, for $n \geq 2$, let $\Phi_n^m = \cup_{j=m+2}^{m+n}(P_{n,-j} \cup P_{n,j})$. Note that Λ_n^m and Φ_n^m (for $n \geq 2$) are each disjoint unions of two 3-balls, $\Lambda_n^m \cap \Sigma_n^m$ is a pair of disjoint disks, and (for $n \geq 2$) so is $\Lambda_n^m \cap \Phi_n^m$. Define $C_n^m = \Sigma_n^m \cup \Lambda_n^m \cup \Phi_n^m$.

Lemma 4.1. *$\{C_m^m\}$ is an exhaustion for $\widetilde{M}$. Each C^m is a nice quasi-exhaustion.*

Proof. $C_n^m \subseteq int\, C_{n+1}^m$, and $C_n^m \subseteq C_n^{m+1}$. A given compact subset K of $\widetilde{M}$ lies in some $p^{-1}(M_n)$ and thus in a finite union of $V_{n,j}$ and hence in some $\Sigma_n^m \subseteq C_n^m \subseteq C_q^q$, where $q = \max\{m, n\}$. Thus $\{C_m^m\}$ is an exhaustion for $\widetilde{M}$.

C_n^m is a cube with $2m+1$ handles. Let $Y = C_{n+1}^m - int\, C_n^m$. We will show that Y is excellent by successive applications of Lemma 3.6.

Consider a $P_{n+1,j}$ contained in C_{n+1}^m. If $|j| < m$, then it meets C_n^m in a regular neighborhood of the union of the j^{th} copies of all the arcs in P_{n+1}. Thus $Y \cap P_{n+1,j}$ is excellent, and Lemma 3.6 implies that the union of these $Y \cap P_{n+1,j}$ is excellent. For

$|j| \geq m$ some care must be taken so that one is always gluing excellent 3-manifolds along surfaces of the appropriate type. Note that $Y \cap (P_{n+1,m} \cup P_{n+1,m+1} \cup \cdots \cup P_{n+1,m+n-1} \cup P_{n+1,m+n})$ is equal to the exterior of the m^{th} copy of all the arcs but β_1 and δ_2 in $P_{n+1,m}$ together with the exterior of the $(m+1)^{st}$ copy of β_2, δ_2, and θ in $P_{n+1,m+1}$, the exterior of the j^{th} copy of θ in $P_{n+1,j}$ for $m+1 < j < m+n$, and the 3-ball $P_{n+1,m+n}$. This space is homeomorphic to the exterior of the m^{th} copy of all the arcs but β_1, δ_1, and θ_1 in $P_{n+1,m+1}$ together with the exterior of the $(m+1)^{st}$ copy of β_2 and δ_2 in $P_{n+1,m+1}$, and the 3-ball consisting of the union of the $P_{n+1,j}$ for which $m+1 < j \leq m+n$. This can be seen by taking the arc consisting of the m^{th} copy of θ_1 and the j^{th} copy of θ for $m < j < m+n$ and retracting it onto the endpoint in which it meets the rest of the graph. This space is then excellent by Lemma 3.6. Similar remarks apply for $j \leq -m$, so these spaces can be added on to get that $Y \cap \cup_{j=-(m+n)}^{m+n} P_{n+1,j}$ is excellent.

We fill in the remainder of Y by adding the exteriors of the j^{th} copies of κ_1, κ_2, and κ_3 in $H_{n+1,j}$ and $\lambda_1 \cup \lambda_2$ in $R_{n+1,j} - int\, R_{n,j}$ for $|j| \leq m$. Since the first of these spaces is a product the union of the two spaces is homeomorphic to the second space, and Lemma 3.6 applies to complete the proof that Y is excellent.

It remains to show that each ∂C_n^m is incompressible in $\widetilde{M} - int\, C_n^m$. Since each $C_{n+s+1}^m - int\, C_{m+s}^m$ is ∂-irreducible we have that ∂C_n^m is incompressible in $C_{n+q}^m - int\, C_n^m$ for each $q \geq 1$. $p^{-1}(M_{n+q})$ is the union of C_{n+q}^m and the sets $C^- = \cup_{j<-m} V_{n+q,j}$ and $C^+ = \cup_{j>m} V_{n+q,j}$. We have that $C^- \cap C_{n+q}^m$ and $C^+ \cap C_{n+q}^m$ are disjoint disks, while $C^- \cap C^+ = \emptyset$. It follows that ∂C_n^m is incompressible in $p^{-1}(M_{n+q}) - int\, C_n^m$. Since $\widetilde{M}$ is the nested union of the $p^{-1}(M_{n+q})$ over all $q \geq 1$ we have the desired result. □

5. The surface group case

Let F be a closed, connected surface other than S^2 or $\mathbf{P}^2$. Let $n \geq 1$. Regard F as being obtained from a $2k$-gon E, $k \geq 2$, by identifying sides s_i and s_i', $1 \leq i \leq k$. This induces an identification of the lateral sides $S_i = s_i \times [-n, n]$ and $S_i' = s_i \times [-n, n]$ of the **prism** $P_n = E \times [-n, n]$ which yields $Q_n = F \times [-n, n]$. Let R_n be a solid torus and H_n a 1-handle $D \times [0, 1]$. Let $V_n = P_n \cup H_n \cup R_n$, where $P_n \cap R_n = D \times \{1\}$ is a disk in ∂R_n, and $P_n \cap R_n = \emptyset$. We again call V_n an **eyebolt**. It is a solid torus whose image under the identification is $M_n = Q_n \cup H_n \cup R_n$, a space homeomorphic to the ∂-connected sum of $F \times [-n, n]$ and a solid torus.

We define an open 3-manifold M by specifying an embedding of M_n in the interior of M_{n+1} and letting M be the direct limit. The inclusion $[-n, n] \subseteq [-(n+1), n+1]$ induces $P_n \subseteq P_{n+1}$ and hence $Q_n \subseteq Q_{n+1}$. We let $R_n \subseteq int\, R_{n+1}$ be any null-homotopic embedding. Again the interesting part of the embedding will be that of H_n in M_{n+1}. It will be the regular neighborhood of a certain arc ω in $M_{n+1} - int\, (Q_n \cup R_n)$ joining ∂Q_n to ∂R_n.

The arc ω is the union of $4k+7$ arcs any two of which are either disjoint or have one common endpoint. The $4k+2$ arcs α_0, α_i, β_i, γ_i, δ_i, $1 \leq i \leq k$, and ε lie in $E \times [n, n+1]$ and are identified with their images in Q_{n+1}; the three arcs κ_1, κ_2, and κ_3 lie in H_{n+1}, and the two arcs λ_1 and λ_2 lie in R_{n+1}. These arcs will have special properties to be described later. We first describe their combinatorics. The arcs in P_{n+1} are all proper arcs in $E \times [n, n+1]$. α_0 runs from $(int\, E) \times \{n\}$ to $int\, S_1$. For $1 \leq i < k$, α_i runs from $int\, S_i$ to $int\, S_{i+1}$. α_k runs from $int\, S'_k$ to $int\, (P_{n+1} \cap H_{n+1})$. For $1 \leq i \leq k$, β_i and δ_i each run from $int\, S'_i$ to itself, while γ_i runs from $int\, S_i$ to itself. These arcs are chosen so that under the identification their endpoints match up in such a way as to give an arc which follows the sequence α_0, β_1, γ_1, δ_1, α_1, $\ldots$, β_k, γ_k, δ_k, α_k. We require ε to run from $int\, (P_{n+1} \cap H_{n+1})$ to itself. κ_1, κ_2, and κ_3 are product arcs in H_{n+1} lying in $(int\, D) \times [0, 1]$. λ_1 and λ_2 are proper arcs in $R_{n+1} - int\, R_n$, with λ_1 running from $int\, (H_{n+1} \cap R_{n+1})$ to itself and λ_2 running from $int\, (H_{n+1} \cap R_{n+1})$ to ∂R_n. These arcs are chosen so as to fit together into the sequence κ_1, λ_1, κ_2, ε, κ_3, λ_2 with the endpoint of κ_1 other than $\kappa_1 \cap \lambda_1$ being the same as the endpoint of α_k other than $\alpha_k \cap \delta_k$. This gives ω.

We now describe the special properties required of these arcs. We require that $\alpha_0 \cup \beta_1 \cup \gamma_1 \cup \delta_1 \cup \alpha_1 \cup \cdots \cup \beta_k \cup \gamma_k \cup \delta_k \cup \varepsilon$ be a poly-excellent $(4k+2)$-tangle in $E \times [n, n+1]$ and $\lambda_1 \cup \lambda_2$ to be an excellent 1-manifold in $R_{n+1} - int\, R_n$.

We now consider the universal covering map $p : \widetilde{M} \to M$. Our goal is to construct a sequence $\{C^m\}$ of nice quasi-exhaustions whose diagonal $\{C^m_m\}$ is an exhaustion for $\widetilde{M}$.

The universal covering space $\widetilde{F}$ of F is tesselated by copies E_j of E. We fix one such copy E_1. We inductively define an exhaustion $\{F_m\}$ for $\widetilde{F}$ as follows. $F_1 = E_1$. F_{m+1} is the union of F_m and all those E_j which meet it. Each F_m is a disk (which we call a **star**). The **inner corona** I_m of F_m is the annulus $F_{m+1} - int\, F_m$. Each vertex on ∂F_m lies in either one or two of those E_j contained in F_m. Each E_j in I_m meets meets F_m in either an edge or a vertex; in both cases it meets exactly two adjacent E_ℓ of I_m, and each of these intersections is an edge. For $n \geq 2$ we define the **outer** n**-corona** O^m_n to be the annulus $F_{m+n} - int\, F_{m+1}$; we define $O^m_1 = \emptyset$. Let σ_2 be a proper arc in F_2 consisting of three edges of the polygons in F_2. . Inductively define a proper arc σ_{m+1} in F_{m+1} by adjoining to σ_m two arcs spanning I_m which are edges of polygons in I_m. Thus each σ_m is an edge path in F_m splitting it into two unions of polygons F'_m and F''_m.

We now consider the structure of $\widetilde{M}$. For $n \geq 1$, $p^{-1}(Q_n) = p^{-1}(P_n)$ is the union of prisms $P_{n,j} = E_j \times [-n, n]$ meeting along their lateral sides to form $\widetilde{F} \times [-n, n]$. $p^{-1}(R_n)$ is a disjoint union of solid tori $R_{n,j}$. $p^{-1}(H_n)$ is a disjoint union of 1-handles $H_{n,j}$ running from $E_j \times \{n\}$ to $\partial R_{n,j}$; these are regular neighborhoods of lifts ω_j of ω. Now $p^{-1}(M_n) = p^{-1}(V_n)$ is the union of $p^{-1}(P_n)$, $p^{-1}(H_n)$, and $p^{-1}(R_n)$. It can be expressed as the union of the eyebolts $V_{n,j} = P_{n,j} \cup H_{n,j} \cup R_{n,j}$ meeting along the lateral sides of the $P_{n,j}$. Finally $\widetilde{M}$ is the nested union of the $p^{-1}(M_n)$.

Let Σ_n^m be the union of those $V_{n,j}$ such that E_j is in the star F_m. Let Λ_n^m be the union of those $P_{n,j}$ such that E_j is in the inner corona I_m. Let Φ_n^m be the union of those $P_{n,j}$ such that E_j is in the outer n-corona O_n^m. Note that Λ_n^m and Φ_n^m (for $n \geq 2$) are solid tori, $\Lambda_n^m \cap \Sigma_n^m$ is an annulus which goes around Λ_n^m once longitudinally and consists of those lateral sides of the prisms in Σ_n^m which lie on $\partial\Sigma_n^m$, and (for $n \geq 2$) $\Lambda_n^m \cap \Phi_n^m$ is an annulus which goes around each of these solid tori once longitudinally. We now define $C_n^m = \Sigma_n^m \cup \Lambda_n^m \cup \Phi_n^m$.

Lemma 5.1. *$\{C_m^m\}$ is an exhaustion for $\widetilde{M}$. Each C^m is a nice quasi-exhaustion.*

Proof. Note that $C_n^m \subseteq int\, C_{n+1}^m$, and $C_n^m \subseteq C_n^{m+1}$. Suppose K is some compact subset of $\widetilde{M}$. Then K lies in some $p^{-1}(M_n)$ and thus in a finite union of $V_{n,j}$ and hence in some $\Sigma_n^m \subseteq C_n^m \subseteq C_q^q$, where $q = \max\{m, n\}$. Thus $\{C_m^m\}$ is an exhaustion for $\widetilde{M}$.

Each C_n^m is a cube with handles, so is irreducible. The number of handles is at least one, so ∂C_n^m has positive genus. Let $Y = C_{n+1}^m - int\, C_n^m$. We will prove that Y is excellent by successive applications of Lemma 3.6. Let $P_{n+1,j}^+$ and $P_{n+1,j}^-$ denote, respectively, $E_j \times [n, n+1]$ and $E_j \times [-(n+1), -n]$.

Consider a $P_{n+1,j}$ contained in Σ_{n+1}^m. It meets C_n^m in $P_{n+1,j}$ together with regular neighborhoods of certain arcs in $P_{n+1,j}^+$. These arcs consist at least of the j^{th} copies of the α_i, the γ_i, and ε which are part of the lift ω_j of ω. If another prism $P_{n+1,\ell}$ in Σ_{n+1}^m meets $P_{n+1,j}$ in a common lateral side, then either ω_j or ω_ℓ will meet this side; in the latter case this contributes a β_i and δ_i to the subsystem of arcs in $P_{n+1,j}^+$. Since the full system of arcs was chosen to be poly-excellent this subsystem of arcs is excellent and so has excellent exterior $Y \cap P_{n+1,j}^+$. Let U' be the union of those $Y \cap P_{n+1,j}^+$ such that $E_j \subseteq F_m'$. This space can be built up inductively by gluing on one $Y \cap P_{n+1,j}^+$ at a time, with the gluing being done along either a disk with two holes (when $P_{n+1,j}$ is glued along one lateral side) or a disk with four holes (when $P_{n+1,j}$ is glued along two adjacent lateral sides). No component of the complement of this surface in the boundary of either manifold is a disk, hence this surface is incompressible in each manifold. It follows that U' is excellent. Similar remarks apply to the space U'' associated with F_m''.

Next consider a $P_{n+1,j}^+$ contained in Λ_{n+1}^m. If $E_j \subseteq F_{m+1}$ and meets F_m' in an edge of $E_\ell \subseteq F_m'$, then either ω_ℓ misses $P_{n+1,j}^+$ or meets it in copies of β_i and δ_i. Thus enlarging U' by adding $Y \cap P_{n+1,j}^+$ either adds a 3-ball along a disk in its boundary, giving a space homeomorphic to U' or gives a new excellent 3-manifold. We adjoin all such $Y \cap P_{n+1,j}^+$ to U'. Then we consider those E_j which meet F_m' in a vertex. Then $P_{n+1,j}^+ = Y \cap P_{n+1,j}^+$, and one can successively adjoin these 3-balls along disks in their boundaries. We denote the enlargement of U' from all these additions again by U'. Similar remarks apply to U''.

Now $(F'_{m+n+1} - int\, F'_{m+1}) \times [n, n+1]$ is a 3-ball which meets U' in a disk, so we adjoin it to U' to get a new U' homeomorphic to the old one. We then adjoin the 3-ball $(F'_{m+n+1} - int\, F'_{m+n}) \times [-n, n]$ which meets this space along a disk to obtain our final U'. The same construction gives U''.

Now U' and U'' are each excellent. $U' \cap U''$ is an annulus with a positive number of disks removed from its interior corresponding to its intersection with arcs passing from $F'_m \times [n, n+1]$ to $F''_m \times [n, n+1]$. No component of the complement of this surface in $\partial U'$ or in $\partial U''$ is a disk; this corresponds to the fact that $F'_m \times \{n\}$, $F''_m \times \{n\}$, $F'_m \times \{n+1\}$, and $F''_m \times \{n+1\}$ each meet some ω_j. Thus this surface is incompressible in both U' and U'', so $U' \cup U''$ is excellent.

Finally we add on the $Y \cap R_{n,j}$ for $E_j \subseteq F_m$ to $U' \cup U''$ to conclude that Y is excellent.

It remains to show that each ∂C^m_n is incompressible in $\widetilde{M} - int\, C^m_n$. First note that since each $C^m_{n+s+1} - int\, C^m_{n+s}$ is ∂-irreducible we must have that ∂C^m_n is incompressible in $C^m_{n+q} - int\, C^m_n$ for each $q \geq 1$. Now consider the set

$$\widetilde{M}_{n+q} = p^{-1}(M_{n+q}) \cup (\widetilde{F} \times [-(n+q+1), -(n+q)]).$$

It can be obtained from C^m_{n+q} as follows. First add the solid tori $R_{n+1,j} \cup H_{n+q,j}$ in $p^{-1}(M_{n+q})$ for which $E_j \subseteq F_{m+q+n}$; these meet C^m_{n+q} in disks. Then add

$$(F_{m+q+n} \times [-(n+q+1), -(n+q)]) \cup (\widetilde{F} - int\, F_{m+q+n}) \times [-(n+q+1), n+q]).$$

This is a space homeomorphic to $\mathbf{R}^2 \times [0,1]$ which meets C^m_{n+q} in the disk

$$(F_{m+q+n} \times \{-(n+q)\}) \cup ((\partial F_{m+q+n}) \times [-(n+q), n+q]).$$

Lastly add all the remaining solid tori $R_{n+q,j} \cup H_{n+q,j}$, where $E_j \subseteq \widetilde{F} - int\, F_{m+q+n}$; these do not meet C^m_{n+q}. This description shows that $C^m_{n+q} \cap (\widetilde{M}_{n+q} - int\, C^m_{n+q})$ consists of (finitely many) disjoint disks, and therefore ∂C^m_n is incompressible in $\widetilde{M}_{n+q} - int\, C^m_n$. Finally since $\widetilde{M}$ is the nested union of the $\widetilde{M}_{n+q}$ over all $q \geq 1$ we have that ∂C^m_n is incompressible in $\widetilde{M} - int\, C^m_n$. $\square$

6. The general case

Suppose $G_1, \ldots, G_k$ are infinite cyclic groups and infinite closed surface groups. For $i = 1, \ldots, k$ let P^i_n be a pillbox or prism, as appropriate, with quotient Q^i_n a solid torus or product I-bundle over a closed surface, respectively. We let H^i_n be a 1-handle attached to P^i_n as before. We let R_n be a common solid torus to which we attach the other ends of all the H^i_n. The union of the Q^i_n and H^i_n with R_n is called M_n. As before we choose arcs in the P^i_n, H^i_n, and R_n and use them to define an embedding of M_n into the interior of M_{n+1}.

The choice of arcs in $R_{n+1} - int\, R_n$, as well as the embedding $R_n \subseteq int\, R_{n+1}$, requires some discussion, since we will want this family λ of arcs to be poly-excellent. Choose a poly-excellent $(2k+2)$-tangle λ^+ in a 3-ball B, with components λ^i_t, $1 \leq i \leq k+1$, $t = 1, 2$. Construct a graph in B by sliding one endpoint of each λ^i_2, $1 \leq i \leq k$, so that it lies on $int\, \lambda^{k+1}_2$. Thus these λ^i_2 now join ∂B to distinct points on $int\, \lambda^{k+1}_2$; all the other λ^i_t still join ∂B to itself. Now choose disjoint disks E_1 and E_2 in ∂B such that E_t meets the graph in $\partial\lambda^{k+1}_t \cap int\, E_t$. Glue E_1 to E_2 so that B becomes a solid torus R_{n+1} and $\lambda^{k+1}_1 \cup \lambda^{k+1}_2$ becomes a simple closed curve. The regular neighborhood of this simple closed curve is our embedding of R_n in the interior of R_{n+1}. Clearly R_n is null-homotopic in R_{n+1}. By Lemma 3.6 its exterior is excellent as is the exterior of the union of R_n with any of the λ^i_t, $1 \leq i \leq k$, $t = 1, 2$.

Let $p : \widetilde{M} \to M$ be the universal covering map. Then $p^{-1}(R_n)$ consists of disjoint solid tori whose union separates $p^{-1}(M_n)$ into components with closures $L^{i,\mu}_n$, where $L^{i,\mu}_n$ is a component of $p^{-1}(Q^i_n \cup H^i_n)$. Let $Z^{i,\mu}_n$ be the union of $L^{i,\mu}_n$ and all those components of $p^{-1}(R_n)$ which meet it. Then $Z^{i,\mu} = \cup_{n\geq 1} Z^{i,\mu}_n$ is an open subset of $\widetilde{M}$ which has a family $\{C^{i,\mu,m}\}$ of quasi-exhaustions as previously described. We will develop from these families an appropriate family $\{C^m\}$ of quasi-exhaustions of $\widetilde{M}$.

We start by choosing a component $\widehat{R}_1$ of $p^{-1}(R_1)$. For each n there is then a unique component $\widehat{R}_n$ of $p^{-1}(R_n)$ which contains $\widehat{R}_1$. We define C^1_n to be the union of $\widehat{R}_n$ and the (finitely many) $C^{i,\mu,1}_n$ which contain it; this space is a solid torus which meets each of these $C^{i,\mu,1}_n$ in a 3-ball which is either a pillbox and a 1-handle or a prism and a 1-handle. Suppose C^m_n has been defined and that it is the union of the $C^{i,\mu,m}_n$ for which $C^m_n \cap L^{i,\mu}_n \neq \emptyset$. We define C^{m+1}_n in two steps. We first take the union C' of all the $C^{i,\mu,m}_n$ such that $C^{i,\mu,m}_n \subseteq C^m_n$. This is just the union of the n^{th} elements of the $(m+1)^{st}$ quasi-exhaustions for those $Z^{i,\mu}_n$ such that $\{i, \mu\}$ is in the current index set. The second step is to enlarge the index set by adding those $\{i, \nu\}$ for which $C' \cap L^{i,\nu}_n \neq \emptyset$ and then adjoin the $C^{i,\nu,m+1}_n$ to C' in order to obtain C^{m+1}_n. One can observe that the $L^{i,\mu}_n$ and $p^{-1}(R_n)$ give $p^{-1}(M_n)$ a tree-like structure and that the passage from C^m_n to C^{m+1}_n goes out further along this tree.

Lemma 6.1. *$\{C^m_m\}$ is an exhaustion for $\widetilde{M}$. C^m is a nice quasi-exhaustion.*

Proof. Again we have $C^m_n \subseteq int\, C^m_{n+1}$ and $C^m_n \subseteq C^{m+1}_n$ with the result that $\{C^m_m\}$ is an exhaustion for $\widetilde{M}$.

As regards the excellence of $C^m_{n+1} - int\, C^m_n$ we note that the only thing new takes place in those components of $p^{-1}(R_{n+1})$ contained in C^m_{n+1}. Instead of two arcs λ_1 and λ_2 as before we have λ^i_1 and λ^i_2 as i ranges over some non-empty subset of $\{1, \ldots, k\}$ We then apply the poly-excellence of the full set of λ^i_t.

The incompressibility of ∂C^m_n in $\widetilde{M} - int\, C^m_n$ follows as before. We first note that ∂C^m_n is incompressible in $C^m_{n+q} - int\, C^m_n$ for each $q \geq 1$. Now define $\widetilde{M}_{n+q}$ to be the union of $p^{-1}(M_{n+q})$ and, for each of the surface group factors G_i of G, the copy

$\widetilde{F}^{i,\mu} \times [-(n+q+1), -(n+q)]$ of $\widetilde{F}^i \times [-(n+q+1), -(n+q)]$ contained in $Z^{i,\mu}$, where $\widetilde{F}^i$ is the universal covering space of the surface F^i with $\pi_1(F^i) \cong G_i$. Then the exterior of C^m_{n+q} in $\widetilde{M}_{n+q}$ meets it in a collection of disjoint disks, from which it follows that ∂C^m_n is incompressible in $\widetilde{M}_{n+q} - int\, C^m_n$, thus is incompressible in $\widetilde{M} - int\, C^m_n$. $\square$

7. Uncountably many examples

We now describe how to get uncountably many examples for a given group G. We will use a trick introduced in [7]. Let $\{X_{n,s}\}$ be a family of exteriors of non-trivial knots in S^3 indexed by $n \geq 2$ and $s \in \{0,1\}$; they are chosen to be anannular, atoroidal, and pairwise non-homeomorphic. (One such family is that of non-trivial, non-trefoil twist knots.) One chooses a function $\varphi(n)$ with values in $\{0,1\}$, i.e. a sequence of 0's and 1's indexed by n, and constructs a 3-manifold $M[\varphi]$ by embedding $X_{n,\varphi(n)}$ in $M_n - int\, M_{n-1}$ so that $\partial X_{n,\varphi(n)}$ in incompressible in $M_n - int\, M_{n-1}$ (but is compressible in M_n). The idea is to do this in such a way that for "large" compact sets C in $\widetilde{M}[\varphi]$ one has components of $p^{-1}(X_{n,\varphi(n)})$ which lie in $\widetilde{M} - C$ and have incompressible boundary in $\widetilde{M} - C$ for "large" values of n; moreover, every knot exterior having these properties should be homeomorphic to some $X_{n,\varphi(n)}$. Thus if $\widetilde{M}[\varphi]$ and $\widetilde{M}[\psi]$ are homeomorphic one must have $\varphi(n) = \psi(n)$ for "large" n. One then notes that there are uncountably many functions which are pairwise inequivalent under this relation.

We proceed to the details. First assume φ is fixed, so we can write $s = \varphi(n)$. The most innocuous place to embed $X_{n,s}$ is in $R_n - int\, R_{n-1}$ since this space is common to all our constructions. Recall that this space contains arcs λ_1, λ_2 or, if G is a non-trivial free product, arcs λ^i_1, λ^i_2, $1 \leq i \leq k$; call this collection of arcs λ. We wish $X_{n,s}$ to lie in the complement of λ in such a way that it is poly-excellent in $R_n - int\,(R_{n-1} \cup X_{n,s})$. We revise the construction of λ from section 6 as follows. Let B_0 and B_1 be 3-balls. Choose disjoint disks D_r and D'_r in ∂B_r. Let ζ_r be a simple closed curve in $\partial B_r - (D_r \cup D'_r)$ which separates D_r from D'_r. Let A_r and A'_r be the annuli into which ζ_r splits the annulus $\partial B_r - int\,(D_r \cup D'_r)$, with the notation chosen so that $A_r \cap D_r = \emptyset$. Let τ_r be a poly-excellent $(4k+4)$-tangle in B_r which is the union of $(2k+2)$-tangles ρ_r and ρ'_r satisfying the following conditions. Each component of ρ_0 runs from $int\, D_0$ to $int\, A'_0$. Each component of ρ_1 runs from $int\, A'_1$ to $int\, D'_1$. Each component of ρ'_0 runs from $int\, D'_0$ to itself. Each component of ρ'_1 runs from $int\, D'_1$ to $int\, D_1$. We then glue A'_0 to A'_1 and D'_0 to D'_1 so as to obtain a space homeomorphic to a 3-ball minus the interior of an unknotted solid torus contained in the interior of the 3-ball. The 2-sphere boundary component is $D_0 \cup D_1$; the torus boundary component is $A_0 \cup A_1$. The gluing is done so that the endpoints of the arcs match up to give a system λ^+ of $2k+2$ arcs. Each arc in this system consists of an arc of ρ_0 followed by an arc of ρ_1 followed by an arc of ρ_0' followed by an arc of ρ_1'. We then glue $X_{n,s}$ to this space along their torus boundaries so as to obtain a 3-ball B. We then apply the construction of section 6 to λ^+ to get a poly-excellent system λ of arcs in $R_n - int\, R_{n-1}$. It is easily seen that this 3-manifold is nice and that $\partial X_{n,s}$ is, up to isotopy, the unique incompressible non-∂-parallel torus in it; $\partial X_{n,s}$ is also, up to isotopy, the unique incompressible torus in the exterior K_σ of any non-empty union σ of components of λ.

Lemma 7.1. *If $\widetilde{M}[\varphi]$ and $\widetilde{M}[\psi]$ are homeomorphic then there is an index N such that $\varphi(n) = \psi(n)$ for all $n \geq N$.*

Proof. Consider $\widetilde{M}$. $Y = C_n^m - int\, C_{n-1}^m$ contains copies of K_σ for various choices of σ. The closure of the complement in Y of these copies consists of excellent 3-manifolds which meet the copies along incompressible planar surfaces. It follows that the various copies of $\partial X_{n,s}$ in Y are, up to isotopy and for $n \geq 3$, the unique incompressible tori in Y. The incompressibility of ∂C_n^m in $\widetilde{M} - int\, C_n^m$ implies that these tori are also incompressible in $\widetilde{M} - int\, C_{n-1}^m$.

Suppose T is an incompressible torus in $\widetilde{M} - int\, C_{n-1}^m$. Then T lies in $\widetilde{M}_{n+q}$ for some $q \geq 0$. The exterior of C_{n+q}^m in $\widetilde{M}_{n+q}$ consists of copies of $D \times \mathbf{R}$ and $\mathbf{R}^2 \times [0,1]$ to which disjoint 1-handles have been attached. It meets C_{n+q}^m in a set of disjoint disks. It follows that T can be isotoped into $C_{n+q}^m - int\, C_{n-1}^m$. Since ∂C_{n+u}^m for $1 \leq u < q$ is not a torus it is easily seen that T can be isotoped into some $C_v^m - int\, C_{v-1}^m$ and thus is isotopic to some copy of $\partial X_{v,\varphi(v)}$. Thus any knot exterior X incompressibly embedded in $\widetilde{M} - int\, C_{n-1}^m$ is homeomorphic to some $X_{v,\varphi(v)}$.

Now consider two different functions φ and ψ. We will show that if $\widetilde{M}[\varphi]$ and $\widetilde{M}[\psi]$ are homeomorphic then there is an N such that $\varphi(n) = \psi(n)$ for all $n \geq N$. Suppose $h : \widetilde{M}[\varphi] \to \widetilde{M}[\psi]$ is a homeomorphism. Distinguish the various submanifolds arising in the construction of these two manifolds by appending $[\varphi]$ and $[\psi]$, respectively. For $n \geq 2$ there are incompressibly embedded copies $\widetilde{X}_{n,\varphi(n)}$ of $X_{n,\varphi(n)}$ in $\widetilde{M}[\varphi] - int\, C_1^1[\varphi]$. There is an index ℓ such that $h(C_1^1[\varphi]) \subseteq int\, C_\ell^\ell[\psi]$. By construction $\cup_{n\geq 2}\widetilde{X}_{n,\varphi(n)}$ is end-proper in $\widetilde{M}[\varphi]$, so there is an index N such that for all $n \geq N$ we have $h(\widetilde{X}_{n,\varphi(n)}) \subseteq \widetilde{M}[\psi] - int\, C_\ell^\ell[\psi]$. Since $h(\partial \widetilde{X}_{n,\varphi(n)})$ is incompressible in $\widetilde{M}[\psi] - int\, h(C_1^1[\psi])$ it is incompressible in the smaller set $\widetilde{M}[\psi] - int\, C_\ell^\ell[\psi]$. Thus it is homeomorphic to $X_{v,\psi(v)}$ for some $v > \ell$. Since the knot exteriors are pairwise non-homeomorphic we must have $n = v$ and $\varphi(n) = \psi(v) = \psi(n)$. $\square$

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