SUBSPACE ARRANGEMENTS OVER FINITE FIELDS: COHOMOLOGICAL AND ENUMERATIVE ASPECTS

ANDERS BJÖRNER AND TORSTEN EKEDAHL

Abstract. The enumeration of points on (or off) the union of some linear or affine subspaces over a finite field is dealt with in combinatorics via the characteristic polynomial and in algebraic geometry via the zeta function. We discuss the basic relations between these two points of view. Counting points is also related to the ℓ-adic cohomology of the arrangement (as a variety). We describe the eigenvalues of the Frobenius map acting on this cohomology, which corresponds to a finer decomposition of the zeta function. The ℓ-adic cohomology groups and their decomposition into eigenspaces are shown to be fully determined by combinatorial data. Finally, it is shown that the zeta function is determined by the topology of the corresponding complex variety in some important cases.

1. Introduction

This paper is concerned with subspace arrangements over general (in particular finite) fields, and with their enumerative and cohomological invariants. In this introduction we will summarise the main results. We begin with a review of some algebraic geometry.

Let V be a d-dimensional projective variety over the field $\mathbb{F}_q$ of order $q = p^\alpha$. (We will not assume that a variety is irreducible.) For each extension field $\mathbb{F}_{q^s}$, $s \geq 1$, let N_s be the number of points on V over $\mathbb{F}_{q^s}$. The zeta function of V is the formal power series

$$Z(V;t) = \exp\left(\sum_{s\geq 1} N_s \frac{t^s}{s}\right).$$

Let ℓ be a prime number, $\ell \neq p$, and $\mathbb{Q}_\ell$ the field of ℓ-adic numbers. Let $H^i(V,\mathbb{Q}_\ell)$ be the ℓ-adic étale cohomology groups of Grothendieck [D4, G4, G5]. These are finite-dimensional vector spaces over $\mathbb{Q}_\ell$, a field of characteristic zero. Furthermore, $H^i(V,\mathbb{Q}_\ell) = 0$ unless $0 \leq i \leq 2d$.

The Frobenius map $F\colon V \to V$, defined by $(x_1,\dots,x_n) \mapsto (x_1^q,\dots,x_n^q)$, induces a linear action $F : H^i(V,\mathbb{Q}_\ell) \to H^i(V,\mathbb{Q}_\ell)$ for each $0 \leq i \leq 2d$. Let α_{ij} be the eigenvalues of this map. The Grothendieck-Lefschetz fixed point formula [G5] implies that $N_s = \sum_{i,j}(-1)^i \alpha_{i,j}^s$, which is equivalent to the following decomposition of the

Björner was partially supported by the Institute for Advanced Study (Princeton, NJ) and the Mathematical Sciences Research Institute (Berkeley, CA). Ekedahl was supported by the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine.

zeta function:

$$Z(V;t) = \frac{P_1(t)P_3(t)\dots P_{2d-1}(t)}{P_0(t)P_2(t)\dots P_{2d}(t)} \quad , \qquad \text{where } P_i(t) = \prod_j (1-\alpha_{ij}t).$$

Note that $P_i(t) \in \mathbb{Q}_\ell[t]$, and this polynomial might, seen a priori, depend on ℓ. However, this is not the case and the coefficients of $P_i(t)$ are in fact algebraic integers. Deligne [D1] showed that $|\alpha_{ij}| = q^{i/2}$ if V is smooth.

In this paper we determine the polynomials $P_i(t)$ for the case when V is a union of linear subspaces. It turns out that $P_i(t)$ is then determined by combinatorial data in the following way.

Let $\mathcal{A} = \{K_1, \dots, K_t\}$ be an arrangement of linear subspaces in projective $(n-1)$-space over $\mathbb{F}_q$, and let $V_\mathcal{A}$ denote their union (a singular projective variety). Let $L_\mathcal{A}$ denote the partially ordered set of all nonempty intersections $K_{i_1} \cap \dots \cap K_{i_m}$, $1 \le i_1 < \dots < i_m \le t$, ordered by reverse inclusion. Let $\beta_i^{\ge j}$ denote the rational i-th Betti number of the (simplicial) order complex homology of the subposet $\{x \in L_\mathcal{A} \mid j \le \dim(x) < n-1\}$.

Theorem 1.1. *For the union of a d-dimensional projective subspace arrangement $\mathcal{A}$ over $\mathbb{F}_q$ we have*

$$P_i(t) = \prod_{j=0}^{d} \left(1 - q^j t\right)^{\beta_{i-2j}^{\ge j}}.$$

The formula can be seen as a sharpening or Frobenius-equivariant version of a recent formula for $H^i(V_\mathcal{A}, \mathbb{Q}_\ell)$ due to Yan [Ya]. Theorem 1.1 will be proved with a cohomological argument that is valid for arbitrary fields. In particular, a unified setting will be given for earlier results such as Yan's theorem and the complex version of Ziegler and Živaljevic [ZZ].

One can also consider ℓ-adic cohomology and eigenvalues of Frobenius on the *complement* (rather than the union) of a projective arrangement. Or one could consider these questions for arrangements of affine subspaces of affine n-space over $\mathbb{F}_q$. Of the four possible combinations, affine/projective and union/complement, we have chosen here to concentrate on unions in the projective case and complements in the affine case. Formulas covering all cases, and also punctured affine arrangements, are given in the paper.

We will now state the result in the affine case. Let $\mathcal{A}$ be an arrangement in $\mathbb{A}^n = \mathbb{F}_q^n$ with union $V_\mathcal{A}$ and complement $M_\mathcal{A} = \mathbb{F}_q^n \setminus V_\mathcal{A}$. Let α_{ij} be the eigenvalues of the induced Frobenius map on ℓ-adic cohomology with compact supports $F : H_c^i(M_\mathcal{A}, \mathbb{Q}_\ell) \to H_c^i(M_\mathcal{A}, \mathbb{Q}_\ell)$. Let $P_i(t) = \prod_j (1 - \alpha_{ij} t)$ as before. Let $\beta_i^{\oplus j} = \sum \beta_i(\hat{0}, x)$, where $\beta_i(\hat{0}, x)$ denotes the i-th rational Betti number of the order complex homology of the open interval $(\hat{0}, x)$ in $L_\mathcal{A}$, and the sum is over all $x \in L_\mathcal{A}$ such that $\dim(x) = j$.

Theorem 1.2. *For the complement of a d-dimensional affine subspace arrangement $\mathcal{A}$ over $\mathbb{F}_q$ we have*

$$P_i(t) = \prod_{j=0}^{d} \left(1 - q^j t\right)^{\beta^{\oplus j}_{i-2j-2}}.$$

A corresponding decomposition of $H^i_c(M_\mathcal{A}, \mathbb{Q}_\ell)$ without the eigenvalue information was given by Yan [Ya], and in the real case earlier by Goresky and MacPherson [GM]. For the special case of hyperplane arrangements Theorem 1.2 specialises to say that Frobenius acts on $H^i_c(M_\mathcal{A}, \mathbb{Q}_\ell)$, for $n \le i \le 2n$, with only one eigenvalue, namely q^{i-n}, and for all other i these cohomology groups vanish; a result earlier obtained by Lehrer [Le] (see also Kim [Ki]).

The paper is organised as follows: Some definitions and facts about subspace arrangements and the combinatorics of intersection semilattices are reviewed in Section 2. In Section 3 we discuss counting points on (or off) subspace arrangements over finite fields. Some formulas relating zeta functions to the characteristic polynomials of intersection semilattices are given along with some related facts. The proofs of the main cohomological results outlined above are given in Section 4. Specifically, Theorems 1.1 and 1.2 appear as part of Theorems 4.8 and 4.9. For arrangements $\mathcal{A}$ defined by forms with integer coefficients we can consider both the arrangement $\mathcal{A}_q$ over $\mathbb{F}_q$ ($q = p^\alpha$), obtained by reduction modulo p, and the complex arrangement $\mathcal{A}_\mathbb{C}$. In the final section we discuss connections between the zeta function of $\mathcal{A}_q$ and the Betti numbers of $\mathcal{A}_\mathbb{C}$, showing that in some important cases they mutually determine each other.

Valuable conversations with P. Deligne and K.S. Sarkaria are gratefully acknowledged.

2. Preliminaries

Let $\mathbb{F}$ be a field. By an *affine subspace arrangement* we mean a finite collection of affine subspaces in $\mathbb{F}^n$. Similarly, by a *projective subspace arrangement* we mean a finite collection of linear subspaces in projective $(n-1)$-space $\mathbb{FP}^{n-1}$. An arrangement of either kind is *essential* if the intersection of all subspaces is empty. An affine arrangement is called *central* if all subspaces contain the origin. There is an obvious one-to-one correspondence between central arrangements in $\mathbb{F}^n$ and projective arrangements in $\mathbb{FP}^{n-1}$. An arrangement (of either kind) is *d-dimensional* if d is the maximal dimension of one of its subspaces.

For an arrangement $\mathcal{A} = \{K_1, \dots, K_t\}$ we denote by $V_\mathcal{A}$ the *variety* $K_1 \cup \cdots \cup K_t$. Also, we let $L_\mathcal{A} = \{K_{i_1} \cap \cdots \cap K_{i_m} \neq \varnothing\}$ be the *intersection semilattice*, the family of all nonempty intersections of subarrangements ordered by reverse inclusion. The semilattice $L_\mathcal{A}$ has a least element $\hat{0}$ (equal to the whole space $\mathbb{F}^n$ or $\mathbb{FP}^{n-1}$, as the case may be), and if $\mathcal{A}$ is central (or equivalent to a central arrangement) then there is also a greatest element $\hat{1}$ (equal to $K_1 \cap \cdots \cap K_t$).

We refer to [Bj] for a general introduction to the theory of subspace arrangements.

Let P be a poset (short for "partially ordered set") and $x, y \in P, x < y$. Then $[x, y] = \{z \in P \mid x \leq x \leq y\}$ and $(x, y) = \{z \in P \mid x < z < y\}$ are the *closed* and *open intervals*. Also, for $p \in P$ let $P_{<p}$ resp. $P_{\leq p}$ be the subset consisting of those elements of P which are less than p (resp. less than or equal to p). We assume familiarity with the *Möbius function* $\mu(x, y)$ of P, see [S2].

With a poset P we associate its *order complex* $\Delta(P)$ consisting of all chains $x_0 < x_1 < \cdots < x_k$. This is a simplicial complex, so we obtain (simplicial) homology groups $H_i(P) = H_i(\Delta(P); \mathbb{Z})$, Betti numbers $\beta_i(P) = \mathrm{rank}_{\mathbb{Z}} H_i(P)$, Euler characteristic $\chi(P)$, etc.

A tilde will always denote reduced homology (Betti numbers, Euler characteristic): $\widetilde{H}_i, \widetilde{\beta}_i, \widetilde{\chi}$. Recall Ph. Hall's theorem [S2, p. 120].

$$\mu(x, y) = \widetilde{\chi}(\Delta(x, y)), \tag{2.1}$$

for all $x < y$ in P.

We will sometimes use the following arrangements, the "k-equal arrangements of type A, B and D" [BL, BSag, BWe], to provide examples:

$$\begin{aligned}
\mathcal{A}_{n,k} &= \{x_{i_1} = x_{i_2} = \cdots = x_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\} \\
\mathcal{D}_{n,k} &= \{\varepsilon_1 x_{i_1} = \varepsilon_2 x_{i_2} = \cdots = \varepsilon_k x_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n, \varepsilon_i \in \{+1, -1\}\} \\
\mathcal{B}_{n,k} &= \mathcal{D}_{n,k} \cup \{x_{j_1} = \cdots = x_{j_{k-1}} = 0 \mid 1 \leq j_1 < \cdots < j_{k-1} \leq n\}.
\end{aligned}$$

3. Characteristic polynomial and zeta function

In this section we will develop some of the basic facts about counting points on (or off) arrangements over finite fields. The tool for this in combinatorics is the characteristic polynomial of the arrangement and in algebraic geometry the zeta function. We will make the relationship between these notions explicit and prove a few related facts.

The following is a version of the combinatorial "principle of inclusion-exclusion".

Proposition 3.1. *Let $\{H_i\}_{i \in I}$ be a family of subsets of a finite set E. Let L be the semilattice of nonempty intersections of subfamilies, ordered by reverse inclusion. Then*

$$\mathrm{card}\left(E - \bigcup_{i \in I} H_i\right) = \sum_{x \in L} \mu\left(\hat{0}, x\right) \mathrm{card}(x).$$

PROOF: For each $e \in E$ let x_e be the intersection of all H_i containing e. If $e \in H_i$ for some $i \in I$, then $x_e \neq \hat{0}$ and the right hand side will count e altogether $\sum_{\hat{0} \leq x \leq x_e} \mu\left(\hat{0}, x\right) = 0$ times. Otherwise, $x_e = \hat{0}$ and e will be counted $\mu\left(\hat{0}, \hat{0}\right) = 1$ time. □

For an affine arrangement $\mathcal{A}$ in $\mathbb{F}^n$ let

$$P_{\mathcal{A}}(t) = \sum_{x \in L_{\mathcal{A}}} \mu\left(\hat{0}, x\right) t^{\dim(x)} \tag{3.1}$$

define the *characteristic polynomial* of $\mathcal{A}$. Such polynomials have long been studied for hyperplane arrangements [OT, Za] and for other graded posets [S2]; they were generalised to subspace arrangements in [Bj, BL].

Proposition 3.2. *Let q be a prime power, and let $\mathcal{A}$ be an affine subspace arrangement in $\mathbb{F}_q^n$. Then:*

$$P_{\mathcal{A}}(q) = \operatorname{card}\left(\mathbb{F}_q^n \setminus V_{\mathcal{A}}\right).$$

PROOF: If x is an affine subspace then $\operatorname{card}(x) = q^{\dim(x)}$, so this is a special case of Proposition 3.1. □

Remark: This result is well-known for hyperplane arrangements, see the "critical problem" in [CR] and also [OT, p. 51]. The extension to subspace arrangements has independently been found by Athanasiadis [At]. A similar result (in a somewhat different setting) appears in Blass and Sagan [BlS].

Now, let $\mathcal{A}$ be a projective arrangement in $\mathbb{FP}^{n-1}$ and define

$$P^*_{\mathcal{A}}(t) = \sum_{x \in L_{\mathcal{A}}} \mu\left(\hat{0}, x\right)\left(1 + t + t^2 + \cdots + t^{\dim(x)}\right). \tag{3.2}$$

Proposition 3.3. *Let $\mathcal{A}$ be a projective arrangement in $\mathbb{F}_q\mathbb{P}^{n-1}$. Then:*

$$P^*_{\mathcal{A}}(q) = \operatorname{card}\left(\mathbb{F}_q\mathbb{P}^{n-1} \setminus V_{\mathcal{A}}\right).$$

PROOF: This follows from Proposition 3.1, since $\operatorname{card}(x) = 1 + q + \cdots + q^{\dim(x)}$ for projective subspaces x. □

Note that for projective $\mathcal{A}$, if $\overline{\mathcal{A}}$ is the corresponding central arrangement in $\mathbb{F}^n$, then

$$P_{\overline{\mathcal{A}}}(t) = (t-1)P^*_{\mathcal{A}}(t). \tag{3.3}$$

This follows from the preceding propositions, since every point in the complement of $\mathcal{A}$ corresponds to $q-1$ points in the complement of $\overline{\mathcal{A}}$.

We will call $P^*_{\mathcal{A}}(t)$ the *reduced characteristic polynomial* of a projective arrangement $\mathcal{A}$. Its coefficients are (up to sign) the reduced Euler characteristics of the *j-truncations* of the intersection semilattice $L_{\mathcal{A}}$, defined by

$$L^{\geq j}_{\mathcal{A}} = \{x \in L_{\mathcal{A}} \setminus \{\hat{0}\} \mid \dim(x) \geq j\}.$$

Proposition 3.4. *Let $P^*_{\mathcal{A}}(t) = \sum_{j=0}^{n-1} c_j t^j$. Then $c_j = -\widetilde{\chi}\left(L^{\geq j}_{\mathcal{A}}\right)$.*

PROOF: Relations (2.1) and (3.2) imply that

$$\widetilde{\chi}\left(L^{\geq j}_{\mathcal{A}}\right) = -\sum_{\dim(x) \geq j} \mu\left(\hat{0}, x\right) = -c_j.$$

□

It follows from the definition that the characteristic polynomial of a d-dimensional affine arrangement in $\mathbb{F}^n$ has the structure

$$P_{\mathcal{A}}(t) = c_0 + c_1 t + \cdots + c_d t^d + t^n, \qquad \text{with } c_d < 0. \tag{3.4}$$

Similarly, the reduced characteristic polynomial of a d-dimensional projective arrangement in $\mathbb{FP}^{n-1}$ has the structure

$$P^*_{\mathcal{A}}(t) = c_0 + c_1 t + \cdots + c_d t^d + t^{d+1} + \cdots + t^{n-1}, \qquad \text{with } c_d \leq 0. \tag{3.5}$$

Hence, there are only $d+1$ essential coefficients $c_0, \ldots, c_d$ in each case.

Example: The 3-equal arrangement $\mathcal{A} = \mathcal{A}_{6,3}$ gives for its 4-dimensional affine version

$$P_{\mathcal{A}}(t) = -26t^2 + 45t^3 - 20t^4 + t^6,$$

and for its 3-dimensional projective version

$$P^*_{\mathcal{A}}(t) = 26t^2 - 19t^3 + t^4 + t^5.$$

Let $\mathcal{A}$ be an affine subspace arrangement in $\mathbb{F}_q^n$, for some prime power q. Consider the field extensions $\mathbb{F}_q \subset \mathbb{F}_{q^s} \subset \overline{\mathbb{F}}_q$, where $\overline{\mathbb{F}}_q$ denotes an algebraic closure of $\mathbb{F}_q$. Then $\mathcal{A}$ can also be considered as an arrangement in $\overline{\mathbb{F}}_q^n$, defined by equations with coefficients in $\mathbb{F}_q$, and its intersection with $\mathbb{F}_{q^s}^n$ for all the intermediate fields is well defined.

For arbitrary subsets $X \subseteq \overline{\mathbb{F}}_q^n$ define the *zeta function* of X as the formal power series

$$Z(X;t) = Z_q(X;t) = \exp\left(\sum_{s\geq 1} \operatorname{card}\left(X \cap \mathbb{F}_{q^s}^n\right) \frac{t^s}{s}\right).$$

The analogous definition is made in the projective case of the *zeta function* for subsets $X \subseteq \mathbb{F}_q\mathbb{P}^{n-1}$:

$$Z(X;t) = Z_q(X;t) = \exp\left(\sum_{s\geq 1} \operatorname{card}\left(X \cap \mathbb{F}_{q^s}\mathbb{P}^{n-1}\right) \frac{t^s}{s}\right).$$

The following lemma will be useful. It is no doubt well known but we lack an appropriate reference.

Lemma 3.5. *If* $P(t) = \sum_{j=0}^d c_j t^j$, *then* $\exp\left(\sum_{s\geq 1} P(q^s)\frac{t^s}{s}\right) = \prod_{j=0}^d \left(1 - q^j t\right)^{-c_j}$.

PROOF: Taking the logarithm and the derivative of both sides reduces the identity to the summing of a linear combination of geometric series. The constant lost by taking the derivative is determined by putting t equal to 0. □

There is the following connection between characteristic polynomials and zeta functions.

Theorem 3.6. (i) *Let $\mathcal{A}$ be an affine arrangement in $\mathbb{F}_q^n$, with characteristic polynomial $P_{\mathcal{A}}(t) = \sum_{j=0}^{n} c_j t^j$. Then*

$$Z\left(\overline{\mathbb{F}}_q^n \setminus V_{\mathcal{A}}; t\right) = \prod_{j=0}^{n} \left(1 - q^j t\right)^{-c_j}.$$

(ii) *Let $\mathcal{A}$ be a d-dimensional projective arrangement in $\mathbb{F}_q\mathbb{P}^{n-1}$, with reduced characteristic polynomial $P_{\mathcal{A}}^*(t) = \sum_{j=0}^{n-1} c_j t^j$. Then*

$$Z(V_{\mathcal{A}}; t) = \prod_{j=0}^{d} \left(1 - q^j t\right)^{c_j - 1}.$$

PROOF: Note that the intersection semilattice of $\mathcal{A}$ is the same over all fields $\mathbb{F}_{q^s}$ (cf. the proof of Lemma 5.1). Hence, part (i) is an immediate consequence of Proposition 3.2 and Lemma 3.5.

Similarly, for part (ii) Proposition 3.3 gives

$$\operatorname{card}\left(V_{\mathcal{A}} \cap \mathbb{F}_{q^s}\mathbb{P}^{n-1}\right) = \frac{q^{sn} - 1}{q^s - 1} - P_{\mathcal{A}}^*\left(q^s\right) = \sum_{j=0}^{n-1} (1 - c_j) q^{sj}.$$

Now, by (3.5) we have that $c_j = 1$ for $j = d+1, \dots, n-1$. Hence the result follows from Lemma 3.5. □

The zeta function for the union of a d-dimensional projective arrangement can because of Proposition 3.4 also be stated in this form:

$$Z(V_{\mathcal{A}}; t) = \prod_{j=0}^{d} \left(1 - q^j t\right)^{-\chi\left(L_{\mathcal{A}}^{\geq j}\right)}. \tag{3.6}$$

It is known from the work of Dwork [Dw] that the zeta function of *any* affine or projective variety over $\mathbb{F}_q$ is a rational power series with coefficients in $\mathbb{N}$. In the theory of formal power series there is a slightly stronger concept. A series is called $\mathbb{N}$-*rational* if it can be produced from $\mathbb{N}$-polynomials by finitely many times applying the operations of addition, multiplication and quasi-inverse (this is defined as $f + f^2 + f^3 + \dots$ for series f such that $f(0) = 0$). The point is that subtraction is never allowed. See Eilenberg [Ei] and Reutenauer [Re] for more about this concept. Reutenauer points out that it follows from the work of Deligne [D1] that the zeta function of a smooth variety is in fact $\mathbb{N}$-rational. The same can be said about the zeta functions considered here, indeed Reutenauer's observation is true without the assumption of smoothness.

Theorem 3.7. *The zeta function of any algebraic variety X defined over the finite field $\mathbb{F}_q$ is $\mathbb{N}$-rational.*

PROOF: A theorem of Soittola [So] says that a rational series with coefficients in $\mathbb{N}$ is $\mathbb{N}$-rational if it has a real pole $\alpha > 0$ such that $|\beta| > \alpha$ for all other poles

β. Assume first that X is *absolutely irreducible*, i.e., is irreducible over an algebraic closure of $\mathbb{F}_q$. Then the estimate by Lang and Weil [LW, Theorem 1] gives that the number N_s of points of X over an extension field of cardinality q^s fulfills a uniform estimate $|N_s - q^{ds}| \leq O(q^{s(d-1/2)})$, where d is the dimension of X. This gives that $Z(X;t) - 1/(1-q^d t)$ is holomorphic in the disc $|t| < q^{1/2-d}$, and hence all the poles of $Z(X;t)$ except q^{-d} have absolute value larger than q^{-d}.

In the more general case when X is irreducible but not necessarily absolutely irreducible, the functions on X algebraic over $\mathbb{F}_q$ form a finite extension field $\mathbb{F}_{q^n}$. As then X has points over a field $\mathbb{F}_{q^m}$ only if $n|m$ we see that $Z(X;t) = Z(X';t^n)$, where X' is X considered as a variety over $\mathbb{F}_{q^n}$. Since the substitution $t \mapsto t^n$ preserves $\mathbb{N}$-rationality we reach the conclusion also in this case.

In the most general case we note that the zeta function is multiplicative over disjoint unions of subvarieties and that any variety is the disjoint union of finitely many irreducible varieties. $\square$

Remark: Instead of using the result of Lang-Weil we could have made an appeal to Deligne's theorem. However, the former predates the latter and is considerably more elementary.

It is a well known fact for affine and projective hyperplane arrangements, originally due to G.-C. Rota, that the coefficients of the corresponding characteristic polynomial "alternate in sign", meaning that $c_j c_{j+1} \leq 0$ for all j, cf. [S2, p. 126]. This has via Theorem 3.6 consequences for the structure of the zeta function. We will now show that the same is true for certain subspace arrangements.

Let us say that an intersection semilattice $L_{\mathcal{A}}$ is *hereditary* if whenever $x \in L_{\mathcal{A}} \setminus \{\hat{0}\}$ and $\dim(x) > 0$ there exists $y \in L_{\mathcal{A}}$ such that $y > x$ and $\dim(y) = \dim(x) - 1$. We will say that $L_{\mathcal{A}}$ is *mod-2-pure* if either all maximal chains are of even length or all maximal chains are of odd length. We will consider the concept of CL-shellability known, see [BW1, BW2] for all details.

Theorem 3.8. *Suppose $L_{\mathcal{A}}$ is hereditary, mod-2-pure and CL-shellable.*

(i) *If $\mathcal{A}$ is affine and d-dimensional with characteristic polynomial $P_{\mathcal{A}}(t) = \sum_{j=0}^{d} c_j t^j + t^n$, then $(-1)^{d-j} c_j \leq 0$ for all $0 \leq j \leq d$.*

(ii) *If $\mathcal{A}$ is projective and d-dimensional with reduced characteristic polynomial $P^*_{\mathcal{A}}(t) = \sum_{j=0}^{d} c_j t^j + \sum_{j=d+1}^{n-1} t^j$, then $(-1)^{d-j} c_j \leq 0$ for all $0 \leq j \leq d$.*

PROOF: Suppose that all maximal chains in $L_{\mathcal{A}}$ are of even length. The case of odd length is handled similarly.

(i) Let $x \neq \hat{0}$. The interval $[\hat{0}, x]$ is CL-shellable [BW1, Lemma 5.6], and being hereditary implies that the length of any maximal chain in $[\hat{0}, x]$ has the same parity as $\dim(x)$. Hence, by [BW1, Proposition 5.7] $(-1)^{\dim(x)} \mu\left(\hat{0}, x\right) \geq 0$. Hence, $(-1)^j c_j \geq 0$ for all $0 \leq j < n$. Combine this with the fact (3.4) that $c_d < 0$.

(ii) Let $0 \leq j \leq d$. The truncation $L_{\mathcal{A}}^{\geq j}$ is CL-shellable because of being hereditary and [BW2, Theorem 10.11], and the length of any maximal chain in $L_{\mathcal{A}}^{\geq j} \cup \{\hat{0}\}$ has the same parity as j. Hence, by Hall's theorem (2.1), [BW1, Proposition 5.7] and Proposition 3.4: $(-1)^{j+1}\widetilde{\chi}(L_{\mathcal{A}}^{\geq j}) = (-1)^j c_j \geq 0$ for all $0 \leq j \leq d$. Finally, we know (3.5) that $c_d \leq 0$.

□

Remark: i) Examples of subspace arrangements covered by Theorem 3.8 are hyperplane arrangements, the k-equal arrangements $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$ for *even* k [BSag, BW1], and several of the orbit arrangements $\mathcal{A}_\lambda$ shown to be CL-shellable by Kozlov [Ko].

ii) Theorem 3.8 is not true for general subspace arrangements. For example, take a planar arrangement $\mathcal{A}$ of two intersecting lines and two points not incident with these lines. In the affine version this has characteristic polynomial $P_{\mathcal{A}}(t) = t^2 - 2t - 1$, and in the projective version we get $P_{\mathcal{A}}^*(t) = -t - 2$. Note that in these cases $L_{\mathcal{A}}$ is not mod-2-pure but has the other two required properties.

iii) If the condition "mod-2-pure" is strengthened to "pure" (all maximal chains have the same length), then Theorem 3.8 would remain valid with "CL-shellable" relaxed to "Cohen-Macaulay". The proof is similar, using well-known results about Cohen-Macaulay complexes. This version of the theorem would however not cover $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$ which are not in general pure.

4. The $\mathbb{Q}_\ell$-cohomology of subspace arrangements

We will now consider the computation of the cohomology of a subspace arrangement, and in particular its étale cohomology. Most of the results that we will prove are well-known in the case of an arrangement over the real or complex numbers, and are at least partially to be found in the literature (cf. [Ya]) over a general field (including positive characteristic). The new contribution is that we keep track of the action of the Galois group of the base field, which has important arithmetic significance. As the general "philosophy of weights" (cf. [D3]) would predict, we can use the same argument to get the mixed Hodge structure in the complex case, a result which seems to be new (except for the case of hyperplane configurations which is due to Kim [Ki]). In this paper we will also be concerned only with results on rational cohomology. In that case one can use the action of the Galois group (or the rational mixed Hodge structure) on cohomology to very quickly get to the desired result.

As we are dealing with varieties over arbitrary fields (our main interest being the case of finite fields) we are forced to deal with étale cohomology of algebraic varieties instead of classical cohomology, since the latter make sense only over the real or complex numbers. Its construction is based on the realisation that to define the usual cohomology one needs access not to the topological space itself but only the category of sheaves on it. Though neither the topological space underlying a complex algebraic variety nor its category of sheaves can be constructed algebraically, a category with properties very similar to this category of sheaves can be constructed in a purely algebraic fashion. In general this category is most definitely not equivalent to the category of sheaves on a topological space, and Grothendieck and his collaborators [G4] introduced an axiomatisation under the name of *topos* that covers both these new categories and the category of sheaves on a topological space. The category associated to an algebraic variety (or more generally a scheme) then goes under the name *étale topos*. Our technical results would be most naturally formulated in terms of toposes and diagrams of them, but in the interest of concreteness we will confine ourselves to algebraic varieties (and implicitly their étale toposes).

Remark: It should be noted that in the case of a reasonable topological space the topological space itself can be recovered from the category of sheaves on it, hence not only is knowledge of the category of sheaves on a (reasonable) topological space enough to be able to compute its cohomology, it actually is equivalent to knowledge about the topological space itself. For the reader interested in details we can add that "reasonable" in this context means that every *irreducible* (not the union of two non-empty closed subsets) closed subset is the closure of a unique point – a condition almost always fulfilled in practice.

The construction of étale cohomology is quite involved. The original work of Grothendieck and his collaborators [D4, G4, G4a, G5] is still the only place where a detailed treatment of its technical properties can be found. The monographs [FK] and [Mi] are easier to approach, but deal mainly with the case of smooth varieties. We will make a thumbnail sketch of how the ℓ-adic étale cohomology groups $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ that we shall use are constructed.

The analogy between the étale topos and the category of sheaves on the space underlying a complex algebraic variety is a very close one, though there are some definite differences. The most important is that the "étale topology" is not fine enough to capture the ordinary cohomology with integer coefficients; one has to use finite coefficients. This is not an artifact of the étale topos but depends on the fact that one wants an algebraically defined cohomology. Consider for instance the fact that the first cohomology group, with integer coefficients, of $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is $\mathbb{Z}$. This reflects the fact that there is a non-trivial covering space of $\mathbb{C}^*$ with structure group $\mathbb{Z}$, given by $\exp\colon \mathbb{C} \to \mathbb{C}^*$. As the exponential function is transcendental this makes no algebraic sense, whereas the first cohomology group with $\mathbb{Z}/n\mathbb{Z}$-coefficients describes covering spaces with structure group $\mathbb{Z}/n\mathbb{Z}$. In the case of $\mathbb{C}^*$ these are described using n'th roots, which are eminently algebraic functions. One is therefore forced, to begin with, to work with cohomology with finite coefficients. In that case one obtains a theory very close to the classical topological one. In fact, a basic theorem [G4a, Exp. XVI, Thm. 4.2] says that for any algebraic variety X over the complex numbers and any finite group A we have a natural isomorphism of abelian groups $H^i_{\text{ét}}(X, A) \cong H^i_{\text{cl}}(X, A)$, where the subscripts cl (as in "classical") and ét denote respectively the ordinary cohomology of the topological space underlying X and the étale cohomology. (This isomorphism is in fact induced by a map from the topos of sheaves on the topological space of X to the étale topos and hence preserves supplementary structures such as cup products.)

Having étale cohomology for finite coefficients one then *defines*, for an algebraic variety X over an algebraically closed field, $H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$, ℓ a prime, to be the inverse limit $\varprojlim_n H^i_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z})$. Using the result above on equality of étale and classical cohomology for complex varieties X and the universal coefficient theorem, one then gets $H^i_{\text{ét}}(X, \mathbb{Z}_\ell) \cong H^i_{\text{cl}}(X, \mathbb{Z}) \bigotimes \mathbb{Z}_l$. If X is defined over a field $\mathbb{F}$ which is not algebraically closed, then (using a not quite standard notation) we define $H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$ to be the étale cohomology of X considered as an algebraic variety over some algebraic closure of the base field. The fact that X is defined over $\mathbb{F}$ is then reflected in the fact that we have a natural action of the Galois group of $\mathbb{F}$ on $H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$, of which we will see examples later on. Finally, we put $H^i_{\text{ét}}(X, \mathbb{Q}_\ell) := H^i_{\text{ét}}(X, \mathbb{Z}_\ell) \bigotimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, which then is a finite dimensional vector space over the field $\mathbb{Q}_\ell$. We will also normally dispense with the ét-subscript (the comparison theorem guarantees that confusion should only rarely result.)

Remark: When the base field is not algebraically closed one may also consider the étale cohomology of X as a variety over the base field. That cohomology will be an appropriate mixture of the étale cohomology of X as an algebraic variety over an algebraic closure and the Galois cohomology of the base field. As we will not be interested in it we have chosen to use $H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$ to denote the object which interests us.

If we consider an arrangement $\mathcal{A}$ of subspaces, $V_{\mathcal{A}}$ is by definition their union. If there are only two of them we would have the Mayer-Vietoris long exact sequence relating the cohomology of the arrangement, the two linear spaces covering it and their intersection. In the general case one gets a Mayer-Vietoris spectral sequence.

The closest analogue of the Mayer-Vietoris long exact sequence would be a spectral sequence starting with an E_1-term. We prefer to start at the E_2-term which, as usual, is more intrinsic. Starting from a covering of an algebraic variety by closed subvarieties, one may consider the cohomology of these subvarieties and their intersections. It forms a diagram of abelian groups over the ordered set of intersections of covering subvarieties. (We will follow the convention of [ZZ] in that an ordered set is considered as a category with morphisms $p \to q$ iff $p \geq q$, so that a diagram $\{X_p\}$ over the poset has morphisms $X_p \to X_q$ when $p \geq q$). The E_2-term will involve the inverse limit and its right derived functors of this diagram, and we begin by recalling a standard way of computing such limits.

Lemma 4.1 *Let $\mathcal{C}$ be a category and $F_{\cdot}$ a diagram of abelian groups. Then the groups $\varprojlim^*_{\mathcal{C}} F_{\cdot}$ are the cohomology groups of the complex $S^*(F)$ whose i'th component is the product*

$$\prod_{X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} X_i} F_{X_0}$$

and whose differential is the alternating sum of the maps obtained by composing two subsequent morphisms and from the structure map $f_0^*\colon F_{X_0} \to F_{X_1}$.

PROOF: This is shown in [JER]. □

Lemma 4.2 *Let* $\{X_p\}$ *be a covering, closed under intersections, of an algebraic variety* X *by closed subvarieties. Let* P *be the poset of these subvarieties ordered by reverse inclusion. Then there is the Mayer-Vietoris spectral sequence*

$$E_2^{i,j} = \varprojlim_P{}^j H^i(X_p, A) \Rightarrow H^{i+j}(X, A) \tag{4.1}$$

for any finite abelian group A.

PROOF: Let $i_p\colon X_p \to X$ be the inclusion map. We may consider the complex

$$0 \to A \to \prod_p i_{p*}A \to \prod_{p \geq q} i_{p*}A \to \dots$$

of sheaves on X, where A is considered as the constant sheaf on X and on the subvarieties X_p. To show that this is an exact sequence it is enough to show that it is exact on all fibres. For a given point $x \in X$ the fibre at x of this complex is the (extended) cochain complex with values in A of the (abstract) simplex with vertices the set of those p for which $x \in X_p$. The simplex being contractible, this is exact. Now, using that $H^i(X, i_{p*}A) = H^i(X_p, A)$, as i_p is a closed embedding, and the spectral sequence of a resolution, we arrive at the Mayer-Vietoris spectral sequence. □

We will now apply this result to the rational cohomology of a subspace arrangement. For this we need to recall some known facts about the action of the Galois group on étale cohomology. The best control on the Galois action is obtained when one ignores torsion, so we will look only at étale cohomology with $\mathbb{Q}_\ell$-coefficients (as defined above). It turns out that in positive characteristic the properties of this cohomology is quite pathological when ℓ is equal to the characteristic of the base field, so from now on we will assume that *the prime* ℓ *is invertible in the base field.* Furthermore, the properties of the Galois action on cohomology is simplest to formulate when the base field is finite, so for the moment we will make that assumption and let q denote its cardinality. Then the Galois group is topologically cyclic (meaning that it has a dense subgroup generated by one element) with a canonical generator called the *Frobenius element.* It is the inverse of the map which raises an element of an algebraic closure to its q'th power.

Remark: Often it is this map itself rather than its inverse that is called the Frobenius element, in matters cohomological the present choice is the more suitable however. The situation is somewhat confusing since the definition of the Frobenius map in cohomology could appear to give the opposite impression. However, there is a subtle distinction between the q'th power as a generator of the Galois group of $\mathbb{F}_q$, and the q'th power as an algebraic map on, for instance, affine space. More precisely, both induce actions on the cohomology of a variety defined over $\mathbb{F}_q$ and these actions are each other's inverses. For a more thorough discussion of this relation see [D4, pp. 76–81].

We have seen that the action of the Galois group on $\mathbb{Q}_\ell$-cohomology of a variety defined over $\mathbb{F}_q$ is given by a single linear map, usually called the *Frobenius map*. As a first invariant of such a map one may look at the eigenvalues (defined over some algebraically closed overfield and counted with the multiplicity in which it appears as zeros of the characteristic polynomial). The following definition may look very strong.

Definition 4.3 *Let F be a linear map of a finite dimensional vector space V over a field of characteristic zero, and let q be a positive integer.*

i) F is said to be pure of weight n (wrt to q) *if all of its eigenvalues are algebraic numbers, all of whose algebraic conjugates have (complex) absolute value $q^{n/2}$.*

ii) F is said to be mixed *if V has a filtration by F-stable subspaces such that F is pure of some weight on each successive subquotient of the filtration (where the weight may depend on the subquotient). The set of the weights of these subquotients will be called the* weights of F.

Remark: i) The condition that all the algebraic conjugates of an algebraic number have the same absolute value is very strong. For instance, if one bounds the degree and the absolute value then there are only finitely many such numbers. This is seen by bounding the coefficients of the minimal polynomial, coefficients that are also integers.

ii) It is implicit in the definition that the set of weights of a mixed linear operator is independent of the choice of filtration. This is obvious as the set of weights can be immediately read off from the eigenvalues of F.

A deep result of Deligne ([D1a, 3.3.8]) says that if X is smooth and proper (over a finite field of cardinality q) then its degree n $\mathbb{Q}_\ell$-cohomology is pure of weight n, and without any assumptions the cohomology is mixed.

Example: i) Affine space is the simplest example, we have $H^0(\mathbb{A}^n, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and the rest of the cohomology groups are equal to zero. The Frobenius map acts as the identity on this vector space and is hence pure of weight zero.

ii) For the projective line we have $H^1(\mathbb{P}^1, \mathbb{Q}_\ell) = 0$, and $H^0(\mathbb{P}^1, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$, with F acting again as the identity, whereas $H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$ is more interesting. As a $\mathbb{Q}_\ell$-vector space it is 1-dimensional. The Galois group of the base field acts by the inverse of the *cyclotomic character*. The cyclotomic character is the character of the Galois group for which an element σ acts by multiplication by the ℓ-adic number by which σ acts on roots of unity of order any power of ℓ. In particular, when the base field is finite (of cardinality q) the Frobenius element acts by multiplication by q on $H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$. Therefore the weight is indeed 2.

In general we denote by $\mathbb{Q}_\ell(i)$ the 1-dimensional $\mathbb{Q}_\ell$-vector space on which the Galois group acts by multiplication by the i'th power of the cyclotomic character.

Then from the fact (which can be proved by some of the methods used in the classical case) that the cohomology of N-dimensional projective space is a truncated polynomial ring and the fact that the Frobenius map preserves multiplication, one sees that $H^{2i}(\mathbb{P}^N, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-i)$ when $i \leq N$ (and all other cohomology groups are zero), which confirms that it is indeed pure of weight $2i$.

iii) Let us consider the multiplicative group $\mathbf{G}_m := \mathbb{A}^1 \setminus \{0\}$. Then $H^1(\mathbf{G}_m, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-1)$ which is not pure of weight 1 but rather of weight 2. This does not contradict Deligne's theorem as $\mathbf{G}_m$, though smooth, is not proper.

More generally, punctured affine space $\mathbb{A}^n \setminus \{0\}$ behaves cohomologically as an odd-dimensional sphere with $H^0(\mathbb{A}^n \setminus \{0\}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$, $H^{2n-1}(\mathbb{A}^n \setminus \{0\}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-n)$, and the other cohomology groups equal to zero.

iv) One can also introduce étale cohomology with compact support, see below. Intuitively this is the reduced cohomology of the one-point compactification — only that this doesn't make sense in our setting since the one-point compactification, even if defined, is not usually an algebraic variety. For the affine spaces one has that there is only a single non-zero cohomology group for cohomology with compact support: $H^{2n}_c(\mathbb{A}^n, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-n)$.

In all of these examples each weight was associated to only one cohomology group. In the case of projective space that follows from Deligne's theorem, in the others it seems to be more of an accident. In any case, they are all situations to which the following theorem applies. Before going into its formulation we want to comment on the cohomology of 1-point compactifications, which occurs prominently in the study of subspace arrangements.

The 1-point compactification of a complex algebraic variety X need not be an algebraic variety. If one is interested only in its cohomology a substitute may be found, the *cohomology with compact support*. For this one chooses some realisation of X as an open subset of some proper variety $j\colon X \hookrightarrow \bar{X}$ and then one considers the étale cohomology $H^i(\bar{X}, j_!A)$, where $j_!A$ is the sheaf on $\bar{X}$ that is equal to A on X and has fibre 0 at all points of $\bar{X}$ outside of X ("the extension by zero"). This turns out to be independent of the choice of j and computes in the case that the base field is the complex numbers the reduced cohomology of the 1-point compactification. For these cohomology groups the notation $H^i_c(X, A)$ is used. (Properly speaking we should also add the subscript $_{\text{ét}}$, as the cohomology with compact support makes excellent sense also in the classical case. In the interest of readability we will dispense with that.) There is now an analogue of the spectral sequence of Lemma 4.2 for cohomology with compact support, the proof is the same (the essential point is that when one embeds X as an open subset of a proper variety one gets at the same time a compactification of all the X_p by taking their closures in the compactification of X).

Theorem 4.4 *Let X be an algebraic variety that is the union of a family of closed subvarieties $\{X_p\}$, closed under intersection. Suppose that there is a function ϕ from $\mathbb{N}$, the natural numbers, to subsets of the integers such that different numbers are taken to disjoint sets, and that the degree i cohomology of each X_p is mixed with* *weights in $\phi(i)$. Then, with the notations of Lemma 4.2, the spectral sequence of (loc. cit.) degenerates at the E_2-term. The same is true if instead cohomology with compact support is considered.*

PROOF: Let us first assume that the base field is finite. If we can prove that $E_2^{i,j}$ is mixed with weights in $\phi(i)$ we are finished, since then all the differentials $d_k^{i,j}$ at the E_k-term, for $k \geq 2$, will be between spaces of disjoint weights. However, Lemma 4.1 presents $E_2^{i,j}$ as a subquotient of spaces with weights in $\phi(i)$.

For the case of a general base field there are standard techniques for reducing to the case of a finite base field, for which we refer to for instance [BBD, 6.1] rather than repeating them here. Very quickly described, one first uses that base extension from one algebraically closed field to an algebraically closed overfield does not change cohomology to reduce to the case where the base field is finitely generated over the prime field. Then there is a specialisation to a finite field, which again does not change cohomology. □

Remark: i) The idea that one could use weights to show that spectral sequences degenerate is not new. One of its first uses can be found in [D2], where it is applied to the study of the cohomology of the complement of a divisor with normal crossings in a smooth and projective variety. However, there one is using the mixed Hodge structure on classical cohomology rather than the Galois action on étale cohomology. The arguments of (loc. cit.) were one of the major inspirations for our theorem.

ii) The theorem applies to the cohomology (including cohomology with compact support) of an affine subspace arrangement, as there only the cohomology of affine spaces are involved and we have seen that they fulfill the required condition. It also applies to projective subspace arrangements, again the cohomology of projective spaces fulfills the condition. Another case is a punctured central arrangement, where one considers the arrangement minus a central point (this is the algebraic analogue of the spherical arrangement associated to a central arrangement over the reals).

To apply this result to the various cases of subspace arrangements we need to compute the cohomology of some diagrams of abelian groups. Recall from Section 2 the definition of the order complex $\Delta(P)$ of a poset P, and of subposets of type $P_{<p}$ and $P_{\leq p}$.

Proposition 4.5 *Let P be a finite poset and let A be an abelian group.*

i) Let Q be an order ideal (i.e., a subset of P such that any element of P less than an element of Q is also in Q), and let $\mathcal{F}_{A,Q}$ be the diagram which is 0 outside of Q and constant with value A on Q. Then we have a natural isomorphism

$$\varprojlim{}^j \mathcal{F}_{A,Q} \cong H^j(\Delta(Q), A).$$

ii) Let $p \in P$ and let $\mathcal{F}_{A,p}$ be the diagram with value A on p and 0 elsewhere. Then we have a natural isomorphism

$$\varprojlim{}^j \mathcal{F}_{A,p} \cong \widetilde{H}^{j-1}(\Delta(P_{<p}), A).$$

(For this formula, recall that the reduced cohomology of the empty complex is A in degree -1 and 0 otherwise.)

PROOF: For part i) we simply use Lemma 4.1, which shows that the higher inverse limits can be computed using a complex which is also the cochain complex of $\Delta(Q)$ with values in A. As for ii), we have a natural inclusion $\mathcal{F}_{A,p} \hookrightarrow \mathcal{F}_{A,P_{\leq p}}$, whose quotient is $\mathcal{F}_{A,P_{<p}}$. Using the long exact sequence of higher inverse limits, part i) and the fact that $\Delta(P_{\leq p})$ is contractible, we immediately reach the desired conclusion. □

Example: i) Consider the cohomology of an affine arrangement $\mathcal{A}$. The only non-trivial cohomology group of affine spaces is $H^0(-,A) = A$, so the spectral sequence degenerates to the isomorphism $H^i(V_{\mathcal{A}}, A) \cong H^i(\Delta(L_{\mathcal{A}} \setminus \{\hat{0}\}), A)$.

ii) If $\mathcal{A}$ is central we may remove the central point to get an arrangement of punctured affine spaces (in the real or complex case it is homotopic to the associated spherical arrangement). Again the condition of Theorem 4.4 is fulfilled. Furthermore, $H^{2i}(-,\mathbb{Q}_\ell)$ is zero for $i > 0$ and $\mathbb{Q}_\ell$ for $i = 0$, and $H^{2i-1}(-,\mathbb{Q}_\ell)$ is $\mathbb{Q}_\ell(-i)$ on i-dimensional elements of the intersection lattice and zero otherwise. This then is a direct sum of diagrams of the type considered in the proposition. Hence, letting $P = L_{\mathcal{A}} \setminus \{\hat{0}\}$ we get

$$\begin{aligned} \varprojlim{}^j H^0(-,\mathbb{Q}_\ell) &= H^j(\Delta(P),\mathbb{Q}_\ell) \\ \varprojlim{}^j H^{2i-1}(-,\mathbb{Q}_\ell) &= \textstyle\bigoplus_{\dim p=i} \widetilde{H}^{j-1}(\Delta(P_{<p}),\mathbb{Z}) \bigotimes \mathbb{Q}_\ell(-i). \end{aligned} \tag{4.2}$$

iii) If $\mathcal{A}$ is a projective arrangement, then $H^{2*+1}(-,A) = 0$ and $H^{2i}(-,A)$ is the constant diagram on the elements of dimension greater than or equal to i. Thus, we may again use the proposition to compute the higher inverse limits and get the result

$$\varprojlim{}^j H^{2i}(-,\mathbb{Q}_\ell) = H^j(\Delta(P^{\geq i}),\mathbb{Z}) \bigotimes \mathbb{Q}_\ell(-i), \tag{4.3}$$

where $P = L_{\mathcal{A}} \setminus \{\hat{0}\}$ and $P^{\geq i} := \{p \in P : dim(p) \geq i\}$.

iv) Once more let $\mathcal{A}$ be an affine arrangement, but this time consider cohomology with compact support. As has been noted, we get a spectral sequence also in that case, and from the computation of the cohomology with compact support of affine space we get that $H_c^{2*+1}(-,\mathbb{Q}_\ell) = 0$ and that $H_c^{2i}(-,\mathbb{Q}_\ell)$ is $\mathbb{Q}_\ell(-i)$ on i-dimensional elements of the intersection lattice and zero otherwise. As in the central affine case we get

$$\varprojlim{}^j H_c^{2i}(-,\mathbb{Q}_\ell) = \bigoplus_{\dim p=i} \widetilde{H}^{j-1}(\Delta(P_{<p}),\mathbb{Z}) \bigotimes \mathbb{Q}_\ell(-i). \tag{4.4}$$

Even if one sticks to the case of the base field being the complex numbers there are advantages to considering diagrams of algebraic varieties. For algebraic varieties over the complex numbers there is an additional structure on its cohomology alluded to previously — its *mixed Hodge structure*.

To give the definition of this notion we first recall that a *Hodge structure of weight* n consists of a finitely generated abelian group $H_{\mathbb{Z}}$ and a decreasing finite filtration F^m of $H_{\mathbb{Z}} \bigotimes \mathbb{C}$ by complex sub-vector spaces such that $\bar{F}^m$, the complex conjugate of of F^m (the complex conjugation being induced by that of the second factor in the tensor product), is a complementary subspace to F^{n-m+1}. We then recall [D2, 2.3.1] that a *mixed Hodge structure* is a finitely generated abelian group $H_{\mathbb{Z}}$ together with one increasing finite filtration W_p of sub-vector spaces of $H_{\mathbb{Q}} := H_{\mathbb{Z}} \bigotimes_{\mathbb{Z}} \mathbb{Q}$ and one decreasing finite filtration F^m by $\mathbb{C}$-sub-vector spaces of $H_{\mathbb{C}} := H_{\mathbb{Z}} \bigotimes_{\mathbb{Z}} \mathbb{C}$, such that for every i the filtration induced by F^m on $W_i/W_{i-1} \bigotimes \mathbb{C}$ forms a Hodge structure of weight i. The class (with the obvious morphisms) of mixed Hodge structures form an abelian category. We also use the term *set of weights* of a mixed Hodge structure for the set of integers for which $W_i \neq W_{i-1}$. If instead one looks at only a rational vector space $H_{\mathbb{Q}}$ without a choice of $H_{\mathbb{Z}}$ one speaks about a *rational Hodge structure*.

Example: Let $\mathbb{Z}(i)$ be the mixed Hodge structure with $H_{\mathbb{Z}} = \mathbb{Z}$, $0 = W_{2i-1} \subset W_{2i} = \mathbb{Q}$, and $0 = F^{i+1} \subset F^i = \mathbb{C}$; and similarly for $\mathbb{Q}(i)$, a rational Hodge structure. This notation will be used to describe the mixed Hodge structures relevant to subspace arrangements after the next theorem.

There is a very strong analogy between mixed Hodge structure and the action of the Galois group on étale cohomology. Parts of the analogy can actually be proven — for instance, if one considers the cohomology of a complex algebraic variety, then the filtration on rational cohomology induced from the Hodge structure coincides with the weight filtration with respect to the Galois action. We illustrate this analogy by giving another proof of the degeneration of the Mayer-Vietoris spectral sequence when the base field is the complex numbers.

Theorem 4.6 *Let X be a complex algebraic variety that is the union of a family of closed subvarieties $\{X_p\}$, closed under intersection. Suppose that there is a function ϕ from $\mathbb{N}$, the natural numbers, to subsets of the integers such that different numbers are taken to disjoint sets, and that the degree i cohomology of each X_p has weights, with respect to its mixed Hodge structure, in $\phi(i)$. Then, with the notation of Lemma 4.2, the spectral sequence of (loc. cit.) with $\mathbb{Q}$-coefficients degenerates at the E_2-term.*

PROOF: We first need to prove that the spectral sequence is a spectral sequence of rational mixed Hodge structures. For this we note another way of constructing it. Namely, we consider the simplicial complex variety $sX_.$ for which sX_j is the disjoint union of the X_{i_0} over the index set $\{i_0 \geq i_1 \geq \cdots \geq i_j\}$ with the obvious structure maps. The spectral sequence (cf. [D2a, 5.3.3.3]) applied to the constant sheaf $\mathbb{Z}$ of this simplicial variety converges to the cohomology of X and has an E_1-term which is the standard complex for computing $\varprojlim_P\{H^i(-,\mathbb{Z})\}$, and hence gives our spectral sequence from the E_2-term on. According to [D2a, 8.3.5] this is a spectral sequence of mixed Hodge structures which becomes a spectral sequence of rational mixed Hodge structures when tensored with $\mathbb{Q}$.

If we can prove that $E_2^{i,j}$ is mixed with weights in $\phi(i)$ we are finished, since then all the differentials $d_k^{i,j}$, for $k \geq 2$, will be between rational mixed Hodge structures of disjoint weights. However, Lemma 4.1 presents $E_2^{i,j}$ as a subquotient of spaces with weights in $\phi(i)$. □

Remark: We have the following computations of the mixed Hodge structure on the cohomology of affine space, puctured affine space and projective space, completely analogous to the action of the Frobenius on étale cohomology:

$$\begin{array}{lcl} H^0(\mathbb{A}^n, \mathbb{Z}) & = & \mathbb{Z}(0), \\ H^0(\mathbb{A}^n \setminus \{0\}, \mathbb{Z}) & = & \mathbb{Z}(0), \\ H^{2n-1}(\mathbb{A}^n \setminus \{0\}, \mathbb{Z}) & = & \mathbb{Z}(n), \\ H^{2i}(\mathbb{P}^n, \mathbb{Z}) & = & \mathbb{Z}(i),\ i \leq n. \end{array}$$

Hence the theorem may be applied to subspace arrangements.

Having developed the necessary general tools, we now want to collect our results as applied to subspace arrangements over finite fields. In that case one extra refinement is possible which is given in the following lemma.

Lemma 4.7 *Let assumptions be as in Theorem 4.4 and assume that the base field is finite. Then there is a canonical isomorphism between $H^*(X, \mathbb{Q}_\ell)$ and the E_2-term of the spectral sequence. This isomorphism preserves the action of the fundamental group of the base field.*

PROOF: We may use the action of the Frobenius map to split up $H^*(X, \mathbb{Q}_\ell)$ as a sum of generalised eigenspaces under it. Since each such eigenspace occurs in just one row of the E_2-term we get the canonical isomorphism. □

Remark: It is not possible to conclude from what we have proven so far that this result remains true for a general field or has an analogue for mixed Hodge structures. The reason for this is that it would be possible for the extensions provided by the spectral sequence to be non-trivial, there are indeed non-trivial extensions between the Galois representations (resp. mixed Hodge structures) involved. It will be proved elsewhere that in the case of subspace arrangements these possibilities are not realised and in fact the isomorphisms of the theorem exist for $\mathbb{Z}_\ell$-cohomology (resp. for cohomology with its mixed Hodge structure).

We now collect the various results obtained so far about the cohomology of unions $V_{\mathcal{A}}$ of subspace arrangements over finite fields. To simplify statements of formulas in this and the following theorem we introduce, just as in the classical case, reduced ℓ-adic cohomology $\tilde{H}^*(X, \mathbb{Q}_\ell)$. This differs from ordinary cohomology for all varieties X only in one dimension, namely $\tilde{H}^0(X, \mathbb{Q}_\ell) = H^0(X, \mathbb{Q}_\ell)/\mathbb{Q}_\ell$ when X is non-empty, and $\tilde{H}^{-1}(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ when X is empty.

Theorem 4.8 *Let $\mathcal{A}$ be a subspace arrangement over a finite field, $d\colon L_{\mathcal{A}} \to \mathbb{Z}$ the dimension function of its intersection semilattice and $P := L_{\mathcal{A}} \setminus \{\hat{0}\}$.*

i) If $\mathcal{A}$ is an affine arrangement then we have a canonical isomorphism

$$H^*(V_{\mathcal{A}}, \mathbb{Z}_\ell) \cong H^*(\Delta(P), \mathbb{Z}_\ell),$$

which respects the action of the Frobenius map if it is assumed to act trivially on the right hand side.

ii) If $\mathcal{A}$ is an affine arrangement then we have a canonical isomorphism

$$H_c^*(V_{\mathcal{A}}, \mathbb{Q}_\ell) \cong \bigoplus_{p \in P} \widetilde{H}^{*-2d(p)-1}(\Delta(P_{<p})) \bigotimes \mathbb{Q}_\ell(-d(p)),$$

which respects the Frobenius action if it is assumed to act trivially on the cohomology of the order complexes $\Delta(P_{<p})$.

iii) If $\mathcal{A}$ is a central arrangement then we have a canonical isomorphism

$$\tilde{H}^*(V_{\mathcal{A}} \setminus \{0\}, \mathbb{Q}_\ell) \cong \bigoplus_{p \in P} \widetilde{H}^{*-2d(p)}(\Delta(P_{<p})) \bigotimes \mathbb{Q}_\ell(-d(p)),$$

which respects the Frobenius action if it is assumed to act trivially on the cohomology of the $\Delta(P_{<p})$.

iv) If $\mathcal{A}$ is a projective arrangement then we have a canonical isomorphism

$$H^*(V_{\mathcal{A}}, \mathbb{Q}_\ell) \cong \bigoplus_{0 \le j} H^{*-2j}(\Delta(P^{\ge j})) \bigotimes \mathbb{Q}_\ell(-j),$$

where $P^{\ge j} := \{p \in P : d(p) \ge j\}$, which respects the Frobenius action if it is assumed to act trivially on the cohomology of the $\Delta(P^{\ge j})$.

PROOF: This follows from the results 4.2, 4.4, 4.5 and 4.7, together with (4.2), (4.3) and (4.4). □

Finally we also collect the consequences of our results for the cohomology of the complements of subspace arrangements over finite fields.

Theorem 4.9 *Let $\mathcal{A}$ be a subspace arrangement in a space of n dimensions over a finite field, $d\colon L_{\mathcal{A}} \to \mathbb{Z}$ the dimension function of its intersection semilattice, and $M_{\mathcal{A}}$ the complement of the union $V_{\mathcal{A}}$. Furthermore, let $P = L_{\mathcal{A}} \setminus \{\hat{0}\}$.*

i) If $\mathcal{A}$ is affine we have canonical isomorphisms

$$H_c^*(M_{\mathcal{A}}, \mathbb{Q}_\ell) \cong \bigoplus_{p \in P} \widetilde{H}^{*-2d(p)-2}(\Delta(P_{<p})) \bigotimes \mathbb{Q}_\ell(-d(p)),$$

when $* \neq 2n$ and $H_c^{2n}(M_{\mathcal{A}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-n)$, and

$$\widetilde{H}^*(M_{\mathcal{A}}, \mathbb{Q}_\ell) \cong \bigoplus_{p \in P} \widetilde{H}_{2n-*-2d(p)-2}(\Delta(P_{<p})) \bigotimes \mathbb{Q}_\ell(d(p) - n).$$

ii) If $\mathcal{A}$ is projective we have canonical isomorphisms

$$H_c^*(M_{\mathcal{A}}, \mathbb{Q}_\ell) \cong \bigoplus_{0 \le j \le n} \widetilde{H}^{*-2j-1}(\Delta(P^{\ge j})) \bigotimes \mathbb{Q}_\ell(-j),$$

when $* \neq 2n$ *and* $H_c^{2n}(M_{\mathcal{A}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-n)$, *and*

$$\widetilde{H}^*(M_{\mathcal{A}}, \mathbb{Q}_\ell) \cong \bigoplus_{0 \le j \le n} \widetilde{H}_{2n-*-2j-1}(\Delta(P^{\ge j})) \bigotimes \mathbb{Q}_\ell(j-n).$$

All these isomorphisms respect the action of Frobenius if it is assumed to act trivially on the order complexes occurring in the right hand sides.

PROOF: One enjoyable property of the cohomology with compact support is "additivity" for a closed subvariety and its complement. More precisely, for a variety Y, a closed subvariety F and its complement U we have (cf. [G4a, Exp. XVII, 5.1.16.3]) a long exact sequence

$$\begin{aligned} 0 \to H_c^0(U, \mathbb{Q}_\ell) \to \ & H_c^0(Y, \mathbb{Q}_\ell) \to \ H_c^0(F, \mathbb{Q}_\ell) \to \\ H_c^1(U, \mathbb{Q}_\ell) \to \ & H_c^1(Y, \mathbb{Q}_\ell) \to \ H_c^1(F, \mathbb{Q}_\ell) \to \dots, \end{aligned}$$

where the maps $H_c^i(Y, \mathbb{Q}_\ell) \to H_c^i(F, \mathbb{Q}_\ell)$ are the restriction maps. If we apply this to an affine arrangement, using the computation of the cohomology of affine space as well as that of the union $V_{\mathcal{A}}$, we get that $H_c^{2n}(M_{\mathcal{A}}, \mathbb{Q}_\ell) = H_c^{2n}(\mathbb{A}^n, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-n)$ and that $H_c^i(M_{\mathcal{A}}, \mathbb{Q}_\ell) = H_c^{i-1}(V_{\mathcal{A}}, \mathbb{Q}_\ell)$ for $i \neq 2n$. Now, $M_{\mathcal{A}}$ is a smooth variety and so we may apply the Poincaré duality theorem [G4a, Exp. XVIII,3.2.6.1], which says that the cup-product $H^i(M_{\mathcal{A}}, \mathbb{Q}_\ell) \bigotimes_{\mathbb{Q}_\ell} H_c^{2n-i}(M_{\mathcal{A}}, \mathbb{Q}_\ell) \to H_c^{2n}(M_{\mathcal{A}}, \mathbb{Q}_\ell)$ composed with the trace map $H_c^{2n}(M_{\mathcal{A}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-n)$ gives a perfect pairing. This gives that $H^i(M_{\mathcal{A}}, \mathbb{Q}_\ell)$ is canonically isomorphic to $H_c^{2n-i}(M_{\mathcal{A}}, \mathbb{Q}_\ell)^* \bigotimes \mathbb{Q}_\ell(-n)$. Using this formula, the relation $H_c^i(M_{\mathcal{A}}, \mathbb{Q}_\ell) = H_c^{i-1}(V_{\mathcal{A}}, \mathbb{Q}_\ell)$ for $i \neq 2n$, Theorem 4.8(ii) and the universal coefficient formula applied to the cohomology of the $\Delta(P_{<p})$, we get the first part of the theorem.

As for the second, we consider again the long exact sequence of cohomology with compact support, using that for a proper variety it is equal to cohomology without compact support, so that we can use Theorem 4.8(iv). Now, it is clear that the restriction map $H^{2i}(\mathbb{P}^n, \mathbb{Q}_\ell) \to H^{2i}(V_{\mathcal{A}}, \mathbb{Q}_\ell)$ maps $\mathbb{Q}_\ell(-i)$ to $1 \otimes \mathbb{Q}_\ell(-i) \subseteq H^0(\Delta(P^{\ge i})) \otimes \mathbb{Q}_\ell(-i)$. This is evidently an injection when $i \le m$, where m is the maximal dimension of subspaces in $\mathcal{A}$, so the long exact sequence splits up into the desired isomorphisms for cohomology with compact support. Using duality gives the formula for cohomology without compact support. □

Remark: Analogs of the formulas in Theorems 4.8 and 4.9 for arrangements over the real and complex numbers were proved by Goresky and MacPherson [GM], Ziegler and Živaljević [ZZ] and others. Some of these formulas in étale cohomology version appear in the paper by Yan [Ya], however without the decomposition into eigenspaces under Frobenius.

5. Arrangements over the integers

In this section we shall be concerned with arrangements specified by integer forms. Let a $\mathbb{Z}$-*arrangement* (*affine* resp. *projective*) mean an arrangement $\mathcal{A} = \{K_1, \ldots, K_t\}$ where each subspace is specified by a certain collection of linear forms (general resp. homogenous) with integer coefficients. Thus, a $\mathbb{Z}$-arrangement is really a list of linear forms over $\mathbb{Z}$ partitioned into t groups. With a $\mathbb{Z}$-arrangement $\mathcal{A}$ we associate on the one hand the complex subspace arrangement $\mathcal{A}_{\mathbb{C}}$ (affine or projective, as the case may be) obtained by interpreting the given $\mathbb{Z}$-forms over $\mathbb{C}$; and on the other hand the subspace arrangement $\mathcal{A}_q$ over the finite field $\mathbb{F}_q$ obtained from the $\mathbb{Z}$-forms by reduction modulo p, for arbitrary prime powers $q = p^\alpha$.

Remark: We could here equally well replace $\mathbb{Z}$ with an arbitrary number ring. Except for trivial notational changes nothing in the arguments to follow would need to be modified.

Lemma 5.1. *Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement and p a prime. Let ε be the identity map on the set of subspaces of $\mathcal{A}$. Then the following conditions are equivalent:*

(i) *ε extends to a dimension-preserving isomorphism $L_{\mathcal{A}_{\mathbb{C}}} \cong L_{\mathcal{A}_p}$;*
(ii) *ε extends to a dimension-preserving isomorphism $L_{\mathcal{A}_{\mathbb{C}}} \cong L_{\mathcal{A}_{p^\alpha}}$, for all $\alpha \geq 1$;*
(iii) $\operatorname{rank}_{\mathbb{C}}\{\ell_1, \ldots, \ell_g\} = \operatorname{rank}_{\mathbb{F}_p}\{\ell_1, \ldots, \ell_g\}$ *for any collection $\ell_1, \ldots, \ell_g$ of linear forms from $\mathcal{A}$, containing for each subspace either all of its defining forms or none of them.*

PROOF: The implications (ii)$\Rightarrow$(i)$\Rightarrow$(iii) are immediate. For (iii)$\Rightarrow$(ii) one checks that the linear algebra in $\mathbb{F}_{p^\alpha}$ of the given forms (reduced modulo p) takes place in the subfield $\mathbb{F}_p$. $\square$

We shall call a prime p *good* with respect to a $\mathbb{Z}$-arrangement $\mathcal{A}$ if it satisfies the conditions of the lemma, otherwise *bad*. Part (iii) shows that for a given $\mathcal{A}$ there is only a finite number of bad primes (these being the divisors of a finite collection of determinants in the ℓ_i's). In the special case when $\mathcal{A}$ is a hyperplane arrangement condition (iii) can be expressed by saying that $\mathcal{A}$ determines the same matroid over $\mathbb{C}$ and over $\mathbb{F}_p$.

Example: The k-equal arrangements defined in Section 2 are $\mathbb{Z}$-arrangements, and $\mathcal{A}_{n,k}$ has no bad primes, while $\mathcal{B}_{n,k}$ and $\mathcal{D}_{n,k}$ have the bad prime 2.

Let $\mathcal{A}$ be a d-dimensional projective $\mathbb{Z}$-arrangement and $q = p^\alpha$, where p is a good prime. Let $L_{\mathcal{A}} = L_{\mathcal{A}_{\mathbb{C}}} \cong L_{\mathcal{A}_q}$ and $L_{\mathcal{A}}^{\geq j} = \{x \in L_{\mathcal{A}} \mid \dim(x) \geq j\} \setminus \{\hat{0}\}$. Define

$$\beta_i^{\geq j} = \dim_{\mathbb{Q}} H_i\left(L_{\mathcal{A}}^{\geq j}, \mathbb{Q}\right). \tag{5.1}$$

These order homology Betti numbers of the j-truncated intersection lattices are possibly nontrivial only in the range $0 \leq i \leq d - j \leq d$. We will call the triangular array $(\beta_i^{\geq j})$ the *beta triangle* of $\mathcal{A}$.

A formula of Ziegler and Živaljević [ZZ, Prop. 2.15] [WZZ, Coroll. 6.7], which is the complex analog of Theorem 4.8(iv), shows that

$$\beta_i^{\mathbb{C}} := \dim_{\mathbb{Q}} H_i\left(V_{\mathcal{A}_{\mathbb{C}}}, \mathbb{Q}\right) = \sum_{j=0}^{d} \beta_{i-2j}^{\geq j}, \tag{5.2}$$

and from formula (3.6) we have that

$$Z\left(V_{\mathcal{A}_q}; t\right) = \prod_{j=0}^{d} \left(1 - q^j t\right)^{\sum (-1)^{i+1} \beta_i^{\geq j}}. \tag{5.3}$$

Thus both the rational Betti numbers of the union of the complex arrangement $\mathcal{A}_{\mathbb{C}}$ and the zeta function of the discrete arrangement $\mathcal{A}_q$ are governed by the same primitive combinatorial data, namely the beta triangle of $\mathcal{A}$.

Example: Here is the beta triangle $(\beta_i^{\geq j})$ of $\mathcal{A}_{6,3}$ in the (i, j) Cartesian plane:

$$\begin{array}{llll} 20 & & & \\ 1 & 26 & & \\ 1 & 10 & 10 & \\ 1 & 0 & 0 & 0 \end{array}$$

It follows that $\mathcal{A}_{6,3}$ has Betti numbers $\beta^{\mathbb{C}} = (1, 0, 1, 10, 11, 26, 20)$ as a complex variety, and zeta function

$$Z\left(V_{\mathcal{A}_{6,3}}; t\right) = \frac{\left(1 - q^2 t\right)^{25}}{(1 - t)(1 - qt)\left(1 - q^3 t\right)^{20}}.$$

as a variety over $\mathbb{F}_q$. Furthermore, from Theorem 1.1 we have that

$$\begin{aligned}
P_0(t) &= 1 - t \\
P_1(t) &= 1 \\
P_2(t) &= 1 - qt \\
P_3(t) &= (1 - qt)^{10} \\
P_4(t) &= (1 - qt)^{10}\left(1 - q^2 t\right) \\
P_5(t) &= \left(1 - q^2 t\right)^{26} \\
P_6(t) &= \left(1 - q^3 t\right)^{20}
\end{aligned}$$

We will now show that for an important class of $\mathbb{Z}$-arrangements the Betti numbers $\beta_i^{\mathbb{C}}$ of the complex variety and the zeta function of the $\mathbb{F}_q$-variety determine each other. This is clearly not true in general.

The intersection semilattice $L_{\mathcal{A}}$ is said to be *rationally Cohen-Macaulay* if for all $x < y$ in $\widehat{L}_{\mathcal{A}} = L_{\mathcal{A}} \cup \{\hat{1}\}$, where $\hat{1}$ is a new top element, we have

$$\widetilde{H}_i\left(\Delta(x, y), \mathbb{Q}\right) = 0, \qquad \text{for all } i < \dim \Delta(x, y).$$

This definition is via a theorem of Reisner equivalent to the Cohen-Macaulayness of the Stanley-Reisner ring of $L_{\mathcal{A}}$. See Stanley [S1] for more about this concept.

We will consider $\mathbb{Z}$-arrangements $\mathcal{A}$ whose semilattice $L_{\mathcal{A}}$ is both Cohen-Macaulay and hereditary (defined in connection with Theorem 3.8). Then every maximal chain in $L_{\mathcal{A}}$ has the form $x_0 > x_1 > \cdots > x_d > \hat{0}$ with $\dim(x_i) = i$ for $0 \leq i \leq d$. Examples are all hyperplane arrangements, many of the orbit arrangements $\mathcal{A}_\lambda$ shown to the shellable by Kozlov [Ko], and the arrangements corresponding to Cohen-Macaulay simplicial complexes considered in Björner and Sarkaria [BSar]. The following generalises the main result of [BSar].

Theorem 5.2. *Let $\mathcal{A}$ be a d-dimensional projective $\mathbb{Z}$-arrangement such that $L_{\mathcal{A}}$ is Cohen-Macaulay and hereditary, and let q be a power of a good prime. Then*

$$Z\left(V_{\mathcal{A}_q};t\right) = \prod_{j=0}^{d}\left(1-q^{d-j}t\right)^{(-1)^{j+1}\beta^{\mathbb{C}}_{2d-j}-\delta_j},$$

where $\delta_j = 1$ if j is odd and $= 0$ otherwise.

PROOF: We will use the fact [S1, Theorem III.4.5] that the truncated posets $L_{\mathcal{A}}^{\geq j}$ are Cohen-Macaulay for all $0 \leq j \leq d$. Thus, the beta triangle $(\beta_i^{\geq j})$ has internal zeros, and ones along the $i=0$ boundary:

$$\beta_i^{\geq j} = \begin{cases} 1, & \text{if } i=0,\ 0 \leq j < d \\ 0, & \text{if } 0 < i < d-j \end{cases}.$$

Therefore formula (5.2) simplifies as follows for $0 \leq j \leq d$:

$$\beta^{\mathbb{C}}_{2d-j} = \begin{cases} \beta_j^{\geq d-j}, & \text{if } j=0 \text{ or } j \text{ is odd} \\ \beta_j^{\geq d-j}+1, & \text{otherwise} \end{cases}.$$

These two formulas imply

$$\sum_i (-1)^{i+1}\beta_i^{\geq d-j} = \begin{cases} \beta^{\mathbb{C}}_{2d-j}-1, & \text{if } j \text{ is odd} \\ -\beta^{\mathbb{C}}_{2d-j}, & \text{otherwise} \end{cases},$$

which because of formula (5.3) is equivalent to the theorem. □

Note that the rational Betti numbers $\beta^{\mathbb{C}}_{2d-j}$ for $0 \leq j \leq d$ appearing in the theorem are the only essential ones, since the structure of the beta triangle in the Cohen-Macaulay case shows that for $0 \leq j < d$:

$$\beta^{\mathbb{C}}_j = \begin{cases} 1, & \text{if } j \text{ is even} \\ 0, & \text{otherwise} \end{cases}.$$

The preceding proof hinges on the very simple structure of the beta triangle $(\beta_i^{\geq j})$ given by the almost total vanishing of Betti numbers in the Cohen-Macaulay case.

The beta triangle has simplified structure also for some other arrangements, including the k-equal arrangements $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$, as we will now show.

Let us say that an intersection semilattice $L_{\mathcal{A}}$ is *mod-m-pure* if the lengths of all maximal chains are congruent mod m.

Theorem 5.3. *Suppose that $L_{\mathcal{A}}$ is hereditary, mod-m-pure and CL-shellable. Then $\widetilde{\beta}_i^{\geq j} = 0$, unless $i + j \equiv d \pmod{m}$.*

PROOF: Let $0 \leq j \leq d$. As in the proof of Theorem 3.8 we conclude that $L_{\mathcal{A}}^{\geq j}$ is CL-shellable and mod-m-pure. Furthermore, since $L_{\mathcal{A}}$ is hereditary and $\dim \mathcal{A} = d$ there is in $L_{\mathcal{A}}^{\geq j}$ a maximal chain $x_j > x_{j+1} > \cdots > x_d$ with $\dim(x_i) = i$ for all $j \leq i \leq d$. Hence, all maximal chains of $L_{\mathcal{A}}^{\geq j}$ have lengths congruent to $d - j$ (mod m), and by [BW1, Theorem 5.9] $\widetilde{\beta}_i^{\geq j} \neq 0$ is possible only if $i \equiv d - j \pmod{m}$. □

The intersection lattices of $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$ satisfy these conditions with $m = k - 2$. The nontrivial part here is the CL-shellability, which was shown in [BW1] and [BSag] respectively.

The material of this section has parallels in the affine case. The results come out in essentially the same way, and we will not repeat the arguments.

References

[At] C. A. Athanasiadis, *Characteristic polynomials of subspace arrangements and finite fields*, Advances in Mathematics **122** (1996), 193–233.

[BBD] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, in "Analyse et topologie sur les espaces singuliers", Astérisque **100**, Soc. Math. de France, 1982.

[Bj] A. Björner, *Subspace arrangements*, in "First European Congress of Mathematics, Paris 1992" (eds. A. Joseph et al), Progress in Math. **119**, Birkhäuser, 1994, pp. 321–370.

[BL] A. Björner and L. Lovász, *Linear decision trees, subspace arrangements and Möbius functions*, Journal Amer. Math. Soc. **7** (1994), 677–706.

[BSag] A. Björner and B. Sagan, *Subspace arrangements of type B_n and D_n*, J. Algebraic Combinatorics **5** (1996), 291–314.

[BSar] A. Björner and K. S. Sarkaria, *The zeta function of a simplicial complex*, Israel J. Math., to appear. (Preprint, 1995)

[BW1] A. Björner and M. Wachs, *Shellable nonpure complexes and posets I*, Trans. Amer. Math. Soc. **348** (1996), 1299–1327.

[BW2] A. Björner and M. Wachs, *Shellable nonpure complexes and posets II*, Trans. Amer. Math. Soc., to appear. (Preprint 1994)

[BWe] A. Björner and V. Welker, *The homology of "k-equal" manifolds and related partition lattices*, Advances in Math. **110** (1995), 277–313.

[BlS] A. Blass and B. E. Sagan, *Characteristic and Ehrhart polynomials*, preprint, 1995.

[CR] H. H. Crapo and G.-C. Rota, *Combinatorial Geometries*, MIT Press, 1970.

[D1] P. Deligne, *La conjecture de Weil. I*, Publ. Math. IHES **43** (1974), 273–307.

[D1a] P. Deligne, *La conjecture de Weil. II*, Publ. Math. IHES **52** (1980), 137–252.

[D2] P. Deligne, *Theorie de Hodge. II*, Publ. Math. IHES **40** (1971), 5–58.

[D2a] P. Deligne, *Theorie de Hodge. III*, Publ. Math. IHES **44** (1974), 5–78.

[D3] P. Deligne, *Poids dans la cohomologie des variétés algébriques*, in "Proc. Intern. Congress Math., Vancouver 1974" Vol. 1, 1975, pp. 79–85.

[D4] P. Deligne, *SGA 4 1/2 – Cohomologie étale*, Lecture Notes in Math. **569**, Springer-Verlag, 1977.

[Dw] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631–648.

[Ei] S. Eilenberg, *Automata, Languages and Machines, Vol. A*, Academic Press, New York, 1974.

[FK] E. Freitag and R. Kiehl, *Étale cohomology and the Weil conjecture*, Ergebnisse Series, Band 13, Springer-Verlag, 1988.

[GM] M. Goresky and R. D. MacPherson, *Stratified Morse Theory*, Ergebnisse Series, Band 14, Springer-Verlag, 1988.

[G4] A. Grothendieck, *SGA 4*, Lecture Notes in Math. **269**, Springer-Verlag, 1972.

[G4a] A. Grothendieck, *SGA 4, Tome 3*, Lecture Notes in Math. **305**, Springer-Verlag, 1973.

[G5] A. Grothendieck, *SGA 5*, Lecture Notes in Math. **589**, Springer-Verlag, 1977.

[Ki] M. Kim, *Weights in cohomology groups arising from hyperplane arrangements*, Proc. Amer. Math. Soc. **120** (1994), 697–703.

[Ko] D. N. Kozlov, *General lexicographic shellability and orbit arrangements*, preprint, 1996.

[LW] S. Lang and A. Weil, *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827.

[Le] G. I. Lehrer, *The l-adic cohomology of hyperplane complements*, Bull. London Math. Soc. **24** (1992), 76–82.

[Mi] J. S. Milne, *Étale cohomology*, Princeton Univ. Press, 1980.

[OT] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, 1992.

[Re] C. Reutenauer, *$\mathbb{N}$-rationality of zeta functions*, preprint, 1995.

[JER] Jan-Erik Roos, *Sur les foncteurs dérivés de $\varprojlim$*, Comptes Rendus Acad. Sci. Paris **252** (1961), 3702–3704.

[So] M. Soittola, *Positive rational sequences*, Theoretical Computer Science **2** (1976), 317–322.

[S1] R. P. Stanley, *Combinatorics and Commutative Algebra*, Second Edition, Birkhäuser, Boston, 1996.

[S2] R. P. Stanley, *Enumerative Combinatorics, Volume I*, Wadsworth, 1986.

[WZZ] V. Welker, G.M. Zielger and R.T. Živaljević, *Comparison lemmas and applications for diagrams of spaces*, Preprint No. 11, Institut für Experimentelle Mathematik, Essen, 1995.

[Ya] Z. Yan, *An étale analog of Goresky-MacPherson formula for subspace arrangements*, preprint, 1995.

[Za] T. Zaslavsky, *The Möbius function and the characteristic polynomial*, in "Combinatorial Geometries" (ed. N. White), Cambridge Univ. Press, 1987, pp. 114-138.

[ZZ] G. M. Ziegler and R. T. Živaljević, *Homotopy types of subspace arrangements via diagrams of spaces*, Math. Annalen **295** (1993), 527–548.

Anders Björner, Department of Mathematics, Kungl. Tekniska Högskolan, S-100 44 Stockholm, Sweden

E-mail address: `bjorner@math.kth.se`

Torsten Ekedahl, Department of Mathematics, Stockholm University, S-106 91 Stockholm, Sweden

E-mail address: `teke@matematik.su.se`