THE NUMBER OF k-FACES OF A SIMPLE d-POLYTOPE

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ABSTRACT. Consider the question: Given integers $k < d < n$, does there exist a simple d-polytope with n faces of dimension k? We show that there exist numbers $G(d,k)$ and $N(d,k)$ such that for $n > N(d,k)$ the answer is yes if and only if $n \equiv 0 \pmod{G(d,k)}$. Furthermore, a formula for $G(d,k)$ is given, showing that e.g. $G(d,k) = 1$ if $k \geq \lfloor \frac{d+1}{2} \rfloor$ or if both d and k are even, and also in some other cases (meaning that all numbers beyond $N(d,k)$ occur as the number of k-faces of some simple d-polytope).

This question has previously been studied only for the case of vertices ($k = 0$), where Lee [Le] proved the existence of $N(d,0)$ (with $G(d,0) = 1$ or 2 depending on whether d is even or odd), and Prabhu [P2] showed that $N(d,0) \leq cd\sqrt{d}$. We show here that asymptotically the true value of Prabhu's constant is $c = \sqrt{2}$ if d is even, and $c = 1$ if d is odd.

1. INTRODUCTION

An integer n will be called (d,k)-*realizable* if there is a simple d-polytope with n faces of dimension k. For terminology and basic properties of polytopes we refer to the literature, see e.g. [Z].

We show, see Theorem 7, that there exist numbers $G(d,k)$ and $N(d,k)$ such that

- if n is (d,k)-realizable then $G(d,k)$ divides n;
- if $G(d,k)$ divides n and $n > N(d,k)$ then n is (d,k)-realizable.

The $G(d,k)$-divisible numbers that are not (d,k)-realizable will be called (d,k)-*gaps*. Thus there are only finitely many gaps for all $d > k \geq 0$. In this paper we study the numbers $G(d,k)$ and $N(d,k)$. Our proofs rely on the g-theorem.

To give some feeling for the results, let us discuss a few special cases. The parity restrictions that exist for each dimension k are easiest to understand for the case of vertices ($k = 0$). Namely, the graph of a simple d-polytope is d-regular, so if the polytope has n vertices then it has $\frac{dn}{2}$ edges. Hence, if d is odd n must be even. This is in fact the only constraint, and we have

$$G(d,0) = \begin{cases} 1, & d \text{ even} \\ 2, & d \text{ odd.} \end{cases}$$

Both authors were supported by EC grant CHRX-CT93-0400 and by the Mathematical Sciences Research Institute (Berkeley, CA). Research at MSRI is supported in part by NSF grant DMS-9022140.

This result is due to Lee [Le], who initiated the study of properties of vertex-count numbers of simple polytopes. Via the regular graph property this also implies the result for edge-count numbers:

$$G(d,1) = \begin{cases} \frac{d}{2}, & d \text{ even} \\ d, & d \text{ odd.} \end{cases}$$

For $1 < k < \lfloor \frac{d+1}{2} \rfloor$ the situation gets more complicated and the answer is different for k even and k odd. For instance,

$$G(d,2) = \begin{cases} 2, & d \equiv 1 \pmod 4 \\ 1, & \text{otherwise.} \end{cases}$$

The modulus $G(d,k)$ can get arbitrarily large in this range; for instance, $G(d,k) = d-k+1$ whenever k is odd and $d-k+1$ is a prime. Then, for $k \geq \lfloor \frac{d+1}{2} \rfloor$ the situation simplifies again to $G(d,k) = 1$. Theorem 2 gives the general formula for $G(d,k)$.

It is also of interest to study the magnitude of the numbers $N(d,k)$ (defined as the smallest possible ones for which the above statement is true). Again, this has been studied for the case of vertices by Prabhu [P2], who showed that $N(d,0) \leq cd\sqrt{d}$. We prove that asymptotically the true value of Prabhu's constant is $c = \sqrt{2}$ if d is even, and $c = 1$ if d is odd, see Section 5. We also give an upper bound for $N(d,k)$ in the general case, Theorems 10 and 11, but leave open the determination of its true asymptotic growth.

2. Preliminaries

Given a d-dimensional polytope P, we will call $\mathbf{f} := (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of P, where f_i is the number of faces of dimension i.

For any integers $n, s \geq 1$, there is a unique way of writing

$$n = \binom{a_s}{s} + \binom{a_{s-1}}{s-1} + \cdots + \binom{a_i}{i},$$

so that $a_s > a_{s-1} > \cdots > a_i \geq i \geq 1$. Then define:

$$\partial^s(n) := \binom{a_s - 1}{s-1} + \binom{a_{s-1} - 1}{s-2} + \cdots + \binom{a_i - 1}{i}.$$

Also let $\partial^s(0) := 0$.

A nonnegative integer sequence $(n_0, n_1, n_2, \ldots)$ is called an *M-sequence* if

$$n_0 = 1 \quad \text{and} \quad \partial^s(n_s) \leq n_{s-1} \quad \text{for all } s \geq 1.$$

Two simple facts we will need about M-sequences is that if there is a zero in the sequence then all the following entries are also zeros, and that any sequence satisfying $n_0 = 1$ and $n_1 \geq n_2 \geq n_3 \geq \ldots$ is an M-sequence.

An alternative definition of M-sequence, due to Macaulay and Stanley [S1], says that a sequence is a M-sequence if and only if it is the f-vector of a multicomplex.

See [Li] and [Z] for examples of other interpretations of M-sequences. Let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the largest integer less than or equal to x and the smallest integer larger than or equal to x, respectively.

Let $\delta := \lfloor \frac{d}{2} \rfloor$ and let $M_d = (m_{ik})$ be the $(\delta+1) \times d$-matrix with entries

$$m_{ik} = \binom{d+1-i}{k+1} - \binom{i}{k+1}, \quad \text{for } 0 \le i \le \delta,\ 0 \le k \le d-1.$$

For example,

$$M_{10} = \begin{pmatrix} 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 & 11 \\ 9 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \\ 7 & 35 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0 \\ 5 & 25 & 55 & 70 & 56 & 28 & 8 & 1 & 0 & 0 \\ 3 & 15 & 31 & 34 & 21 & 7 & 1 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Our proofs will rely on the g-theorem, conjectured by McMullen. Sufficiency was proved by Billera and Lee [BL], and necessity by Stanley [S2] and later by McMullen [M], see [Z]. We will use the following matrix reformulation of the g-theorem, given by Björner [B1, B2], see also [Z]. We have here reformulated the statement from simplicial polytopes to simple polytopes, which just correspond to reading the f-vector backwards.

Theorem 1. **[The g-theorem]** *The matrix equation*

$$\mathbf{f} = \mathbf{g} \cdot M_d$$

gives a one-to-one correspondence between f-vectors $\mathbf{f}$ of simple d-polytopes and M-sequences $\mathbf{g} = (g_0, g_1, \ldots, g_\delta)$. □

3. The modulus $G(d,k)$

The modulus mentioned in the introduction is defined as follows

$$G(d,k) := \gcd(m_{1,k}, m_{2,k}, \ldots, m_{\delta,k}), \tag{1}$$

the greatest common divisor for the elements in the k-th column and below the top row of the matrix M_d. In this section we will give simple and explicit formulas for $G(d,k)$. The role of $G(d,k)$ as the period for the possible numbers of k-faces of d-polytopes will be shown in the next section.

Theorem 2. (i) *If $k \ge \lfloor \frac{d+1}{2} \rfloor$, then $G(d,k) = 1$.*

(ii) *If $k < \lfloor \frac{d+1}{2} \rfloor$ is even, let e be the integer such that $2^e \le k+1 < 2^{e+1}$. Then*

$$G(d,k) = \begin{cases} 2, & \text{if } d-k+1 \equiv 0 \pmod{2^{e+1}} \\ 1, & \text{otherwise.} \end{cases}$$

(iii) *If* $k < \left\lfloor \frac{d+1}{2} \right\rfloor$ *is odd, let* $p_1, \ldots, p_t$ *be the primes smaller than or equal to* $k+1$*, and let* $e_i \geq 1$ *be the integers such that* $p_i^{e_i} \leq k+1 < p_i^{e_i+1}$*. Then*

$$G(d,k) = \frac{d-k+1}{\gcd(d-k+1, p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t})}.$$

For the proof we will need some facts about binomial coefficients modulo powers of a prime, that will be developed in a sequence of lemmas.

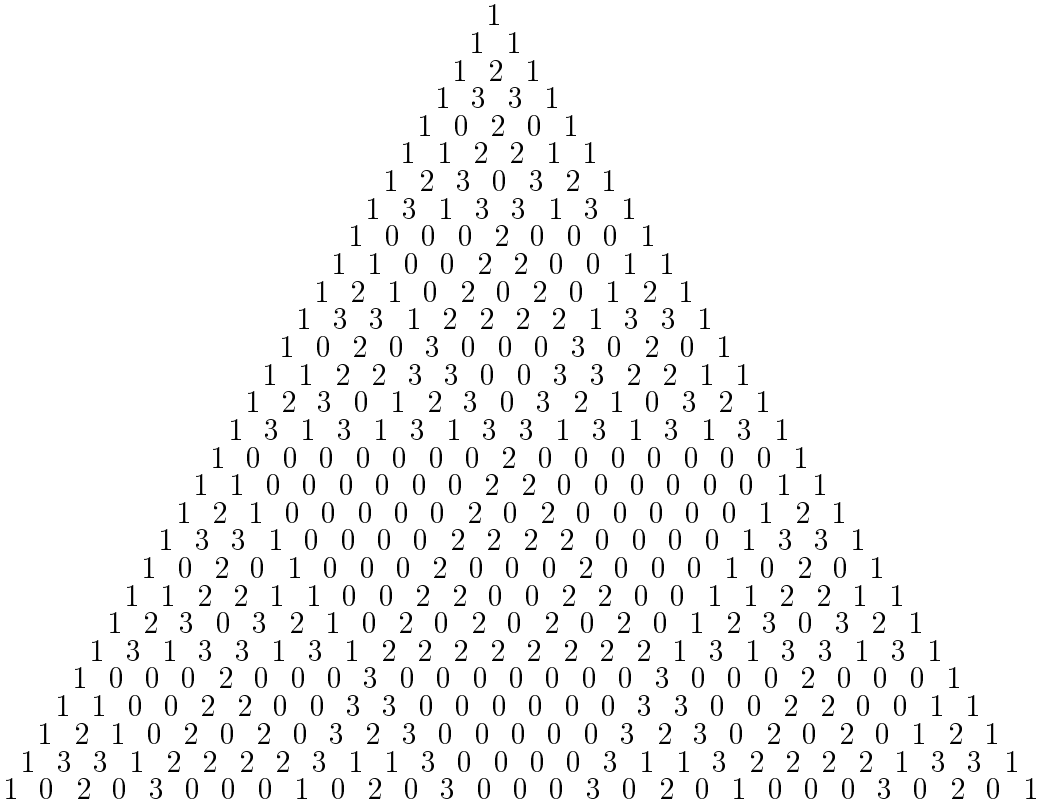
Lemma 3. *Let* $k, e \geq 0$ *and let* p *be a prime such that* $p^e \leq k+1 < p^{e+1}$*. Then for all* $r \geq 1$ *we have*

(i) $d \equiv d' \pmod{p^{e+r}}$ *implies* $\binom{d}{k+1} \equiv \binom{d'}{k+1} \pmod{p^r}$;
(ii)

$$\binom{p^{e+r}+k-i}{k+1} \equiv (-1)^{k+1} \binom{i}{k+1} \pmod{p^r},$$

for all $i = 0, 1, \ldots, p^{e+r}+k$*;*
(iii) *the unique longest run of zeros in the period is* $\binom{d}{k+1} \equiv_{p^r} 0$ *for all* $p^{e+r} \leq d \leq p^{e+r}+k$*.*

Part (*ii*) shows that if k is odd then the period extended by k is symmetric (mod p^r), and if k is even then it is antisymmetric. The lemma is illustrated by the following modular Pascal triangle.



Pascal's triangle (mod 4).

For each prime p define the valuation $v_p : \mathbb{Z} \setminus \{0\} \to \mathbb{N}$ by $v_p(n) = s$, where p^s is the highest power of p that is a divisor to n. We will frequently use that

$$v_p(n+m) = v_p(n) \quad \text{if } v_p(n) < v_p(m); \tag{2}$$

in particular, $v_p(n + p^x) = v_p(n)$ if $|n| < p^x$.

Lemma 4. *Let k, e and p be as in Lemma 3. Then*

$$v_p\left[(k+1)\binom{k}{j}\right] \leq e, \quad \textit{for all } 0 \leq j \leq k.$$

Proof. The proof hinges on the following fact: *Among all products of $x \leq p^e(p-1)$ consecutive integers in the interval $1, 2, \ldots, p^{e+1}-1$ the maximum valuation is attained by the string that starts with p^e.* To show this, assume that $r, r+1, \ldots, r+x-1$ is such a string of integers. If $r > p^e$, say $ap^e < r \leq (a+1)p^e$, then the string beginning with $r - ap^e$ has the same valuation. Thus we may assume that $r \leq p^e$.

If $r < p^e$, let s be the least number such that $r < s \leq p^e$ and $v_p(r) < v_p(s)$. Then it is easy to see that $v_p(r(r+1)\cdots(r+x-1)) \leq v_p(s(s+1)\cdots(s+x-1)$, and the claim follows.

We may assume that $j \leq \frac{k}{2}$. Then $j+1 \leq p^e(p-1)$, and what was just shown implies that $v_p((k-j+1)(k-j+2)\cdots(k+1)) \leq v_p(p^e(p^e+1)\cdots(p^e+j)) = e+v_p(j!)$, which is equivalent to the stated formula. □

Proof of Lemma 3 We know from Pascal's triangle that $\binom{d}{k+1} \pmod{p^r}$ is completely determined by the values of $\binom{j}{0} = 1, j \geq 0$, and $\binom{d'+i}{i} \pmod{p^r}$ for any $d' \geq 0$ and all $i = 1, \ldots, k+1$. Therefore it suffices to show that

$$\binom{p^{e+r}-1+i}{i} \equiv_{p^r} 0, \tag{3}$$

for all $i = 1, \ldots, p^{e+1}-1$, to establish the first part of the lemma. We have that $v_p(p^{e+r}+s) = v_p(s)$ for all $s = 1, \ldots, p^{e+r}-1$. So the expansion of the binomial coefficient

$$\binom{p^{e+r}-1+i}{i} = \frac{(p^{e+r}+i-1)(p^{e+r}+i-2)\cdots p^{e+r}}{i(i-1)(i-2)\cdots 2\cdot 1}$$

gives $v_p(\binom{p^{e+r}-1+i}{i}) = e+r-v_p(i) \geq r$, if $i < p^{e+1}$, which proves (3).

The second part of the lemma is obvious when $0 \leq i \leq k$, since both sides are zero (for the left-hand side this follows from (i)). Therefore, assume that $k < i < p^{e+r}$. For each $j \neq 0$ write $j = p^{\min\{v_p(j),e\}}q_j$. Then, for $0 < j < p^{e+r}$:

$$q_{p^{e+r}-j} \equiv_{p^r} q_{-j} = -q_j. \tag{4}$$

We have the equality

$$\binom{i}{k+1}\frac{(k+1)!}{p^{v_p((k+1)!)}} = \frac{i(i-1)\cdots(i-k)}{p^{v_p((k+1)!)}} =$$

$$p^{\sum_{j=0}^{k}\min\{v_p(i-j),e\}-v_p((k+1)!)}\prod_{j=0}^{k}q_{i-j} = p^{\alpha}\prod_{j=0}^{k}q_{i-j},$$

and similarly

$$\binom{p^{e+r}+k-i}{k+1}\frac{(k+1)!}{p^{v_p((k+1)!)}} = \frac{(p^{e+r}-i)(p^{e+r}-i+1)\cdots(p^{e+r}-i+k)}{p^{v_p((k+1)!)}} =$$

$$p^{\sum_{j=0}^{k}\min\{v_p(p^{e+r}-(i-j)),e\}-v_p((k+1)!)}\prod_{j=0}^{k}q_{p^{e+r}-(i-j)} = p^{\beta}\prod_{j=0}^{k}q_{p^{e+r}-(i-j)}.$$

We claim that

$$\beta = \alpha \geq 0.$$

The equality follows from (2), and the inequality will soon be proved. The two identities therefore give, using (4):

$$\binom{i}{k+1}\frac{(k+1)!}{p^{v_p((k+1)!)}} \equiv_{p^r} (-1)^{k+1}\binom{p^{e+r}+k-i}{k+1}\frac{(k+1)!}{p^{v_p((k+1)!)}}.$$

Since $\frac{(k+1)!}{p^{v_p((k+1)!)}}$ is invertible in $\mathbb{Z}_{p^r}$ this implies (ii).

It remains to show that $\alpha \geq 0$. If $v_p(i-j) \leq e$ for all $j = 0, \ldots, k$, then $\alpha = v_p(\binom{i}{k+1}) \geq 0$. If not, then since $k+1 < p^{e+1}$ there is exactly one s, with $i-k \leq s \leq i$, such that $v_p(s) > e$. In that case we have

$$\alpha + v_p((k+1)!) = v_p((i-k)\cdots s \cdots i) - v_p(s) + e =$$

$$v_p((i-k)\cdots(s-1)) + v_p((s+1)\cdots i) + e = v_p((s-(i-k))!) + v_p((i-s)!) + e,$$

where the last equality uses (2) twice. Thus, using Lemma 4,

$$\alpha = -v_p((k-(i-s)+1)(k-(i-s)+2)\cdots(k+1)) + v_p((i-s)!) + e =$$

$$-v_p\left((k+1)\binom{k}{i-s}\right) + e \geq 0.$$

To prove (iii) assume that $\binom{d+i}{k+1} \equiv_{p^r} 0$, for some $d \geq k+1$ and all $i = 0, \ldots, k$. Then $\binom{d}{j} \equiv_{p^r} 0$, for $j = 1, \ldots, k+1$. Especially, $\binom{d}{p^s} \equiv_{p^r} 0$ for $0 \leq s \leq e$, which gives that $v_p(\binom{d}{p^s}) \geq r$. In particular, $v_p(d) \geq r$. We will now show that $v_p(d) \geq r+s$ for all $0 \leq s \leq e$, by induction on s. Assume that $v_p(d) \geq r+s-1$. Then $r \leq v_p(\binom{d}{p^s}) = v_p(d(d-1)\cdots(d-(p^s-1))) - v_p(1\cdot 2\cdots(p^s-1)p^s) = v_p(d) + v_p((p^s-1)!) - v_p((p^s-1)!) - v_p(p^s) = v_p(d) - s$. Hence, a run of $k+1$ consecutive zeros must begin with $\binom{d}{k+1}$ for some d divisible by p^{e+r}. On the other hand, the 1's along the left boundary of Pascal's triangle show that there cannot be a run of more than $k+1$ zeros of the form $\binom{i}{k+1}$. This proves the lemma. □

We can now proceed toward the proof of Theorem 2.

Lemma 5. *For each $k < \lfloor \frac{d+1}{2} \rfloor$, $G(d,k)$ is a divisor of $\frac{d-k+1}{\gcd(d-k+1, p_1^{e_1}p_2^{e_2}\cdots p_t^{e_t})}$, where $p_1, \ldots, p_t$ are the primes $\leq k+1$ and $p_i^{e_i} \leq k+1 < p_i^{e_i+1}$.*

Proof. Take a prime p dividing $G(d,k)$ and let $x := v_p(G(d,k)) \geq 1$. Write $k+1$ in base p, $k+1 = k_0 + k_1 p + \cdots + k_e p^e$, where $0 \leq k_i < p$ and $k_e \neq 0$. Notice that

$$v_p\left(\frac{d-k+1}{\gcd(d-k+1, p_1^{e_1}p_2^{e_2}\cdots p_t^{e_t})}\right) = \begin{cases} v_p(d-k+1) - e & \text{if } v_p(d-k+1) \geq e, \\ 0 & \text{otherwise,} \end{cases}$$

so it will suffice to show that $v_p(d-k+1) - e \geq x$ in the first case and obtain a contradiction in the second.

Since $p^x | G(d,k)$ we get that $\binom{d+1-i}{k+1} \equiv_{p^x} \binom{i}{k+1}$, for all $i = 1, \ldots, \delta$. Especially we must have $\binom{d+1-i}{k+1} \equiv_{p^x} 0$, for $i = 1, \ldots, k$ and $\binom{d-k}{k+1} \equiv_{p^x} 1$. From

$$(d-2k)\binom{d+1-k}{k+1} = \binom{d-k}{k+1}(d-k+1),$$

we get $v_p(d-k+1) - v_p(d-2k) = v_p(\binom{d+1-k}{k+1}) \geq x \geq 1$. Hence by (2), $v_p(k+1) = v_p(d-k+1-(d-2k)) = v_p(d-2k) < v_p(d-k+1)$.

There are now two cases: First assume that $v_p(d-k+1) \geq e$. If $k+1 = k_e p^e$ we are done, since we have shown that $v_p(d-k+1) - v_p(k+1) \geq x$. Assume that $k+1 > k_e p^e$. From $d-k+1 \geq k+1 > k_e p^e$ we conclude that $v_p(d-k+1-k_e p^e) \geq e$, which implies $v_p(d-k+1-k_e p^e - i) = v_p(i)$, for all $i = 1, \ldots, k+1-k_e p^e < p^e$. This in turn implies that

$$v_p\left(\frac{(d-k-k_e p^e)!}{(d-2k)!}\right) = v_p((k-k_e p^e)!) = v_p\left(\frac{(d+1-k_e p^e)!}{(d+1-k)!}\right).$$

Using the equality

$$\frac{(d-k-k_e p^e)!}{(d-2k)!}\binom{d+1-k_e p^e}{k+1} = \binom{d+1-k}{k+1}\frac{(d+1-k_e p^e)!}{(d+1-k)!},$$

we get that $v_p(\binom{d+1-k_e p^e}{k+1}) = v_p(\binom{d+1-k}{k+1})$. This together with the identity

$$(d-k+1-k_e p^e)\binom{d+2-k_e p^e}{k+1} = \binom{d+1-k_e p^e}{k+1}(d+2-k_e p^e)$$

gives

$$x \leq v_p(m_{k_e p^e - 1, k}) = v_p\left(\binom{d+2-k_e p^e}{k+1}\right) \leq$$

$$v_p\left(\binom{d+1-k_e p^e}{k+1}\right) - e + v_p(d+2-k_e p^e) =$$

$$v_p\left(\binom{d-k+1}{k+1}\right) - e + v_p(k+1-k_e p^e) = v_p(d-k+1) - e.$$

Here the last equality comes from $v_p(k+1-k_e p^e) = v_p(k+1) = v_p(d-2k)$ and $v_p(\binom{d-k+1}{k+1}) = v_p(d-k+1) - v_p(d-2k)$, established above.

The second case is if $a := v_p(d-k+1) < e$. The same argument can be applied again; however, now replacing $k_e p^e$ everywhere by $k_a p^a + \cdots + k_e p^e$ and replacing e by a. We then get $x \leq v_p(d-k+1) - a = 0$, a contradiction. □

Lemma 6. *$G(d,k)$ is a divisor to $m_{0,k} = \binom{d+1}{k+1}$.*

Proof. If for a prime p we have that p^r divides $G(d,k)$ and $p^e \leq k+1 < p^{e+1}$, then Lemma 5 implies that p^{r+e} divides $d-k+1$. Hence $v_p(\binom{d+1}{k+1}) = v_p(\binom{d+1}{k}) + v_p(d-k+1) - v_p(k+1) \geq r$. □

Proof of Theorem 2. The first statement follows from the fact that $m_{d-k,k} = 1$ for $k \geq \left\lfloor \frac{d+1}{2} \right\rfloor$.

Let $k < \left\lfloor \frac{d+1}{2} \right\rfloor$. We have from the definition of $m_{i,k}$ that for every prime p and every $r \geq 1$:

$$p^r | G(d,k) \Longleftrightarrow \binom{d+1-i}{k+1} \equiv_{p^r} \binom{i}{k+1}, \text{for } i = 0, 1, \ldots, \left\lfloor \tfrac{d+1}{2} \right\rfloor. \tag{5}$$

Actually, the definition supports this only for $i = 1, \ldots, \delta = \left\lfloor \frac{d}{2} \right\rfloor$ on the right hand side, but $i = 0$ can be added because of Lemma 6 and $i = \left\lfloor \frac{d+1}{2} \right\rfloor$ (for d odd) gives a trivially true identity.

Case 1: k even. Assume that $G(d,k) \neq 1$, and that $p^r | G(d,k)$. Since by Lemma 3 there is a unique longest run of $k+1$ zeros in the period of $\binom{i}{k+1} \pmod{p^{e+r}}$ we get from (5) that $d-k+1 \equiv_{p^{e+r}} 0$. Therefore, Lemma 3 and (5) give

$$\binom{k+1}{k+1} \equiv_{p^r} \binom{d-k}{k+1} \equiv_{p^r} \binom{p^{e+r}-1}{k+1} \equiv_{p^r} -\binom{k+1}{k+1},$$

which implies $p = 2$ and $r = 1$. Hence, $G(d,k) = 2$, and this happens only if $d-k+1 \equiv_{2^{e+1}} 0$. On the other hand, if $d-k+1 \equiv_{2^{e+1}} 0$ then $2 | G(d,k)$ can be concluded from Lemma 3 and (5).

Case 2: k odd. Let p^r be a divisor of $\frac{d-k+1}{\gcd(d-k+1, p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t})}$. By Lemma 5 it suffices to show that $p^r | G(d,k)$. The assumption implies that p^{r+e} divides $d-k+1$, where as usual e is defined by $p^e \leq k+1 < p^{e+1}$. Hence by Lemma 3

$$\binom{d+1-i}{k+1} \equiv_{p^r} \binom{i}{k+1}, \qquad \text{for all } i = 0, \ldots, d+1,$$

which via (5) shows that $p^r | G(d,k)$.

This finishes the proof of the theorem. □

Example We want to calculate $G(116, 9)$. Since $k = 9$ is odd we calculate the greatest common divisor of $116 - 9 + 1 = 108$ and $2^3 \cdot 3^2 \cdot 5 \cdot 7$ which is 36. We get $G(116,9) = 118/36 = 3$. □

4. Periodicity of (d,k)-realizable numbers

We will now show the general theorem about the ultimately stable periodic distribution of the (d,k)-realizable numbers.

Theorem 7. *Fix $0 \leq k < d$, and let $G(d,k)$ be the number defined in equation (1). Then there exists an integer N such that for all $n > N$:*

n is the number of k-faces of a simple d-polytope

$$\Longleftrightarrow$$

$$n \equiv 0 \pmod{G(d,k)}$$

Proof. We will prove the theorem with the last statement replaced by $n \equiv m_{0,k} \pmod{G(d,k)}$. Lemma 6 shows that $m_{0,k}$ is divisible by $G(d,k)$, so this reformulation is equivalent.

$\Longrightarrow$ This direction is clear from Theorem 1.

$\Longleftarrow$ Write

$$G(d,k) = \sum_{i=1}^{\delta} \lambda_i m_{i,k}, \qquad \lambda_i \in \mathbb{Z}.$$

Suppose $m_{1,k} = C \cdot G(d,k)$. Define

$$g_\delta := \begin{cases} (C-1)|\lambda_\delta|, & \text{if } \lambda_\delta < 0 \\ 0, & \text{otherwise;} \end{cases}$$

and recursively

$$g_i := g_{i+1} + (C-1)(|\lambda_i| + |\lambda_{i+1}|), \qquad 0 < i < \delta.$$

Let $N := m_{0,k} + \sum_{i=1}^{\delta} g_i m_{i,k}$, and let $g_i^{(p)} = g_i + p\lambda_i$, for $p = 0, 1, \ldots$

Then $g^{(p,q)} = (1, g_1^{(p)} + q, g_2^{(p)}, \ldots, g_\delta^{(p)})$ is nonnegative and decreasing after the first entry for all $q \geq 0$ and all $0 \leq p < C$, and hence is an M-sequence. The f_k values corresponding to these g-vectors are

$$f_k^{(p,q)} = N + qCG(d,k) + pG(d,k).$$

It is clear from the construction that all numbers $N + j \cdot G(d,k), j = 0, 1, \ldots$ are of the form $f_k^{(p,q)}$ for suitable $q \geq 0$ and $0 \leq p < C$. □

Corollary 8. *If the $m_{i,k}$ are relatively prime, then all numbers from some point on are (d,k)-realizable. Furthermore, Theorem 2 shows that this happens precisely in the following cases:*

(i) *if $k \geq \left\lfloor \frac{d+1}{2} \right\rfloor$;*
(ii) *if $k < \left\lfloor \frac{d+1}{2} \right\rfloor$ is even, unless $d - k + 1 \equiv 0 \pmod{2^{e+1}}$;*
(iii) *if $k < \left\lfloor \frac{d+1}{2} \right\rfloor$ is odd, unless $d - k + 1$ fails to divide $p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$.* □

Now, define $N(d,k)$ to be the least number N for which Theorem 7 is true. Note that $N(d, d-1) = d$, so in what follows we may assume that $k < d - 1$.

What can be said about the magnitude of $N(d,k)$? We will here give a general upper bound, and then we will determine the exact asymptotic growth for the special case $N(d,0)$ in the following section.

Define

$$L(d,k) := \min \max_{i=1}^{\delta} |\lambda_i|,$$

with the minimum taken over all ways to represent $G(d,k)$ on the form

$$G(d,k) = \sum_{i=1}^{\delta} \lambda_i m_{i,k}, \qquad \lambda_i \in \mathbb{Z}.$$

Lemma 9. *For all $0 \le k \le d-2$, we have $L(d,k) < m_{1,k}$.*

Proof. Assume $G(d,k) = \sum_{i=1}^{\delta} \lambda_i m_{i,k}$, with $|\lambda_s| \ge m_{1,k}$ and $m_{s,k} \neq 0$. By symmetry we may assume that λ_s is positive, that is $\lambda_s \ge m_{1,k}$. Since $G(d,k) < m_{1,k} m_{s,k}$, there has to be a t such that $\lambda_t < 0$ and $m_{t,k} \neq 0$. Let

$$\lambda_i' = \begin{cases} \lambda_s - m_{t,k} & \text{if } i = s, \\ \lambda_t + m_{s,k} & \text{if } i = t, \\ \lambda_i & \text{otherwise.} \end{cases}$$

We get $G(d,k) = \sum_{i=1}^{\delta} \lambda_i' m_{i,k}$. Since $|\lambda_s'| < |\lambda_s|$, $|\lambda_t'| < |\lambda_t|$ or else $|\lambda_t'| < m_{s,k}$, and all other $|\lambda_i'|$ are unchanged, we can continue this process until $|\lambda_i| < m_{1,k}$, for all i. □

Theorem 10. $N(d,k) < \frac{1}{2} d^2 \binom{d}{k+1}^3$

Proof. Referring to the proof of Theorem 7, with an optimal choice of the λ_i-s, we have

$$N(d,k) \le \sum_{i=0}^{\delta} g_i m_{i,k} \le \binom{d+1}{k+1} + L(d,k)(C-1) \sum_{i=1}^{\delta} (2\delta + 1 - 2i) \binom{d+1-i}{k+1}$$

$$\le \binom{d+1}{k+1} + L(d,k)(C-1)\delta(2\delta-1)\binom{d}{k+1} < 2L(d,k)\delta^2 \binom{d}{k+1}^2.$$

□

For example, let $k = 0$. The general bound specializes to $N(d,0) \le \frac{1}{2} d^5$. This should be compared to the true asymptotic value $N(d,0) \sim c d^{3/2}$, which will be proved in the next section.

For $k \ge \lfloor \frac{d+1}{2} \rfloor$ we can improve on the general bound significantly.

Theorem 11. *Suppose $k \ge \lfloor \frac{d+1}{2} \rfloor$. Then $N(d,k) < \binom{d+1}{d-k}(d-k)(k+1)(d+1)/2$.*

Since $G(d,k)=1$ for such k this implies that for every $n \geq \binom{d+1}{d-k}(d-k)(k+1)(d+1)/2$ there is a simple d-polytope with n faces of dimension k.

To prove this we need a more technical construction than before. First we extend the definition of ∂^s. Define for $p \leq s$:

$$\partial_p^s(n) := \binom{a_s - p}{s-p} + \binom{a_{s-1}-p}{s-1-p} + \cdots + \binom{a_i - p}{i-p},$$

where n is written in the unique expansion

$$n = \binom{a_s}{s} + \binom{a_{s-1}}{s-1} + \cdots + \binom{a_i}{i},$$

as in Section 2. Also let $\partial_p^s(0) := 0$. We will allow p to be negative, which corresponds to the natural "inverse" of ∂_p^s for positive p. Thus, for $p>0$, $\partial_{-p}^s(n)$ is the greatest number such that $\partial_p^s(\partial_{-p}^s(n)) = n$. We will continue to write just ∂^s for ∂_1^s.

Now, fix d and $k \geq \lfloor \frac{d+1}{2} \rfloor$. Define a vector $\mathbf{g} := (g_0, g_1, \ldots, g_{d-k})$ inductively as follows.

- Let $g_{d-k} := 0$.
- Assume we have defined $g_{d-k}, g_{d-k-1}, \ldots, g_i$, for some $0 < i \leq d-k$. Let

$$g_{i-1} := \partial^i(x_i), \qquad \text{where } x_i \text{ is the smallest integer such that } x_i \geq g_i \text{ and}$$

$$\sum_{s=i}^{d-k} \left(\partial_{i-s}^i(x_i) - g_s\right) m_{s,k} \geq m_{i-1,k} - 1. \tag{6}$$

This is an M-sequence by construction.

Lemma 12. *Given the g-vector above, define $N := \sum_{i=0}^{d-k} g_i m_{i,k}$. Then there are no (d,k)-gaps larger than or equal to N.*

Proof. Adding any positive integer to g_1 in an M-sequence gives another M-sequence. Thus we only have to prove that it is possible to form all the $m_{1,k}-1$ integers following N with legal g-vectors. This will imply the lemma.

We will think of the elements in column k of M_d as weights which we combine to get the correct total weight.

Consider first the choice of $g_{d-k-1} := \partial^{d-k}(x_{d-k})$, where $x_{d-k} = m_{d-k-1,k} - 1$. All the vectors $(g_0, g_1, \ldots, g_{d-k-1}, i)$, $i = 0, \ldots m_{d-k-1,k} - 1$ are M-sequences, producing $N, N+1, \ldots, N+m_{d-k-1,k}-1$ k-faces respectively. We here use the fact that $m_{d-k,k} = 1$. Similarly $(g_0, g_1, \ldots, g_{d-k-1} + j, i)$, for fixed j and $i = 0, \ldots m_{d-k-1,k} - 1$ gives $N + j m_{d-k-1,k}, N + j m_{d-k-1,k} + 1, \ldots, N + (j+1)m_{d-k-1,k} - 1$ k-faces. The definition of g_{d-k-2} allows us to have j sufficiently large to get all the numbers at least up to and including $N + m_{d-k-2,k} - 1$.

Assuming inductively that we can form the sequence $N, N+1, \ldots, N + m_{i,k} - 1$ by increasing only coordinates $i+1, \ldots, d-k$, the definition of $\mathbf{g}$ gives that we can

form all the numbers $N, N+1, \ldots, N+m_{i-1,k}-1$ by increasing only coordinates $i, \ldots, d-k$ of $\mathbf{g}$. This proves the lemma □

EXAMPLE Take $d=10$ and $k=6$. We see from the matrix M_{10}, displayed in Section 2, that the weights are $330, 120, 36, 8$ and 1. We get $\mathbf{g}=(1,4,6,6,0)$ which gives $N(10,6)<1074$, showing that every $n \geq 1074$ is $(10,6)$-realizable.

PROOF OF THEOREM 11 First we show that $g_i \leq (d-k-i)(k+1)$ by reverse induction. It is trivially true for g_{d-k}. Assume it is true for g_i. Since $g_{i-1}=\partial^i(x_i) \leq x_i$, it suffices to bound x_i. The inequality (6) is true if $(x_i-g_i)m_{i,k} \geq m_{i-1,k}-1$. Since x_i is chosen to be minimal we get that

$$x_i \leq g_i + \left\lceil \frac{m_{i-1,k}-1}{m_{i,k}} \right\rceil = g_i + \left\lceil \frac{\binom{d+2-i}{k+1}-1}{\binom{d+1-i}{k+1}} \right\rceil \leq$$

$$\leq g_i + \left\lceil \frac{d+2-i}{d+1-i-k} - \frac{1}{\binom{d+1-i}{k+1}} \right\rceil \leq g_i + k + 1 \leq (k+1)(d-k-i+1).$$

Now,

$$N \leq \binom{d+1}{k+1} + (k+1)\sum_{i=1}^{d-k}(d-k-i)\binom{d+1-i}{k+1} \leq$$

$$\binom{d+1}{k+1} + (k+1)\binom{d}{k+1}\binom{d-k}{2} \leq \binom{d+1}{k+1}\left(1+\frac{(k+1)(d-k-1)(d+1)}{2}\right).$$

This proves the theorem. □

Note that $m_{0,k}+1=\binom{d+1}{d-k}+1$ is never (d,k)- realizable for $k<d-1$. This gives a trivial lower bound for $N(d,k)$ to be compared with the upper bounds in Theorems 10 and 11.

5. The case of vertices

The only case of (d,k)-realizability that seems to have been previously studied is for $k=0$, i.e., the number of vertices. We will make a more exact analysis of that case.

In [Le, Corollary 4.4.15], Lee shows that for each dimension d all sufficiently large numbers are $(d,0)$-realizable (with parity restrictions, see the Introduction). Prabhu [P1, P2] strengthened the result and proved that there exists a constant c such that every $n > cd\sqrt{d}$ is $(d,0)$-realizable (with parity restrictions). This gives an upper bound on the size $N(d,0)$ of the largest gap in each dimension — we are not aware of any published non-trivial lower bound. The exact result is previously known only for small dimensions, see [Le] where Lee lists all $(d,0)$-gaps for $d \leq 9$.

We will sharpen Prabhu's result in both directions and prove that one can use $c = \sqrt{2} + \epsilon$ as constant in his theorem for any $\epsilon > 0$ and sufficiently large even d. However, the statement is not true for $c < \sqrt{2}$.

Theorem 13. *If $d \geq 4$ is even, then there does not exist a simple d-polytope with $n = (d-1)(\lceil\sqrt{2d-4}\rceil - 2) + 4$ vertices.* □

Theorem 14. *For every even $d \geq 2$ and every $n > (d-1)(\sqrt{2d} + 2\sqrt{2\sqrt{2d}} + 5)$, there exists a simple d-polytope with n vertices.* □

If we restrict our attention to d odd, the true value for c is bounded by $1 \leq c \leq 1+\epsilon$.

Theorem 15. *If $d \geq 3$ is odd, then there does not exist a simple d-polytope with $n = (d-1)(\lceil\sqrt{d-2}\rceil - 2) + 4$ vertices, but for every even integer $n > (d-1)(\sqrt{d} + 2\sqrt{2\sqrt{2d}} + 5)$, there exists a simple d-polytope with n vertices.* □

Proof of Theorem 13 Let $d = 2\delta \geq 6$, so the first column of M_d will be $2\delta+1, 2\delta-1, \ldots, 3, 1$. We will look for the lowest possible value for f_0 such that $f_0 \equiv 4 \pmod{2\delta-1}$. The entries of the first column will be the weights by which we seek to create the value of f_0. They are $2, 0, -2, -4, \ldots, -(2\delta-2) \pmod{2\delta-1}$. By the properties of M-sequences we have to take at least $k+2$ weights to obtain $f_0 \equiv 4 \equiv -2\delta + 5 \pmod{2\delta-1}$, where

$$2 + \sum_{i=0}^{k} -2i \leq -(2\delta-1) - (2\delta-5),$$

corresponding to the M-sequence $\mathbf{g} = (1, 1, \ldots, 1, 1)$ with $k+2$ 1's. This is equivalent to

$$k(k+1) \geq 4\delta - 4.$$

Now, choose k such that $k \leq \sqrt{4\delta-4} < k+1$. We then get that

$$f_0 \geq 2\delta + 1 + \sum_{i=0}^{k} 2\delta - 1 - 2i = 2\delta + 1 + (2\delta - k - 1)(k+1) >$$

$$2\delta + 1 + (2\delta - 1)\left\lceil\sqrt{4\delta-4}\right\rceil - (\left\lfloor\sqrt{4\delta-4}\right\rfloor^2 + \left\lfloor\sqrt{4\delta-4}\right\rfloor) >$$

$$(d-1)(\left\lceil\sqrt{2d-4}\right\rceil - 2) + 4.$$

Hence, $(d-1)(\lceil\sqrt{2d-4}\rceil - 2) + 4$ is a gap.

The result is easily seen to be true also for $d = 4$. □

Proof of Theorem 14 Let $d = 2\delta \geq 4$ (the case $d = 2$ is easily checked). As above the first column of M_d will be $2\delta+1, 2\delta-1, \ldots, 3, 1$. First we note that if

$n+1, n+2, \ldots, n+d-1$ are all realizable then every integer larger then n is realizable since we can just add 1 to g_1 in the corresponding M-sequences.

As in the previous proof we let k_1 be such that $k_1 \leq \sqrt{4\delta - 4} < k_1 + 1$. We consider the M-sequences $1 = g_0 = g_1 = \cdots = g_i$ and $0 = g_{i+1} = g_{i+2} = \ldots$, for $0 \leq i \leq k_1 + 1$. The corresponding values for f_0 constitute one sequence of odd residues and one sequence of even residues modulo $d-1$, with no distance being larger than $2(k_1 - 1)$. Now we choose k_2 such that

$$\sum_{i=0}^{k_2} -2i \leq -(2k_1 - 1) \Longleftrightarrow k_2 + 1 > \sqrt{2(k_1 - 1)}.$$

It is clear that the M-sequences $1 = g_0, 2 = g_1 = \cdots = g_j$, $1 = g_{j+1} = g_{j+2} = \cdots = g_i$ and $0 = g_{i+1} = g_{i+2} = \ldots$, for $0 \leq j < i \leq k_1 + 1$ and $j \leq k_2$, give values for $f_0 \pmod{d-1}$ where no residue is more than $2(k_2 - 1)$ away from another residue of the same parity. Continuing this process, we choose integers $k_1, k_2, \ldots, k_s$, as small as possible such that $\sqrt{2(k_{i-1} - 1)} < k_i + 1$, for $2 \leq i \leq s$. We stop when we have reached $k_s = 1$. Hence, every possible value for $f_0 \pmod{d-1}$ can be obtained with an M-sequence that has coordinates satisfying $g_i \leq j$, whenever $k_{j+1} + 1 < i$.

So if f_0 is a gap then we must have

$$f_0 < 2\delta + 1 + \sum_{i=0}^{k_1}(2\delta - 1 - 2i) + \sum_{i=0}^{k_2}(2\delta - 1 - 2i) + \cdots + \sum_{i=0}^{k_s}(2\delta - 1 - 2i) <$$

$$2\delta + 1 + (k_1 + 1 + k_2 + 1 + \cdots + k_s + 1)(2\delta - 1) < [\text{by induction}]$$

$$2\delta + 1 + (k_1 + 1 + 2(k_2 + 1))(2\delta - 1) < (d-1)(\sqrt{2d-4} + 2\sqrt{2\sqrt{2d-4} - 2} + 5).$$

This estimate suffices to show the theorem. □

Proof of Theorem 15 The proof can be carried out in the same manner as the two previous proofs. □

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