FLOER HOMOLOGY OF BRIESKORN HOMOLOGY SPHERES : SOLUTION TO ATIYAH'S PROBLEM

NIKOLAI SAVELIEV

Abstract. In this paper we answer the question posed by M. Atiyah, see [12], and give an explicit formula for Floer homology of Brieskorn homology spheres in terms of their branching sets over the 3–sphere. We further show how Floer homology is related to other invariants of knots and 3–manifolds, among which are the $\bar{\mu}$–invariant of W. Neumann and L. Siebenmann and the Jones polynomial. Essential progress is made in proving the homology cobordism invariance of our own ν–invariant, see [15].

Let p, q, and r be pairwise coprime positive integers. The Brieskorn homology 3–sphere $\Sigma(p,q,r)$ is the link of the singularity of $f^{-1}(0)$, where $f : \mathbb{C}^3 \to \mathbb{C}$ is a map of the form $f(x,y,z) = x^p + y^q + z^r$. The complex conjugation in $\mathbb{C}^3$ acts on $\Sigma(p,q,r)$ turning it into a double branched cover of S^3 branched over a Montesinos knot $k(p,q,r)$. In this paper, we express the Floer instanton homology groups $I_n(\Sigma(p,q,r))$, see [5], in terms of the Casson's λ–invariant of $\Sigma(p,q,r)$ and the signature of the knot $k(p,q,r)$.

Theorem 1. *The Floer homology group $I_n(\Sigma(p,q,r))$ is trivial if n is odd, and is a free abelian group of the rank*

$$\operatorname{rank} I_n(\Sigma(p,q,r)) = \frac{1}{16}\left(8\lambda(\Sigma(p,q,r)) - (-1)^{n/2}\operatorname{sign} k(p,q,r)\right)$$

if n is even.

The groups $I_*(\Sigma(p,q,r))$ were studied by R. Fintushel and R. Stern in [6] (we use the Floer index convention of this paper). They gave an algorithm which for any given set of positive coprime integers p,q,r computes the ranks of $I_*(\Sigma(p,q,r))$. However, their approach does not give a closed form formula for $I_*(\Sigma(p,q,r))$. It should be mentioned though that the vanishing of the groups $I_n(\Sigma(p,q,r))$ for odd n follows from [6]; we give an alternative proof in Section 3.

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In [15] we introduced an invariant ν, which for Brieskorn homology spheres takes the form

$$\nu(\Sigma(p,q,r)) = \frac{1}{2}\sum_{n=0}^{3}(-1)^{n+1}\operatorname{rank} I_{2n}(\Sigma(p,q,r)). \tag{1}$$

Note that if the ranks in (1) were added without plus or minus signs one would get the Casson invariant $\lambda(\Sigma(p,q,r))$. We conjectured in [15] that the ν–invariant equals the $\bar\mu$–invariant of W. Neumann [13] and L. Siebenmann [17] for all Seifert fibered homology spheres. We prove that our conjecture is true. The first step is given by the following theorem.

Theorem 2. *For any Brieskorn homology 3–sphere $\Sigma(p,q,r)$, the ν–invariant defined by (1) and the $\bar\mu$–invariant coincide, $\nu(\Sigma(p,q,r)) = \bar\mu(\Sigma(p,q,r))$.*

It can be easily seen that Theorem 2 is in fact equivalent to Theorem 1 due to the following two observations: first, $\bar\mu(\Sigma(p,q,r)) = 1/8\cdot\operatorname{sign} k(p,q,r)$, see [17], and second, $I_n(\Sigma(p,q,r)) = I_{n+4}(\Sigma(p,q,r))$ for all n. Second step in proving our conjecture makes use of the so called splicing additivity proven for both the $\bar\mu$– and ν–invariants in [15]. Namely, $\bar\mu(\Sigma(a_1,\dots,a_n)) = \bar\mu(\Sigma(a_1,\dots,a_j,p))+\bar\mu(\Sigma(q,a_{j+1},\dots,a_n))$, where j is any integer between 2 and $n-2$, and the integers $q = a_1\cdots a_j$ and $p = a_{j+1}\cdots a_n$ are the products of the first j and the last $(n-j)$ Seifert invariants, respectively. The same additivity holds for the invariant ν.

Theorem 3. *For any Seifert fibered homology 3–sphere $\Sigma(a_1,\dots,a_n)$, the invariants ν and $\bar\mu$ coincide, $\nu(\Sigma(a_1,\dots,a_n)) = \bar\mu(\Sigma(a_1,\dots,a_n))$.*

Another reformulation of Theorem 1 establishes links between Floer homology and Jones polynomial. The result below follows from Theorem 1 and D. Mullins formula for the Casson invariant of two-fold branched covers [11].

Theorem 4. *Let $\Sigma(p,q,r)$ be a Brieskorn homology sphere then*

$$\operatorname{rank} I_0(\Sigma(p,q,r)) = -\frac{1}{12}\cdot\left.\frac{d}{dt}\right|_{t=-1}\ln V_k(t)$$

where V_k is the Jones polynomial of the Montesinos knot $k = k(p,q,r)$.

Theorem 3 has the following application to the homology 4–cobordism. In [15] and [16] we proved that $\bar\mu$ is a homology cobordism invariant for some classes of Seifert fibered homology 3–spheres. The corresponding results for the ν–invariant hold due to the identification $\bar\mu = \nu$ in Theorem 3.

Theorem 5. *(1) Let $\Sigma(a_1,\dots,a_n)$ be a Seifert fibered homology sphere homology cobordant to zero. Then $\nu(\Sigma(a_1,\dots,a_n)) \geq 0$.*

(2) Let $\Sigma = \Sigma(p,q,pqm\pm 1)$ be a surgery on a (p,q)–torus knot, and suppose that Σ is homology cobordant to zero. Then $\nu(\Sigma) = 0$.

(3) For all known Seifert fibered homology 3–spheres Σ which are known to be homology cobordant to zero including the lists of Casson-Harer [3] *and Stern* [18], $\nu(\Sigma) = 0$.

We will prove that $\nu(\Sigma(p,q,r)) = 1/8 \cdot \operatorname{sign} k(p,q,r)$, which is equivalent to proving Theorem 1. The idea is shortly as follows. In [19], C. Taubes proved that, for any homology 3–sphere Σ,

$$\lambda(\Sigma) = 1/2 \cdot \chi(I_*(\Sigma)). \tag{2}$$

The Casson's λ–invariant on the left is defined topologically using a Heegaard splitting of Σ and SU(2)–representation spaces. The number on the right is the Euler characteristic of Floer homology $I_*(\Sigma)$; it can be defined using SU(2) gauge theory as an infinite dimensional generalization of the classical Euler characteristic.

Let now $\Sigma = \Sigma(p,q,r)$ be endowed with the involution σ induced on $\Sigma \subset \mathbb{C}^3$ by the complex conjugation. We work out σ–invariant versions of both invariants in (2) in this particular situation. We use a σ–invariant Heegaard splitting of Σ and the corresponding σ–invariant SU(2)–representation spaces to define an invariant $\lambda^\rho(\Sigma)$ in a manner similar to that of A. Casson. On the other hand, a σ–invariant gauge theory produces a σ–invariant Euler characteristic which we denote $\chi^\rho(\Sigma)$. Note that the latter can be defined without actually working out any Floer homology. Then, a σ–invariant version of the Taubes result (2) is that

$$\lambda^\rho(\Sigma(p,q,r)) = 1/2 \cdot \chi^\rho(\Sigma(p,q,r)). \tag{3}$$

Our next step is to show that $\lambda^\rho(\Sigma(p,q,r)) = 1/8 \cdot \operatorname{sign} k(p,q,r)$. We achieve this by pushing σ–invariant representations down to the quotients. We do this for all the manifolds in a σ–invariant Heegaard splitting and note that the push-down representations are only defined on the complements of the fixed point sets, and they map all the meridians to trace-free matrices in SU(2). After that we are in position to identify our λ^ρ–invariant with the Casson–Lin invariant $1/8 \cdot \operatorname{sign} k(p,q,r)$ of [8]. The crucial observation is that *all* the representations of $\pi_1(\Sigma(p,q,r))$ are in fact σ–invariant, and the λ^ρ–invariant just counts them with signs different from those defined by Casson for his λ–invariant.

Finally, we identify $1/2 \cdot \chi^\rho(\Sigma)$ with $\nu(\Sigma)$ as follows. We choose a σ–invariant metric on Σ and an almost complex structure on $\Sigma \times \mathbb{R}$ so that the involution $\sigma \times 1$ on $\Sigma \times \mathbb{R}$ is anti-holomorphic. At the level of tangent spaces, the ± 1–eigenspaces of $\sigma \times 1$ are then isomorphic to each other and hence have the same dimension. Therefore, the spectral flow used in [19] to define $\chi(\Sigma)$ splits in halves, and the ν–invariant becomes an "honest" σ–invariant Euler characteristic equal to $1/2 \cdot \chi^\rho(\Sigma)$.

The paper begins with an introduction to the relevant topology of Brieskorn homology spheres and Montesinos knots. In Section 2 the invariant $\lambda^\rho(\Sigma(p,q,r))$ is introduced and the equality $\lambda^\rho(\Sigma(p,q,r)) = 1/8 \cdot \operatorname{sign} k(p,q,r)$ is proven. The invariant

$\chi^\rho(\Sigma(p,q,r))$ is defined in Section 3; the equality $1/2\cdot\chi^\rho(\Sigma(p,q,r)) = \nu(\Sigma(p,q,r))$ is proven in the same section. Section 4 is devoted to the proof of the fact that $\lambda^\rho(\Sigma(p,q,r)) = 1/2\cdot\chi^\rho(\Sigma(p,q,r))$. Each section has its own introduction and is further subdivided.

It should be pointed out that most of the results in the paper hold in a more general situation. A description of the relevant results will appear elsewhere, as will a more detailed version of this preprint.

1. Topology of Brieskorn homology spheres

Let p,q,r be relatively prime positive integers greater than or equal to 2. The Brieskorn homology sphere $\Sigma(p,q,r)$ is defined as the algebraic link

$$\Sigma(p,q,r) = \{\,(x,y,z)\in\mathbb{C}^3 \mid x^p+y^q+z^r=0\,\}\cap S^5_1,$$

where S^5_1 is the unit sphere in $\mathbb{C}^3$. This is a smooth naturally oriented 3–manifold with $H_*(\Sigma(p,q,r)) = H_*(S^3)$. Moreover, $\Sigma(p,q,r)$ is Seifert fibered, see [14], with the Seifert invariants $\{\,b,(p,b_1),(q,b_2),(r,b_3)\,\}$ such that

$$b_1qr+b_2pr+b_3pq+bpqr=1. \tag{4}$$

The complex conjugation on $\mathbb{C}^3$ obviously acts on $\Sigma(p,q,r)$ as

$$\sigma : \Sigma(p,q,r)\to\Sigma(p,q,r), \quad (x,y,z)\mapsto(\bar{x},\bar{y},\bar{z}). \tag{5}$$

The fixed point set of this action is never empty. The quotient of $\Sigma(p,q,r)$ by the involution σ is S^3, with the branching set the so called Montesinos knot $k(p,q,r)$, see [10], [2], or [17]. The knot $k(p,q,r)$ can be described by the following diagram

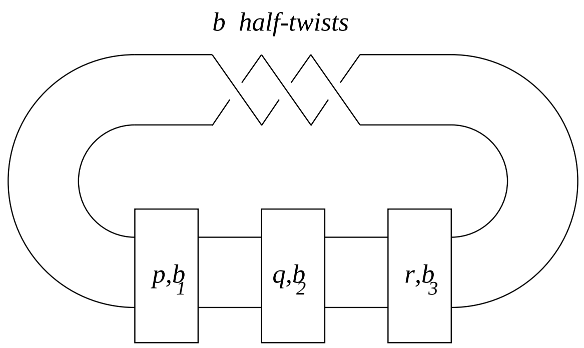

Figure 1

where a box with α, β in it stands for a rational (α, β)–tangle, see [2], Fig. 12.9. The parameters $b, (p, b_1), (q, b_2), (r, b_3)$ are the Seifert invariants of the corresponding $\Sigma(p, q, r)$. According to [2], Theorem 12.28, these parameters together with (4) determine the knot $k(p, q, r)$ uniquely up to isotopy.

2. The invariant λ^ρ

In this section we first shortly recall the definition of Casson's λ–invariant. Our λ^ρ–invariant for $\Sigma(p, q, r)$ will be a σ–invariant version of λ. To define it, we introduce a σ–invariant Heegaard splitting of $\Sigma(p, q, r)$. Then we define the corresponding σ–invariant representation spaces and investigate closely the representation space of $\pi_1(\Sigma(p, q, r))$. It turns out that all the representations in the latter space are σ–invariant. After computing the necessary dimensions and checking the transversality condition, we define the λ^ρ–invariant as an intersection number of σ–invariant representation spaces. Finally, we check that $\lambda^\rho(\Sigma(p, q, r)) = 1/8 \cdot \operatorname{sign} k(p, q, r)$.

1. Casson invariant. Let M be an oriented homology 3–sphere with a Heegard splitting $M = M_1 \cup M_2$ where M_1 and M_2 are handlebodies of genus $g \geq 2$ glued along their common boundary, a Riemann surface M_0. Let

$$\mathcal{R}(M_i) = \operatorname{Hom}^*(\pi_1(M_i), \mathrm{SU}(2))/\operatorname{ad}\mathrm{SU}(2), \;\; i = 0, 1, 2, \emptyset,$$

be the set of conjugacy classes of irreducible representations of $\pi_1(M_i)$ is $\mathrm{SU}(2)$. Each $\mathcal{R}(M_i)$, $i = 0, 1, 2$, is naturally an oriented manifold. The dimension of $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ is $3g-3$, and that of $\mathcal{R}(M_0)$ is $6g-6$. The inclusions $M_0 \subset M_i$, $i = 1, 2$, induce embeddings $\mathcal{R}(M_i) \subset \mathcal{R}(M_0)$. The points of intersection of $\mathcal{R}(M_1)$ with $\mathcal{R}(M_2)$ are in one-to-one correspondence with $\mathcal{R}(M)$. If the intersection is transversal we define the Casson's λ–invariant as

$$\lambda(M) = \frac{1}{2} \sum_{\alpha \in \mathcal{R}(M)} \varepsilon_\alpha$$

where $\varepsilon_\alpha = \pm 1$ is a sign obtained by comparing the orientations on $T_\alpha\mathcal{R}(M_1) \oplus T_\alpha\mathcal{R}(M_2)$ and $T_\alpha\mathcal{R}(M_0)$. Note that the intersection *is* transversal if $\Sigma = \Sigma(p, q, r)$.

2. The invariant Heegaard splitting. We are going to construct an invariant Heegaard splitting of $\Sigma(p, q, r)$. Let us first fix notations. By Σ we denote a Brieskorn homology sphere $\Sigma(p, q, r)$ and by $\sigma : \Sigma \to \Sigma$ the involution constructed in (5). The projection on the quotient space will be denoted by π, so $\pi : \Sigma \to \Sigma/\sigma = S^3$. The projection π maps the fixed point set $\operatorname{Fix}(\sigma) \subset \Sigma$ onto the Montesinos knot $k = k(p, q, r) \subset S^3$.

The knot $k \subset S^3$ can be represented as the closure of a braid β on n strings. We think about k as consisting of the braid β and n untangled arcs in S^3 forming its closure, see Figure 2. Let $S \subset S^3$ be an embedded 2–sphere in S^3 splitting S^3 in two 3–balls, B_1 and B_2, with common boundary S, and such that the entire braid β belongs to $\operatorname{int} B_1$. The intersection $k \cap S$ consists of $2n$ points $P_1, \dots, P_{2n}$.

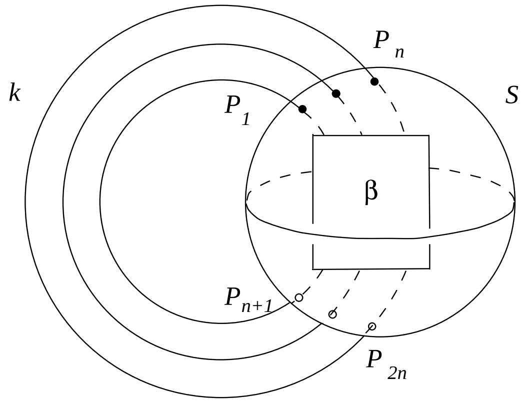

Figure 2

Now, we define a Heegaard splitting $\Sigma = M_1 \cup_{M_0} M_2$ as follows:

$$M_1 = \pi^{-1}(B_1), \quad M_2 = \pi^{-1}(B_2), \quad M_0 = \pi^{-1}(S). \tag{6}$$

Obviously, M_1 and M_2 are handlebodies branched over the braid β and the n untangled arcs, respectively. Their common boundary is M_0, which is a closed surface branched over the points $P_1, \dots, P_{2n} \in S$. The genus g of M_0 can be figured out by comparing Euler characteristics,

$$2 \cdot \chi(S) = \chi(M_0) + \chi(\operatorname{Fix}(\sigma) \cap S),$$

where $\chi(S) = 2$, $\chi(\operatorname{Fix}(\sigma) \cap S) = 2n$, and $\chi(M_0) = 2 - 2g$. Therefore, $g = n - 1$. The Heegaard splitting we just defined is σ–invariant in the sense tnat $\sigma(M_i) = M_i$ for $i = 0, 1, 2$.

3. Representations of Brieskorn homology spheres. Let $\Sigma(p, q, r)$ be a Brieskorn homology 3–sphere, and

$$\mathcal{R}(\Sigma(p,q,r)) = \operatorname{Hom}^*(\pi_1(\Sigma(p,q,r)), \mathrm{SU}(2)) / \operatorname{ad} \mathrm{SU}(2) \tag{7}$$

the space of the conjugacy classes of irreducible representations of its fundamental group in $\mathrm{SU}(2)$. In this subsection we are concerned with describing the space $\mathcal{R}(\Sigma(p,q,r))$ and the involution σ^* induced on the representation space by σ.

We will follow [6] and first identify the group SU(2) with the group S^3 of unit quaternions in the usual way, so that

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Under this identification, the trace $\operatorname{tr}(A)$ of an element $A \in \mathrm{SU}(2)$ coincides with the number $2\operatorname{Re} A,\ A \in S^3$, and the mapping $r : S^3 \to [0, \pi],\ r(A) = \arccos(\operatorname{Re} A)$, is a complete invariant of the conjugacy class of the element A. This conjugacy class is a copy of S^2 in S^3 unless $A = \pm 1$, in which case the conjugacy class consists of just one point.

The fundamental group $\pi_1(\Sigma(p, q, r))$ has the following presentation, see [2],

$$\pi_1(\Sigma(p, q, r)) = \langle x, y, z, h \mid h \text{ central },\, x^p = h^{-b_1}, y^q = h^{-b_2}, z^r = h^{-b_3}, xyz = h^{-b} \rangle.$$

Specifying an irreducible representation $\alpha : \pi_1(\Sigma(p, q, r)) \to \mathrm{SU}(2)$ amounts to specifying a set $\{\alpha(h), \alpha(x), \alpha(y), \alpha(z)\}$ of unit quaternions. In fact, we only need to specify the first three quaternions in this set because $\alpha(z)$ will then be expressed in their terms as $\alpha(z) = (\alpha(x)\alpha(y))^{-1}\alpha(h)^{-b}$. Since h is central and the representation α is irreducible, $\alpha(h) = \pm 1$. Let us denote $\varepsilon_i = \alpha(h)^{b_i} = \pm 1,\ i = 1, 2$, and $\varepsilon_3 = \alpha(h)^{b_3 - rb} = \pm 1$. Then the relations $x^p = h^{-b_1}, y^q = h^{-b_2}$ and $(xy)^r = h^{b_3 - rb}$ imply the following restrictions on $\alpha(x)$ and $\alpha(y)$:

$$r(\alpha(x)) = \pi\ell_1/p, \quad r(\alpha(y)) = \pi\ell_2/q, \quad r(\alpha(x)\alpha(y)) = \pi\ell_3/r,$$

where ℓ_i is even if $\varepsilon_i = 1$, ℓ_i is odd if $\varepsilon_i = -1$, and $0 < \ell_1 < p,\ 0 < \ell_2 < q,\ 0 < \ell_3 < r$.

After conjugation, we may assume that $\alpha(x) = e^{\pi i \ell_1/p}$. The quaternions $\alpha(y)$ and $\alpha(x)\alpha(y)$ should lie in their respective conjugacy classes, $S_2 = r^{-1}(\pi\ell_2/q)$ and $S^3 = r^{-1}(\pi\ell_3/r)$. On the other hand, $\alpha(x)\alpha(y)$ lies in $\alpha(x) \cdot S_2$, therefore, in order for $\alpha(x)$ and $\alpha(y)$ to define a representation, the intersection $\alpha(x) \cdot S_2 \cap S_3$ must be non-empty. Since $\alpha(x) \cdot S_2$ is a 2-sphere centered at $\alpha(x)$, the intersection $\alpha(x) \cdot S_2 \cap S_3$ in S^3 (if non-empty) is a circle. This circle parametrizes a whole collection of representations α' coming together with α such that $r(\alpha'(x)) = r(\alpha(x)),\ r(\alpha'(y)) = r(\alpha(y))$, and $r(\alpha'(x)\alpha'(y)) = r(\alpha(x)\alpha(y))$. In fact, all these representations are conjugate to each other by simultaneous conjugation of $\alpha(x)$ and $\alpha(y)$ by the complex circle $S^1 \subset S^3$. This can be seen from the following technical lemma.

Lemma 6. *Let α and β be irreducible representations of the group $\pi_1(\Sigma(p, q, r))$ in* SU(2) *such that*

(1) $\alpha(h) = \beta(h)$ *and* $\alpha(x) = \beta(x) \in \mathbb{C}$,
(2) $r(\alpha(y)) = r(\beta(y))$, *and*
(3) $r(\alpha(x)\alpha(y)) = r(\beta(x)\beta(y))$.

Then the representations α and β are conjugate to each other, that is there exists a unit quaternion c such that

$$\beta(t) = c \cdot \alpha(t) \cdot c^{-1} \quad \textit{for all} \quad t \in \pi_1(\Sigma(p, q, r)).$$

Moreover, the quaternion c may be chosen to be a complex number, $c \in \mathbb{C}$.

Corollary 7. *(Compare [6]). The representation space $\mathcal{R}(\Sigma(p,q,r))$ is finite.*

The involution $\sigma : \Sigma(p,q,r) \to \Sigma(p,q,r)$ induces the involution on the fundamental group,

$$\sigma_* : \pi_1(\Sigma(p,q,r)) \to \pi_1(\Sigma(p,q,r)),$$
$$h \mapsto h^{-1}, \quad x \mapsto x^{-1}, \quad y \mapsto xy^{-1}x^{-1}, \quad z \mapsto xyz^{-1}y^{-1}x^{-1},$$

see [2], Proposition 12.30, which in turn induces an involution on the corresponding representation space ($[\cdot,\cdot]$ stands for conjugacy class),

$$\sigma^* : \mathcal{R}(\Sigma(p,q,r)) \to \mathcal{R}(\Sigma(p,q,r)), \quad [\alpha] \mapsto [\alpha'], \tag{8}$$
$$\text{where} \quad \alpha'(t) = \alpha(\sigma_*(t)), \quad t \in \pi_1(\Sigma(p,q,r)). \tag{9}$$

Lemma 8. *If $\alpha' : \pi_1(\Sigma(p,q,r)) \to \mathrm{SU}(2)$ is a representation defined by the formula (9) then there exists an element $\rho \in \mathrm{SU}(2)$ such that $\rho^2 = -1$ and*

$$\alpha'(t) = \rho \cdot \alpha(t) \cdot \rho^{-1} \quad \textit{for all} \quad t \in \pi_1(\Sigma(p,q,r)).$$

The element ρ is defined uniquely up to multiplication by ± 1, and the elements ρ corresponding to different representations α are conjugate to each other. In particular, the action (8) on the space $\mathcal{R}(\Sigma(p,q,r))$ of the conjugacy classes of irreducible representations of $\pi_1(\Sigma(p,q,r))$ in $\mathrm{SU}(2)$ is trivial.

Proof. After conjugation we may assume that $\alpha(x)$ is a unit complex number. Then $\alpha'(x) = \alpha(\sigma_*(x)) = \alpha(x^{-1}) = \alpha(x)^{-1}$ is a complex number as well. Let us consider another representation, β, defined as a conjugate of α by the unit quaternion j,

$$\beta(t) = j^{-1} \cdot \alpha'(t) \cdot j \quad \text{for all} \quad t \in \pi_1(\Sigma(p,q,r)).$$

Next we want to verify that the representations α and β satisfy the conditions of Lemma 6. We have

$$\begin{aligned}\beta(x) &= j^{-1} \cdot \alpha'(x) \cdot j \\ &= j^{-1} \cdot \alpha(x)^{-1} \cdot j \\ &= j^{-1} \cdot \overline{\alpha(x)} \cdot j \\ &= j^{-1} \cdot j \cdot \alpha(x), \qquad \text{since} \quad \alpha(x) \in \mathbb{C}, \\ &= \alpha(x).\end{aligned}$$

Note that, for any unit quaternion a, we have $r(a) = r(\bar{a}) = r(a^{-1})$. Therefore,

$$\begin{aligned}r(\beta(y)) &= r(j^{-1} \cdot \alpha'(y) \cdot j) = r(\alpha'(y)) \\ &= r(\alpha(xy^{-1}x^{-1})) = r(\alpha(x)\alpha(y)^{-1}\alpha(x)^{-1}) \\ &= r(\alpha(y)^{-1}) = r(\alpha(y)),\end{aligned}$$

and similarly,

$$\begin{aligned} r(\beta(x)\beta(y)) &= r(j^{-1}\cdot\alpha'(x)\alpha'(y)\cdot j) = r(\alpha'(x)\alpha'(y)) \\ &= r(\alpha(x)^{-1}\alpha(x)\alpha(y)^{-1}\alpha(x)^{-1}) = r((\alpha(x)\alpha(y))^{-1}) \\ &= r(\alpha(x)\alpha(y)). \end{aligned}$$

Since $r(\alpha(h))$ is equal to $r(\beta(h))$ we can apply Lemma 6 to the representations α and β to find a complex number $c \in \mathbb{C}$ such that $\beta(t) = c \cdot \alpha(t) \cdot c^{-1}$ for all $t \in \pi_1(\Sigma(p,q,r))$. Since $\beta(t) = j^{-1}\cdot\alpha'(t)\cdot j = c\cdot\alpha(t)\cdot c^{-1}$, we get that $\alpha'(t) = \rho\cdot\alpha(t)\cdot\rho^{-1}$ with $\rho = jc$.

If we apply σ^* twice to the representation α, we will get α again because σ^* is an involution. Therefore, α is conjugate to itself by the element ρ^2. But the representation α is irreducible, therefore, $\rho^2 = \pm 1$. The following easy computation shows that ρ^2 is in fact -1,

$$\rho^2 = jcjc = jj\bar{c}c = -|c|^2 = -1.$$

The uniqueness of ρ up to ± 1 follows from the irreducibility of α. The fact that the elements ρ defined by different representations are conjugate follows from the fact that $\operatorname{tr}(jc) = 0$ for any complex number c. □

4. The definition of λ^ρ. Let $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere with a σ–invariant Heegaard splitting

$$\Sigma = M_1 \cup_{M_0} M_2 \tag{10}$$

constructed in (6). According to Lemma 8, for any representation $\alpha : \pi_1(\Sigma(p,q,r)) \to \mathrm{SU}(2)$, there exists an element $\rho \in \mathrm{SU}(2)$ such that $\alpha \circ \sigma_* = \rho\alpha\rho^{-1}$ and $\rho^2 = -1$. Generally speaking, ρ depends on α but it follows from Lemma 8 that the elements ρ corresponding to different α's are conjugate to each other. Therefore, the following definition makes sense.

For each manifold M_i, $i = 0,1,2$, in (10) we define the so called ρ–invariant representation space,

$$\begin{aligned} \mathcal{R}^\rho(M_i) = \{\, \alpha : \pi_1(M_i) \to \mathrm{SU}(2) \mid \quad & \alpha \text{ is irreducible and} \\ & \alpha(\sigma_*(t)) = \rho\alpha(t)\rho^{-1}, \quad t \in \pi_1(M_i)\,\}/\operatorname{ad}\mathrm{SU}(2)^\rho, \end{aligned} \tag{11}$$

as the space of all irreducible ρ–invariant representations modulo the adjoint action of the 1–dimensional Lie group $\mathrm{SU}(2)^\rho \subset \mathrm{SU}(2)$ consisting of all $g \in \mathrm{SU}(2)$ such that $\rho g = \pm g\rho$. Note that whenever $\rho' = \pm h\rho h^{-1}$, $h \in \mathrm{SU}(2)$, we have that $\mathcal{R}^\rho(M_i) = \mathcal{R}^{\rho'}(M_i)$, hence each of the spaces (11) only depends on the conjugacy class of ρ and not on ρ itself. We do not need to define a ρ–invariant representation space $\mathcal{R}^\rho(\Sigma(p,q,r))$ because all the representations of $\pi_1(\Sigma(p,q,r))$ in SU(2) are ρ–invariant, see Lemma 8, and therefore $\mathcal{R}^\rho(\Sigma(p,q,r)) = \mathcal{R}(\Sigma(p,q,r))$.

Due to van Kampen's theorem, we have the following commutative diagrams of the fundamental groups

$$\begin{array}{ccc} \pi_1(\Sigma(p,q,r)) & \longleftarrow & \pi_1(M_1) \\ \uparrow & & \uparrow \\ \pi_1(M_2) & \longleftarrow & \pi_1(M_0) \end{array}$$

and of the ρ–invariant representation spaces

$$\begin{array}{ccc} \mathcal{R}(\Sigma(p,q,r)) & \longrightarrow & \mathcal{R}^\rho(M_1) \\ \downarrow & & \downarrow \\ \mathcal{R}^\rho(M_2) & \longrightarrow & \mathcal{R}^\rho(M_0) \end{array}$$

The maps in the latter diagram are injective, so we can think about $\mathcal{R}^\rho(\Sigma(p,q,r))$ as the intersection of $\mathcal{R}^\rho(M_1)$ and $\mathcal{R}^\rho(M_2)$ inside $\mathcal{R}^\rho(M_0)$.

Proposition 9. *The ρ–invariant representation varieties $\mathcal{R}^\rho(M_i)$, $i = 0,1,2$, are smooth manifolds of dimensions $\mathcal{R}^\rho(M_i) = 2g-1$, $i = 1,2$, and $\dim \mathcal{R}^\rho(M_0) = 4g-2$ where g is the genus of a σ–invariant Heegaard splitting (10). The maps in the second diagram are embeddings. The submanifolds $\mathcal{R}^\rho(M_1)$ and $\mathcal{R}^\rho(M_2)$ of $\mathcal{R}^\rho(M_0)$ intersect transversally, and their intersection is in one-to-one correspondence with $\mathcal{R}(\Sigma(p,q,r))$.*

The manifolds $\mathcal{R}^\rho(M_i)$, $i = 0,1,2$, can be oriented as follows. We start with $\mathcal{R}^\rho(M_0)$. Since the quotient M_0/σ is a 2–sphere and the branching set consists of $2n$ points, $n = g+1$, one can choose generators $a_1, b_1, \dots, a_g, b_g$ in $\pi_1(M_0)$ such that $\sigma_*(a_i) = a_i^{-1}$, $\sigma_*(b_i) = b_i^{-1}$. Then $\mathcal{R}^\rho(M_0) \subset \operatorname{Hom}^\rho(F_{2g}, \mathrm{SU}(2))/\operatorname{ad}\mathrm{SU}(2)^\rho$ where F_{2g} is a free group on the generators $a_1, b_1, \dots, a_g, b_g$, and

$$\operatorname{Hom}^\rho(F_{2g}, \mathrm{SU}(2)) = \{\ \alpha : F_{2g} \to \mathrm{SU}(2) \mid \alpha(a_i)^{-1} = \rho\alpha(a_i)\rho^{-1},\ \alpha(b_i)^{-1} = \rho\alpha(b_i)\rho^{-1},\ i = 1,\dots,g\ \}.$$

Lemma 10. *Let $\rho \in \mathrm{SU}(2)$ be such that $\rho^2 = -1$. Then the subset S^ρ of $\mathrm{SU}(2)$ consisting of all $a \in \mathrm{SU}(2)$ such that $a^{-1} = \rho a \rho^{-1}$ is a two-dimensional sphere $S^2 \subset \mathrm{SU}(2)$.*

Proof. Any element ρ of $\mathrm{SU}(2)$ with $\rho^2 = -1$ has zero trace. Therefore, there exists $x \in \mathrm{SU}(2)$ such that $\rho = xjx^{-1}$ (remember that we identify $\mathrm{SU}(2)$ with the group of unit quaternions), and then the map $a \mapsto x^{-1}ax$ establishes a diffeomorphism between S^j and S^ρ. Now, S^j consists of all $b \in \mathrm{SU}(2)$ such that $\bar{b}j = jb$ and $|b|^2 = 1$, hence $b = u + vi + wk$ for some real u, v, w with $u^2 + v^2 + w^2 = 1$. This of course determines a 2–sphere. $\square$

Our choice of $a_1, b_1, \dots, a_g, b_g$ establishes a diffeomorphism

$$\operatorname{Hom}^\rho(F_{2g}, \mathrm{SU}(2)) = (S^2)^{2g}.$$

By choosing an orientation on S^2, we orient $\operatorname{Hom}^\rho(F_{2g}, \mathrm{SU}(2))$ as a product. This also orients $\mathcal{R}(M_0)$ by the usual "base–fiber" convention. Note that the orientation on $\mathcal{R}^\rho(M_0)$ is independent of our choices of orientations on S^2 and $\mathrm{SU}(2)^\rho$ or our choice of a basis $a_1, b_1, \dots, a_g, b_g$.

The orientation on $\mathcal{R}^\rho(M_1)$ can be defined as follows. First, we choose a basis $x_1, \dots, x_g$ of the free group $\pi_1(M_1)$ so that $\sigma_*(x_i) = x_i^{-1}$, $i = 1, \dots, g$. Then $\mathcal{R}^\rho(M_1) \subset \operatorname{Hom}^\rho(F_g, \mathrm{SU}(2))/\operatorname{ad}\mathrm{SU}(2)^\rho$, where F_g is a free group on the generators $x_1, \dots, x_g$, and

$$\begin{aligned}\operatorname{Hom}^\rho(F_g, \mathrm{SU}(2)) &= \{\ \alpha : F_{2g} \to \mathrm{SU}(2) \mid \alpha(\sigma_*(x)) = \rho\alpha(x)\rho^{-1}\} \\ &= \{\alpha : F_g \to \mathrm{SU}(2) \mid \alpha(x_i)^{-1} = \rho\alpha(x_i)\rho^{-1},\ i = 1, \dots, g\ \}.\end{aligned}$$

Our choice of $x_1, \dots, x_n$ establishes a diffeomorphism

$$\operatorname{Hom}^\rho(F_g, \mathrm{SU}(2)) = (S^2)^g.$$

We choose an orientation on S^2 and orient $\operatorname{Hom}^\rho(F_g, \mathrm{SU}(2))$ as a product. Note that this orientation changes by $(-1)^g$ when the orientation on S^2 is changed; however, it does not depend on the choice of a basis in $\pi_1(M_1) = F_g$. The orientation of $\operatorname{Hom}^\rho(F_g, \mathrm{SU}(2))$ and an orientation of $\mathrm{SU}(2)^\rho$ orient $\mathcal{R}(M_1)$.

The ρ–invariant representation space $\mathcal{R}^\rho(M_2)$ is oriented in a completely similar way, the orientation of $\mathrm{SU}(2)^\rho$ having already been fixed.

If the orientations on S^2 or $\mathrm{SU}(2)^\rho$ are changed, the spaces $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ change their orientations simultaneously; therefore, their algebraic intersection number in $\mathcal{R}^\rho(M_0)$ does not change (though *a priori* it may depend on the chosen Heegaard decomposition). We define our λ^ρ–invariant as one half of this algebraic intersection number, so that

$$\lambda^\rho(\Sigma(p,q,r)) = 1/2 \cdot \sum_{\alpha \in \mathcal{R}(\Sigma)} \varepsilon_\alpha \tag{12}$$

where ε_α equals ± 1 depending on whether the orientations on the tangent spaces $T_\alpha\mathcal{R}^\rho(M_1) \oplus T_\alpha\mathcal{R}^\rho(M_2)$ and $T_\alpha\mathcal{R}^\rho(M_0)$ agree.

Proposition 11. *For any Brieskorn homology sphere $\Sigma(p,q,r)$ the invariant $\lambda^\rho(\Sigma(p,q,r))$ is well-defined, in particular, it does not depend on the choice of a σ–invariant Heegaard splitting. Moreover,*

$$\lambda^\rho(\Sigma(p,q,r)) = 1/8 \cdot \operatorname{sign} k(p,q,r)$$

where sign *stands for the knot signature.*

This proposition is proven below by pushing invariant representations down to a knot $k(p,q,r)$ complement in S^3 and applying a result from [8] on trace-free SU(2)–representations of knot groups.

Thus our definition of λ^ρ–invariant is modelled on the Casson's definition of the λ–invariant; the difference is that we use *ρ–invariant* representation spaces. Surprisingly enough, we get an invariant which is *different* from λ. For example, $\lambda(\Sigma(2,3,5)) = \lambda(\Sigma(2,3,7)) = 1$ while $\lambda^\rho(\Sigma(2,3,5)) = -1$ and $\lambda^\rho(\Sigma(2,3,7)) = 1$.

5. Casson-Lin invariant. Let B_n be the braid group of rank n with the standard generators $\beta_1, \dots, \beta_{n-1}$ represented in a free group F_n on symbols $x_1, \dots, x_n$ as follows:

$$\begin{aligned} \beta_i : \quad & x_i && \mapsto && x_i x_{i+1} x_i^{-1} \\ & x_{i+1} && \mapsto && x_i \\ & x_j && \mapsto && x_j, \quad \text{if} \quad j \neq i, i+1. \end{aligned}$$

If $\beta \in B_n$ then the automorphism of F_n representing β maps each x_i to a conjugate of some x_j and preserves the product $x_1 \cdots x_n$.

Let $k \subset S^3$ be a knot represented as the closure of a braid $\beta \in B_n$. Let us fix an embedding of k into S^3 as shown in Figure 2, the sphere S separating S^3 in two 3–balls, B_1 and B_2, with the braid β inside B_1 and n untangled arcs inside B_2. The fundamental group $\pi_1(K)$ of the knot k complement $K = S^3 - nbd\,(k)$, has the presentation

$$\pi_1(K) = \langle\, x_1, \dots, x_n \mid \quad x_i = \beta(x_i),\ i = 1, \dots, n \,\rangle,$$

the generators $x_1, \dots, x_n$ being represented by the meridians of β.

The knot complement K can be now decomposed as $K = M_1' \cup_{M_0'} M_2'$ where $M_0' = S \cap K$ and $M_i' = B_i \cap K,\ i = 1, 2$. The manifolds M_1' and M_2' are handlebodies of genus n, and M_0' is a 2–sphere with $2n$ small 2–discs removed around the points $P_1, \dots, P_{2n}$, see Figure 2. Due to van Kampen's theorem, we get the following commutative diagram of fundamental groups

$$\begin{array}{ccc} \pi_1(K) & \longleftarrow & \pi_1(M_1') \\ \uparrow & & \uparrow \\ \pi_1(M_2') & \longleftarrow & \pi_1(M_0') \end{array}$$

where

$$\pi_1(M_0') = \langle\, x_1, \dots, x_n, y_1, \dots, y_n \mid x_1 \cdots x_n = y_1 \cdots y_n \,\rangle,$$
$$\pi_1(M_1') = \langle\, x_1, \dots, x_n \mid \quad \rangle, \quad \pi_1(M_2') = \langle\, y_1, \dots, y_n \mid \quad \rangle$$

are isomorphic to free groups. To continue an analogy with the definition of the Casson invariant, we need to define SU(2)–representation spaces of the groups $\pi_1(K)$ and $\pi_1(M_i)$, $i = 0, 1, 2$, and compute the corresponding intersection number. To make things work, we impose the extra condition on the representations that all the meridians go to trace-free matrices in SU(2). Thus, following [8], we define

$$\begin{aligned}
\hat{H}_n &= \{\, \alpha : \pi_1(M_0') \to \mathrm{SU}(2) \mid \alpha \text{ is irreducible, } \operatorname{tr} x_i = \operatorname{tr} y_i = 0 \,\} / \operatorname{ad} \mathrm{SU}(2), \\
\hat{\Gamma}_\beta &= \{\, \alpha : \pi_1(M_1') \to \mathrm{SU}(2) \mid \alpha \text{ is irreducible, } \operatorname{tr} x_i = 0 \,\} / \operatorname{ad} \mathrm{SU}(2), \\
\hat{\Lambda}_n &= \{\, \alpha : \pi_1(M_2') \to \mathrm{SU}(2) \mid \alpha \text{ is irreducible, } \operatorname{tr} y_i = 0 \,\} / \operatorname{ad} \mathrm{SU}(2), \\
\mathcal{R}^0(K) &= \{\, \alpha : \pi_1(K) \to \mathrm{SU}(2) \mid \alpha \text{ is irreducible, } \operatorname{tr} x_i = 0 \,\} / \operatorname{ad} \mathrm{SU}(2).
\end{aligned}$$

The first three are smooth manifolds of dimensions $\dim \hat{H}_n = 4n - 6$, and $\dim \hat{\Gamma}_\beta = \dim \hat{\Lambda}_n = 2n - 3$. The commutative diagram of fundamental groups induces the following commutative diagram

$$\begin{tikzcd}
\mathcal{R}^0(K) \arrow[r] \arrow[d] & \hat{\Gamma}_\beta \arrow[d] \\
\hat{\Lambda}_n \arrow[r] & \hat{H}_n
\end{tikzcd}$$

In particular, the irreducible trace-free representations of $\pi_1(K)$ in SU(2) are in one-to-one correspondence with the intersection points of $\hat{\Lambda}_n$ with $\hat{\Gamma}_\beta$ in $\hat{H}_n$. The manifolds $\hat{\Lambda}_n, \hat{\Gamma}_\beta$, and $\hat{H}_n$ are naturally oriented, see [8], and (possibly after a perturbation to make the intersection transversal) one can define

$$h(k) = \sum_{\alpha \in \mathcal{R}^0(K)} \varepsilon'_\alpha \tag{13}$$

where $\varepsilon'_\alpha = \pm 1$ is a sign obtained by comparing the orientations on $T_\alpha \hat{\Lambda}_n \oplus T_\alpha \hat{\Gamma}_\beta$ and $T_\alpha \hat{H}_n$. The invariant $h(k)$ only depends on the knot k and not on the choices in the definition. X.-S. Lin in [8] further proves that $h(k) = 1/2 \cdot \operatorname{sign} k$.

6. Representations of a knot $k(p, q, r)$ complement. Throughout this subsection $\Sigma = \Sigma(p, q, r)$ is a Brieskorn homology sphere, and $k = k(p, q, r)$ is the corresponding Montesinos knot.

Let $E \to \Sigma$ be a (necessarily trivial) SU(2)–vector bundle over Σ. Let us fix a Riemannian metric on Σ and consider the Banach manifold

$$\mathcal{B}^* = \mathcal{A}^* / \mathcal{G}$$

of the L^2_2–gauge equivalence classes of irreducible L^2_1–connections in E. It is a classical result in differential geometry that the holonomy map establishes a one-to-one correspondence between the gauge equivalence classes of irreducible flat connections and the conjugacy classes of irreducible SU(2)–representations of $\pi_1(\Sigma)$.

The involution σ can be lifted to a bundle endomorphism of E. Any endomorphism of E clearly induces an action on $\mathcal{A}^*$ by pull-back, and an action on $\mathcal{B}^*$ as well. Since any two liftings of σ differ by a gauge transformation, we have a well-defined action $\sigma^* : \mathcal{B}^* \to \mathcal{B}^*$. Denote by $\mathcal{B}^\sigma$ the space of connections invariant with respect to σ^*.

Let $\rho \in \mathrm{SU}(2)$ be such that $\rho^2 = -1$. The formula

$$(x, \xi) \mapsto (\sigma(x), \rho\xi\rho^{-1}) \tag{14}$$

defines a lifting of $\sigma : \Sigma \to \Sigma$ on E, which will again be denoted by $\rho : E \to E$. Let $\mathcal{B}^\rho \subset \mathcal{B}^\sigma$ consist of the gauge equivalence classes of irreducible connections A in E such that $\rho^* A = A$. Due to Lemma 8, all irreducible flat connections on Σ belong to $\mathcal{B}^\rho$; in particular, $\mathcal{B}^\rho$ is non-empty. The following three lemmas are easy corollaries of Propositions 1 and 17 and Theorem 18 in [20].

Lemma 12. *For any $\rho, \rho' \in \mathrm{SU}(2)$ such that $\rho^2 = \rho'^2 = -1$, $\mathcal{B}^\rho = \mathcal{B}^{\rho'}$. Furthermore, as a set, $\mathcal{B}^\rho$ is bijective to $\mathcal{A}^\rho/\overline{\mathcal{G}}^\rho$ where $\overline{\mathcal{G}} = \mathcal{G}/\pm 1$ and $\mathcal{A}^\rho = \{\, A \in \mathcal{A}^* \mid \rho^* A = A \,\}$.*

The quotient of $\Sigma(p, q, r)$ by σ is S^3. Since $\rho \neq \pm 1$, it is impossible to define the quotient bundle of E over S^3. One can though define it away from the image of the fixed point set of σ, that is, on the knot complement $K = S^3 \setminus k$. Given an irreducible flat ρ–invariant SU(2)–connection $A \in \mathcal{A}^\rho$, its push-down A' is an irreducible flat SO(3)–connection on K. In other words, A' is an irreducible flat SO(3)–connection singular along k in the sense of P. Kronheimer and T. Mrowka, see [7].

Lemma 13. *The* SO(3)*–connection A' has holonomy 1/2 around k.*

According to $H^1(K; \mathbb{Z}/2) = \mathbb{Z}/2$ and $H^2(K; \mathbb{Z}/2) = 0$, there are two different ways to lift the SO(3)–connection A' to a (singular) SU(2)–connection. Both SU(2)–liftings have holonomy $1/4$ around k; in other words, their holonomy around k is trace–free.

Lemma 14. *Through the push-down map, the representation space $\mathcal{R}(\Sigma)$ is in one-to-two correspondence with the representation space $\mathcal{R}^0(K)$ of irreducible trace–free* SU(2)*–representations of $\pi_1(K)$.*

Similar one-to-two identifications through push-down hold for the pairs $\mathcal{R}^\rho(M_0)$ and $\hat{H}_n$, $\mathcal{R}^\rho(M_1)$ and $\hat{\Gamma}_\beta$, $\mathcal{R}^\rho(M_2)$ and $\hat{\Lambda}_n$, where $g = n - 1$ is the genus of the σ–invariant Heegaard splitting, see Figure 2.

Proof of Proposition 11. Remember that throughout this subsection, $\Sigma = \Sigma(p, q, r)$ and $k = k(p, q, r)$. The invariants $\lambda^\rho(\Sigma)$ and $h(k) = 1/2 \cdot \operatorname{sign} k$ were defined in (12) and (13) as an algebraic count of points in the corresponding (finite) representation spaces, $\mathcal{R}(\Sigma)$ and $\mathcal{R}^0(K)$. These spaces are in one-to-two correspondence by Lemma 14. A routine comparison of the orientations shows that $\varepsilon_\alpha = \varepsilon'_{\alpha'}$ where α' is a push-down of α. Therefore,

$$\begin{aligned}\lambda^\rho(\Sigma) &= 1/2 \cdot \sum \varepsilon_\alpha \\ &= 1/4 \cdot \sum \varepsilon'_{\alpha'} \\ &= 1/8 \cdot \operatorname{sign} k.\end{aligned}$$

This completes the proof of Proposition 11. □

3. Gauge theory for Brieskorn homology spheres

Our next step is to give an "analytic" definition of the invariant λ^ρ. In [19], C. Taubes gave a gauge-theoretical interpretation of the Casson's λ–invariant in terms of (what later became known as) Floer homology. Our gauge-theoretical interpretation of λ^ρ will follow the same lines with the only difference that everything will be "ρ–invariant".

1. Casson invariant via gauge theory. Let us recall shortly the Taubes construction [19] for a homology 3–sphere Σ. Let $E \to \Sigma$ be a trivial SU(2)–bundle, and $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ the space of gauge equaivalence classes of irreducible connections in E. It is a classical differential geometry result that flat connections in E (modulo gauge equivalence) are in one-to-one correspondence with SU(2)–representations of $\pi_1(\Sigma)$ (modulo conjugation). Given two irreducible representations α, β, we may think of them as points in $\mathcal{A}^*$ or $\mathcal{B}^*$.

For any $A \in \mathcal{A}^*$, define the following elliptic differential operator

$$K_A = \begin{vmatrix} 0 & d_A^* \\ d_A & *d_A \end{vmatrix} : (\Omega^0 \oplus \Omega^1)(\Sigma, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^1)(\Sigma, \mathfrak{su}(2)), \tag{15}$$

where d_A stands for the operator of covariant differentiation with respect to A, and $*$ is the Hodge operator associated with a Riemannian metric on Σ. In a proper Sobolev completion of $(\Omega^0 \oplus \Omega^1)(\Sigma, \mathfrak{su}(2))$, the operator K_A is Fredholm. Let $A(t)$, $0 \le t \le 1$, be a path in $\mathcal{A}^*$ connecting α with β, so that $A(0) = \alpha, A(1) = \beta$. Associated with $A(t)$ is a path of Fredholm operators, $K_{A(t)}$. Let us assume that $\ker K_{A(0)} = \ker K_{A(1)}$, as is the case for a Brieskorn homology sphere $\Sigma = \Sigma(p, q, r)$. We define $sf(\alpha, \beta)$ as the spectral flow of the path $K_{A(t)}$ between α and β, see [1]. This number is well-defined modulo 8, and Taubes defines an infinite dimensional generalization $\chi(\Sigma)$ of the Euler characteristic as

$$\chi(\Sigma) = \varepsilon_\alpha \cdot \sum_{\beta \in \mathcal{R}(\Sigma)} (-1)^{sf(\alpha,\beta)} \tag{16}$$

where ε_α is figured out from the spectral flow between the trivial connection θ and α (the number $\chi(\Sigma)$ turns out to be independent of α). It is proven in [19] that $\lambda(\Sigma) = 1/2 \cdot \chi(\Sigma)$. Later, A. Floer showed in [5] that there are well–defined $\mathbb{Z}/8$–graded instanton homology groups $I_*(\Sigma)$ such that $\chi(\Sigma) = \chi(I_*(\Sigma))$.

Let now $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere. R. Fintushel and R. Stern showed in [6] (see also Proposition 15 below) that for any pair $\alpha, \beta \in \mathcal{R}(\Sigma(p,q,r))$,

$$sf(\alpha,\beta) = 0 \mod 2,$$

and that all the signs involved in computing $\chi(\Sigma(p,q,r))$ are positive, so χ simply counts the irreducible representations.

2. Floer index for Brieskorn homology spheres. Let α and β be irreducible flat connections in a trivial SU(2)–bundle E over $\Sigma = \Sigma(p,q,r)$. By pull-back, we can extend E to a trivial bundle (which we also denote by E) over the infinite cylinder $\Sigma \times \mathbb{R}$. Let us choose a σ–invariant Riemannian metric on Σ, and the corresponding product metric on $\Sigma \times \mathbb{R}$. Let further $\rho : E \to E$ be the lifting of σ defined in (14), and $A(t) \in \mathcal{A}^\rho$ a path of connections forming an invariant connection A over $\Sigma \times \mathbb{R}$ vanishing in the $\mathbb{R}$–direction and equal to respectively α and β near the ends of $\Sigma \times \mathbb{R}$. Denote by d_A the operator of covariant differentiation with respect to A.

The (relative) Floer index of the pair (α,β) equals the Fredholm index of the following elliptic complex, see [1],

$$\Omega^0(\Sigma \times \mathbb{R}, \mathrm{ad}\, E) \xrightarrow{d_A} \Omega^1(\Sigma \times \mathbb{R}, \mathrm{ad}\, E) \xrightarrow{d_A^-} \Omega^2_+(\Sigma \times \mathbb{R}, \mathrm{ad}\, E) \tag{17}$$

where $\mathrm{ad}\, E$ is the adjoint bundle of E over $\Sigma \times \mathbb{R}$, and $d_A^- = P_- \circ d_A$ where P_- is the projection onto the self dual forms with respect to the fixed product metric on $\Sigma \times \mathbb{R}$. The Sobolev norms on the spaces in (17) are specified as usual, see [5].

The product metric and the connection A on $\Sigma \times \mathbb{R}$ are ρ–invariant, therefore, one can define the following ρ–invariant elliptic subcomplex of (17),

$$\Omega^0(\Sigma \times \mathbb{R}, \mathrm{ad}\, E)^\rho \xrightarrow{d_A} \Omega^1(\Sigma \times \mathbb{R}, \mathrm{ad}\, E)^\rho \xrightarrow{d_A^-} \Omega^2_+(\Sigma \times \mathbb{R}, \mathrm{ad}\, E)^\rho \tag{18}$$

Proposition 15. *For any irreducible flat connections α, β on $\Sigma = \Sigma(p,q,r)$, the index of the elliptic complex (17) is even. In fact, it equals twice the index of the ρ–invariant complex (18).*

In order to prove the proposition we need the following two technical lemmas.

Lemma 16. *There exist a Riemannian metric on $\Sigma(p,q,r)$ and an almost complex structure J on $\Sigma(p,q,r) \times \mathbb{R}$ such that*

(1) *The product metric on $\Sigma \times \mathbb{R}$ is $(\sigma \times 1)$–invariant;*
(2) *J is compatible with this product metric; and*
(3) *The involution $\sigma \times 1$ is anti-holomorphic with respect to J, that is $(\sigma \times 1)_* J = -J(\sigma \times 1)_*$ on the tangent bundle.*

Proof. Let us consider the algebraic variety

$$V(p,q,r) = \{\, (x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0 \,\} \subset \mathbb{C}^3,$$

which is a non-singular complex surface except perhaps at the origin. The homology sphere $\Sigma(p,q,r)$ in question is the intersection of $V(p,q,r)$ with the unit 5-sphere S^5_1,

$$\Sigma(p,q,r) = V(p,q,r) \cap S^5_1,$$

and the variety $V(p,q,r)$ is in fact a cone over $\Sigma(p,q,r)$ with the vertex at the origin.

Let $V^0(p,q,r)$ be the variety $V(p,q,r)$ with the origin removed. The Riemann metric induced on $V^0(p,q,r)$ from the standard flat metric on $\mathbb{C}^3$ is a cone metric given by

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{19}$$

in the spherical coordinates (r,θ). Here $d\theta^2 = \sum h_{ij} d\theta_i d\theta_j$ is a metric on $\Sigma(p,q,r)$, and r is the distance from the origin (in $\mathbb{C}^3$). Obviously, both the complex conjugation σ' and the almost complex structure J induced on $V^0(p,q,r)$ from $\mathbb{C}^3$ are compatible with the metric (19).

Let us now form the conformally equivalent metric

$$ds^2/r^2 = dr^2/r^2 + d\theta^2 \tag{20}$$

on $V^0(p,q,r)$. The substitution $r = e^{-\tau}$ gives coordinates in which ds^2/r^2 is the standard product metric

$$d\tilde{s}^2 = d\tau^2 + d\theta^2 \tag{21}$$

on the cylinder $\Sigma(p,q,r) \times \mathbb{R}$. Since the metrics (19) and (21) are conformally equivalent, both σ' and J are compatible with the product metric (21). Moreover, σ' preserves all the spheres S^5_r, $0 < r < \infty$, in $\mathbb{C}^3$, therefore, it has the form $\sigma' = \sigma \times 1$. Finally, the involution $\sigma \times 1$ is antiholomorphic with respect to J because this is the case in $\mathbb{C}^3$. $\square$

Lemma 17. *There exists a differential operator $\tilde{d}^-_A : \Omega^1(\Sigma \times \mathbb{R}, \operatorname{ad} E) \to \Omega^2_+(\Sigma \times \mathbb{R}, \operatorname{ad} E)$ such that*

(1) *The difference $\tilde{d}^-_A - d^-_A$ is a compact operator,*
(2) $\operatorname{coker} \tilde{d}^-_A = 0$, *and*
(3) *The operator $\tilde{d}^-_A$ is ρ–invariant.*

Proof. This is essentially the transversality result from [5], Proposition 2c.2. The key difference is that we have to bound ourselves to invariant perturbations which, generally speaking, may be not generic among the perturbations used by Floer. Following [5], Section 1b, consider a collection $S(m)$ of m circles smoothly embedded in

$\mathbb{R}^3$ which intersect precisely at the origin and have the same tangent there. Let

$$\gamma : \bigvee_{i=1}^{m} S^1_i \times D^2 \to \Sigma,$$

be a map which restricts to smooth embeddings $\gamma_\theta : S(m) \to \Sigma$ for each $\theta \in D^2$ and $\gamma_i : S^1_i \times D^2 \to \Sigma$ for each i. By adding extra circles, one can choose γ so that its image in Σ will be σ–invariant. Now, γ defines a family of holonomy maps

$$\begin{aligned} \gamma_\theta : \mathcal{B} \to L_m &= \mathrm{SU}(2)^m / \operatorname{ad}\mathrm{SU}(2), \\ A &\mapsto (\mathrm{hol}_A(\gamma_1), \dots , \mathrm{hol}_A(\gamma_m)) \end{aligned} \tag{22}$$

Let $C^\infty_{sym}(L_m, \mathbb{R})$ be defined as the set of smooth $\operatorname{ad}\mathrm{SU}(2)$–invariant real valued functions on $\mathrm{SU}(2)^m$ with the additional property that for any $h \in C^\infty_{sym}(L_m, \mathbb{R})$,

$$h(x_{\tau(1)}, \dots , x_{\tau(m)}) = h(x_1, \dots , x_m)$$

for any permutation τ on m symbols. For example, one can start with an arbitrary $\operatorname{ad}\mathrm{SU}(2)$–invariant function on $\mathrm{SU}(2)^m$ and take h to be its symmetrization.

Given such γ and h, we define the function

$$h_\gamma : \mathcal{B} \to \mathbb{R}, \quad h_\gamma(A) = \int_{D^2} h(\gamma_\theta(A))\, d^2\theta \tag{23}$$

where $d^2\theta$ is a smooth compactly supported volume form on the interior of D^2. The following argument shows that this function is invariant with respect to the induced σ^*–action on $\mathcal{B}$:

$$\begin{aligned} h_\gamma(\sigma^* A) &= \int_{D^2} h(\,\mathrm{hol}_{\sigma^* A}(\gamma_1), \dots , \mathrm{hol}_{\sigma^* A}(\gamma_m))\, d^2\theta \\ &= \int_{D^2} h(\,\mathrm{hol}_A(\sigma(\gamma_1)), \dots , \mathrm{hol}_A(\sigma(\gamma_m)))\, d^2\theta \\ &= \int_{D^2} h(\,\mathrm{hol}_A(\gamma_{\tau(1)}), \dots , \mathrm{hol}_A(\gamma_{\tau(m)}))\, d^2\theta \\ &= \int_{D^2} h(\,\mathrm{hol}_A(\gamma_1), \dots , \mathrm{hol}_A(\gamma_m))\, d^2\theta = h_\gamma(A). \end{aligned}$$

The class of maps we defined in (23) is large enough for the set

$$\{\, \operatorname{grad} h_\gamma(A) \mid h_\gamma \ \text{ defined by (23)} \,\} \subset T_A\mathcal{B}^*$$

to be dense in the tangent space $T_A\mathcal{B}^*$. Now the proof of Proposition 2c.2 in [5] goes through, which provides a function h_γ such that a linearization of its gradient is a compact invariant perturbation of the operator d^-_A with the desired properties. □

Sketch of the proof of Proposition 15. In Lemma 16 we defined a σ–invariant Riemannian metric on $\Sigma(p,q,r)$ and an almost complex structure J on the manifold $\Sigma(p,q,r)\times\mathbb{R}$ compatible with the product metric such that the involution $\sigma\times 1:\Sigma\times\mathbb{R}\to\Sigma\times\mathbb{R}$ is anti-holomorphic with respect to J. We can use J to identify the ±1–eigenspaces of the involution induced by $\sigma\times 1$ on the space of 1–forms in (17), and make sure that (possibly after perturbation provided by Lemma 17) $\ker d_A = 0$ and $\operatorname{coker} d_A^- = 0$. This splits the index of the complex (17) and hence the corresponding spectral flow in halves. $\square$

Proposition 18. *(Compare* [6] *). Let $\Sigma(p,q,r)$ be a Brieskorn homology sphere. Then the Floer homology groups $I_n(\Sigma(p,q,r))$ vanish for n odd.*

Proof. We only need to fix the sign ε_α in (16). It can be achieved by the requirement that for $\varepsilon_\alpha = (-1)^{\eta(\alpha)}$,

$$-3-\eta(\alpha) = \operatorname{index} D_A^+ - 3\,(1+b_2^+)(W),$$

where W is any smooth simply connected non-compact 4–manifold with a single end of the form $\Sigma(p,q,r)\times\mathbb{R}_+$, A is any SU(2)–connection on W with limiting value α, and

$$D_A^+ = d_A\oplus d_A^+ : \Omega^1(W,\mathfrak{su}(2))\to\Omega^0(W,\mathfrak{su}(2))\oplus\Omega^2_+(W,\mathfrak{su}(2))$$

is the ASD–operator, compare [4]. Let W be the Milnor fiber of the singularity of $f^{-1}(0)$ where $f(x,y,z)=x^p+y^q+z^r$ so that $\partial W=\Sigma(p,q,r)$, see [9]. The involution (5) extends to an antiholomorphic involution on W, and an argument similar to that in the proof of Proposition 15 proves that $\operatorname{index} D_A^+$ is even. On the other hand, one can easily show that modulo 2, $b_2^+ = 1/2\cdot b_2(W) = 1/2\cdot(p-1)(q-1)(r-1) = 0$. Therefore, $\varepsilon_\alpha = 1$, and the Floer homology groups of $\Sigma(p,q,r)$ vanish in odd dimensions. $\square$

3. The definition of χ^ρ.

Let $\rho: E\to E$ be the lifting of $\sigma:\Sigma(p,q,r)\to\Sigma(p,q,r)$ defined by (14). It induces an involution ρ^* on the $\mathfrak{su}(2)$–valued differential forms on $\Sigma(p,q,r)$. Therefore, for any $A\in\mathcal{A}^\rho$, the differential operator K_A, see (15), can be restricted to the $+1$–eigenspaces of the involution ρ^* to give a new elliptic differential operator K_A^ρ,

$$K_A^\rho = \begin{vmatrix} 0 & d_A^* \\ d_A & *d_A\end{vmatrix} : (\Omega^0\oplus\Omega^1)^\rho(\Sigma,\mathfrak{su}(2))\to(\Omega^0\oplus\Omega^1)^\rho(\Sigma,\mathfrak{su}(2)).$$

For any pair $\alpha,\beta\in\mathcal{R}(\Sigma(p,q,r))$, let us denote by $sf^\rho(\alpha,\beta)$ the spectral flow of a family of operators $K^\rho_{A(t)}$, where $A(t)$ is a path in $\mathcal{A}^\rho$ connecting α to β. The number $sf^\rho(\alpha,\beta)$ is well-defined modulo 4. The invariant $\chi^\rho(\Sigma(p,q,r))$ is now defined by the formula

$$\chi^\rho(\Sigma) = \varepsilon_\alpha \cdot \sum_{\beta \in \mathcal{R}(\Sigma)} (-1)^{sf^\rho(\alpha,\beta)}, \tag{24}$$

compare to (16). The following result is an easy consequence of Proposition 15.

Proposition 19. *Let $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere. Then $1/2 \cdot \chi^\rho(\Sigma) = \nu(\Sigma)$ where the invariants $\chi^\rho(\Sigma)$ and $\nu(\Sigma)$ are defined by (24) and (1).*

4. The invariant λ^ρ via gauge theory

Proposition 20. *Let $\Sigma = \Sigma(p,q,r)$ be a Brieskorn homology sphere, and λ^ρ and χ^ρ the invariant defined by (12) and (24). Then $\lambda^\rho(\Sigma) = 1/2 \cdot \chi^\rho(\Sigma)$.*

This result is a straightforward application of the Taubes argument in [19] in our ρ–invariant setting.

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: saveliev@math.lsa.umich.edu