GEOMETRY AND TOPOLOGY OF COMPLEX HYPERBOLIC AND CR-MANIFOLDS

Boris Apanasov

ABSTRACT. We study geometry, topology and deformation spaces of noncompact complex hyperbolic manifolds (geometrically finite, with variable negative curvature), whose properties make them surprisingly different from real hyperbolic manifolds with constant negative curvature. This study uses an interaction between Kähler geometry of the complex hyperbolic space and the contact structure at its infinity (the one-point compactification of the Heisenberg group), in particular an established structural theorem for discrete group actions on nilpotent Lie groups.

1. Introduction

This paper presents recent progress in studying topology and geometry of complex hyperbolic manifolds M with *variable* negative curvature and spherical Cauchy-Riemannian manifolds with Carnot-Caratheodory structure at infinity M_∞.

Among negatively curved manifolds, the class of complex hyperbolic manifolds occupies a distinguished niche due to several reasons. First, such manifolds furnish the simplest examples of negatively curved Kähler manifolds, and due to their complex analytic nature, a broad spectrum of techniques can contribute to the study. Simultaneously, the infinity of such manifolds, that is the spherical Cauchy-Riemannian manifolds furnish the simplest examples of manifolds with contact structures. Second, such manifolds provide simplest examples of negatively curved manifolds not having *constant* sectional curvature, and already obtained results show surprising differences between geometry and topology of such manifolds and corresponding properties of (real hyperbolic) manifolds with constant negative curvature, see [BS, BuM, EMM, Go1, GM, Min, Yu1]. Third, such manifolds occupy a remarkable place among rank-one symmetric spaces in the sense of their deformations: they enjoy the flexibility of low dimensional real hyperbolic manifolds (see [Th, A1, A2] and §7) as well as the rigidity of quaternionic/octionic hyperbolic manifolds and higher-rank locally symmetric spaces [MG1, Co2, P]. Finally, since

1991 *Mathematics Subject Classification.* 57, 55, 53, 51, 32, 22, 20.

Key words and phrases. Complex hyperbolic geometry, Cauchy-Riemannian manifolds, discrete groups, geometrical finiteness, nilpotent and Heisenberg groups, Bieberbach theorems, fiber bundles, homology cobordisms, quasiconformal maps, structure deformations, Teichmüller spaces.

Research in MSRI was supported in part by NSF grant DMS-9022140.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

its inception, the theory of smooth 4-manifolds has relied upon complex surface theory to provide its basic examples. Nowadays it pays back, and one can study complex analytic 2-manifolds by using Seiberg-Witten invariants, decomposition of 4-manifolds along homology 3-spheres, Floer homology and new (homology) cobordism invariants, see [W, LB, BE, FS, S, A9] and §5.

Complex hyperbolic geometry is the geometry of the unit ball $\mathbb{B}^n_{\mathbb{C}}$ in $\mathbb{C}^n$ with the Kähler structure given by the Bergman metric (compare [CG, Go3], whose automorphisms are biholomorphic automorphisms of the ball, i.e., elements of $PU(n,1)$. (We notice that complex hyperbolic manifolds with non-elementary fundamental groups are complex hyperbolic in the sense of S.Kobayashi [Kob].) Here we study topology and geometry of complex hyperbolic manifolds by using spherical Cauchy-Riemannian geometry at their infinity. This CR-geometry is modeled on the one point compactification of the (nilpotent) Heisenberg group, which appears as the sphere at infinity of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. In particular, our study exploits a structural Theorem 3.1 about actions of discrete groups on nilpotent Lie groups (in particular on the Heisenberg group $\mathcal{H}_n$), which generalizes a Bieberbach theorem for Euclidean spaces [Wo] and strengthens a result by L.Auslander [Au].

Our main assumption on a complex hyperbolic n-manifold M is the geometrical finiteness condition on its fundamental group $\pi_1(M) = G \subset PU(n,1)$, which in particular implies that G is finitely generated [Bow] and even finitely presented, see Corollary 4.5. The original definition of a geometrically finite manifold M (due to L.Ahlfors [Ah]) came from an assumption that M may be decomposed into a cell by cutting along a finite number of its totally geodesic hypersurfaces. The notion of geometrical finiteness has been essentially used in the case of real hyperbolic manifolds (of constant sectional curvature), where geometric analysis and ideas of Thurston have provided powerful tools for understanding of their structure, see [BM, MA, Th, A1, A3]. Some of those ideas also work in spaces with pinched negative curvature, see [Bow]. However, geometric methods based on consideration of finite sided fundamental polyhedra cannot be used in spaces of variable curvature, see §4, and we base our geometric description of geometrically finite complex hyperbolic manifolds on a geometric analysis of their "thin" ends. This analysis is based on establishing a fiber bundle structure on Heisenberg (in general, non-compact) manifolds which remind Gromov's almost flat (compact) manifolds, see [Gr1, BK].

As an application of our results on geometrical finiteness, we are able to find finite coverings of an arbitrary geometrically finite complex hyperbolic manifold such that their parabolic ends have the simplest possible structure, i.e., ends with either Abelian or 2-step nilpotent holonomy (Theorem 4.9). In another such an application, we study an interplay between topology and Kähler geometry of complex hyperbolic n-manifolds, and topology and Cauchy-Riemannian geometry of their boundary $(2n-1)$-manifolds at infinity, see our homology cobordism Theorem 5.4. In that respect, the problem of geometrical finiteness is very different in complex dimension two, where it is quite possible that complex surfaces with finitely generated fundamental groups and "big" ends at infinity are in fact geometrically finite. We also note that such non-compact geometrically finite complex hyperbolic surfaces

have infinitely many smooth structures, see [BE].

The homology cobordism Theorem 5.4 is also an attempt to control the boundary components at infinity of complex hyperbolic manifolds. Here the situation is absolutely different from the real hyperbolic one. In fact, due to Kohn–Rossi analytic extension theorem in the compact case [EMM] and to D.Burns theorem in the case when only one boundary component at infinity is compact (see also [NR1, Th.4.4], [NR2]), the whole boundary at infinity of a complex hyperbolic manifold M of infinite volume is connected (and the manifold itself is geometrically finite if $\dim_{\mathbb{C}} M \geq 3$) if one of the above compactness conditions holds. However, if boundary components of M are non-compact, the boundary $\partial_\infty M$ may have arbitrarily many components due to our construction in Theorems 5.2 and 5.3.

The results on geometrical finiteness are naturally linked with the Sullivan's stability of discrete representations of $\pi_1(M)$ into $PU(n,1)$, deformations of complex hyperbolic manifolds and Cauchy-Riemannian manifolds at their infinity, and equivariant (quasiconformal or quasisymmetric) homeomorphisms inducing such deformations and isomorphisms of discrete subgroups of $PU(n,1)$. Results in these directions are discussed in the last two sections of the paper.

First of all, complex hyperbolic and CR-structures are very interesting due to properties of their deformations, rigidity versus flexibility. Namely, finite volume complex hyperbolic manifolds are rigid due to Mostow's rigidity [Mo1] (for all locally symmetric spaces of rank one). Nevertheless their *constant curvature* analogue, real hyperbolic manifolds are flexible in low dimensions and in the sense of quasi-Fuchsian deformations (see our discussion in §7). Contrasting to such a flexibility, complex hyperbolic manifolds share the super-rigidity of quaternionic/octionic hyperbolic manifolds (see Pansu's [P] and Corlette's [Co1-2] rigidity theorems, analogous to Margulis's [MG1] super-rigidity in higher rank). Namely, due to Goldman's [Go1] local rigidity theorem in dimension $n = 2$ and its extension [GM] for $n \geq 3$, every nearby discrete representation $\rho : G \to PU(n,1)$ of a cocompact lattice $G \subset PU(n-1,1)$ stabilizes a complex totally geodesic subspace $\mathbb{H}^{n-1}_{\mathbb{C}}$ in $\mathbb{H}^n_{\mathbb{C}}$, and for $n \geq 3$, this rigidity is even global due to a celebrated Yue's theorem [Yu1].

One of our goals here is to show that, in contrast to that rigidity of complex hyperbolic non-Stein manifolds, complex hyperbolic Stein manifolds are not rigid in general. Such a flexibility has two aspects. Firstly, we point out that the rigidity condition that the group $G \subset PU(n,1)$ preserves a complex totally geodesic hyperspace in $\mathbb{H}^n_{\mathbb{C}}$ is essential for local rigidity of deformations only for co-compact lattices $G \subset PU(n-1,1)$. This is due to the following our result [ACG]:

Theorem 7.1. *Let $G \subset PU(1,1)$ be a co-finite free lattice whose action in $\mathbb{H}^2_{\mathbb{C}}$ is generated by four real involutions (with fixed mutually tangent real circles at infinity). Then there is a continuous family $\{f_\alpha\}$, $-\epsilon < \alpha < \epsilon$, of G-equivariant homeomorphisms in $\overline{\mathbb{H}^2_{\mathbb{C}}}$ which induce non-trivial quasi-Fuchsian (discrete faithful) representations $f^*_\alpha : G \to PU(2,1)$. Moreover, for each $\alpha \neq 0$, any G-equivariant homeomorphism of $\overline{\mathbb{H}^2_{\mathbb{C}}}$ that induces the representation f^*_α cannot be quasiconformal.*

This also shows the impossibility to extend the Sullivan's quasiconformal stability

theorem [Su2] to that situation, as well as provides the first continuous (topological) deformation of a co-finite Fuchsian group $G \subset PU(1,1)$ into quasi-Fuchsian groups $G_\alpha = f_\alpha G f_\alpha^{-1} \subset PU(2,1)$ with the (arbitrarily close to one) Hausdorff dimension $\dim_H \Lambda(G_\alpha) > 1$ of the limit set $\Lambda(G_\alpha)$, $\alpha \neq 1$, compare [Co1].

Secondly, we point out that the noncompactness condition in our non-rigidity theorem is not essential, either. Namely, complex hyperbolic Stein manifolds homotopy equivalent to their closed totally *real* geodesic surfaces are not rigid, too. Namely, in complex dimension $n = 2$, we provide a canonical construction of continuous quasi-Fuchsian deformations of complex surfaces fibered over closed Riemannian surfaces, which we call "complex bendings" along simple close geodesics. This is the first such deformations (moreover, quasiconformally induced ones) of complex analytic fibrations over a compact base:

Theorem 7.2. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a given (non-elementary) discrete group. Then, for any simple closed geodesic α in the Riemann 2-surface $S = H^2_{\mathbb{R}}/G$ and a sufficiently small $\eta_0 > 0$, there is a holomorphic family of G-equivariant quasiconformal homeomorphisms $F_\eta : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$, $-\eta_0 < \eta < \eta_0$, which defines the bending (quasi-Fuchsian) deformation $\mathcal{B}_\alpha : (-\eta_0, \eta_0) \to \mathcal{R}_0(G)$ of the group G along the geodesic α, $\mathcal{B}_\alpha(\eta) = F^*_\eta$.*

The constructed deformations depend on many parameters described by the Teichmüller space $\mathcal{T}(M)$ of isotopy classes of complex hyperbolic structures on M, or equivalently by the Teichmüller space $\mathcal{T}(G) = \mathcal{R}_0(G)/PU(n,1)$ of conjugacy classes of discrete faithful representations $\rho \in \mathcal{R}_0(G) \subset \mathrm{Hom}(G, PU(n,1))$ of $G = \pi_1(M)$:

Corollary 7.3. *Let $S_p = \mathbb{H}^2_{\mathbb{R}}/G$ be a closed totally real geodesic surface of genus $p > 1$ in a given complex hyperbolic surface $M = \mathbb{H}^2_{\mathbb{C}}/G$, $G \subset PO(2,1) \subset PU(2,1)$. Then there is an embedding $\pi \circ \mathcal{B} : B^{3p-3} \hookrightarrow \mathcal{T}(M)$ of a real $(3p-3)$-ball into the Teichmüller space of M, defined by bending deformations along disjoint closed geodesics in M and by the projection $\pi : \mathcal{D}(M) \to \mathcal{T}(M) = \mathcal{D}(M)/PU(2,1)$ in the development space $\mathcal{D}(M)$.*

As an application of the constructed deformations, we answer a well known question about cusp groups on the boundary of the Teichmüller space $\mathcal{T}(M)$ of a (Stein) complex hyperbolic surface M fibering over a compact Riemann surface of genus $p > 1$ [AG]:

Corollary 7.12. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a uniform lattice isomorphic to the fundamental group of a closed surface S_p of genus $p \geq 2$. Then there is a continuous deformation $R : \mathbb{R}^{3p-3} \to \mathcal{T}(G)$ (induced by G-equivariant quasiconformal homeomorphisms of $\mathbb{H}^2_{\mathbb{C}}$) whose boundary group $G_\infty = R(\infty)(G)$ has $(3p-3)$ non-conjugate accidental parabolic subgroups.*

Naturally, all constructed topological deformations are in particular geometric realizations of the corresponding (type preserving) discrete group isomorphisms,

see Problem 6.1. However, as Example 6.7 shows, not all such type preserving isomorphisms are so good. Nevertheless, as the first step in solving the geometrization Problem 6.1, we prove the following geometric realization theorem [A7]:

Theorem 6.2. *Let $\phi : G \to H$ be a type preserving isomorphism of two non-elementary geometrically finite groups $G, H \subset PU(n,1)$. Then there exists a unique equivariant homeomorphism $f_\phi : \Lambda(G) \to \Lambda(H)$ of their limit sets that induces the isomorphism ϕ. Moreover, if $\Lambda(G) = \partial\mathbb{H}^n_{\mathbb{C}}$, the homeomorphism f_ϕ is the restriction of a hyperbolic isometry $h \in PU(n,1)$.*

We note that, in contrast to Tukia [Tu] isomorphism theorem in the real hyperbolic geometry, one might suspect that in general the homeomorphism f_ϕ has no good metric properties, compare Theorem 7.1. This is still one of open problems in complex hyperbolic geometry (see §6 for discussions).

2. Complex hyperbolic and Heisenberg manifolds

We recall some facts concerning the link between nilpotent geometry of the Heisenberg group, the Cauchy-Riemannian geometry (and contact structure) of its one-point compactification, and the Kähler geometry of the complex hyperbolic space (compare [GP1, Go3, KR]).

One can realize the complex hyperbolic geometry in the complex projective space,

$$\mathbb{H}^n_{\mathbb{C}} = \{[z] \in \mathbb{CP}^n \ : \langle z, z\rangle < 0\,,\ z \in \mathbb{C}^{n,1}\}\,,$$

as the set of negative lines in the Hermitian vector space $\mathbb{C}^{n,1}$, with Hermitian structure given by the indefinite $(n,1)$-form $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n - z_{n+1}\overline{w}_{n+1}$. Its boundary $\partial\mathbb{H}^n_{\mathbb{C}} = \{[z] \in \mathbb{CP}^{n,1} \ : \langle z, z\rangle = 0\}$ consists of all null lines in $\mathbb{CP}^n$ and is homeomorphic to the (2n-1)-sphere S^{2n-1}.

The full group $\mathrm{Isom}\,\mathbb{H}^n_{\mathbb{C}}$ of isometries of $\mathbb{H}^n_{\mathbb{C}}$ is generated by the group of holomorphic automorphisms (= the projective unitary group $PU(n,1)$ defined by the group $U(n,1)$ of unitary automorphisms of $\mathbb{C}^{n,1}$) together with the antiholomorphic automorphism of $\mathbb{H}^n_{\mathbb{C}}$ defined by the $\mathbb{C}$-antilinear unitary automorphism of $\mathbb{C}^{n,1}$ given by complex conjugation $z \mapsto \bar{z}$. The group $PU(n,1)$ can be embedded in a linear group due to A.Borel [Bor] (cf. [AX1, L.2.1]), hence any finitely generated group $G \subset PU(n,1)$ is residually finite and has a finite index torsion free subgroup. Elements $g \in PU(n,1)$ are of the following three types. If g fixes a point in $\mathbb{H}^n_{\mathbb{C}}$, it is called *elliptic*. If g has exactly one fixed point, and it lies in $\partial\mathbb{H}^n_{\mathbb{C}}$, g is called *parabolic*. If g has exactly two fixed points, and they lie in $\partial\mathbb{H}^n_{\mathbb{C}}$, g is called *loxodromic*. These three types exhaust all the possibilities.

There are two common models of complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ as domains in $\mathbb{C}^n$, the unit ball $\mathbb{B}^n_{\mathbb{C}}$ and the Siegel domain $\mathfrak{S}_n$. They arise from two affine patches in projective space related to $\mathbb{H}^n_{\mathbb{C}}$ and its boundary. Namely, embedding $\mathbb{C}^n$ onto the affine patch of $\mathbb{CP}^{n,1}$ defined by $z_{n+1} \neq 0$ (in homogeneous coordinates) as $A : \mathbb{C}^n \to \mathbb{CP}^n, z \mapsto [(z,1)]$, we may identify the unit ball $\mathbb{B}^n_{\mathbb{C}}(0,1) \subset \mathbb{C}^n$ with $\mathbb{H}^n_{\mathbb{C}} = A(\mathbb{B}^n_{\mathbb{C}})$. Here the metric in $\mathbb{C}^n$ is defined by the standard Hermitian form $\langle\langle\ ,\ \rangle\rangle$,

and the induced metric on $\mathbb{B}^n_{\mathbb{C}}$ is the Bergman metric (with constant holomorphic curvature -1) whose sectional curvature is between -1 and -1/4.

The Siegel domain model of $\mathbb{H}^n_{\mathbb{C}}$ arises from the affine patch complimentary to a projective hyperplane H_∞ which is tangent to $\partial\mathbb{H}^n_{\mathbb{C}}$ at a point $\infty \in \partial\mathbb{H}^n_{\mathbb{C}}$. For example, taking that point ∞ as $(0', -1, 1)$ with $0' \in \mathbb{C}^{n-1}$ and $H_\infty = \{[z] \in \mathbb{CP}^n \, : \, z_n + z_{n+1} = 0\}$, one has the map $\mathbf{S} : \mathbb{C}^n \to \mathbb{CP}^n \backslash H_\infty$ such that

$$\begin{pmatrix} z' \\ z_n \end{pmatrix} \longmapsto \begin{bmatrix} z' \\ \frac{1}{2} - z_n \\ \frac{1}{2} + z_n \end{bmatrix} \quad \text{where} \quad z' = \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \in \mathbb{C}^{n-1} \, .$$

In the obtained affine coordinates, the complex hyperbolic space is identified with the *Siegel domain*

$$\mathfrak{S}_n = \mathbf{S}^{-1}(\mathbb{H}^n_{\mathbb{C}}) = \{z \in \mathbb{C}^n \, : \, z_n + \overline{z}_n > \langle\langle z', z' \rangle\rangle\} \, ,$$

where the Hermitian form is $\langle \mathbf{S}(z), \mathbf{S}(w) \rangle = \langle\langle z', w' \rangle\rangle - z_n - \overline{w}_n$. The automorphism group of this affine model of $\mathbb{H}^n_{\mathbb{C}}$ is the group of affine transformations of $\mathbb{C}^n$ preserving $\mathfrak{S}_n$. Its unipotent radical is the *Heisenberg group* $\mathcal{H}_n$ consisting of all *Heisenberg translations*

$$T_{\xi,v} \, : \, (w', w_n) \mapsto \left(w' + \xi, w_n + \langle\langle \xi, w' \rangle\rangle + \frac{1}{2}(\langle\langle \xi, xi \rangle\rangle - iv) \right) \, ,$$

where $w', \xi \in \mathbb{C}^{n-1}$ and $v \in \mathbb{R}$.

In particular $\mathcal{H}_n$ acts simply transitively on $\partial\mathbb{H}^n_{\mathbb{C}} \backslash \{\infty\}$, and one obtains the *upper half space model* for complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ by identifying $\mathbb{C}^{n-1} \times \mathbb{R} \times [0, \infty)$ and $\overline{\mathbb{H}^n_{\mathbb{C}}} \backslash \{\infty\}$ as

$$(\xi, v, u) \longmapsto \begin{bmatrix} \xi \\ \frac{1}{2}(1 - \langle\langle \xi, \xi \rangle\rangle - u + iv) \\ \frac{1}{2}(1 + \langle\langle \xi, \xi \rangle\rangle + u - iv) \end{bmatrix} \, ,$$

where $(\xi, v, u) \in \mathbb{C}^{n-1} \times \mathbb{R} \times [0, \infty)$ are the horospherical coordinates of the corresponding point in $\overline{\mathbb{H}^n_{\mathbb{C}}} \backslash \{\infty\}$ (with respect to the point $\infty \in \partial\mathbb{H}^n_{\mathbb{C}}$, see [GP1]).

We notice that, under this identification, the horospheres in $\mathbb{H}^n_{\mathbb{C}}$ centered at ∞ are the horizontal slices $H_t = \{(\xi, v, u) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+ \, : \, u = t\}$, and the geodesics running to ∞ are the vertical lines $c_{\xi,v}(t) = (\xi, v, e^{2t})$ passing through points $(\xi, v) \in \mathbb{C}^{n-1} \times \mathbb{R}$. Thus we see that, via the geodesic perspective from ∞, various horospheres correspond as $H_t \to H_u$ with $(\xi, v, t) \mapsto (\xi, v, u)$.

The "boundary plane" $\mathbb{C}^{n-1} \times \mathbb{R} \times \{0\} = \partial\mathbb{H}^n_{\mathbb{C}} \backslash \{\infty\}$ and the horospheres $H_u = \mathbb{C}^{n-1} \times \mathbb{R} \times \{u\}$, $0 < u < \infty$, centered at ∞ are identified with the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$. It is a 2-step nilpotent group with center $\{0\} \times \mathbb{R} \subset \mathbb{C}^{n-1} \times \mathbb{R}$, with the isometric action on itself and on $\mathbb{H}^n_{\mathbb{C}}$ by left translations:

$$T_{(\xi_0, v_0)} \, : \, (\xi, v, u) \longmapsto (\xi_0 + \xi \, , v_0 + v + 2 \, \mathrm{Im} \langle\langle \xi_0, \xi \rangle\rangle \, , u) \, ,$$

and the inverse of (ξ, v) is $(\xi, v)^{-1} = (-\xi, -v)$. The unitary group $U(n-1)$ acts on $\mathcal{H}_n$ and $\mathbb{H}^n_{\mathbb{C}}$ by rotations: $A(\xi, v, u) = (A\xi\,, v\,, u)$ for $A \in U(n-1)$. The semidirect product $\mathcal{H}(n) = \mathcal{H}_n \rtimes U(n-1)$ is naturally embedded in $U(n,1)$ as follows:

$$A \longmapsto \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(n,1) \quad \text{for} \quad A \in U(n-1)\,,$$

$$(\xi, v) \longmapsto \begin{pmatrix} I_{n-1} & \xi & \xi \\ -\bar{\xi}^t & 1 - \frac{1}{2}(|\xi|^2 - iv) & -\frac{1}{2}(|\xi|^2 - iv) \\ \bar{\xi}^t & \frac{1}{2}(|\xi|^2 - iv) & 1 + \frac{1}{2}(|\xi|^2 - iv) \end{pmatrix} \in U(n,1)$$

where $(\xi, v) \in \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$ and $\bar{\xi}^t$ is the conjugate transpose of ξ.

The action of $\mathcal{H}(n)$ on $\overline{\mathbb{H}^n_{\mathbb{C}}} \backslash \{\infty\}$ also preserves the Cygan metric ρ_c there, which plays the same role as the Euclidean metric does on the upper half-space model of the real hyperbolic space $\mathbb{H}^n$ and is induced by the following norm:

$$||(\xi, v, u)||_c = |\,||\xi||^2 + u - iv|^{1/2}\,, \quad (\xi, v, u) \in \mathbb{C}^{n-1} \times \mathbb{R} \times [0, \infty)\,. \tag{2.1}$$

The relevant geometry on each horosphere $H_u \subset \mathbb{H}^n_{\mathbb{C}}$, $H_u \cong \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$, is the spherical CR-geometry induced by the complex hyperbolic structure. The geodesic perspective from ∞ defines CR-maps between horospheres, which extend to CR-maps between the one-point compactifications $H_u \cup \infty \approx S^{2n-1}$. In the limit, the induced metrics on horospheres fail to converge but the CR-structure remains fixed. In this way, the complex hyperbolic geometry induces CR-geometry on the sphere at infinity $\partial\mathbb{H}^n_{\mathbb{C}} \approx S^{2n-1}$, naturally identified with the one-point compactification of the Heisenberg group $\mathcal{H}_n$.

3. Discrete actions on nilpotent groups and Heisenberg manifolds

In order to study the structure of Heisenberg manifolds (i.e., the manifolds locally modeled on the Heisenberg group $\mathcal{H}_n$) and cusp ends of complex hyperbolic manifolds, we need a Bieberbach type structural theorem for isometric discrete group actions on $\mathcal{H}_n$, originally proved in [AX1]. It claims that each discrete isometry group of the Heisenberg group $\mathcal{H}_n$ preserves some left coset of a connected Lie subgroup, on which the group action is cocompact.

Here we consider more general situation. Let N be a connected, simply connected nilpotent Lie group, C a compact group of automorphisms of N, and Γ a discrete subgroup of the semidirect product $N \rtimes C$. Such discrete groups are the holonomy groups of parabolic ends of locally symmetric rank one (negatively curved) manifolds and can be described as follows.

Theorem 3.1. *There exist a connected Lie subgroup V of N and a finite index normal subgroup Γ^* of Γ with the following properties:*

1. *There exists $b \in N$ such that $b\Gamma b^{-1}$ preserves V.*
2. *$V/b\Gamma b^{-1}$ is compact.*
3. *$b\Gamma^* b^{-1}$ acts on V by left translations and this action is free.*

Remark 3.2. (1) It immediately follows that any discrete subgroup $\Gamma \subset N \rtimes C$ is virtually nilpotent because it has a finite index subgroup $\Gamma^* \subset \Gamma$ isomorphic to a lattice in $V \subset N$.

(2) Here, compactness of C is an essential condition because of Margulis [MG2] construction of nonabelian free discrete subgroups Γ of $R^3 \rtimes GL(3,R)$.

(3) This theorem generalizes a Bieberbach theorem for Euclidean spaces, see [Wo], and strengthens a result by L.Auslander [Au] who claimed those properties not for whole group Γ but only for its finite index subgroup. Initially in [AX1], we proved this theorem for the Heisenberg group $\mathcal{H}_n$ where we used Margulis Lemma [MG1, BGS] and geometry of $\mathcal{H}_n$ in order to extend the classical arguments in [Wo]. In the case of general nilpotent groups, our proof uses different ideas and goes as follows (see[AX2] for details).

Sketch of Proof. Let $p : \Gamma \to C$ be the composition of the inclusion $\Gamma \subset N \rtimes C$ and the projection $N \rtimes C \to C$, G the identity component of $\overline{\Gamma N}$, and $\Gamma_1 = G \cap \Gamma$. Due to compactness of C, G has finite index in $\overline{\Gamma N}$, so Γ_1 has finite index in Γ. Let $W \subset N$ be the analytic subgroup pointwise fixed by $p(\Gamma_1)$. Due to [Au], for all $\gamma = (w,c) \in \Gamma_1$, w lies in W. Thus Γ preserves W and, by replacing N with W, we may assume that $\Phi = p(\Gamma)$ is finite.

Consider $\Gamma^* = \ker(p)$ which is a discrete subgroup of N and has finite index in Γ. Let V be the connected Lie subgroup of N in which Γ^* is a lattice. Then the conjugation action of Γ on Γ^* induces a Γ-action on V. We form the semi-direct product $V \rtimes \Gamma$ and let $K = \{(a^{-1},(a,1)) \in V \rtimes \Gamma : (a,1) \in \Gamma^*\}$. Obviously, K is a normal subgroup of $V \rtimes \Gamma$. Defining the maps $i : V \to V \rtimes \Gamma/K$ by $i(v) = (v,(1,1))K$ and $\pi : V \rtimes \Gamma/K \to \Phi$ by $\pi(v,(a,A)) = A$, we get a short exact sequence

$$1 \longrightarrow V \xrightarrow{\ i\ } V \rtimes \Gamma/K \xrightarrow{\ \pi\ } \Phi \longrightarrow 1\,.$$

Since any extension of a finite group by a simply connected nilpotent Lie group splits, there is a homomorphism $s : \Phi \to V \rtimes \Gamma/K$ such that $\pi \circ s = id_\Phi$. For each $A \in \Phi$, we fix an element $(f(A),(g(A),A)) \in V \rtimes \Gamma$ representing $s(A)$. Since s is a homomorphism, we have

$$g(AB)^{-1}f(AB)^{-1} = A\left(g(B)^{-1}f(B)^{-1}\right)g(A)^{-1}f(A)^{-1} \quad \text{for} \quad A,B \in \Phi\,. \tag{4.3}$$

Define $h : \Phi \to N$ by $h(A) = g(A)^{-1}f(A)^{-1}$. Then (2.4) shows that h is a cocycle. Since Φ is finite and N is a simply connected nilpotent Lie group, $H^1(\Phi,N) = 0$ due to [LR]. Thus there exists $b \in N$ such that $h(A) = A(b^{-1})b$ for all $A \in \Phi$.

On the other hand, $\pi((1,(a,A))K) = \pi((f(A),(g(A),A))K) = A$ for any $\gamma = (a,A) \in \Gamma$. It follows that there is $v_0 \in V$ such that $a^{-1}v_0 = h(A)$. This and (4.3) imply that $a^{-1}v_0 = A(b^{-1})b$, and hence $baA(b^{-1}) = bv_0b^{-1}$.

Now consider the group $b\Gamma b^{-1}$ which acts on bVb^{-1}., For any $\gamma = (a,A) \in \Gamma$, the action of the element $b\gamma b^{-1} = (baA(b^{-1}),A)$ on bVb^{-1} is as follows:

$$((baA(b^{-1}),A),v') \to baA(b^{-1})A(v')(baA(b^{-1}))^{-1}\,.$$

In particular, $baA(b^{-1})\cdot A(bVb^{-1})\cdot(baA(b^{-1}))^{-1} = bVb^{-1}$. Therefore, $A(bVb^{-1}) = bVb^{-1}$ because of $baA(b^{-1}) = bv_0b^{-1} \in bVb^{-1}$, and hence $b\gamma b^{-1}$ preserves bVb^{-1}. □

Now we can apply our description of discrete group actions on a nilpotent group (Theorem 3.1) to study the structure of Heisenberg manifolds. Such manifolds are locally modeled on the $(\mathcal{H}_n, \mathcal{H}(n))$-geometry and each of them can be represented as the quotient $\mathcal{H}_n/G$ under a discrete, free isometric action of its fundamental group G on $\mathcal{H}_n$, i.e., the isometric action of a torsion free discrete subgroup of $\mathcal{H}(n) = \mathcal{H}_n \rtimes U(n-1)$. Actually, we establish fiber bundle structures on all noncompact Heisenberg manifolds:

Theorem 3.3. *Let $\Gamma \subset \mathcal{H}_n \rtimes U(n-1)$ be a torsion-free discrete group acting on the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$ with non-compact quotient. Then the quotient $\mathcal{H}_n/\Gamma$ has zero Euler characteristic and is a vector bundle over a compact manifold. Furthermore, this compact manifold is finitely covered by a nil-manifold which is either a torus or the total space of a circle bundle over a torus.*

The proof of this claim (see [AX1]) is based on two facts due to Theorem 3.1. First, that the discrete holonomy group $\Gamma \cong \pi_1(M)$ of any noncompact Heisenberg manifold $M = \mathcal{H}_n/\Gamma$, $\Gamma \subset \mathcal{H}(n)$, has a proper Γ-invariant subspace $\mathcal{H}_\Gamma \subset \mathcal{H}_n$. And second, the compact manifold $\mathcal{H}_\Gamma/\Gamma$ is finitely covered by $\mathcal{H}_\Gamma/\Gamma^*$ where Γ^* acts on $\mathcal{H}_\Gamma$ by translations. The structure of the covering manifold $\mathcal{H}_G/G^*$ is given in the following lemma.

Lemma 3.4. *Let V be a connected Lie subgroup of the Heisenberg group $\mathcal{H}_n$ and $G \subset V$ a discrete co-compact subgroup of V. Then the manifold V/G is*

1. *a torus if V is Abelian;*
2. *the total space of a torus bundle over a torus if V is not Abelian.*

Though noncompact Heisenberg manifolds M are vector bundles $\mathcal{H}_n/\Gamma \to \mathcal{H}_\Gamma/\Gamma$, simple examples show [AX1] that such vector bundles may be non-trivial in general. However, up to finite coverings, they are trivial [AX1]:

Theorem 3.5. *Let $\Gamma \subset \mathcal{H}_n \rtimes U(n-1)$ be a discrete group and $\mathcal{H}_\Gamma \subset \mathcal{H}_n$ a connected Γ-invariant Lie subgroup on which Γ acts co-compactly. Then there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ such that the vector bundle $\mathcal{H}_n/\Gamma_0 \to \mathcal{H}_\Gamma/\Gamma_0$ is trivial. In particular, any Heisenberg orbifold $\mathcal{H}_n/\Gamma$ is finitely covered by the product of a compact nil-manifold $\mathcal{H}_\Gamma/\Gamma_0$ and an Euclidean space.*

We remark that in the case when $\Gamma \subset \mathcal{H}_n \rtimes U(n-1)$ is a lattice, that is the quotient $\mathcal{H}_n/\Gamma$ is compact, the existence of such finite cover of $\mathcal{H}_n/\Gamma$ by a closed nilpotent manifold $\mathcal{H}_n/\Gamma_0$ is due to Gromov [Gr] and Buser-Karcher [BK] results for almost flat manifolds.

Our proof of Theorem 3.5 has the following scheme. Firstly, passing to a finite index subgroup, we may assume that the group Γ is torsion-free. After that, we shall find a finite index subgroup $\Gamma_0 \subset \Gamma$ whose rotational part is "good". Then we

shall express the vector bundle $\mathcal{H}_n/\Gamma_0 \to \mathcal{H}_\Gamma/\Gamma_0$ as the Whitney sum of a trivial bundle and a fiber product. We finish the proof by using the following criterion about the triviality of fiber products:

Lemma 3.6. *Let $F \times_H V$ be a fiber product and suppose that the homomorphism $\rho : H \to GL(V)$ extends to a homomorphism $\rho : F \to GL(V)$. Then $F \times_H V$ is a trivial bundle, $F \times_H V \cong F/H \times V$.*

Proof. The isomorphism $F \times_H V \cong F/H \times V$ is given by $[f, v] \to (Hf, \rho(f)^{-1}(v))$. □

4. Geometrical finiteness in complex hyperbolic geometry

Our main assumption on a complex hyperbolic n-manifold M is the geometrical finiteness of its fundamental group $\pi_1(M) = G \subset PU(n,1)$, which in particular implies that the discrete group G is finitely generated.

Here a subgroup $G \subset PU(n,1)$ is called *discrete* if it is a discrete subset of $PU(n,1)$. The *limit set* $\Lambda(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$ of a discrete group G is the set of accumulation points of (any) orbit $G(y)$, $y \in \mathbb{H}^n_{\mathbb{C}}$. The complement of $\Lambda(G)$ in $\partial\mathbb{H}^n_{\mathbb{C}}$ is called the *discontinuity set* $\Omega(G)$. A discrete group G is called *elementary* if its limit set $\Lambda(G)$ consists of at most two points. An infinite discrete group G is called *parabolic* if it has exactly one fixed point $\mathrm{fix}(G)$; then $\Lambda(G) = \mathrm{fix}(G)$, and G consists of either parabolic or elliptic elements. As it was observed by many authors (cf. [MaG]), parabolicity in the variable curvature case is not as easy a condition to deal with as it is in the constant curvature space. However the results of §2 simplify the situation, especially for geometrically finite groups.

Geometrical finiteness has been essentially used for real hyperbolic manifolds, where geometric analysis and ideas of Thurston provided powerful tools for understanding of their structure. Due to the absence of totally geodesic hypersurfaces in a space of variable negative curvature, we cannot use the original definition of geometrical finiteness which came from an assumption that the corresponding real hyperbolic manifold $M = \mathbb{H}^n/G$ may be decomposed into a cell by cutting along a finite number of its totally geodesic hypersurfaces, that is the group G should possess a finite-sided fundamental polyhedron, see [Ah]. However, we can define *geometrically finite* groups $G \subset PU(n,1)$ as those ones whose limit sets $\Lambda(G)$ consist of only conical limit points and parabolic (cusp) points p with compact quotients $(\Lambda(G)\backslash\{p\})/G_p$ with respect to parabolic stabilizers $G_p \subset G$ of p, see [BM, Bow]. There are other definitions of geometrical finiteness in terms of ends and the minimal convex retract of the noncompact manifold M, which work well not only in the real hyperbolic spaces $\mathbb{H}^n$ (see [Mar, Th, A1, A3]) but also in spaces with variable pinched negative curvature [Bow].

Our study of geometrical finiteness in complex hyperbolic geometry is based on analysis of geometry and topology of thin (parabolic) ends of corresponding manifolds and parabolic cusps of discrete isometry groups $G \subset PU(n,1)$.

Namely, suppose a point $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ is fixed by some parabolic element of a given discrete group $G \subset PU(n,1)$, and G_p is the stabilizer of p in G. Conjugating G by

an element $h_p \in PU(n,1)\,, h_p(p) = \infty$, we may assume that the stabilizer G_p is a subgroup $G_\infty \subset \mathcal{H}(n)$. In particular, if p is the origin $0 \in \mathcal{H}_n$, the transformation h_0 can be taken as the Heisenberg inversion $\mathcal{I}$ in the hyperchain $\partial\mathbb{H}^{n-1}_{\mathbb{C}}$. It preserves the unit Heisenberg sphere $S_c(0,1) = \{(\xi, v) \in \mathcal{H}_n \,:\, ||(\xi, v)||_c = 1\}$ and acts in $\mathcal{H}_n$ as follows:

$$\mathcal{I}(\xi, v) = \left(\frac{\xi}{|\xi|^2 - iv}\,,\, \frac{-v}{v^2 + |\xi|^4} \right) \quad \text{where } (\xi, v) \in \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}\,. \tag{4.1}$$

For any other point p, we may take h_p as the Heisenberg inversion $\mathcal{I}_p$ which preserves the unit Heisenberg sphere $S_c(p,1) = \{(\xi, v) \,:\, \rho_c(p, (\xi, v)) = 1\}$ centered at p. The inversion $\mathcal{I}_p$ is conjugate of $\mathcal{I}$ by the Heisenberg translation T_p and maps p to ∞.

After such a conjugation, we can apply Theorem 3.1 to the parabolic stabilizer $G_\infty \subset \mathcal{H}(n)$ and get a connected Lie subgroup $\mathcal{H}_\infty \subseteq \mathcal{H}_n$ preserved by G_∞ (up to changing the origin). So we can make the following definition.

Definition 4.2. A set $U_{p,r} \subset \overline{\mathbb{H}^n_{\mathbb{C}}} \backslash \{p\}$ is called a *standard cusp neighborhood of radius* $r > 0$ at a parabolic fixed point $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ of a discrete group $G \subset PU(n,1)$ if, for the Heisenberg inversion $\mathcal{I}_p \in PU(n,1)$ with respect to the unit sphere $S_c(p,1)$, $\mathcal{I}_p(p) = \infty$, the following conditions hold:

(1) $U_{p,r} = \mathcal{I}_p^{-1}\left(\{x \in \mathbb{H}^n_{\mathbb{C}} \cup \mathcal{H}_n \,:\, \rho_c(x, \mathcal{H}_\infty) \geq 1/r\}\right)$;
(2) $U_{p,r}$ is precisely invariant with respect to $G_p \subset G$, that is:

$$\gamma(U_{p,r}) = U_{p,r} \quad \text{for } \gamma \in G_p \quad \text{and} \quad g(U_{p,r}) \cap U_{p,r} = \emptyset \quad \text{for } g \in G \backslash G_p\,.$$

A parabolic point $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ of $G \subset PU(n,1)$ is called a *cusp point* if it has a cusp neighborhood $U_{p,r}$.

We remark that some parabolic points of a discrete group $G \subset PU(n,1)$ may not be cusp points, see examples in §5.4 of [AX1]. Applying Theorem 3.1 and [Bow], we have:

Lemma 4.3. *Let $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ be a parabolic fixed point of a discrete subgroup G in $PU(n,1)$. Then p is a cusp point if and only if $(\Lambda(G)\backslash\{p\})/G_p$ is compact.*

This and finiteness results of Bowditch [B] allow us to use another equivalent definitions of geometrical finiteness. In particular it follows that a discrete subgroup G in $PU(n,1)$ is *geometrically finite* if and only if its quotient space $M(G) = [\mathbb{H}^n_{\mathbb{C}} \cup \Omega(G)]/G$ has finitely many ends, and each of them is a cusp end, that is an end whose neighborhood can be taken (for an appropriate $r > 0$) in the form:

$$U_{p,r}/G_p \approx (S_{p,r}/G_p) \times (0, 1]\,, \tag{4.4}$$

where

$$S_{p,r} = \partial_H U_{p,r} = \mathcal{I}_p^{-1}\left(\{x \in H^n_{\mathbb{C}} \cup \mathcal{H}_n \,:\, \rho_c(x, \mathcal{H}_\infty) = 1/r\}\right)\,.$$

Now we see that a geometrically finite manifold can be decomposed into a compact submanifold and finitely many cusp submanifolds of the form (4.4). Clearly, each of such cusp ends is homotopy equivalent to a Heisenberg $(2n-1)$-manifold and moreover, due to Theorem 3.3, to a compact k-manifold, $k \leq 2n-1$. From the last fact, it follows that the fundamental group of a Heisenberg manifold is finitely presented, and we get the following finiteness result:

Corollary 4.5. *Geometrically finite groups $G \subset PU(n,1)$ are finitely presented.*

In the case of variable curvature, it is problematic to use geometric methods based on consideration of finite sided fundamental polyhedra, in particular, Dirichlet polyhedra $D_y(G)$ for $G \subset PU(n,1)$ bounded by bisectors in a complicated way, see [Mo2, GP1, FG]. In the case of discrete parabolic groups $G \subset PU(n,1)$, one may expect that the Dirichlet polyhedron $D_y(G)$ centered at a point y lying in a G-invariant subspace has finitely many sides. It is true for real hyperbolic spaces [A1] as well as for cyclic and dihedral parabolic groups in complex hyperbolic spaces. Namely, due to [Ph], Dirichlet polyhedra $D_y(G)$ are always two sided for any cyclic group $G \subset PU(n,1)$ generated by a Heisenberg translation. Due the main result in [GP1], this finiteness also holds for a cyclic ellipto-parabolic group or a dihedral parabolic group $G \subset PU(n,1)$ generated by inversions in asymptotic complex hyperplanes in $\mathbb{H}^n_{\mathbb{C}}$ if the central point y lies in a G-invariant vertical line or $\mathbb{R}$-plane (for any other center y, $D_y(G)$ has infinitely many sides). Our technique easily implies that this finiteness still holds for generic parabolic cyclic groups [AX1]:

Theorem 4.6. *For any discrete group $G \subset PU(n,1)$ generated by a parabolic element, there exists a point $y_0 \in \mathbb{H}^n_{\mathbb{C}}$ such that the Dirichlet polyhedron $D_{y_0}(G)$ centered at y_0 has two sides.*

Proof. Conjugating G and due to Theorem 3.1, we may assume that G preserves a one dimensional subspace $\mathcal{H}_\infty \subset \mathcal{H}_n$ as well as $\mathcal{H}_\infty \times \mathbb{R}_+ \subset \mathbb{H}^n_{\mathbb{C}}$, where G acts by translations. So we can take any point $y_0 \in \mathcal{H}_\infty \times \mathbb{R}_+$ as the central point of (two-sided) Dirichlet polyhedron $D_{y_0}(G)$ because its orbit $G(y_0)$ coincides with the orbit $G'(y_0)$ of a cyclic group generated by the Heisenberg translation induced by G. □

However, the behavior of Dirichlet polyhedra for parabolic groups $G \subset PU(n,1)$ of rank more than one can be very bad. It is given by our construction [AX1], where we have evaluated intersections of Dirichlet bisectors with a 2-dimensional slice:

Theorem 4.7. *Let $G \subset PU(2,1)$ be a discrete parabolic group conjugate to the subgroup $\Gamma = \{(m,n) \in \mathbb{C} \times \mathbb{R} : m,n \in \mathbb{Z}\}$ of the Heisenberg group $\mathcal{H}_2 = \mathbb{C} \times \mathbb{R}$. Then any Dirichlet polyhedron $D_y(G)$ centered at any point $y \in \mathbb{H}^2_{\mathbb{C}}$ has infinitely many sides.*

Despite the above example, the below application of Theorem 3.1 provides a construction of fundamental polyhedra $P(G) \subset \mathbb{H}^n_{\mathbb{C}}$ for arbitrary discrete parabolic

groups $G \subset PU(n,1)$, which are bounded by finitely many hypersurfaces (different from Dirichlet bisectors). This result may be seen as a base for extension of Apanasov's construction [A1] of finite sided pseudo-Dirichlet polyhedra in $\mathbb{H}^n$ to the case of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$.

Theorem 4.8. *For any discrete parabolic group $G \subset PU(n,1)$, there exists a finite-sided fundamental polyhedron $P(G) \subset \mathbb{H}^n_{\mathbb{C}}$.*

Proof. After conjugation, we may assume that $G \subset \mathcal{H}_n \rtimes U(n-1)$. Let $\mathcal{H}_\infty \subset \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$ be the connected G-invariant subgroup given by Theorem 3.1. For a fixed $u_0 > 0$, we consider the horocycle $V_{u_0} = \mathcal{H}_\infty \times \{u_0\} \subset \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+ = \mathbb{H}^n_{\mathbb{C}}$. For distinct points $y, y' \in V_{u_0}$, the bisector $C(y,y') = \{z \in \mathbb{H}^n_{\mathbb{C}} : d(z,y) = d(z,y')\}$ intersects V_{u_0} transversally. Since V_{u_0} is G-invariant, its intersection with a Dirichlet polyhedron

$$D_y(G) = \bigcap_{g \in G \setminus \{id\}} \{w \in \mathbb{H}^n_{\mathbb{C}} : d(w,y) < d(w,g(y))\}$$

centered at a point $y \in V_{u_0}$ is a fundamental polyhedron for the G-action on V_{u_0}. The polyhedron $D_y(G) \cap V_{u_0}$ is compact due to Theorem 3.3, and hence has finitely many sides. Now, considering G-equivariant projections [AX1]:

$$\pi : \mathcal{H}_n \to \mathcal{H}_\infty\,, \quad \pi' : \mathbb{H}^n_{\mathbb{C}} = \mathcal{H}_n \times \mathbb{R}_+ \to V_{u_0}\,, \quad \pi'(x,u) = (\pi(x), u_0)\,,$$

we get a finite-sided fundamental polyhedron ${\pi'}^{-1}(D_y(G) \cap V_{u_0})$ for the action of G in $\mathbb{H}^n_{\mathbb{C}}$.

□

Another important application of Theorem 3.1 shows that cusp ends of a geometrically finite complex hyperbolic orbifolds M have, up to a finite covering of M, a very simple structure:

Theorem 4.9. *Let $G \subset PU(n,1)$ be a geometrically finite discrete group. Then G has a subgroup G_0 of finite index such that every parabolic subgroup of G_0 is isomorphic to a discrete subgroup of the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$. In particular, each parabolic subgroup of G_0 is free Abelian or 2-step nilpotent.*

The proof of this fact [AX1] is based on the residual finiteness of geometrically finite subgroups in $PU(n,1)$ and the following two lemmas.

Lemma 4.10. *Let $G \subset \mathcal{H}_n \rtimes U(n-1)$ be a discrete group and $\mathcal{H}_G \subset \mathcal{H}_n$ a minimal G-invariant connected Lie subgroup (given by Theorem 3.1). Then G acts on $\mathcal{H}_G$ by translations if G is either Abelian or 2-step nilpotent.*

Lemma 4.11. *Let $G \subset \mathcal{H}_n \rtimes U(n-1)$ be a torsion free discrete group, F a finite group and $\phi : G \longrightarrow F$ an epimorphism. Then the rotational part of $\ker(\phi)$ has strictly smaller order than that of G if one of the following happens:*

1. *G contains a finite index Abelian subgroup and F is not Abelian;*
2. *G contains a finite index 2-step nilpotent subgroup and F is not a 2-step nilpotent group.*

We remark that the last Lemma generalizes a result of C.S.Aravinda and T.Farrell [AF] for Euclidean crystallographic groups.

We conclude this section by pointing out that the problem of geometrical finiteness is very different in complex dimension two. Namely, it is a well known fact that any finitely generated discrete subgroup of $PU(1,1)$ or $PO(2,1)$ is geometrically finite. This and Goldman's [Go1] local rigidity theorem for cocompact lattices $G \subset U(1,1) \subset PU(2,1)$ allow us to formulate the following conjecture:

Conjecture 4.12. *All finitely generated discrete groups $G \subset PU(2,1)$ with non-empty discontinuity set $\Omega(G) \subset \partial\mathbb{H}^2_{\mathbb{C}}$ are geometrically finite.*

5. Complex homology cobordisms and the boundary at infinity

The aim of this section is to study the topology of complex analytic "Kleinian" manifolds $M(G) = [\mathbb{H}^n_{\mathbb{C}} \cup \Omega(G)]/G$ with geometrically finite holonomy groups $G \subset PU(n,1)$. The boundary of this manifold, $\partial M = \Omega(G)/G$, has a spherical CR-structure and, in general, is non-compact.

We are especially interested in the case of complex analytic surfaces, where powerful methods of 4-dimensional topology may be used. It is still unknown what are suitable cuts of 4-manifolds, which (conjecturally) split them into geometric blocks (alike Jaco-Shalen-Johannson decomposition of 3-manifolds in Thurston's geometrization program; for a classification of 4-dimensional geometries, see [F, Wa]). Nevertheless, studying of complex surfaces suggests that in this case one can use integer homology 3-spheres and "almost flat" 3-manifolds (with virtually nilpotent fundamental groups). Actually, as Sections 3 and 4 show, the latter manifolds appear at the ends of finite volume complex hyperbolic manifolds. As it was shown by C.T.C.Wall [Wa], the assignment of the appropriate 4-geometry (when available) gives a detailed insight into the intrinsic structure of a complex surface. To identify complex hyperbolic blocks in such a splitting, one can use Yau's uniformization theorem [Ya]. It implies that every smooth complex projective 2-surface M with positive canonical bundle and satisfying the topological condition that $\chi(M) = 3\,\mathrm{Signature}(M)$, is a complex hyperbolic manifold. The necessity of homology sphere decomposition in dimension four is due to M.Freedman and L.Taylor result ([FT]):

Let M be a simply connected 4-manifold with intersection form q_M which decomposes as a direct sum $q_M = q_{M_1} \oplus q_{M_2}$, where M_1, M_2 are smooth manifolds. Then the manifold M can be represented as a connected sum $M = M_1 \#_\Sigma M_2$ along a homology sphere Σ.

Let us present an example of such a splitting, $M = X \#_\Sigma Y$, of a simply connected complex surface M with the intersection form Q_M into smooth manifolds (with boundary) X and Y, along a $\mathbb{Z}$-homology 3-sphere Σ such that $Q_M = Q_X \oplus Q_Y$. Here one should mention that though X and Y are no longer closed manifolds, the intersection forms Q_X and Q_Y are well defined on the second cohomology and are unimodular due to the condition that Σ is a $\mathbb{Z}$-homology 3-sphere.

Example 5.1. *Let M be the Kummer surface*

$$K3 = \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \, : \, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \, .$$

Then there are four disjointly embedded (Seifert fibered) $\mathbb{Z}$-homology 3-spheres in M, which split the Kummer surface into five blocks:

$$K3 = X_1 \cup_\Sigma Y_1 \cup_{\Sigma'} Y_2 \cup_{-\Sigma'} Y_3 \cup_{-\Sigma} X_2 \, ,$$

with intersection forms Q_{X_j} and Q_{Y_i} equal E_8 and H, respectively:

$$E_8 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix} , \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Here the $\mathbb{Z}$-homology spheres Σ and Σ' are correspondingly the Poincaré homology sphere $\Sigma(2,3,5)$ and Seifert fibered homology sphere $\Sigma(2,3,7)$; the minus sign means the change of orientation.

Scheme of splitting. Due to J.Milnor [Mil] (see also [RV]), all Seifert fibered homology 3-spheres Σ can be seen as the boundaries at infinity of (geometrically finite) complex hyperbolic orbifolds $\mathbb{H}^2_{\mathbb{C}}/\Gamma$, where the fundamental groups $\pi_1(\Sigma) = \Gamma \subset PU(2,1)$ act free in the sphere at infinity $\partial\mathbb{H}^2_{\mathbb{C}} = \overline{\mathcal{H}_2}$. In particular, the Seifert fibered homology sphere $\Sigma' = \Sigma(2,3,7)$ is diffeomorphic to the quotient $[(\mathbb{C} \times \mathbb{R}) \backslash (\{0\} \times \mathbb{R}]/\Gamma(2,3,7)$. Here $[(\mathbb{C} \times \mathbb{R}) \backslash (\{0\} \times \mathbb{R})]$ is the complement in the 3-sphere $\overline{\mathcal{H}_2} = \partial B^2_{\mathbb{C}}$ to the boundary circle at infinity of the complex geodesic $B^2_{\mathbb{C}} \cap (\mathbb{C} \times \{0\})$, and the group $\Gamma(2,3,7) \subset PU(2,1)$ acts on this complex geodesic as the standard triangle group $(2,3,7)$ in the disk Poincaré model of the hyperbolic 2-plane $\mathbb{H}^2_{\mathbb{R}}$.

This homology 3-sphere Σ' embeds in the $K3$-surface M, splitting it into submanifolds with intersection forms $E_8 \oplus H$ and $E_8 \oplus 2H$. This embedding is described in [Lo] and [FS1]. One can keep decomposing the obtained two manifolds as in [FS2] and finally split it into five pieces. Among additional embedded homology spheres, there is the only one known homology 3-sphere with finite fundamental group, the Poincaré homology sphere $\Sigma = \Sigma(2,3,5)$. One can introduce a spherical geometry on Σ by representing $\pi_1(\Sigma)$ as a finite subgroup $\Gamma(2,3,5)$ of the orthogonal group $O(4)$ acting free on $S^3 = \partial B^2_{\mathbb{C}}$. Then $\Sigma(2,3,5) = S^3/\Gamma(2,3,5)$ can obtained by identifying the opposite sides of the spherical dodecahedron whose dihedral angles are $2\pi/3$, see [KAG]. $\square$

However we note that it is unknown whether the obtained blocks may support some homogeneous 4-geometries classified by Filipkiewicz [F] and (from the point of view of Kähler structures) C.T.C.Wall [Wa]. This raises a question whether homogeneous geometries or splitting along homology spheres (important from the topological point of view) are relevant for a geometrization of smooth 4-dimensional manifolds. For example, neither of Y_i blocks in Example 5.1 (with the intersection form H) can support a complex hyperbolic structure (which is a natural geometric candidate since Σ has a spherical CR-structure) because each of them has two compact boundary components.

In fact, in a sharp contrast to the real hyperbolic case, for a compact manifold $M(G)$ (that is for a geometrically finite group $G \subset PU(n,1)$ without cusps), an application of Kohn-Rossi analytic extension theorem shows that the boundary of $M(G)$ is connected, and the limit set $\Lambda(G)$ is in some sense small (see [EMM] and, for quaternionic and Caley hyperbolic manifolds [C, CI]). Moreover, according to a recent result of D.Burns (see also Theorem 4.4 in [NR1]), the same claim about connectedness of the boundary $\partial M(G)$ still holds if only a boundary component is compact. (In dimension $n \geq 3$, D.Burns theorem based on [BuM] uses the last compactness condition to prove geometrical finiteness of the whole manifold $M(G)$, see also [NR2].)

However, if no component of $\partial M(G)$ is compact and we have no finiteness condition on the holonomy group of the complex hyperbolic manifold $M(G)$, the situation is completely different due to our construction [AX1]:

Theorem 5.2. *In any dimension $n \geq 2$ and for any integers k, k_0, $k \geq k_0 \geq 0$, there exists a complex hyperbolic n-manifold $M = \mathbb{H}^n_{\mathbb{C}}/G$, $G \subset PU(n,1)$, whose boundary at infinity splits up into k connected manifolds, $\partial_\infty M = N_1 \cup \cdots \cup N_k$. Moreover, for each boundary component N_j, $j \leq k_0$, its inclusion into the manifold $M(G)$, $i_j : N_j \subset M(G)$, induces a homotopy equivalence of N_j to $M(G)$.*

For a torsion free discrete group $G \subset PU(n,1)$, a connected component Ω_0 of the discontinuity set $\Omega(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$ with the stabilizer $G_0 \subset G$ is contractible and G-invariant if and only if the inclusion $N_0 = \Omega_0/G_0 \subset M(G)$ induces a homotopy equivalence of N_0 to $M(G)$ [A1, AX1]. It allows us to reformulate Theorem 5.2 as

Theorem 5.3. *In any complex dimension $n \geq 2$ and for any natural numbers k and k_0, $k \geq k_0 \geq 0$, there exists a discrete group $G = G(n,k,k_0) \subset PU(n,1)$ whose discontinuity set $\Omega(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$ splits up into k G-invariant components, $\Omega(G) = \Omega_1 \cup \cdots \cup \Omega_k$, and the first k_0 components are contractible.*

Sketch of Proof. To prove this claim (see [AX1] for details), it is crucial to construct a discrete group $G \subset PU(n,1)$ whose discontinuity set consists of two G-invariant topological balls. To do that, we construct an infinite family Σ of disjoint closed Heisenberg balls $B_i = B(a_i, r_i) \subset \partial\mathbb{H}^n_{\mathbb{C}}$ such that the complement of their closure, $\partial\mathbb{H}^n_{\mathbb{C}} \setminus \overline{\bigcup_i B(a_i, r_i)} = P_1 \cup P_2$, consists of two topological balls, P_1 and P_2. In our construction of such a family Σ of $\mathcal{H}$-balls B_j, we essentially relie on the contact structure of the Heisenberg group $\mathcal{H}_n$. Namely, Σ is the disjoint union of

finite sets Σ_i of closed $\mathcal{H}$-balls whose boundary $\mathcal{H}$-spheres have "real hyperspheres" serving as the boundaries of $(2n-2)$-dimensional cobordisms N_i. In the limit, these cobordisms converge to the set of limit vertices of the polyhedra P_1 and P_2 which are bounded by the $\mathcal{H}$-spheres $S_j = \partial B_j$, $B_j \in \Sigma$. Then the desired group $G = G(n,2,2) \subset PU(n,1)$ is generated by involutions $\mathcal{I}_j$ which preserve those real $(2n-3)$-spheres lying in $S_j \subset \partial P_1 \cup \partial P_2$, see Fig.1.

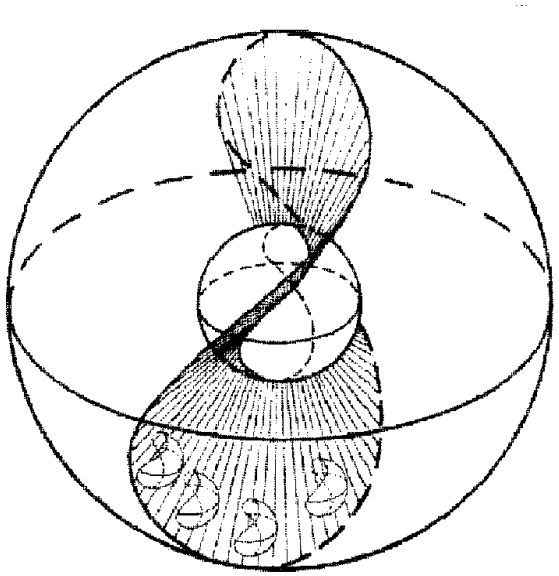

FIGURE 1. Cobordism N_0 in $\mathcal{H}$ with two boundary real circles

We notice that, due to our construction, the intersection of each $\mathcal{H}$-sphere S_j and each of the polyhedra P_1 and P_2 in the complement to the balls $B_j \in \Sigma$ is a topological $(2n-2)$-ball which splits into two sides, A_j and A'_j, and $\mathcal{I}_i(A_i) = A'_i$. This allows us to define our desired discrete group $G = G_\Sigma \subset PU(n,1)$ as the discrete free product, $G_\Sigma = *_j \, \Gamma_j = *_i \, \langle \mathcal{I}_j \rangle$, of infinitely many cyclic groups Γ_j generated by involutions $\mathcal{I}_j$ with respect to the $\mathcal{H}$-spheres $S_j = \partial B_j$. So $P_1 \cup P_2$ is a fundamental polyhedron for the action of G in $\partial \mathbb{H}^n_{\mathbb{C}}$, and sides of each of its connected components, P_1 or P_2, are topological balls pairwise equivalent with respect to the corresponding generators $\mathcal{I}_j \in G$. Applying standard arguments (see [A1], Lemmas 3.7, 3.8), we see that the discontinuity set $\Omega(G) \subset \mathcal{H}_n$ consists of two G-invariant topological balls Ω_1 and Ω_2, $\Omega_k = \text{int}\left(\bigcup_{g \in G} g(\overline{P_k})\right)$, $k = 1, 2$. The fact that Ω_k is a topological ball follows from the observation that this domain is the union of a monotone sequence,

$$V_0 = \text{int}(\overline{P_k}) \subset V_1 = \text{int}\left(\overline{P_k} \cup \mathcal{I}_0(\overline{P_k})\right) \subset V_2 \subset \dots,$$

of open topological balls, see [Br]. Note that here we use the property of our construction that V_i is always a topological ball.

In the general case of $k \geq k_0 \geq 0$, $k \geq 3$, we can apply the above infinite free products and our cobordism construction of infinite families of $\mathcal{H}$-balls with preassigned properties in order to (sufficiently closely) "approximate" a given hypersurface in $\mathcal{H}_n$ by the limit sets of constructed discrete groups. For such hypersurfaces, we use the so called "tree-like surfaces" which are boundaries of regular neighborhoods of trees in $\mathcal{H}_n$. This allows us to generalize A.Tetenov's [T1, KAG] construction of discrete groups G on the m-dimensional sphere S^m, $m \geq 3$, whose discontinuity sets split into any given number k of G-invariant contractible connected components. □

Although, in the general case of complex hyperbolic manifolds M with finitely generated $\pi_1(M) \cong G$, the problem on the number of boundary components of $M(G)$ is still unclear, we show below that the situation described in Theorem 5.3 is impossible if M is geometrically finite. We refer the reader to [AX1] for more precise formulation and proof of this cobordism theorem:

Theorem 5.4. *Let $G \subset PU(n,1)$ be a geometrically finite non-elementary torsion free discrete group whose Kleinian manifold $M(G)$ has non-compact boundary $\partial M = \Omega(G)/G$ with a component $N_0 \subset \partial M$ homotopy equivalent to $M(G)$. Then there exists a compact homology cobordism $M_c \subset M(G)$ such that $M(G)$ can be reconstructed from M_c by gluing up a finite number of open collars $M_i \times [0,\infty)$ where each M_i is finitely covered by the product $E_k \times B^{2n-k-1}$ of a closed (2n-1-k)-ball and a closed k-manifold E_k which is either flat or a nil-manifold (with 2-step nilpotent fundamental group).*

In connection to this cobordism theorem, it is worth to mention another interesting fact due to Livingston–Myers [My] construction. Namely, any $\mathbb{Z}$-homology 3-sphere is homology cobordant to a real hyperbolic one. However, it is still unknown whether one can introduce a geometric structure on such a homology cobordism, or a CR-structure on a given real hyperbolic 3-manifold (in particular, a homology sphere) or on a $\mathbb{Z}$-homology 3-sphere of plumbing type. We refer to [S, Mat] for recent advances on homology cobordisms, in particular, for results on Floer homology of homology 3-spheres and a new Saveliev's (presumably, homology cobordism) invariant based on Floer homology.

6. Homeomorphisms induced by group isomorphisms

As another application of the developed methods, we study the following well known problem of geometric realizations of group isomorphisms:

Problem 6.1. *Given a type preserving isomorphism $\varphi : G \to H$ of discrete groups $G, H \subset PU(n,1)$, find subsets $X_G, X_H \subset \overline{\mathbb{H}^n_{\mathbb{C}}}$ invariant for the action of groups G and H, respectively, and an equivariant homeomorphism $f_\varphi : X_G \to X_H$ which induces the isomorphism φ. Determine metric properties of f_φ, in particular, whether it is either quasisymmetric or quasiconformal.*

Such type problems were studied by several authors. In the case of lattices G and H in rank 1 symmetric spaces X, G.Mostow [Mo1] proved in his celebrated rigidity theorem that such isomorphisms $\varphi : G \to H$ can be extended to inner isomorphisms of X, provided that there is no analytic homomorphism of X onto $PSL(2,\mathbb{R})$. For that proof, it was essential to prove that φ can be induced by a quasiconformal homeomorphism of the sphere at infinity ∂X which is the one point compactification of a (nilpotent) Carnot group N (for quasiconformal mappings in Heisenberg and Carnot groups, see [KR, P]).

If geometrically finite groups $G, H \subset PU(n,1)$ have parabolic elements and are neither lattices nor trivial, the only results on geometric realization of their isomorphisms are known in the real hyperbolic space [Tu]. Generally, those methods cannot be used in the complex hyperbolic space due to lack of control over convex hulls (where the convex hull of three points may be 4-dimensional), especially nearby cusps. Another (dynamical) approach due to C.Yue [Yu2, Cor.B] (and the Anosov-Smale stability theorem for hyperbolic flows) can be used only for convex cocompact groups G and H, see [Yu3]. As a first step in solving the general Problem 6.1, we have the following isomorphism theorem [A7]:

Theorem 6.2. *Let $\phi : G \to H$ be a type preserving isomorphism of two non-elementary geometrically finite groups $G, H \subset PU(n,1)$. Then there exists a unique equivariant homeomorphism $f_\phi : \Lambda(G) \to \Lambda(H)$ of their limit sets that induces the isomorphism ϕ. Moreover, if $\Lambda(G) = \partial\mathbb{H}^n_{\mathbb{C}}$, the homeomorphism f_ϕ is the restriction of a hyperbolic isometry $h \in PU(n,1)$.*

Proof. To prove this claim, we consider the Cayley graph $K(G,\sigma)$ of a group G with a given finite set σ of generators. This is a 1-complex whose vertices are elements of G, and such that two vertices $a, b \in G$ are joined by an edge if and only if $a = bg^{\pm 1}$ for some generator $g \in \sigma$. Let $|*|$ be the word norm on $K(G,\sigma)$, that is, $|g|$ equals the minimal length of words in the alphabet σ representing a given element $g \in G$. Choosing a function ρ such that $\rho(r) = 1/r^2$ for $r > 0$ and $\rho(0) = 1$, one can define the length of an edge $[a,b] \subset K(G,\sigma)$ as $d_\rho(a,b) = \min\{\rho(|a|), \rho(|b|)\}$. Considering paths of minimal length in the sense of the function $d_\rho(a,b)$, one can extend it to a metric on the Cayley graph $K(G,\sigma)$. So taking the Cauchy completion $\overline{K(G,\sigma)}$ of that metric space, we have the definition of the group completion $\overline{G}$ as the compact metric space $\overline{K(G,\sigma)} \backslash K(G,\sigma)$, see [Fl]. Up to a Lipschitz equivalence, this definition does not depend on σ. It is also clear that, for a cyclic group $\mathbb{Z}$, its completion $\overline{\mathbb{Z}}$ consists of two points. Nevertheless, for a nilpotent group G with one end, its completion $\overline{G}$ is a one-point set [Fl].

Now we can define a proper equivariant embedding $F : K(G,\sigma) \hookrightarrow \mathbb{H}^n_{\mathbb{C}}$ of the Cayley graph of a given geometrically finite group $G \subset PU(n,1)$. To do that we may assume that the stabilizer of a point, say $0 \in \mathbb{H}^n_{\mathbb{C}}$, is trivial. Then we set $F(g) = g(0)$ for any vertex $g \in K(G,\sigma)$, and F maps any edge $[a,b] \subset K(G,\sigma)$ to the geodesic segment $[a(0), b(0)] \subset \mathbb{H}^n_{\mathbb{C}}$.

Proposition 6.3. *For a geometrically finite discrete group $G \subset PU(n,1)$, there*

are constants $K, K' > 0$ such that the following bounds hold for all elements $g \in G$ with $|g| \geq K'$:

$$\ln(2|g| - K)^2 - \ln K^2 \leq d(0, g(0)) \leq K|g| \,. \tag{6.4}$$

The proof of this claim is based on a comparison of the Bergman metric $d(*,*)$ and the path metric $d_0(*,*)$ on the following subset $bh_0 \subset \mathbb{H}^n_{\mathbb{C}}$. Let $C(\Lambda(G)) \subset \mathbb{H}^n_{\mathbb{C}}$ be the convex hull of the limit set $\Lambda(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$, that is the minimal convex subset in $\mathbb{H}^n_{\mathbb{C}}$ whose closure in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ contains $\Lambda(G)$. Clearly, it is G-invariant, and its quotient $C(\Lambda(G))/G$ is the minimal convex retract of $\mathbb{H}^n_{\mathbb{C}}/G$. Since G is geometrically finite, the complement in $M(G)$ to neighbourhoods of (finitely many) cusp ends is compact and correspond to a compact subset in the minimal convex retract, which can be taken as $\mathbb{H}_0/G$. In other words, $\mathbb{H}_0 \subset C(\Lambda(G))$ is the complement in the convex hull to a G-invariant family of disjoint horoballs each of which is strictly invariant with respect to its (parabolic) stabilizer in G, see [AX1, Bow], cf. also [A1, Th. 6.33]. Now, having co-compact action of the group G on the domain $\mathbb{H}_0$ whose boundary includes some horospheres, we can reduce our comparison of distances $d = d(x, x')$ and $d_0 = d_0(x, x')$ to their comparison on a horosphere. So we can take points $x = (0, 0, u)$ and $x' = (\xi, v, u)$ on a "horizontal" horosphere $S_u = \mathbb{C}^{n-1} \times \mathbb{R} \times \{u\} \subset \mathbb{H}^n_{\mathbb{C}}$. Then the distances d and d_0 are as follows [Pr2]:

$$\cosh^2 \frac{d}{2} = \frac{1}{4u^2} \left(|\xi|^4 + 4u|\xi|^2 + 4u^2 + v^2 \right) , \quad d_0^2 = \frac{|\xi|^2}{u} + \frac{v^2}{4u^2} \,. \tag{6.5}$$

This comparison and the basic fact due to Cannon [Can] that, for a co-compact action of a group G in a metric space X, its Cayley graph can be quasi-isometrically embedded into X, finish our proof of (6.4).

Now we apply Proposition 6.3 to define a G-equivariant extension of the map F from the Cayley graph $K(G, \sigma)$ to the group completion $\overline{G}$. Since the group completion of any parabolic subgroup $G_p \subset G$ is either a point or a two-point set (depending on whether G_p is a finite extension of cyclic or a nilpotent group with one end), we get

Theorem 6.6. *For a geometrically finite discrete group $G \subset PU(n, 1)$, there is a continuous G-equivariant map $\Phi_G : \overline{G} \to \Lambda(G)$. Moreover, the map Φ_G is bijective everywhere but the set of parabolic fixed points $p \in \Lambda(G)$ whose stabilizers $G_p \subset G$ have rank one. On this set, the map Φ_G is two-to-one.*

Now we can finish our proof of Theorem 6.2 by looking at the following diagram of maps:

$$\Lambda(G) \xleftarrow{\Phi_G} \overline{G} \xrightarrow{\overline{\phi}} \overline{H} \xrightarrow{\Phi_H} \Lambda(H) \,,$$

where the homeomorphism $\overline{\phi}$ is induced by the isomorphism ϕ, and the continuous maps Φ_G and Φ_H are defined by Theorem 6.6. Namely, one can define a map $f_\phi = \Phi_H \overline{\phi} \Phi_G^{-1}$. Here the map Φ_G^{-1} is the right inverse to Φ_G, which exists due to Theorem 6.6. Furthermore, the map Φ_G^{-1} is bijective everywhere but the set of parabolic fixed points $p \in \Lambda(G)$ whose stabilizers $G_p \subset G$ have rank one, where

it is 2-to-1. Hence the composition map f_ϕ is bijective and G-equivariant. Its uniqueness follows from its continuity and the fact that the image of the attractive fixed point of an loxodromic element $g \in G$ must be the attractive fixed point of the loxodromic element $\phi(g) \in H$ (such loxodromic fixed points are dense in the limit set, see [A1]).

The last claim of the Theorem 6.2 directly follows from the Mostow rigidity theorem [Mo1] because a geometrically finite group $G \subset PU(n,1)$ with $\Lambda(G) = \partial\mathbb{H}^n_{\mathbb{C}}$ is co-finite: $\mathrm{vol}\,(\mathbb{H}^n_{\mathbb{C}}/G) < \infty$.

□

Remark 6.7. Our proof of Theorem 6.2 can be easily extended to the general situation, that is, to construct equivariant homeomorphisms $f_\phi : \Lambda(G) \to \Lambda(H)$ conjugating the actions (on the limit sets) of isomorphic geometrically finite groups $G, H \subset \mathrm{Isom}\, X$ in a (symmetric) space X with pinched negative curvature K, $-b^2 \le K \le -a^2 < 0$. Actually, bounds similar to (6.4) in Prop. 6.3 (crucial for our argument) can be obtained from a result due to Heintze and Im Hof [HI, Th.4.6] which compares the geometry of horospheres $S_u \subset X$ with that in the spaces of constant curvature $-a^2$ and $-b^2$, respectively. It gives, that for all $x, y \in S_u$ and their distances $d = d(x,y)$ and $d_u = d_u(x,y)$ in the space X and in the horosphere S_u, respectively, one has that $\frac{2}{a}\sinh(a \cdot d/2) \le d_u \le \frac{2}{b}\sinh(b \cdot d/2)$.

Upon existence of such homeomorphisms f_φ inducing given isomorphisms φ of discrete subgroups of $PU(n,1)$, the Problem 6.1 can be reduced to the questions whether f_φ is quasisymmetric with respect to the Carnot-Carathéodory (or Cygan) metric, and whether there exists its G-equivariant extension to a bigger set (to the sphere at infinity ∂X or even to the whole space $\overline{\mathbb{H}^n_{\mathbb{C}}}$) inducing the isomorphism φ. For convex cocompact groups obtained by nearby representations, this may be seen as a generalization of D.Sullivan stability theorem [Su2], see also [A9].

However, in a deep contrast to the real hyperbolic case, here we have an interesting effect related to possible noncompactness of the boundary $\partial M(G) = \Omega(G)/G$. Namely, even for the simplest case of parabolic cyclic groups $G \cong H \subset PU(n,1)$, the homeomorphic CR-manifolds $\partial M(G) = \mathcal{H}^n/G$ and $\partial M(H) = \mathcal{H}^n/H$ may be not quasiconformally equivalent, see [Min]. In fact, among such Cauchy-Riemannian 3-manifolds (homeomorphic to $\mathbb{R}^2 \times S^1$), there are exactly two quasiconformal equivalence classes whose representatives have the holonomy groups generated correspondingly by a vertical $\mathcal{H}$-translation by $(0,1) \in \mathbb{C} \times \mathbb{R}$ and a horizontal $\mathcal{H}$-translation by $(1,0) \in \mathbb{C} \times \mathbb{R}$.

Theorem 7.1 presents a more sophisticated topological deformation $\{f_\alpha\}$, $f_\alpha : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$, of a "complex-Fuchsian" co-finite group $G \subset PU(1,1) \subset PU(2,1)$ to quasi-Fuchsian discrete groups $G_\alpha = f_\alpha G f_\alpha^{-1} \subset PU(2,1)$. It deforms pure parabolic subgroups in G to subgroups in G_α generated by Heisenberg "screw translations". As we point out, any such G-equivariant conjugations of the groups G and G_α cannot be contactomorphisms because they must map some poli of Dirichlet bisectors to non-poli ones in the image-bisectors; moreover, they cannot be quasiconformal, either. This shows the impossibility of the mentioned extension of

Sullivan's stability theorem to the case of groups with rank one cusps.

Also we note that, besides the metrical (quasisymmetric) part of the geometrization Problem 6.1, there are some topological obstructions for extensions of equivariant homeomorphisms f_φ, $f_\varphi : \Lambda(G) \to \Lambda(H)$. It follows from the next example.

Example 6.7. *Let $G \subset PU(1,1) \subset PU(2,1)$ and $H \subset PO(2,1) \subset PU(2,1)$ be two geometrically finite (loxodromic) groups isomorphic to the fundamental group $\pi_1(S_g)$ of a compact oriented surface S_g of genus $g > 1$. Then the equivariant homeomorphism $f_\varphi : \Lambda(G) \to \Lambda(H)$ cannot be homeomorphically extended to the whole sphere $\partial\mathbb{H}^2_{\mathbb{C}} \approx S^3$.*

Proof. The obstruction in this example is topological and is due to the fact that the quotient manifolds $M_1 = \mathbb{H}^2_{\mathbb{C}}/G$ and $M_2 = \mathbb{H}^2_{\mathbb{C}}/H$ are not homeomorphic. Namely, these complex surfaces are disk bundles over the Riemann surface S_g and have different Toledo invariants: $\tau(\mathbb{H}^2_{\mathbb{C}}/G) = 2g-2$ and $\tau(\mathbb{H}^2_{\mathbb{C}}/H) = 0$, see [To].

The complex structures of the complex surfaces M_1 and M_2 are quite different, too. The first manifold M_1 has a natural embedding of the Riemann surface S_g as a holomorphic totally geodesic closed submanifold, and hence M_1 cannot be a Stein manifolds. The second manifolds M_2 is a Stein manifold due to a result by Burns–Shnider [BS]. Moreover due to Goldman [Go1], since the surface $S_p \subset M_1$ is closed, the manifold M_1 is locally rigid in the sense that every nearby representation $G \to PU(2,1)$ stabilizes a complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$ and is conjugate to a representation $G \to PU(1,1) \subset PU(2,1)$. In other words, there are no non-trivial "quasi-Fuchsian" deformations of G and M_1. On the other hand, as we show in the next section (cf. Theorem 7.1), the second manifold M_2 has plentiful enough Teichmüller space of different "quasi-Fuchsian" complex hyperbolic structures.

□

7. Deformations of complex hyperbolic and CR-structures: flexibility versus rigidity

Since any real hyperbolic n-manifold can be (totally geodesically) embedded to a complex hyperbolic n-manifold $\mathbb{H}^n_{\mathbb{C}}/G$, flexibility of the latter ones is evident starting with hyperbolic structures on a Riemann surface of genus $g > 1$, which form Teichmüller space, a complex analytic $(3g-3)$-manifold. Strong rigidity starts in real dimension 3. Namely, due to the Mostow rigidity theorem [M1], hyperbolic structures of finite volume and (real) dimension at least three are uniquely determined by their topology, and one has no continuous deformations of them. Yet hyperbolic 3-manifolds have plentiful enough infinitesimal deformations and, according to Thurston's hyperbolic Dehn surgery theorem [Th], noncompact hyperbolic 3-manifolds of finite volume can be approximated by compact hyperbolic 3-manifolds.

Also, despite their hyperbolic rigidity, real hyperbolic manifolds M can be deformed as conformal manifolds, or equivalently as higher-dimensional hyperbolic manifolds $M \times (0,1)$ of infinite volume. First such quasi-Fuchsian deformations were given by the author [A2] and, after Thurston's "Mickey Mouse" example [Th], they

were called bendings of M along its totally geodesic hypersurfaces, see also [A1, A2, A4-A6, JM, Ko, Su1]. Furthermore, all these deformations are quasiconformally equivalent showing a rich supply of quasiconformal G-equivariant homeomorphisms in the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. In particular, the limit set $\Lambda(G) \subset \partial\mathbb{H}^{n+1}_{\mathbb{R}}$ deforms continuously from a round sphere $\partial\mathbb{H}^n_{\mathbb{R}} = S^{n-1} \subset S^n = \mathbb{H}^{n+1}_{\mathbb{R}}$ into nondifferentiably embedded topological $(n-1)$-spheres quasiconformally equivalent to S^{n-1}.

Contrasting to the above flexibility, "non-real" hyperbolic manifolds seem much more rigid. In particular, due to Pansu [P], quasiconformal maps in the sphere at infinity of quaternionic/octionic hyperbolic spaces are necessarily automorphisms, and thus there cannot be interesting quasiconformal deformations of corresponding structures. Secondly, due to Corlette's rigidity theorem [Co2], such manifolds are even super-rigid – analogously to Margulis super-rigidity in higher rank [MG1]. Furthermore, complex hyperbolic manifolds share the above rigidity of quaternionic/octionic hyperbolic manifolds. Namely, due to the Goldman's local rigidity theorem in dimension $n = 2$ [G1] and its extension for $n \geq 3$ [GM], every nearby discrete representation $\rho : G \to PU(n,1)$ of a cocompact lattice $G \subset PU(n-1,1)$ stabilizes a complex totally geodesic subspace $\mathbb{H}^{n-1}_{\mathbb{C}}$ in $\mathbb{H}^n_{\mathbb{C}}$. Thus the limit set $\Lambda(\rho G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$ is always a round sphere S^{2n-3}. In higher dimensions $n \geq 3$, this local rigidity of complex hyperbolic n-manifolds M homotopy equivalent to their closed complex totally geodesic hypersurfaces is even global due to a recent Yue's rigidity theorem [Yu1].

Our goal here is to show that, in contrast to rigidity of complex hyperbolic (non-Stein) manifolds M from the above class, complex hyperbolic Stein manifolds M are not rigid in general (we suspect that it is true for all Stein manifolds with "big" ends at infinity). Such a flexibility has two aspects.

First, we point out that the condition that the group $G \subset PU(n,1)$ preserves a complex totally geodesic hyperspace in $\mathbb{H}^n_{\mathbb{C}}$ is essential for local rigidity of deformations only for co-compact lattices $G \subset PU(n-1,1)$. This is due to the following our result [ACG]:

Theorem 7.1. *Let $G \subset PU(1,1)$ be a co-finite free lattice whose action in $\mathbb{H}^2_{\mathbb{C}}$ is generated by four real involutions (with fixed mutually tangent real circles at infinity). Then there is a continuous family $\{f_\alpha\}$, $-\epsilon < \alpha < \epsilon$, of G-equivariant homeomorphisms in $\overline{\mathbb{H}^2_{\mathbb{C}}}$ which induce non-trivial quasi-Fuchsian (discrete faithful) representations $f^*_\alpha : G \to PU(2,1)$. Moreover, for each $\alpha \neq 0$, any G-equivariant homeomorphism of $\overline{\mathbb{H}^2_{\mathbb{C}}}$ that induces the representation f^*_α cannot be quasiconformal.*

This and an Yue's [Yu2] result on Hausdorff dimension show that there are deformations of a co-finite Fuchsian group $G \subset PU(1,1)$ into quasi-Fuchsian groups $G_\alpha = f_\alpha G f_\alpha^{-1} \subset PU(2,1)$ with Hausdorff dimension of the limit set $\Lambda(G_\alpha)$ strictly bigger than one.

Secondly, we point out that the noncompactness condition in the above non-rigidity is not essential, either. Namely, complex hyperbolic Stein manifolds M homotopy equivalent to their closed totally *real* geodesic surfaces are not rigid, too. Namely, we give a canonical construction of continuous non-trivial quasi-Fuchsian

deformations of manifolds M, $\dim_{\mathbb{C}} M = 2$, fibered over closed Riemann surfaces, which are the first such deformations of fibrations over compact base (for a non-compact base corresponding to an ideal triangle group $G \subset PO(2,1)$, see [GP2]).

Our construction is inspired by the approach the author used for bending deformations of real hyperbolic (conformal) manifolds along totally geodesic hypersurfaces ([A2, A4]) and by an example of M.Carneiro–N.Gusevskii [Gu] constructing a non-trivial discrete representation of a surface group into $PU(2,1)$. In the case of complex hyperbolic (and Cauchy-Riemannian) structures, the constructed "bendings" work however in a different way than in the real case. Namely our complex bending deformations involve simultaneous bending of the base of the fibration of the complex surface M as well as bendings of each of its totally geodesic fibers (see Remark 7.9). Such bending deformations of complex surfaces are associated to their real simple closed geodesics (of real codimension 3), but have nothing common with the so called cone deformations of real hyperbolic 3-manifolds along closed geodesics, see [A6, A9].

Furthermore, there are well known complications in constructing equivariant homeomorphisms in the complex hyperbolic space and in Cauchy-Riemannian geometry, which are due to necessary invariantness of the Kähler and contact structures (correspondingly in $\mathbb{H}^n_{\mathbb{C}}$ and at its infinity, $\overline{\mathcal{H}_n}$). Despite that, the constructed complex bending deformations are induced by equivariant homeomorphisms of $\overline{\mathbb{H}^n_{\mathbb{C}}}$, which are in addition quasiconformal:

Theorem 7.2. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a given (non-elementary) discrete group. Then, for any simple closed geodesic α in the Riemann 2-surface $S = H^2_{\mathbb{R}}/G$ and a sufficiently small $\eta_0 > 0$, there is a holomorphic family of G-equivariant quasiconformal homeomorphisms $F_\eta : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$, $-\eta_0 < \eta < \eta_0$, which defines the bending (quasi-Fuchsian) deformation $\mathcal{B}_\alpha : (-\eta_0, \eta_0) \to \mathcal{R}_0(G)$ of the group G along the geodesic α, $\mathcal{B}_\alpha(\eta) = F^*_\eta$.*

We notice that deformations of a complex hyperbolic manifold M may depend on many parameters described by the Teichmüller space $\mathcal{T}(M)$ of isotopy classes of complex hyperbolic structures on M. One can reduce the study of this space $\mathcal{T}(M)$ to studying the variety $\mathcal{T}(G)$ of conjugacy classes of discrete faithful representations $\rho : G \to PU(n,1)$ (involving the space $\mathcal{D}(M)$ of the developing maps, see [Go2, FG]). Here $\mathcal{T}(G) = \mathcal{R}_0(G)/PU(n,1)$, and the variety $\mathcal{R}_0(G) \subset \mathrm{Hom}(G, PU(n,1))$ consists of discrete faithful representations ρ of the group G with infinite co-volume, $\mathrm{Vol}(\mathbb{H}^n_{\mathbb{C}}/G) = \infty$. In particular, our complex bending deformations depend on many independent parameters as it can be shown by applying our construction and Élie Cartan [Car] angular invariant in Cauchy-Riemannian geometry:

Corollary 7.3. *Let $S_p = \mathbb{H}^2_{\mathbb{R}}/G$ be a closed totally real geodesic surface of genus $p > 1$ in a given complex hyperbolic surface $M = \mathbb{H}^2_{\mathbb{C}}/G$, $G \subset PO(2,1) \subset PU(2,1)$. Then there is an embedding $\pi \circ \mathcal{B} : B^{3p-3} \hookrightarrow \mathcal{T}(M)$ of a real $(3p-3)$-ball into the Teichmüller space of M, defined by bending deformations along disjoint closed geodesics in M and by the projection $\pi : \mathcal{D}(M) \to \mathcal{T}(M) = \mathcal{D}(M)/PU(2,1)$ in the*

development space $\mathcal{D}(M)$.

Basic Construction (Proof of Theorem 7.2). Now we start with a totally real geodesic surface $S = \mathbb{H}^2_{\mathbb{R}}/G$ in the complex surface $M = \mathbb{H}^2_{\mathbb{C}}/G$, where $G \subset PO(2,1) \subset PU(2,1)$ is a given discrete group, and fix a simple closed geodesic α on S. We may assume that the loop α is covered by a geodesic $A \subset \mathbb{H}^2_{\mathbb{R}} \subset \mathbb{H}^2_{\mathbb{C}}$ whose ends at infinity are ∞ and the origin of the Heisenberg group $\mathcal{H} = \mathbb{C} \times \mathbb{R}$, $\overline{\mathcal{H}} = \partial\mathbb{H}^2_{\mathbb{C}}$. Furthermore, using quasiconformal deformations of the Riemann surface S (in the Teichmüller space $\mathcal{T}(S)$, that is, by deforming the inclusion $G \subset PO(2,1)$ in $PO(2,1)$ by bendings along the loop α, see Corollary 3.3 in [A10]), we can assume that the hyperbolic length of α is sufficiently small and the radius of its tubular neighborhood is big enough:

Lemma 7.4. *Let g_α be a hyperbolic element of a non-elementary discrete group $G \subset PO(2,1) \subset PU(2,1)$ with translation length ℓ along its axis $A \subset \mathbb{H}^2_{\mathbb{R}}$. Then any tubular neighborhood $U_\delta(A)$ of the axis A of radius $\delta > 0$ is precisely invariant with respect to its stabilizer $G_0 \subset G$ if $\sinh(\ell/4) \cdot \sinh(\delta/2) \leq 1/2$. Furthermore, for sufficiently small ℓ, $\ell < 4\delta$, the Dirichlet polyhedron $D_z(G) \subset \mathbb{H}^2_{\mathbb{C}}$ of the group G centered at a point $z \in A$ has two sides a and a' intersecting the axis A and such that $g_\alpha(a) = a'$.*

Then the group G and its subgroups G_0, G_1, G_2 in the free amalgamated (or HNN-extension) decomposition of G have Dirichlet polyhedra $D_z(G_i) \subset \mathbb{H}^2_{\mathbb{C}}$, $i = 0, 1, 2$, centered at a point $z \in A = (0, \infty)$, whose intersections with the hyperbolic 2-plane $\mathbb{H}^2_{\mathbb{R}}$ have the shapes indicated in Figures 2-5.

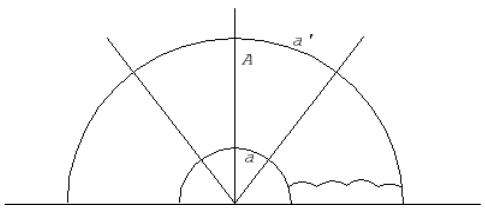

FIGURE 2. $G_1 \subset G = G_1 *_{G_0} G_2$

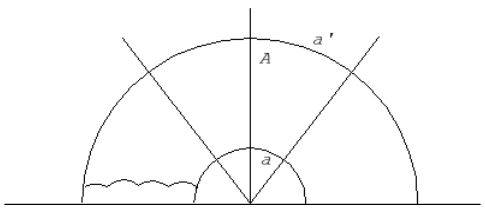

FIGURE 3. $G_2 \subset G = G_1 *_{G_0} G_2$

In particular we have that, except two bisectors $\mathfrak{S}$ and $\mathfrak{S}'$ that are equivalent under the hyperbolic translation g_α (which generates the stabilizer $G_0 \subset G$ of the axis A), all other bisectors bounding such a Dirichlet polyhedron lie in sufficiently small "cone neighborhoods" C_+ and C_- of the arcs (infinite rays) $\mathbb{R}_+$ and $\mathbb{R}_-$ of the real circle $\mathbb{R} \times \{0\} \subset \mathbb{C} \times \mathbb{R} = \mathcal{H}$.

Actually, we may assume that the Heisenberg spheres at infinity of the bisectors $\mathfrak{S}$ and $\mathfrak{S}'$ have radii 1 and $r_0 > 1$, correspondingly. Then, for a sufficiently small ϵ, $0 < \epsilon << r_0 - 1$, the cone neighborhoods $C_+, C_- \subset \overline{\mathbb{H}^2_{\mathbb{C}}} \setminus \{\infty\} = \mathbb{C} \times \mathbb{R} \times [0, +\infty)$ are

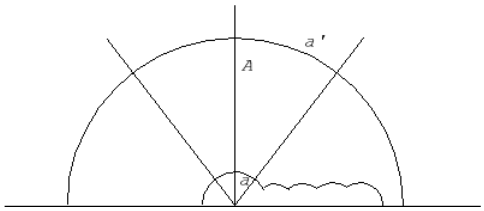

FIGURE 4. $G_1 \subset G = G_1 *_{G_0}$

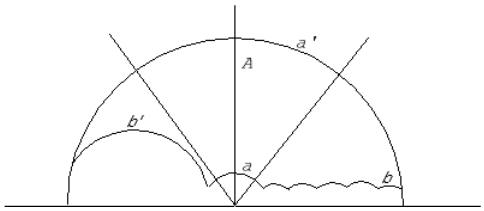

FIGURE 5. $G = G_1 *_{G_0}$

correspondingly the cones of the ϵ-neighborhoods of the points $(1,0,0), (-1,0,0) \in \mathbb{C} \times \mathbb{R} \times [0,+\infty)$ with respect to the Cygan metric ρ_c in $\overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \{\infty\}$, see (2.1).

Clearly, we may consider the length ℓ of the geodesic α so small that closures of all equidistant halfspaces in $\overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \{\infty\}$ bounded by those bisectors (and whose interiors are disjoint from the Dirichlet polyhedron $D_z(G)$) do not intersect the co-vertical bisector whose infinity is $i\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}$. It follows from the fact [Go3, Thm VII.4.0.3] that equidistant half-spaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in $\mathbb{H}^2_{\mathbb{C}}$ are disjoint if and only if the half-planes $\mathfrak{S}_1 \cap \mathbb{H}^2_{\mathbb{R}}$ and $\mathfrak{S}_2 \cap \mathbb{H}^2_{\mathbb{R}}$ are disjoint, see Figures 2-5.

Now we are ready to define a quasiconformal bending deformation of the group G along the geodesic A, which defines a bending deformation of the complex surface $M = \mathbb{H}^2_{\mathbb{C}}/G$ along the given closed geodesic $\alpha \subset S \subset M$.

We specify numbers η and ζ such that $0 < \zeta < \pi/2$, $0 \le \eta < \pi - 2\zeta$ and the intersection $C_+ \cap (\mathbb{C} \times \{0\})$ is contained in the angle $\{z \in \mathbb{C} : |\arg z| \le \zeta\}$. Then we define a bending homeomorphism $\phi = \phi_{\eta,\zeta} : \mathbb{C} \to \mathbb{C}$ which bends the real axis $\mathbb{R} \subset \mathbb{C}$ at the origin by the angle η, see Fig. 6:

$$\phi_{\eta,\zeta}(z) = \begin{cases} z & \text{if } |\arg z| \ge \pi - \zeta \\ z \cdot \exp(i\eta) & \text{if } |\arg z| \le \zeta \\ z \cdot \exp(i\eta(1 - (\arg z - \zeta)/(\pi - 2\zeta))) & \text{if } \zeta < \arg z < \pi - \zeta \\ z \cdot \exp(i\eta(1 + (\arg z + \zeta)/(\pi - 2\zeta))) & \text{if } \zeta - \pi < \arg z < -\zeta\,. \end{cases} \tag{7.5}$$

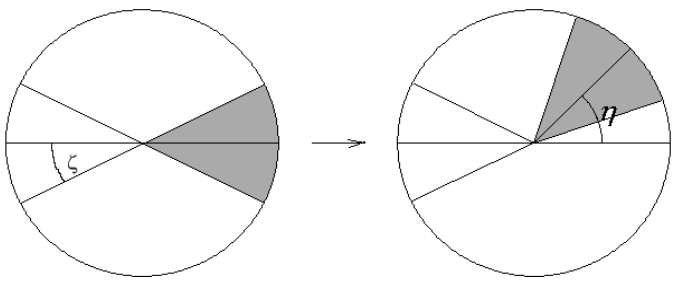

FIGURE 6

For negative η, $2\zeta - \pi < \eta < 0$, we set $\phi_{\eta,\zeta}(z) = \overline{\phi_{-\eta,\zeta}(\overline{z})}$. Clearly, $\phi_{\eta,\zeta}$ is quasiconformal with respect to the Cygan norm (2.1) and is an isometry in the ζ-cone neighborhood of the real axis $\mathbb{R}$ because its linear distortion is given by

$$K(\phi_{\eta,\zeta}, z) = \begin{cases} 1 & \text{if } |\arg z| \geq \pi - \zeta \\ 1 & \text{if } |\arg z| \leq \zeta \\ (\pi - 2\zeta)/(\pi - 2\zeta - \eta) & \text{if } \zeta < \arg z < \pi - \zeta \\ (\pi - 2\zeta + \eta)/(\pi - 2\zeta) & \text{if } \zeta - \pi < \arg z < -\zeta\,. \end{cases} \tag{7.6}$$

Foliating the punctured Heisenberg group $\mathcal{H}\backslash\{0\}$ by Heisenberg spheres $S(0,r)$ of radii $r > 0$, we can extend the bending homeomorphism $\phi_{\eta,\zeta}$ to an elementary bending homeomorphism $\varphi = \varphi_{\eta,\zeta} : \mathcal{H} \to \mathcal{H}$, $\varphi(0) = 0$, $\varphi(\infty) = \infty$, of the whole sphere $S^3 = \overline{\mathcal{H}}$ at infinity.

Namely, for the "dihedral angles" $W_+, W_- \subset \mathcal{H}$ with the common vertical axis $\{0\} \times \mathbb{R}$ and which are foliated by arcs of real circles connecting points $(0,v)$ and $(0,-v)$ on the vertical axis and intersecting the the ζ-cone neighborhoods of infinite rays $\mathbb{R}_+, \mathbb{R}_- \subset \mathbb{C}$, correspondingly, the restrictions $\varphi|_{W_-}$ and $\varphi|_{W_+}$ of the bending homeomorphism $\varphi = \varphi_{\eta,\zeta}$ are correspondingly the identity and the unitary rotation $U_\eta \in PU(2,1)$ by angle η about the vertical axis $\{0\} \times \mathbb{R} \subset \mathcal{H}$, see also [A10, (4.4)]. Then it follows from (7.6) that $\varphi_{\eta,\zeta}$ is a G_0-equivariant quasiconformal homeomorphism in $\overline{\mathcal{H}}$.

We can naturally extend the foliation of the punctured Heisenberg group $\mathcal{H}\backslash\{0\}$ by Heisenberg spheres $S(0,r)$ to a foliation of the hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ by bisectors $\mathfrak{S}_r$ having those $S(0,r)$ as the spheres at infinity. It is well known (see [M2]) that each bisector $\mathfrak{S}_r$ contains a geodesic γ_r which connects points $(0,-r^2)$ and $(0,r^2)$ of the Heisenberg group $\mathcal{H}$ at infinity, and furthermore $\mathfrak{S}_r$ fibers over γ_r by complex geodesics Y whose circles at infinity are complex circles foliating the sphere $S(0,r)$.

Using those foliations of the hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ and bisectors $\mathfrak{S}_r$, we extend the elementary bending homeomorphism $\varphi_{\eta,\zeta} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ at infinity to an elementary bending homeomorphism $\Phi_{\eta,\zeta} : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$. Namely, the map $\Phi_{\eta,\zeta}$ preserves each of bisectors $\mathfrak{S}_r$, each complex geodesic fiber Y in such bisectors, and fixes the intersection points y of those complex geodesic fibers and the complex geodesic connecting the origin and ∞ of the Heisenberg group $\mathcal{H}$ at infinity. We complete our extension $\Phi_{\eta,\zeta}$ by defining its restriction to a given (invariant) complex geodesic fiber Y with the fixed point $y \in Y$. This map is obtained by radiating the circle homeomorphism $\varphi_{\eta,\zeta}|_{\partial Y}$ to the whole (Poincaré) hyperbolic 2-plane Y along geodesic rays $[y,\infty) \subset Y$, so that it preserves circles in Y centered at y and bends (at y, by the angle η) the geodesic in Y connecting the central points of the corresponding arcs of the complex circle ∂Y, see Fig.6.

Due to the construction, the elementary bending (quasiconformal) homeomorphism $\Phi_{\eta,\zeta}$ commutes with elements of the cyclic loxodromic group $G_0 \subset G$. Another most important property of the homeomorphism $\Phi_{\eta,\zeta}$ is the following.

Let $D_z(G)$ be the Dirichlet fundamental polyhedron of the group G centered at a given point z on the axis A of the cyclic loxodromic group $G_0 \subset G$, and $\mathfrak{S}^+ \subset \mathbb{H}^2_{\mathbb{C}}$

be a "half-space" disjoint from $D_z(G)$ and bounded by a bisector $\mathfrak{S} \subset \mathbb{H}^2_{\mathbb{C}}$ which is different from bisectors $\mathfrak{S}_r, r > 0$, and contains a side $\mathfrak{s}$ of the polyhedron $D_z(G)$. Then there is an open neighborhood $U(\overline{\mathfrak{S}^+}) \subset \overline{\mathbb{H}^2_{\mathbb{C}}}$ such that the restriction of the elementary bending homeomorphism $\Phi_{\eta,\zeta}$ to it either is the identity or coincides with the unitary rotation $U_\eta \subset PU(2,1)$ by the angle η about the "vertical" complex geodesic (containing the vertical axis $\{0\} \times \mathbb{R} \subset \mathcal{H}$ at infinity).

The above properties of quasiconformal homeomorphism $\Phi = \Phi_{\eta,\zeta}$ show that the image $D_\eta = \Phi_{\eta,\zeta}(D_z(G))$ is a polyhedron in $\mathbb{H}^2_{\mathbb{C}}$ bounded by bisectors. Furthermore, there is a natural identification of its sides induced by $\Phi_{\eta,\zeta}$. Namely, the pairs of sides preserved by Φ are identified by the original generators of the group $G_1 \subset G$. For other sides $\mathfrak{s}_\eta$ of D_η, which are images of corresponding sides $\mathfrak{s} \subset D_z(G)$ under the unitary rotation U_η, we define side pairings by using the group G decomposition (see Fig. 2-5).

Actually, if $G = G_1 *_{G_0} G_2$, we change the original side pairings $g \in G_2$ of $D_z(G)$-sides to the hyperbolic isometries $U_\eta g U_\eta^{-1} \in PU(2,1)$. In the case of HNN-extension, $G = G_1 *_{G_0} = \langle G_1, g_2 \rangle$, we change the original side pairing $g_2 \in G$ of $D_z(G)$-sides to the hyperbolic isometry $U_\eta g_2 \in PU(2,1)$. In other words, we define deformed groups $G_\eta \subset PU(2,1)$ correspondingly as

$$G_\eta = G_1 *_{G_0} U_\eta G_2 U_\eta^{-1} \quad \text{or} \quad G_\eta = \langle G_1, U_\eta g_2 \rangle = G_1 *_{G_0} \ . \tag{7.7}$$

This shows that the family of representations $G \to G_\eta \subset PU(2,1)$ does not depend on angles ζ and holomorphically depends on the angle parameter η. Let us also observe that, for small enough angles η, the behavior of neighboring polyhedra $g'(D_\eta)$, $g' \in G_\eta$ is the same as of those $g(D_z(G))$, $g \in G$, around the Dirichlet fundamental polyhedron $D_z(G)$. This is because the new polyhedron $D_\eta \subset \mathbb{H}^2_{\mathbb{C}}$ has isometrically the same (tesselations of) neighborhoods of its side-intersections as $D_z(G)$ had. This implies that the polyhedra $g'(D_\eta)$, $g' \in G_\eta$, form a tesselation of $\mathbb{H}^2_{\mathbb{C}}$ (with non-overlapping interiors). Hence the deformed group $G_\eta \subset PU(2,1)$ is a discrete group, and D_η is its fundamental polyhedron bounded by bisectors.

Using G-compatibility of the restriction of the elementary bending homeomorphism $\Phi = \Phi_{\eta,\zeta}$ to the closure $\overline{D_z(G)} \subset \overline{\mathbb{H}^2_{\mathbb{C}}}$, we equivariantly extend it from the polyhedron $\overline{D_z(G)}$ to the whole space $\mathbb{H}^2_{\mathbb{C}} \cup \Omega(G)$ accordingly to the G-action. In fact, in terms of the natural isomorphism $\chi : G \to G_\eta$ which is identical on the subgroup $G_1 \subset G$, we can write the obtained G-equivariant homeomorphism $F = F_\eta : \overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \Lambda(G) \to \overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \Lambda(G_\eta)$ in the following form:

$$\begin{aligned} F_\eta(x) &= \Phi_\eta(x) \quad \text{for} \quad x \in \overline{D_z(G)}, \\ F_\eta \circ g(x) &= g_\eta \circ F_\eta(x) \quad \text{for} \quad x \in \overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \Lambda(G),\ g \in G,\ g_\eta = \chi(g) \in G_\eta \, . \end{aligned} \tag{7.8}$$

Due to quasiconformality of Φ_η, the extended G-equivariant homeomorphism F_η is quasiconformal. Furthermore, its extension by continuity to the limit (real) circle $\Lambda(G)$ coincides with the canonical equivariant homeomorphism $f_\chi : \Lambda(G) \to$

$\Lambda(G_\eta)$ given by the isomorphism Theorem 6.2. Hence we have a G-equivariant quasiconformal self-homeomorphism of the whole space $\overline{\mathbb{H}^2_{\mathbb{C}}}$, which we denote as before by F_η.

The family of G-equivariant quasiconformal homeomorphisms F_η induces representations $F^*_\eta : G \to G_\eta = F_\eta G_2 F_\eta^{-1}$, $\eta \in (-\eta_0, \eta_0)$. In other words, we have a curve $\mathcal{B} : (-\eta_0, \eta_0) \to \mathcal{R}_0(G)$ in the variety $\mathcal{R}_0(G)$ of faithful discrete representations of G into $PU(2,1)$, which covers a nontrivial curve in the Teichmüller space $\mathcal{T}(G)$ represented by conjugacy classes $[\mathcal{B}(\eta)] = [F^*_\eta]$. We call the constructed deformation $\mathcal{B}$ the bending deformation of a given lattice $G \subset PO(2,1) \subset PU(2,1)$ along a bending geodesic $A \subset \mathbb{H}^2_{\mathbb{C}}$ with loxodromic stabilizer $G_0 \subset G$. In terms of manifolds, $\mathcal{B}$ is the bending deformation of a given complex surface $M = \mathbb{H}^2_{\mathbb{C}}/G$ homotopy equivalent to its totally real geodesic surface $S_g \subset M$, along a given simple geodesic α.

□

Remark 7.9. It follows from the above construction of the bending homeomorphism $F_{\eta,\zeta}$, that the deformed complex hyperbolic surface $M_\eta = \mathbb{H}^2_{\mathbb{C}}/G_\eta$ fibers over the pleated hyperbolic surface $S_\eta = F_\eta(\mathbb{H}^2_{\mathbb{R}})/G_\eta$ (with the closed geodesic α as the singular locus). The fibers of this fibration are "singular real planes" obtained from totally real geodesic 2-planes by bending them by angle η along complete real geodesics. These (singular) real geodesics are the intersections of the complex geodesic connecting the axis A of the cyclic group $G_0 \subset G$ and the totally real geodesic planes that represent fibers of the original fibration in $M = \mathbb{H}^2_{\mathbb{C}}/G$.

Proof of Corollary 7.3. Since, due to (7.7), bendings along disjoint closed geodesics are independent, we need to show that our bending deformation is not trivial, and $[\mathcal{B}(\eta)] \neq [\mathcal{B}(\eta')]$ for any $\eta \neq \eta'$.

The non-triviality of our deformation follows directly from (7.7), cf. [A9]. Namely, the restrictions $\rho_\eta|_{G_1}$ of bending representations to a non-elementary subgroup $G_1 \subset G$ (in general, to a "real" subgroup $G_r \subset G$ corresponding to a totally real geodesic piece in the homotopy equivalent surface $S \backsimeq M$) are identical. So if the deformation $\mathcal{B}$ were trivial then it would be conjugation of the group G by projective transformations that commute with the non-trivial real subgroup $G_r \subset G$ and pointwise fix the totally real geodesic plane $\mathbb{H}^2_{\mathbb{R}}$. This contradicts to the fact that the limit set of any deformed group G_η, $\eta \neq 0$, does not belong to the real circle containing the limit Cantor set $\Lambda(G_r)$.

The injectivity of the map $\mathcal{B}$ can be obtained by using Élie Cartan [Car] angular invariant $\mathbb{A}(x)$, $-\pi/2 \leq \mathbb{A}(x) \leq \pi/2$, for a triple $x = (x^0, x^1, x^2)$ of points in $\partial\mathbb{H}^2_{\mathbb{C}}$. It is known (see [Go3]) that, for two triples x and y, $\mathbb{A}(x) = \mathbb{A}(y)$ if and only if there exists $g \in PU(2,1)$ such that $y = g(x)$; furthermore, such a g is unique provided that $\mathbb{A}(x)$ is neither zero nor $\pm\pi/2$. Here $\mathbb{A}(x) = 0$ if and only if x^0, x^1 and x^2 lie on an $\mathbb{R}$-circle, and $\mathbb{A}(x) = \pm\pi/2$ if and only if x^0, x^1 and x^2 lie on a chain ($\mathbb{C}$-circle).

Namely, let $g_2 \in G \backslash G_1$ be a generator of the group G in (4.5) whose fixed point $x^2 \in \Lambda(G)$ lies in $\mathbb{R}_+ \times \{0\} \subset \mathcal{H}$, and $x^2_\eta \in \Lambda(G_\eta)$ the corresponding fixed point

of the element $\chi_\eta(g_2) \in G_\eta$ under the free-product isomorphism $\chi_\eta : G \to G_\eta$. Due to our construction, one can see that the orbit $\gamma(x^2_\eta)$, $\gamma \in G_0$, under the loxodromic (dilation) subgroup $G_0 \subset G \cap G_\eta$ approximates the origin along a ray $(0, \infty)$ which has a non-zero angle η with the ray $\mathbb{R}_- \times \{0\} \subset \mathcal{H}$. The latter ray also contains an orbit $\gamma(x^1)$, $\gamma \in G_0$, of a limit point x^1 of G_1 which approximates the origin from the other side. Taking triples $x = (x^1, 0, x^2)$ and $x_\eta = (x^1, 0, x^2_\eta)$ of points which lie correspondingly in the limit sets $\Lambda(G)$ and $\Lambda(G_\eta)$, we have that $\mathbb{A}(x) = 0$ and $\mathbb{A}(x_\eta) \neq 0, \pm\pi/2$. Due to Theorem 6.2, both limit sets are topological circles which however cannot be equivalent under a hyperbolic isometry because of different Cartan invariants (and hence, again, our deformation is not trivial).

Similarly, for two different values η and η', we have triples x_η and $x_{\eta'}$ with different (non-trivial) Cartan angular invariants $\mathbb{A}(x_\eta) \neq \mathbb{A}(x_{\eta'})$. Hence $\Lambda(G_\eta)$ and $\Lambda(G_{\eta'})$ are not $PU(2,1)$-equivalent.

□

One can apply the above proof to a general situation of bending deformations of a complex hyperbolic surface $M = \mathbb{H}^2_{\mathbb{C}}/G$ whose holonomy group $G \subset PU(2,1)$ has a non-elementary subgroup G_r preserving a totally real geodesic plane $\mathbb{H}^2_{\mathbb{R}}$. In other words, such a complex surfaces M has an embedded totally real geodesic surface with geodesic boundary. In particular all complex surfaces constructed in [GKL] with a given Toledo invariant lie in this class. So we immediately have:

Corollary 7.10. *Let $M = \mathbb{H}^2_{\mathbb{C}}/G$ be a complex hyperbolic surface with embedded totally real geodesic surface $S_r \subset M$ with geodesic boundary, and $\mathcal{B} : (-\eta, \eta) \to \mathcal{D}(M)$ be the bending deformation of M along a simple closed geodesic $\alpha \subset S_r$. Then the map $\pi \circ \mathcal{B} : (-\eta, \eta) \to \mathcal{T}(M) = \mathcal{D}(M)/PU(2,1)$ is a smooth embedding provided that the limit set $\Lambda(G)$ of the holonomy group G does not belong to the G-orbit of the real circle $S^1_{\mathbb{R}}$ and the chain $S^1_{\mathbb{C}}$, where the latter is the infinity of the complex geodesic containing a lift $\tilde{\alpha} \subset \mathbb{H}^2_{\mathbb{C}}$ of the closed geodesic α, and the former one contains the limit set of the holonomy group $G_r \subset G$ of the geodesic surface S_r.*

□

As an application of the constructed bending deformations, we answer a well known question about cusp groups on the boundary of the Teichmüller space $\mathcal{T}(M)$ of a Stein complex hyperbolic surface M fibering over a compact Riemann surface of genus $p > 1$. It is a direct corollary of the following result, see [AG]:

Theorem 7.11. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a non-elementary discrete group S_p of genus $p \geq 2$. Then, for any simple closed geodesic α in the Riemann surface $S = H^2_{\mathbb{R}}/G$, there is a continuous deformation $\rho_t = f_t^*$ induced by G-equivariant quasiconformal homeomorphisms $f_t : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$ whose limit representation ρ_∞ corresponds to a boundary cusp point of the Teichmüller space $\mathcal{T}(G)$, that is, the boundary group $\rho_\infty(G)$ has an accidental parabolic element $\rho_\infty(g_\alpha)$ where $g_\alpha \in G$ represents the geodesic $\alpha \subset S$.*

We note that, due to our construction of such continuous quasiconformal deformations in [AG], they are independent if the corresponding geodesics $\alpha_i \subset S_p$ are disjoint. It implies the existence of a boundary group in $\partial\mathcal{T}(G)$ with "maximal" number of non-conjugate accidental parabolic subgroups:

Corollary 7.12. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a uniform lattice isomorphic to the fundamental group of a closed surface S_p of genus $p \geq 2$. Then there is a continuous deformation $R : \mathbb{R}^{3p-3} \to \mathcal{T}(G)$ whose boundary group $G_\infty = R(\infty)(G)$ has $(3p-3)$ non-conjugate accidental parabolic subgroups.*

Finally, we mention another aspect of the intrigue Problem 4.12 on geometrical finiteness of complex hyperbolic surfaces (see [AX1, AX2]) for which it may perhaps be possible to apply our complex bending deformations:

Problem. *Construct a geometrically infinite (finitely generated) discrete group $G \subset PU(2,1)$ whose limit set is the whole sphere at infinity, $\Lambda(G) = \partial\mathbb{H}^2_{\mathbb{C}} = \overline{\mathcal{H}}$, and which is the limit of convex cocompact groups $G_i \subset PU(2,1)$ from the Teichmüller space $\mathcal{T}(\Gamma)$ of a convex cocompact group $\Gamma \subset PU(2,1)$. Is that possible for a Schottky group Γ?*

REFERENCES

[Ah] Lars V. Ahlfors, Fundamental polyhedra and limit point sets of Kleinian groups. - Proc. Nat. Acad. Sci. USA, **55**(1966), 251-254.

[A1] Boris Apanasov, Discrete groups in Space and Uniformization Problems. - Math. and Appl., **40**, Kluwer Academic Publishers, Dordrecht, 1991.

[A2] _____, Nontriviality of Teichmüller space for Kleinian group in space.- Riemann Surfaces and Related Topics: Proc. 1978 Stony Brook Conference (I.Kra and B.Maskit, eds), Ann. of Math. Studies **97**, Princeton Univ. Press, 1981, 21-31.

[A3] _____, Geometrically finite hyperbolic structures on manifolds. - Ann. of Glob. Analysis and Geom., **1:3**(1983), 1-22.

[A4] _____, Thurston's bends and geometric deformations of conformal structures.- Complex Analysis and Applications'85, Publ. Bulgarian Acad. Sci., Sofia, 1986, 14-28.

[A5] _____, Nonstandard uniformized conformal structures on hyperbolic manifolds. - Invent. Math., **105**(1991), 137-152.

[A6] _____, Deformations of conformal structures on hyperbolic manifolds.- J. Diff. Geom. **35** (1992), 1-20.

[A7] _____, Canonical homeomorphisms in Heisenberg group induced by isomorphisms of discrete subgroups of $PU(n,1)$.- Preprint,1995; Russian Acad.Sci.Dokl.Math., to appear.

[A8] _____, Quasiconformality and geometrical finiteness in Carnot-Carathéodory and negatively curved spaces. - Math. Sci. Res. Inst. at Berkeley, 1996-019.

[A9] _____, Conformal geometry of discrete groups and manifolds. - W. de Gruyter, Berlin-New York, to appear.

[A10] _____, Bending deformations of complex hyperbolic surfaces. - Preprint 1996-062, Math. Sci. Res. Inst., Berkeley, 1996.

[ACG] Boris Apanasov, Mario Carneiro and Nikolay Gusevskii, Some deformations of complex hyperbolic surfaces. - In preparation.

[AG] Boris Apanasov and Nikolay Gusevskii, The boundary of Teichmüller space of complex hyperbolic surfaces. - In preparation.

[AX1] Boris Apanasov and Xiangdong Xie, Geometrically finite complex hyperbolic manifolds.- Preprint, 1995.

[AX2] _____, Manifolds of negative curvature and nilpotent groups.- Preprint, 1995.

[AT] Boris Apanasov and Andrew Tetenov, Nontrivial cobordisms with geometrically finite hyperbolic structures. - J. Diff. Geom., **28**(1988), 407-422.

[BGS] W. Ballmann, M. Gromov and V. Schroeder, Manifolds of Nonpositive Curvature. - Birkhäuser, Boston, 1985.

[BM] A.F. Beardon and B. Maskit, Limit points of Kleinian groups and finite sided fundamental polyhedra. - Acta Math. **132** (1974), 1-12.

[Be] Igor Belegradek, Discrete surface groups actions with accidental parabolics on complex hyperbolic plane.- Preprint, Univ. of Maryland.

[BE] Žarko Bižaca and John Etnyre, Smooth structures on collarable ends of 4-manifolds.- Preprint, 1996.

[Bor] A.Borel, Compact Clifford-Klein forms of symmetric spaces. - Topol.**2**(1962), 111-122.

[Bow] Brian Bowditch, Geometrical finiteness with variable negative curvature. - Duke J. Math., **77** (1995), 229-274.

[Br] M. Brown, The monotone union of open n-cells is an open n-cell. - Proc. Amer. Math. Soc. **12** (1961), 812-814.

[BS] D. Burns and S. Shnider, Spherical hypersurfaces in complex manifolds. - Inv. Math., **33**(1976), 223-246.

[BuM] Dan Burns and Rafe Mazzeo, On the geometry of cusps for $SU(n,1)$. - Preprint, Univ. of Michigan, 1994.

[BK] P.Buser and H.Karcher, Gromov's almost flat manifolds. - Asterisque **81** (1981), 1-148.

[Can] James W. Cannon, The combinatorial structure of cocompact discrete hyperbolic groups. - Geom. Dedicata **16** (1984), 123–148.

[Car] É.Cartan, Sur le groupe de la géométrie hypersphérique.- Comm.Math.Helv.**4**(1932), 158

[CG] S. Chen and L. Greenberg, Hyperbolic spaces.- Contributions to Analysis, Academic Press, New York, 1974, 49-87.

[Co1] Kevin Corlette, Hausdorff dimensions of limit sets. - Invent. Math. **102** (1990), 521-542.

[Co2] _____, Archimedian superrigidity and hyperbolic geometry. - Ann. of Math. **135** (1992), 165-182.

[Cy] J. Cygan, Wiener's test for Brownian motion on the Heisenberg group. - Colloquium Math. **39**(1978), 367-373.

[EMM] C.Epstein, R.Melrose and G.Mendoza, Resolvent of the Laplacian on strictly pseudoconvex domains. - Acta Math. **167**(1991), 1-106.

[F] R.P. Filipkiewicz, Four-dimensional geometries. - Ph.D.Thesis, Univ. of Warwick, 1984.

[FG] Elisha Falbel and Nikolay Gusevskii, Spherical CR-manifolds of dimension 3.- Bol. Soc. Bras. Mat. **25** (1994), 31-56.

[FS1] Ronald Fintushel and Ronald Stern, Homotopy $K3$ surfaces containing $\Sigma(2,3,7)$. - J. Diff. Geom. **34** (1991), 255–265.

[FS2] _____, Using Floer's exact triangle to compute Donaldson invariants. - The Floer memorial volume, Progr. Math. **133**, Birkhäuser, Basel, 1995, 435–444.

[FS3] _____, Surgery in cusp neighborhoods and the geography of irreducible 4-manifolds.- Invent. Math. **117** (1994), 455-523.

[FT] M.H. Freedman and L. Taylor, Λ-splitting 4-manifolds. - Topology **16** (1977), 181-184.

[Go1] William Goldman, Representations of fundamental groups of surfaces.- Geometry and topology, Lect. Notes Math. **1167**, Springer, 1985, 95-117.

[Go2] _____, Geometric structures on manifolds and varieties of representations. - Geometry of Group Representations, Contemp. Math. **74** (1988), 169-198.

[Go3] _____, Complex hyperbolic geometry. - Oxford Univ. Press, to appear.

[GKL] W.Goldman, M.Kapovich and B.Leeb, Complex hyperbolic manifolds homotopy equivalent to a Riemann surface. - Preprint, 1995.

[GM] William Goldman and John Millson, Local rigidity of discrete groups acting on complex hyperbolic space.- Invent. Math. **88**(1987), 495-520.

[GP1] William Goldman and John Parker, Dirichlet polyhedron for dihedral groups acting on complex hyperbolic space. - J. of Geom. Analysis, **2:6** (1992), 517-554.

[GP2] _____, Complex hyperbolic ideal triangle groups.- J.reine angew.Math.**425**(1992),71-86.

[Gr] M. Gromov, Almost flat manifolds. - J. Diff. Geom. **13** (1978), 231-141.

[Gu] Nikolay Gusevskii, Colloquium talk, Univ. of Oklahoma, Norman, December 1995.

[HI] Ernst Heintze and Hans-Christoph Im Hof, Geometry of horospheres. - J. Diff. Geom. **12** (1977), 481–491.

[JM] Dennis Johnson and John Millson, Deformation spaces associated to compact hyperbolic manifolds.- Discrete Groups in Geometry and Analysis,Birkhauser, 1987, 48-106.

[Kob] Shoshichi Kobayashi, Hyperbolic manifolds and holomorphic mappings.- M.Decker, 1970.

[KR] Adam Koranyi and Martin Reimann, Quasiconformal mappings on the Heisenberg group. - Invent. Math. **80** (1985), 309-338.

[Ko] Christos Kourouniotis, Deformations of hyperbolic structures.- Math. Proc. Cambr. Phil. Soc. **98** (1985), 247-261.

[KAG] Samuil Krushkal', Boris Apanasov and Nikolai Gusevskii, Kleinian groups and uniformization in examples and problems. - Trans. Math. Mono. **62**, Amer. Math. Soc., Providence, 1986, 1992.

[LB] Claude LeBrun, Einstein metrics and Mostow rigidity. - Preprint, Stony Brook, 1994.

[Lo] Eduard Looijenga, The smoothing components of a triangle singularity. - Proc. Symp. Pure Math. **40** (1983), 173–184.

[Mar] Albert Marden, The geometry of finitely generated Kleinian groups. - Ann. of Math., **99**(1974), 383-462.

[MG1] Gregory Margulis, Discrete groups of motions of manifolds of nonpositive curvature. - Amer. Math. Soc. Translations, **109**(1977), 33-45.

[MG2] _____, Free properly discontinuous groups of affine transformations.- Dokl. Acad. Sci. USSR, **272** (1983), 937-940.

[Mat] Rostislav Matveyev, A decomposition of smooth simply-connected h-cobordant 4-manifolds. - Preprint, Michigan State Univ. at E.Lansing, 1995.

[Mil] John Milnor, On the 3-dimensional Brieskorn manifolds $M(p,q,r)$.- Knots, groups and 3-manifolds, Ann. of Math. Studies **84**, Princeton Univ. Press, 1975, 175–225.

[Min] Robert Miner, Quasiconformal equivalence of spherical CR manifolds.- Ann. Acad. Sci. Fenn. Ser. A I Math. **19** (1994), 83-93.

[Mo1] G.D. Mostow, Strong rigidity of locally symmetric spaces.- Princeton Univ. Press, 1973.

[Mo2] _____, On a remarkable class of polyhedra in complex hyperbolic space. - Pacific J. Math., **86**(1980), 171-276.

[My] Robert Myers, Homology cobordisms, link concordances, and hyperbolic 3-manifolds.- Trans. Amer. Math. Soc. **278** (1983), 271-288.

[NR1] T. Napier and M. Ramachandran, Structure theorems for complete Kähler manifolds and applications to Lefschetz type theorems.- Geom. Funct. Anal. **5** (1995), 807-851.

[NR2] _____, The L^2 $\bar{\partial}$-method, weak Lefschetz theorems, and topology of Kähler manifolds.- Preprint, 1996.

[P] Pierre Pansu, Métriques de Carnot-Carathéodory et quasiisométries des espaces symmétries de rang un.- Ann. of Math. **129** (1989), 1-60.

[Pr1] John Parker, Shimizu's lemma for complex hyperbolic space. - Intern. J. Math. **3:2** (1992), 291-308.

[Pr2] _____, Private communication, 1995.

[Ph] M.B. Phillips, Dirichlet polyhedra for cyclic groups in complex hyperbolic space. - Proc. AMS, **115**(1992), 221-228.

[RV] F. Raymond and A.T. Vasquez, 3-manifolds whose universal coverings are Lie groups. - Topology Appl. **12** (1981), 161-179.

[Sa] Nikolai Saveliev, Floer homology and 3-manifold invariants.- Thesis, Univ. of Oklahoma at Norman, 1995.

[Su1] Dennis P. Sullivan, Discrete conformal groups and measurable dynamics. - Bull. Amer. Math. Soc. **6** (1982), 57–73.

[Su2] _____, Quasiconformal homeomorphisms and dynamics, II: Structural stability implies hyperbolicity for Kleinian groups. - Acta Math. **155** (1985), 243–260.

[T1] Andrew Tetenov, Infinitely generated Kleinian groups in space. - Siberian Math. J., **21**(1980), 709-717.

[T2] _____, The discontinuity set for a Kleinian group and topology of its Kleinian manifold. - Intern. J. Math., **4:1**(1993), 167-177.

[Th] William Thurston, The geometry and topology of three-manifolds. - Lect. Notes, Princeton Univ., 1981.

[To] Domingo Toledo, Representations of surface groups on complex hyperbolic space.- J. Diff. Geom. **29** (1989), 125-133.

[Tu] Pekka Tukia, On isomorphisms of geometrically finite Kleinian groups.- Publ. Math. IHES **61**(1985), 171-214.

[V] Serguei K. Vodopyanov, Quasiconformal mappings on Carnot groups.- Russian Dokl. Math. **347** (1996), 439-442.

[Wa] C.T.C. Wall, Geometric structures on compact complex analytic surfaces. - Topology, **25** (1986), 119-153.

[Wi] E.Witten, Monopoles and four-manifolds.- Math. Res. Lett. **1** (1994), 769-796.

[Wo] Joseph A. Wolf, Spaces of constant curvature.- Publ. or Perish, Berkeley, 1977.

[Ya] S.T.Yau, Calabi's conjecture and some new results in algebraic geometry.- Proc. Nat. Acad. Sci **74** (1977), 1798-1799.

[Yu1] Chengbo Yue, Dimension and rigidity of quasi-Fuchsian representations.- Ann. of Math. **143** (1996)331-355.

[Yu2] _____, Mostow rigidity of rank 1 discrete groups with ergodic Bowen-Margulis measure. - Invent. Math. **125** (1996), 75-102.

[Yu3] _____, Private communication, Norman/OK, November 1996.

Department of Mathematics, University of Oklahoma, Norman, OK 73019
E-mail address: Apanasov@ou.edu

Sobolev Inst. of Mathematics, Russian Acad. Sci., Novosibirsk, Russia 630090

Mathematical Sciences Research Institute, Berkeley, CA 94720-5070