RETARDATION OF PLATEAU-RAYLEIGH INSTABILITY: A DISTINGUISHING CHARACTERISTIC AMONG PERFECTLY WETTING FLUIDS

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ABSTRACT. We consider a cylindrical film of fluid adhering to a rigid cylinder of fixed radius. The main result is to give the critical (maximum) length for which such a film of given thickness can be stable. The problem is considered both when the cylinder remains stationary and when the fluid and the cylinder are co-rotating at a fixed angular velocity.

1. YOUNG'S EQUATION

The equation of Young asserts that the angle at which a fluid surface in equilibrium meets homogeneous walls containing the fluid is constant (Figure 1(a)). This

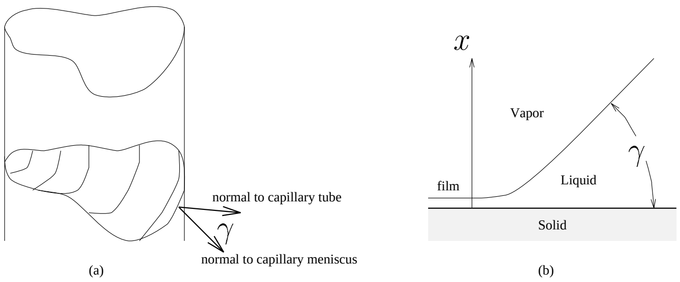

FIGURE 1. Contact Angles

assertion has been the subject of debate since it was introduced; Adamson [1, pg. 360] writes concerning the microscopic contour of a meniscus, "The likely picture for the case of a nonwetting drop on a flat surface is shown in Figure 1(b). There is a region of negative curvature as the drop profile joins the plane of the solid." Presumably, such a film (and perhaps the region of negative curvature) are not subject to variation. Nevertheless, if such a region exists even microscopically, then a mathematical description of Figure 1(b) incorporating the macroscopic contact angle is of obvious interest. We offer such an explanation in an (admittedly) degenerate case

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for which the film *is* subject to variation and (for this reason) does play a role on the macroscopic level.

2. Perfectly Wetting Fluids

Films of certain fluids assert their presence conspicuously due to their ability to transport the fluid. We have, here, helium at low temperature in mind, but there are other examples. From the point of view of equilibria, however, these fluids have typically been treated under the assumptions that the film is negligible and that Young's condition prevails with $\gamma = 0$. Considering the typical thicknesses of such films (20-90 atomic diameters for helium) these assumptions may seem justified. A recent experiment [5] involving rotating nearly cylindrical films indicates otherwise. We now give a brief description of this experiment.

In low gravity [1] an annular container consisting of the region between two coaxial cylinders was partially filled with a perfectly wetting fluid. In this instance the inner cylinder was made of steel and the fluid was silicone oil. Previous applications of the Young condition $\gamma = 0$ and the assumptions described above predicted that under rotation about the axis of the cylinders (at certain speeds with certain volumes of fluid) a stable axially symmetric equilibrium would be observed with cross-section resembling that shown in Figure 2.

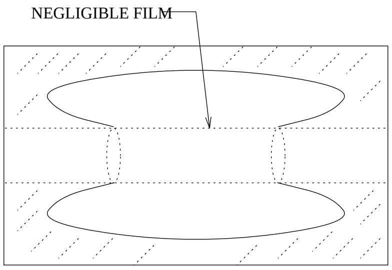

FIGURE 2. Axially Symmetric Doubly Connected Bubble

What was in fact observed was no equilibrium at all, but the nearly cylindrical film on the inner cylinder wall thickened periodically and formed "pendant drops" which were flung outward, returning to the bulk of fluid near the outer cylinder. In this way the fluid was "pumped" by the capillary instability of the "film" on the inner cylinder in a cycle apparently against the centripetal force in part of its course. See Figure 3.

It is this experiment which prompted the work to be presented below.

[1] For this discussion we may assume zero gravity; in the actual experiment low gravity was simulated on the earth's surface.

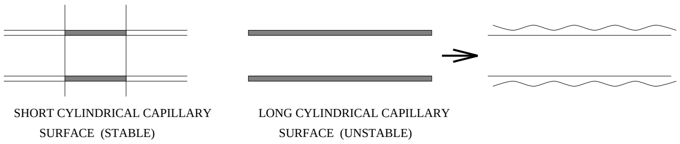

FIGURE 3. Thickening of Cylindrical Film

3. A New Approach

In the standard Young-Laplace theory of equilibrium capillary surfaces, for which the reader is referred to [3], equilibrium surfaces $\mathcal{S}$ and their stability are determined by an energy functional of the form

$$E = \sigma|\mathcal{S}| - \sigma\beta|\mathcal{W}| + \int_X \mathcal{U} \tag{1}$$

where σ is the surface tension, $|\mathcal{S}|$ is the area of the free surface, $|\mathcal{W}|$ is the area of the wetted region, X is the volume of fluid, and $\mathcal{U}$ is a potential depending on position; the Young equation is given by $\cos\gamma = \beta$ where β is the "adhesion coefficient." One might attempt to distinguish between various "zero contact angle" fluids in this framework by choosing an adhesion coefficient $\beta > 1$. It has recently been shown by B. White [8] that such an approach yields no difference (in equilibria nor stability) from the case $\beta = 1$.

In the perfectly wetting case we suggest that the second term in (1) be dropped altogether and a "Van der Waals" potential be incorporated in $\mathcal{U}$. For containers with suitable geometry this additional potential depends only on the distance d to the container and, in any event, is required to become infinite and negative as $d \searrow 0$. In this way, the issue of a contact line or a contact angle need never be addressed because all equilibria must wet every portion of the container.

In the remainder of the paper we carry out an analysis of certain equilibria in a particularly simple container geometry. The results are rather surprising particularly in view of the historical introduction of the next section.

4. Plateau-Rayleigh Instability

Plateau in 1873 determined experimentally that the length at which a cylindrical column of liquid becomes unstable is between 3.13 and 3.18 times the diameter of the column.[2] The theoretical value of π for this multiple has been given by Rayleigh and more recently by Barbosa and do Carmo [6, 2]. We present a simple (and partial) derivation based on the second variation of (1). In this case $|\mathcal{W}| = \mathcal{U} \equiv 0$, and the first variation δE gives rise to the equation of constant mean curvature $2H = \lambda =$ constant to which the cylinder is a solution with $H = 1/2r$.

[2]This experiment was repeated in [4] by G. Mason who obtained a value $3.143 \pm .004$.

Calculating the second variation about the cylindrical solution one obtains

$$\delta^2 E = \left(-\frac{1}{r^2}\phi_{\theta\theta} - \phi_{zz} - \frac{1}{r^2}\phi\right)\phi \tag{2}$$

where θ and z are cylindrical coordinates on the cylinder $\mathcal{S}$. Thus, using the criterion of Barbosa and DoCarmo [2], the cylinder will be unstable when $L\phi = -(1/r^2)\phi_{\theta\theta} - \phi_{zz} - (1/r^2)\phi$ has a non-positive eigenvalue whose eigenfunction satisfies

$$\int_{\mathcal{S}} \phi = 0. \tag{3}$$

This latter condition is the result of constrained volume. Solving $L\phi = \mu\phi$ by a separation of variables $\phi = A(\theta)B(z)$ with the boundary conditions $A(0) = A(2\pi),\ A'(0) = A'(2\pi);\ B(0) = 0 = B(l)$ one obtains eigenvalues

$$\mu_{k,m} = \frac{k^2-1}{r^2} + \frac{m^2\phi^2}{l^2} \tag{4}$$

for $k = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$.

$\mu_{k,m} > 0$ for $k > 0$. Thus, instability can only arise for eigenvalues $\mu_{0,m}$. The smallest of these is $\mu_{0,1} = \pi^2/l^2 - 1/r^2$, but the corresponding eigenfunction $\phi_{0,1} = \sin(\pi/l)z$ does not satisfy (3) and must be discarded. The next smallest is $\mu_{0,2} = 4\pi^2/l^2 - 1/r^2$ whose eigenfunction does satisfy (3) and gives rise to an instability exactly when

$$l \geq 2\pi r. \tag{5}$$

This is a symmetric variation, and at the length indicated in (5) the column of fluid will develop a neck which becomes smaller in radius until the column separates into two pieces. It is of note that in the experiment of G. Mason [4], the column of fluid was centered on a silica rod and the necking apparently proceeded until the column "ruptured" around the rod. According to our model such behavior is excluded for perfectly wetting fluids.

5. Rotating Cylindrical Films

We now carry over the analysis of the previous section to determine the critical length for the onset of instability of a cylindrical film of perfectly wetting fluid on a rotating, coaxial, rigid cylinder.

The rotational potential energy is given by $\mathcal{U}_1 = -(\rho\omega^2/2)r^2$, and we take a model Van der Waals potential approximating that for helium $\mathcal{U}_2 = -\alpha/d^3$ where $d = r - r_0$. A general form for $L\phi$ has been calculated by Wente and others [7]:

$$L\phi = -\Delta\phi - 2(2H^2 - K)\phi + \nabla\mathcal{U} \cdot N\phi \tag{6}$$

where Δ is the intrinsic Laplacian on the free surface $\mathcal{S}$, H and K are the mean and Gauss curvatures of $\mathcal{S}$ and N is the outward normal to $\mathcal{S}$. From this one obtains by separation of variables as before

$$\mu_{k,m} = \frac{\partial \mathcal{U}}{\partial r} + \frac{k^2 - 1}{r^2} + \frac{m^2 \pi^2}{l^2} \tag{7}$$

for $k = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$. The two smallest contributing eigenvalues are

$$\mu_{0,2} = \frac{\partial \mathcal{U}}{\partial r} - \frac{1}{r^2} + \frac{4\pi^2}{l^2} \tag{8}$$

corresponding to the axisymmetric variation of the last section and

$$\mu_{1,1} = \frac{\partial \mathcal{U}}{\partial r} + \frac{\pi^2}{l^2} \tag{9}$$

corresponding to a non-axisymmetric variation. The typical shape of $\partial \mathcal{U}/\partial r$ is shown in Figure 4 and evidently results in three cases.

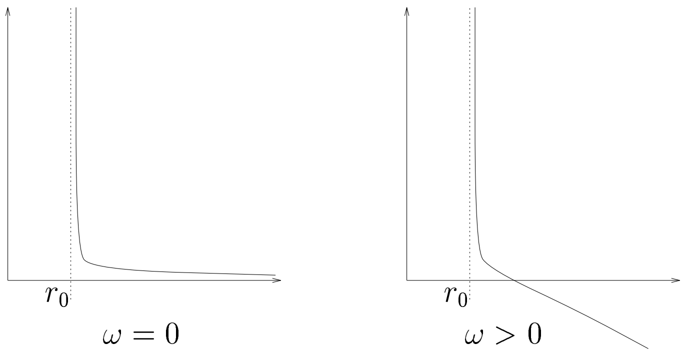

FIGURE 4. The Shape of $\partial \mathcal{U}/\partial r$

VERY THIN FILMS: There is a value r_1 determined by the equation

$$\frac{\partial \mathcal{U}}{\partial r}(r_1) = \frac{1}{r_1^2} \tag{10}$$

such that any cylindrical film (of arbitrary length) satisfying $r_0 < r < r_1$ is stable.

For helium in the absence of rotation (10) becomes

$$\frac{3\alpha}{(r - r_0)^4} = \frac{1}{r^2}. \tag{11}$$

Thus, $r = r_{\text{critical}}$ satisfies

$$r_{\text{critical}} = r_0 + \frac{\sqrt{3\alpha}}{2} + \frac{1}{2}\sqrt{4r_0\sqrt{3\alpha} + 3\alpha}. \tag{12}$$

MEDIUM THICKNESS FILMS: Without rotation all cylindrical films with $r_1 \le r$ are unstable to the axially symmetric variation associated to $\mu_{0,2}$ whenever

$$l \ge l_{0,2} \equiv \frac{2\pi}{\sqrt{\frac{1}{r^2} - \frac{\partial \mathcal{U}}{\partial r}}}.$$

In the presence of rotation there is a value $r_2 > r_1$ satisfying the equation

$$\frac{\partial \mathcal{U}}{\partial r}(r_2) = -\frac{1}{3r_2^2}. \tag{13}$$

If $r_1 \le r < r_2$, then an *axially symmetric* instability sets in at $l = l_{0,2}$.

THICK FILMS (only in the presence of rotation): If $r_2 \le r$, then the cylinder of radius r becomes unstable to the *non-axisymmetric* variation associated to $\mu_{1,1}$ when l exceeds

$$l_{1,1} = \frac{\pi}{\sqrt{-\frac{\partial \mathcal{U}}{\partial r}}}.$$

NOTE: The only properties of the virtual (Van der Waals) potential $\mathcal{U}_2$ which have been used are

: (0) $\lim_{r \searrow r_0} \mathcal{U}_2(r) = +\infty$.

: (i)

$$\frac{\partial^2 \mathcal{U}_2}{\partial r^2} < 0.$$

: (ii) (When $\omega = 0$) $\frac{\partial \mathcal{U}}{\partial r}$ decays faster than $1/r^2$ as $r \to \infty$.

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