SOME REMARKS ON REAL AND COMPLEX OUTPUT FEEDBACK

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ABSTRACT. We provide some new necessary and sufficient conditions which guarantee arbitrary pole placement of a particular linear system over the complex numbers. We exhibit a non-trivial real linear system which is not controllable by real static output feedback and discuss a conjecture from algebraic geometry concerning the existence of real linear systems for which all static feedback laws are real.

1. PRELIMINARIES

Let $\mathbb{F}$ be an arbitrary field and let m, p, n be fixed positive integers. Let A, B, C be matrices with entries in $\mathbb{F}$ of sizes $n \times n$, $n \times m$, and $p \times n$ respectively. Identify the space of monic polynomials having degree n,

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \in \mathbb{F}[s],$$

with the vector space $\mathbb{F}^n$. In its simplest form, the static output pole placement problem asks for conditions on the matrices A, B, C which guarantee that the pole placement map

$$\chi_{(A,B,C)} : \mathbb{F}^{mp} \longrightarrow \mathbb{F}^n, \quad K \longmapsto \det(sI - A - BKC) \tag{1}$$

is surjective. A dimension argument shows the necessity of $mp \geq n$. This question has been studied intensively and we refer to the survey articles [4, 14] and the recent papers [9, 10, 12, 13, 19, 20] for details. We summarize some of the most important results.

A matrix pair (A, B) defined over a field $\mathbb{F}$ is *controllable* if the matrix pencil $[sI - A \mid B]$ is left coprime. Equivalently, if the full size minors of the pencil $[sI - A \mid B]$ have no common non-trivial polynomial factor. Similarly, a matrix pair (A, C) is *observable* if the matrix pencil $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ is right coprime. Then we have:

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Lemma 1.1. *$\chi_{(A,B,C)}$ is surjective only if (A, B) is a controllable pair and (A, C) is an observable pair.*

Proof. The following identity immediately establishes the claim:

$$\det(sI - A - BKC) = \det \begin{bmatrix} sI - A & -B \\ -KC & I \end{bmatrix} = \det \begin{bmatrix} sI - A & -BK \\ -C & I \end{bmatrix}$$

□

The necessary conditions $mp \geq n$, controllability, and observability are not sufficient to guarantee arbitrary pole assignability. When $p = 1$, the following straightforward lemma provides exact conditions for arbitrary pole assignability over any field $\mathbb{F}$.

Lemma 1.2. *Let $p = 1$ and let $d^{-1}(s)(n_1(s), \ldots, n_m(s))$ be a left coprime factorization of the transfer function $C(sI - A)^{-1}B$. Then the pole placement map (1) is surjective if and only if $n_1(s), \ldots, n_m(s)$ span the vector space of polynomials of degree at most $n - 1$.*

One readily establishes a similar result when $m = 1$. Lemma 1.2 gives algebraic conditions on the set of systems parameters. To make this precise, identify the set of matrices (A, B, C) having fixed sizes $n \times n$, $n \times m$, and $p \times n$ with the vector space $V := \mathbb{F}^{n(n+m+p)}$. Recall that a subset $G \subset V$ is *generic* if a non-trivial polynomial vanishes on its complement $V \setminus G$. Thus Lemma 1.2 implies that if $p = 1$ and $m \geq n$, then the set of systems which can be arbitrarily pole assigned forms a generic set.

Since non-controllable systems (A, B, C) cannot be arbitrarily pole assigned, pole placement results are often restricted to a generic class of systems. If the base field $\mathbb{F}$ is the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, then a generic set $G \subset V$ is open and dense with respect to the usual Euclidean topology, and its complement $V \setminus G$ has measure zero.

If the pole placement map χ is surjective for a generic set of systems and some fixed base field $\mathbb{F}$ we will say in short that χ is *generically surjective.*

The major results are as follows:

Theorem 1.3 (Brockett and Byrnes [3])**.** *If the base field $\mathbb{F}$ is algebraically closed and if $mp \geq n$ then χ is generically surjective. Moreover if $mp = n$ then for a generic set of systems the cardinality of $\chi^{-1}(\phi)$ (when counted with multiplicity) is independent of the closed loop polynomial $\phi \in \mathbb{F}^n$ and is equal to*

$$d(m,p) = \frac{1!2!\cdots(p-1)!(mp)!}{m!(m+1)!\cdots(m+p-1)!} \tag{2}$$

Since $mp \geq n$ is necessary for χ to be surjective, Theorem 1.3 gives the best possible bound when the base field $\mathbb{F}$ is algebraically closed.

The number $d(m,p)$ is the degree of the Grassmann variety, which was computed in the last century by Schubert [15]. Although the real numbers $\mathbb{R}$ are not algebraically closed and Theorem 1.3 therefore does not apply one still has the following Corollary:

Corollary 1.4. *If $\mathbb{F} = \mathbb{R}$, $mp = n$, and $d(m,p)$ is odd, then χ is generically surjective.*

Proof. If (A, B, C) are real matrices then the set $\chi^{-1}(\phi)$ is closed under complex conjugation for every closed loop polynomial $\phi \in \mathbb{R}^n$. Therefore, for a generic set of systems, $\chi^{-1}(\phi)$ contains a real point for each ϕ. □

As an example, consider the case $\mathbb{F} = \mathbb{R}$, $m = 2$, $p = 3$ and $n = 6$. Here, $d(2,3) = 5$. At least one of the 5 points $\chi^{-1}(\phi)$ is real, so χ is generically surjective even over the reals.

Berstein determined when $d(m,p)$ is odd.

Proposition 1.5 (Berstein [2])**.** *The number $d(m,p)$ is odd if and only if $\min(m,p) = 1$ or $\min(m,p) = 2$ and $\max(m,p) = 2^t - 1$, where t is a positive integer.*

When $d(m,p)$ is even, the best known sufficiency result over the reals is due to Wang:

Theorem 1.6 (Wang [19])**.** *If $\mathbb{F} = \mathbb{R}$ and $mp > n$, then χ is generically surjective.*

For an elementary direct proof of this important sufficiency result we refer to [12].

For generic surjectivity over the reals, there is a difference of one degree of freedom between sufficiency ($mp > n$) and necessity ($mp \geq n$). As we already noted, $mp \geq n$ is sufficient if $d(m,p)$ is an odd number. One may ask if $mp \geq n$ might be always sufficient?

Proposition 1.7 (Willems and Hesselink [21])**.** *If $\mathbb{F} = \mathbb{R}$ and if $m = p = 2$ and $n = 4$ then there is an open Euclidean neighborhood $U \subset V = \mathbb{R}^{32}$ having the property that $\chi_{(A,B,C)}$ is not surjective if $(A, B, C) \in U$. In particular χ is not generically surjective.*

It has been conjectured by S.-W. Kim that $m = p = 2$, $n = 4$ is the only case where $mp = n$ is not a sufficient condition for χ to be generically surjective over the reals. In the next section we exhibit a counterexample.

2. Main Results

The result by Brockett and Byrnes provides a sufficiency result for a generic set of systems. We provide exact conditions which guarantee that a particular plant $(\bar{A}, \bar{B}, \bar{C})$ is arbitrarily pole assignable. Our approach is geometric, utilizing the central projection of the Grassmann variety induced by the pole placement map [3, 20].

Let $D^{-1}(s)N(s) = C(sI-A)^{-1}B$ be a left coprime factorization of the transfer function having the property that $\det(sI-A) = \det D(s)$. Then the closed loop characteristic polynomial can be written as:

$$\det(sI - A - BKC) = \det \begin{bmatrix} D(s) & N(s) \\ -K & I \end{bmatrix} = \sum_\alpha g_\alpha(s) k_\alpha, \tag{3}$$

where the numbers k_α are the Plücker coordinates (full size minors) of the compensator $[-K\ I]$ inside $\wedge^m \mathbb{F}^{m+p}$ and the polynomials $g_\alpha(s)$ are (up to sign) the corresponding Plücker coordinates of $[D(s)\ N(s)]$. Let $\mathbb{P}^N$ be the projective space $\mathbb{P}(\wedge^m \mathbb{F}^{m+p})$ and let

$$E_{(A,B,C)} := \left\{ k \in \mathbb{P}^N \mid \sum_\alpha g_\alpha(s) k_\alpha = 0 \right\}.$$

Since each $g_\alpha(s)$ has degree at most n, $E_{(A,B,C)}$ has dimension at least $N-n-1$, and its dimension equals $N-n-1$ precisely when the $g_\alpha(s)$ span the vector space of polynomials of degree at most n. In this case, the central projection induced by χ (see [20])

$$L_{(A,B,C)} \ :\ \mathbb{P}^N - E_{(A,B,C)} \longrightarrow \mathbb{P}^n, \qquad k \longmapsto \sum_\alpha g_\alpha(s) k_\alpha \tag{4}$$

is surjective.

By(3), there is a unique Plücker coordinate $\bar\alpha$ with $g_{\bar\alpha}(s)$ of degree n, namely that corresponding to the minor $\det D(s)$ of $[D(s)\ N(s)]$. Moreover, $k_{\bar\alpha} = 1$ and all other $g_\alpha(s)$ have degree at most $n-1$. Identify $\mathbb{F}^N \subset \mathbb{P}^N$ with those points whose $\bar\alpha$th coordinate is 1. Then the central projection $L_{(A,B,C)}$ maps $\mathbb{F}^N$ to the set of monic polynomials of degree n, and its complement $\mathbb{P}^N - \mathbb{F}^N$ to polynomials of degree at most $n-1$.

Every $m \times p$ compensator K defines a m-dimensional linear subspace of $\mathbb{F}^{m+p}$, the row space of $[-K\ I]$ and therefore a point of the Grassmann variety $\mathrm{Grass}(m, \mathbb{F}^{m+p}) \subset \mathbb{P}^N$. The previous paragraph shows this point is in $\mathbb{F}^N$. Conversely, all points in $\mathrm{Grass}(m, \mathbb{F}^{m+p}) \cap \mathbb{F}^N$ are of the form rowspace$[-K\ I]$ (*cf.* [3]).

The main theorem we have is:

Theorem 2.1. *Let $\mathbb{F}$ be algebraically closed and $n \le mp$. Then the pole placement map $\chi_{(\bar A,\bar B,\bar C)}$ is surjective for a particular system $(\bar A, \bar B, \bar C)$ if and only if* $\dim E_{(\bar A,\bar B,\bar C)} = N-n-1$ *and, for any $y \in \mathbb{F}^N - E_{(\bar A,\bar B,\bar C)} \cap \mathbb{F}^N$,*

$$\mathrm{span}\left(E_{(\bar A,\bar B,\bar C)}, y\right) \cap \mathrm{Grass}(m, \mathbb{F}^{m+p}) \ \neq\ E_{(\bar A,\bar B,\bar C)} \cap \mathrm{Grass}(m, \mathbb{F}^{m+p}). \tag{5}$$

Proof. Suppose $\chi_{(\bar A,\bar B,\bar C)}$ is surjective. Then the central projection $L_{(\bar A,\bar B,\bar C)}$ is surjective and so $\dim E_{(\bar A,\bar B,\bar C)} = N-n-1$. If for some $\hat y \in \mathbb{F}^N - E_{(\bar A,\bar B,\bar C)} \cap \mathbb{F}^N$,

$$\mathrm{span}\left(E_{(\bar A,\bar B,\bar C)}, \hat y\right) \cap \mathrm{Grass}(m, \mathbb{F}^{m+p}) \ =\ E_{(\bar A,\bar B,\bar C)} \cap \mathrm{Grass}(m, \mathbb{F}^{m+p}), \tag{6}$$

then there is also equality in(5) for all $y \in \text{span}\left(E_{(\bar{A},\bar{B},\bar{C})}, \hat{y}\right)$. In particular, we see that the set $\chi^{-1}_{(\bar{A},\bar{B},\bar{C})}(L_{(\bar{A},\bar{B},\bar{C})}(\hat{y}))$ is empty, a contradiction.

Conversely, if $\dim E_{(\bar{A},\bar{B},\bar{C})} = N - n - 1$, then $L_{(\bar{A},\bar{B},\bar{C})}$ is surjective. Let $\phi \in \mathbb{P}^n$ be any closed loop polynomial and $y \in \mathbb{P}^N$ satisfy $L_{(\bar{A},\bar{B},\bar{C})}(y) = \phi$. Then necessarily $y \in \mathbb{F}^N$, and condition(5) guarantees that there exists $P \in \text{Grass}(m, \mathbb{F}^{m+p})$ with $L_{(\bar{A},\bar{B},\bar{C})}(P) = \phi$. But then P is the row space of $[-K\ I]$, for some compensator K. Hence $\chi_{(\bar{A},\bar{B},\bar{C})}(K) = \phi$. □

Remark 2.2. A system $(\bar{A}, \bar{B}, \bar{C})$ is *nondegenerate* if $E_{(\bar{A},\bar{B},\bar{C})} \cap \text{Grass}(m, \mathbb{F}^{m+p}) = \emptyset$. In [3] it was shown that nondegenerate systems can be arbitrarily pole assigned and that the set of nondegenerate systems forms a generic set if and only if $mp \leq n$.

The remainder of the paper is concerned with the question of when the condition $mp = n$ is also sufficient for the pole placement map χ to be generically surjective over the reals. If (A, B, C) are real matrices and if $\chi_{(A,B,C)} : \mathbb{R}^{mp} \longrightarrow \mathbb{R}^n$ is the real pole placement map, we let $\tilde{\chi}_{(A,B,C)} : \mathbb{C}^{mp} \longrightarrow \mathbb{C}^n$ denote the corresponding complexified map.

Theorem 2.3. *Let $\mathbb{F} = \mathbb{R}$ and assume that $mp = n$ and $d(m,p)$ is even. Then χ is not generically surjective if and only if there exists a system $(\bar{A}, \bar{B}, \bar{C})$ and a polynomial $\bar{\phi} \in \mathbb{R}[s]$ such that $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi}) \subset \mathbb{C}^{mp}$ consists of $d(m,p)$ different complex points, none of them real.*

Proof. Assume χ is not generically surjective. Then there exists a Euclidean open neighborhood $U \subset \mathbb{R}^{n(n+m+p)}$ for which $\chi_{(A,B,C)}$ is not surjective if $(A, B, C) \in U$. Since U is open, there exists a nondegenerate plant $(\bar{A}, \bar{B}, \bar{C}) \in U$ having the property that $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\phi)$ consists of $d(m,p)$ points independent of ϕ. Choosing a polynomial $\bar{\phi}$ which is not in the image of χ establishes one direction of the proof.

On the other hand, if $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi}) \subset \mathbb{C}^{mp}$ consists of $d(m,p)$ different complex points, then necessarily $(\bar{A}, \bar{B}, \bar{C})$ is a nondegenerate plant. It follows that there exists an open Euclidean neighborhood U of $(\bar{A}, \bar{B}, \bar{C})$ consisting solely of nondegenerate systems, none of which can be assigned the closed loop characteristic polynomial $\bar{\phi}$. □

Theorem 2.3 is interesting since it seeks a geometric configuration where all discrete solutions are purely complex. We use it to show that besides the case of $m = p = 2$ and $n = 4$, there are other situations where $mp = n$ is not sufficient to guarantee that χ is generically surjective over the reals. This disproves the conjecture by S.-W. Kim mentioned in §1.

Example 2.4. If $\mathbb{F} = \mathbb{R}$, $p = 2$, $m = 4$, and $n = 8$ then χ is not generically surjective.

By Lemma 2.5, it suffices to exhibit a real system $(\bar{A}, \bar{B}, \bar{C})$ and a polynomial $\bar{\phi}$ of degree 8 with 8 real roots such that $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi}) \subset \mathbb{C}^8$ consists of exactly $d(4,2) = 14$ purely complex solutions. Here is such a system:

Let $(\bar{A}, \bar{B}, \bar{C})$ be a minimal realization of the system represented through a coprime factorization $D^{-1}(s)N(s)$, where

$$D(s) = \begin{bmatrix} x^4-16x^3+3x^2+11x & -26x^3+10x^2+7x+16 \\ 6x^3-4x^2-9x-5 & x^4+3x^3-x^2-16x-13 \end{bmatrix}$$

$$N(s) = \begin{bmatrix} 9x^3-12x^2+13x-17 & -31x^3-16x^2+43x-23 & x^3-36x^2+8x-13 & 23x^3-x^2+2x-21 \\ 8x^3-6x^2+5x+15 & 26x^3-14x^2-11x+12 & 11x^3+5x^2+11x+33 & -7x^2+11x+5 \end{bmatrix}.$$

Let

$$\bar{\phi}(s) := (s+8)(s+6)(s+4)(s+2)(s-1)(s-2)(s-3)(s-4).$$

We claim that $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ consists of 14 purely complex solutions (displayed below). First, we discuss how we compute $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ for such a system with $n = mp$. Identify $\mathbb{C}^{mp}$ with the set of compensators K. Then the mp polynomial equations

$$\det \begin{bmatrix} D(s) & N(s) \\ -K & I \end{bmatrix} = 0 \tag{7}$$

as s ranges over the roots of $\bar{\phi}$ generate the ideal of $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ in $\mathbb{C}^{mp}$. We used the software package SINGULAR [7] to compute an elimination Gröbner basis [5] of this ideal and verify that $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ is zero-dimensional with degree 14. This calculation on the system(7) requires 59 seconds of CPU on a HP 9000 D250, 800 Series computer.

This Gröbner basis contains a univariate polynomial, the eliminant, whose roots are the values of that variable for the solutions. We used the `realroot` routine of Maple to determine the number of real roots of the eliminant and `fsolve` to compute its roots numerically. Since we only obtain one coordinate of each solution, we repeated this procedure to find the others and to match the coordinates with the solutions.

Here are numerical solutions of the system(7).

$$\begin{bmatrix} -548.1543631072859 \pm 539.02172783574002\sqrt{-1} & -2966.220011381735449 \pm 1301.806890926492508\sqrt{-1} \\ 227.99002317474104 \mp 195.29675914098226\sqrt{-1} & 1189.40572416765385 \mp 428.190112835481936\sqrt{-1} \\ 253.619670619102274 \mp 128.997418066861599\sqrt{-1} & 1192.66093663708038 \mp 127.782426659628597\sqrt{-1} \\ -373.4608141108503 \mp 376.1870941851628\sqrt{-1} & -907.2715490825303837 \mp 2040.657619029875556\sqrt{-1} \end{bmatrix}$$

$$\begin{bmatrix} 182.1974051162797 \pm 1524.2891350121054\sqrt{-1} & -3910.9491667600289 \pm 3319.9425134666556\sqrt{-1} \\ -92.76689536072804 \mp 494.390627883840\sqrt{-1} & 1206.13014159582817 \mp 1171.58923461208352\sqrt{-1} \\ 202.71121387564936 \mp 458.78014215695346\sqrt{-1} & 1652.30669576900037 \mp 280.820983264097575\sqrt{-1} \\ -999.496765955436554 \mp 938.918292576740638\sqrt{-1} & 771.9810394973421 \mp 4516.958140814761213\sqrt{-1} \end{bmatrix}$$

$$\begin{bmatrix} 2792.9110057318105 \mp 969.00549705135278\sqrt{-1} & 3350.9339523791667 \mp 832.762320679797284\sqrt{-1} \\ -338.608141548768 \mp 31.1420684422097\sqrt{-1} & -390.733153481711 \mp 71.9581835765450\sqrt{-1} \\ -858.10666480772375 \pm 463.34803831698071\sqrt{-1} & -1047.08493981311276 \pm 448.52274247532122\sqrt{-1} \\ -1736.0182637110866 \pm 473.54602107116131\sqrt{-1} & -2069.7786151738302 \pm 367.88390311074763\sqrt{-1} \end{bmatrix}$$

$$\begin{bmatrix} 566.14047176252718 \mp 390.1690631954798\sqrt{-1} & 894.7573009772359 \mp 213.7664118348474\sqrt{-1} \\ -28.9144418101747 \mp 8.82325220859399\sqrt{-1} & -31.9889032754154 \mp 25.1286025912621\sqrt{-1} \\ -101.611268377237 \pm 166.198294126534\sqrt{-1} & -207.075559094765 \pm 158.433905818864\sqrt{-1} \\ -433.109410705026 \pm 160.543671922194\sqrt{-1} & -618.358581551134 \mp 8.42746099774335\sqrt{-1} \end{bmatrix}$$

$$\begin{bmatrix} -1328.31492831596508 \pm 780.43146580510958\sqrt{-1} & 2115.8811996413627 \mp 363.25099106004349\sqrt{-1} \\ 277.0599315399026 \mp 134.0101686258348\sqrt{-1} & -426.505631447159 \pm 38.4080785894925\sqrt{-1} \\ 242.753288068855 \mp 128.748683783964\sqrt{-1} & -380.517275415650 \pm 48.1897454846160\sqrt{-1} \\ 809.814164981704 \mp 420.527784827832\sqrt{-1} & -1263.86094232894868 \pm 149.27131835292291\sqrt{-1} \end{bmatrix}$$

$$\begin{bmatrix} -74.07812921055438 \mp 1186.0867962658997\sqrt{-1} & 481.83814937211068 \mp 659.46539248077808\sqrt{-1} \\ 131.85311577768057 \pm 223.6599712395458\sqrt{-1} & -28.4575338243835 \pm 176.018708417247\sqrt{-1} \\ 50.0398731323218 \pm 311.162560564792\sqrt{-1} & -110.484321267527 \pm 186.531966999705\sqrt{-1} \\ 120.94035205524575 \pm 693.23751296762126\sqrt{-1} & -241.138619140528 \pm 419.709352592197\sqrt{-1} \end{bmatrix}$$

$$\begin{bmatrix} -466.3420096818032 \pm 2560.3776496553293\sqrt{-1} & -477.06216348936717 \pm 1505.4573962873226\sqrt{-1} \\ 206.16217936754085 \mp 504.1659905544772\sqrt{-1} & 162.819554092696 \mp 287.715806475160\sqrt{-1} \\ 198.483315335125 \mp 690.317301079547\sqrt{-1} & 172.179197573658 \mp 400.2773514799496\sqrt{-1} \\ 350.2539156691074 \mp 1658.3575908118343\sqrt{-1} & 337.47012412920796 \mp 971.42452550058667 8\sqrt{-1} \end{bmatrix}$$

After discovering this example, we did a systematic search for others. In all, we generated 70 pairs $D(s), N(s)$ with random integral polynomial entries, and, for each of the 70, considered 25 degree 8 polynomials $\bar{\phi}(s)$ with distinct integral roots in $[-12, 12]$. Of the 1750 instances of $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ we tested, *none* had 14 purely complex solutions, and only 3 had the 'opposite' situation of 14 purely real solutions. This suggests that these extreme situations of real systems with real data giving only purely complex (or purely real) solutions are quite rare. Despite this, we believe that it is always possible to find such examples. Specifically:

Conjecture 2.5. *If $d(m,p)$ is even and $n = mp$, then χ is not generically surjective over $\mathbb{R}$.*

Consider now the 'opposite' situation. Namely, for which m, p, n with $n = mp$ does there exist a real system $(\bar{A}, \bar{B}, \bar{C})$ and a polynomial $\bar{\phi}$ all of whose (n) roots are real such that $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ consists of exactly $d(m,p)$ real solutions? Similar questions have recently been of interest in algebraic geometry (see [11, 16, 18] or the survey [17]). In fact, there is a precise conjecture of Shapiro and Shapiro which is relevant to systems theory:

Conjecture 2.6 (Shapiro-Shapiro)**.** *Let $(\bar{A}, \bar{B}, \bar{C})$ be a minimal realization of the system represented through a coprime factorization $D^{-1}(s)N(s)$, where the matrix $[D(s) \mid N(s)]$ has the following form: The first row is*

$$s^{m+p-1}, s^{m+p-2}, \ldots, s^2, s, 1$$

and, for $1 \leq j < p$, the $(j+1)$st row consists of the derivative of the jth row divided by j.

Then the system is nondegenerate, and for any polynomial $\bar{\phi}$ of degree mp with distinct real roots, $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ consists of exactly $d(m,p)$ real solutions.

For example, if $m=p=3$, then we have

$$[D(s) \mid N(s)] \;=\; \left[\begin{array}{ccc|ccc} s^5 & s^4 & s^3 & s^2 & s & 1 \\ 5s^4 & 4s^3 & 3s^2 & 2s & 1 & 0 \\ 10s^3 & 6s^2 & 3s & 1 & 0 & 0 \end{array}\right].$$

For such a system, $\chi(\underline{0}) = s^{mp}$ and $\chi^{-1}(s^{mp}) = \underline{0}$, a real point with multiplicity $d(m,p)$. Here, $\underline{0}$ is the null compensator, the matrix of all 0's. Prior to learning of this conjecture, one of us (Rosenthal) had suggested that it might be possible to perturb s^{mp} and obtain a polynomial $\bar{\phi}$ all of whose roots are real so that $\chi^{-1}(\bar{\phi})$ consists of $d(m,p)$ real solutions.

When p or m is 1, this conjecture follows from Corollary 1.4, and when $m=p=2$, it can be verified by hand. All other cases remain open. There is strong computational evidence in support of this conjecture: In every instance we have checked, $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ consists of exactly $d(m,p)$ real solutions. When $m=4, p=2$ (so that $d(4,2)=14$), we checked about 50 polynomials $\bar{\phi}$. In light of the search described above, we feel this gives overwhelming evidence for this conjecture. In addition, we have considered numerous instances when $m=3, p=2$, and a handful of instances for each of $m=5, p=2$ and $m=3, p=3$. For each of these last two cases, $d(m,p)$ is 42. Unfortunately, the task of computing an elimination Gröbner basis for larger m,p overwhelms the HP 9000 computer we use for these calculations.

There are other methods for solving systems of polynomials which we have not tried, but which should work for larger m,p. When $(m,p) = (5,2), (6,2)$, or $(4,3)$, we can compute a Gröbner basis, and there are linear algebraic methods for solving a polynomial system, given a Gröbner basis [6, §2.4]. Also, homotopy continuation [1] algorithms which are optimized for these systems have been developed [8], and are presently being implemented.

Remark 2.7. The row space of the matrix $[D(s) \mid N(s)]$ of Conjecture 2.6 is a p-plane $H(s)$ which osculates the moment, or rational normal curve in $\mathbb{R}^{m+p}$. The rational normal curve is the image of the map

$$s \;\longmapsto\; (s^{m+p-1}, s^{m+p-2}, \ldots, s^2, s, 1).$$

This observation, together with the fact that all non-degenerate rational curves of degree $m+p-1$ in $\mathbb{P}^{m+p-1}$ are projectively equivalent, show that the conditions of Conjecture 2.6 may be relaxed somewhat to the following:

The row span of the matrix $[D(s) \mid N(s)]$ equals the row span of a matrix $P(s)$ of real polynomials, where

1. The first row of $P(s)$ is a basis for all polynomials of degree at most $m+p-1$ and therefore defines a non-degenerate rational curve of degree $m+p-1$.

2. For $1 \leq j < p$, the $(j+1)$st row of $P(s)$ is the derivative of the jth row of $P(s)$.

Thus the Conjecture of Shapiro and Shapiro proposes a family of real systems $(\bar{A}, \bar{B}, \bar{C})$ for which $\tilde{\chi}^{-1}_{(\bar{A},\bar{B},\bar{C})}(\bar{\phi})$ consists of exactly $d(m,p)$ real solutions, whenever $\bar{\phi}$ has all real roots.

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