KNOTS, LINKS, AND 4-MANIFOLDS

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Abstract. In this paper we investigate the relationship between isotopy classes of knots and links in S^3 and the diffeomorphism types of homeomorphic smooth 4-manifolds. As a corollary of this initial investigation, we begin to uncover the surprisingly rich structure of diffeomorphism types of manifolds homeomorphic to the K3 surface.

1. Introduction

In this paper we investigate the relationship between isotopy classes of knots and links in S^3 and the diffeomorphism types of homeomorphic smooth 4-manifolds. As a corollary of this initial investigation, we begin to uncover the surprisingly rich structure of diffeomorphism types of manifolds homeomorphic to the K3 surface.

In order to state our theorems we need to view the Seiberg-Witten invariant of a smooth 4-manifold as a multivariable (Laurent) polynomial. To do this, recall that the Seiberg-Witten invariant of a smooth closed oriented 4-manifold X with $b_2^+(X) > 1$ is an integer valued function which is defined on the set of *spin*^{c} structures over X, (*cf.* [W], [KM],[Ko1],[T1]). In case $H_1(X,\mathbf{Z})$ has no 2-torsion (which will be the situation in this paper) there is a natural identification of the *spin*^{c} structures of X with the characteristic elements of $H^2(X,\mathbf{Z})$. In this case we view the Seiberg-Witten invariant as

$$\mathrm{SW}_X : \{k \in H^2(X,\mathbf{Z}) | k \equiv w_2(TX) \pmod 2)\} \to \mathbf{Z}.$$

The Seiberg-Witten invariant SW_X is a diffeomorphism invariant whose sign depends on an orientation of $H^0(X,\mathbf{R}) \otimes \det H^2_+(X,\mathbf{R}) \otimes \det H^1(X,\mathbf{R})$. If $\mathrm{SW}_X(\beta) \neq 0$, then we call β a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with

$$\mathrm{SW}_X(-\beta) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}_X(\beta)$$

where $\mathrm{e}(X)$ is the Euler number and $\mathrm{sign}(X)$ is the signature of X.

The first author was partially supported NSF Grant DMS9401032 and the second author by NSF Grant DMS9626330.

Now let $\{\pm\beta_1, \dots, \pm\beta_n\}$ be the set of nonzero basic classes for X. For the purposes of this paper we define the Seiberg-Witten invariant of X to be the formal series

$$\mathrm{SW}_X = b_0 + \sum_{j=1}^{n} b_j(\exp(\beta_j) + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4} \exp(-\beta_j))$$

where $b_0 = \mathrm{SW}_X(0)$ and $b_j = \mathrm{SW}_X(\beta_j)$. Letting $t_j = \exp(\beta_j)$, we have that the Seiberg-Witten invariant is the 'symmetric' Laurent polynomial

$$\mathrm{SW}_X = b_0 + \sum_{j=1}^{n} b_j(t_j + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\, t_j^{-1})$$

in the (formal) variables $t_1, \dots, t_n$.

We refer to any symmetric Laurent polynomial $P(t) = a_0 + \sum_{j=1}^{n} a_j(t^j + t^{-j})$ of one variable with coefficient sum $a_0 + 2\sum_{j=1}^{n} a_j = \pm 1$ as an *A-polynomial*. If, in addition, $a_n = \pm 1$ we refer to $P(t)$ as a *monic* A-polynomial.

Let X be any simply connected smooth 4-manifold with $b^+ > 1$. We define a *cusp* in X to be a PL embedded 2-sphere of self-intersection 0 with a single nonlocally flat point whose neighborhood is the cone on the right-hand trefoil knot. (This agrees with the notion of a cusp fiber in an elliptic surface.) The regular neighborhood N of a cusp in a 4-manifold is a *cusp neighborhood*; it is the manifold obtained by performing 0-framed surgery on a trefoil knot in the boundary of the 4-ball. Since the trefoil knot is a fibered knot with a genus 1 fiber, N is fibered by smooth tori with one singular fiber, the cusp. If T is a smoothly embedded torus representing a *nontrivial* homology class $[T]$, we say that T is *c-embedded* if T is a smooth fiber in a cusp neighborhood N; equivalently, T has two vanishing cycles. Note that a c-embedded torus has self-intersection 0. We can now state our first theorem.

Theorem 1.1. *Let X be any simply connected smooth 4-manifold with $b^+ > 1$. Suppose that X contains a smoothly c-embedded torus T with $\pi_1(X \setminus T) = 1$. Then for any A-polynomial $P(t)$, there is a smooth 4-manifold X_P which is homeomorphic to X and has Seiberg-Witten invariant*

$$SW_{X_P} = SW_X \cdot P(t)$$

where $t = \exp(2[T])$.

The basic classes of X were defined above to be elements of $H^2(X)$. To make sense of the statement of the theorem, we need to replace $[T]$ by its Poincaré dual. Throughout this paper we allow ourselves to pass freely between $H^2(X)$ and $H_2(X)$ without further comment.

As a corollary to the construction of X_P we shall show:

Corollary 1.2. *Suppose further that X is symplectic and that T is symplectically embedded. If $P(t)$ is a monic A-polynomial, then X_P can be constructed as a symplectic manifold.*

Using the work of Taubes [T1–T5] concerning the nature of the Seiberg-Witten invariants of symplectic manifolds, we shall also deduce:

Corollary 1.3. *If $P(t)$ is not monic, then X_P does not admit a symplectic structure. Furthermore, if X contains a surface Σ_g of genus g disjoint from T with $0 \neq [\Sigma_g] \in H_2(X;\mathbf{Z})$ and with $[\Sigma_g]^2 < 2-2g$ if $g>0$, or $[\Sigma_g]^2 \leq 0$ if $g=0$, then X_P with the opposite orientation does not admit a symplectic structure.*

As a corollary, we have many interesting new homotopy K3 surfaces (*i.e.* manifolds homeomorphic to the K3 surface). In particular, since $\mathrm{SW}_{K3}=1$ we have:

Corollary 1.4. *Any A-polynomial $P(t)$ can occur as the Seiberg-Witten invariant of an irreducible homotopy K3 surface. If $P(t)$ is not monic, then the homotopy* K3 *surface does not admit a symplectic structure with either orientation. Furthermore, any monic A-polynomial can occur as the Seiberg-Witten invariant of a symplectic homotopy K3 surface.*

In fact, there are three disjoint c-embedded tori T_1, T_2, T_3 in the K3 surface representing distinct homology classes $[T_j]$, $j=1,2,3$ (*cf.* [GM]). Theorem 1.1 then implies that the product of any three A-polynomials $P_j(t_j)$, $j=1,2,3$, can occur as the Seiberg-Witten invariant of a homotopy K3 surface $K3_{P_1P_2P_3}$ with $t_j=\exp(2[T_j])$. Furthermore, if all three of the polynomials are monic, the resulting homotopy K3 surface can be constructed as a symplectic manifold. If any one of the A-polynomials $P_j(t_j)$ is not monic, then the resulting homotopy K3 surface admits no symplectic structure.

A common method for constructing exotic manifolds is to perform log transforms on c-embedded tori. One might ask whether these new homotopy K3 surfaces $K3_{P_1P_2P_3}$ can be constructed in this fashion. However, it is shown in [FS1] that if X' is the result of performing log transforms of multiplicities $p_1,\dots,p_n$ on c-embedded tori in the K3 surface, then $\mathrm{SW}_{X'}(1,\dots,1)=\pm p_1\cdots p_n$. However, $\mathrm{SW}_{K3_{P_1P_2P_3}}(1,1,1)=P_1(1)P_2(1)P_3(1)=\pm1$; so $K3_{P_1P_2P_3}$ cannot be built using log transforms in this way. (Note that the '1' above is $\exp(0)$ corresponding to the zero class of H_2.)

The A-polynomials appearing in Theorem 1.1 are familiar to knot theorists. It is known that any A-polynomial occurs as the Alexander polynomial $\Delta_K(t)$ of some knot K in S^3. Conversely, the Alexander polynomial of a knot K is an A-polynomial. Furthermore, if the A-polynomial is monic then the knot can be constructed as a fibered knot, and if K is fibered, then $\Delta_K(t)$ is a monic A-polynomial. Indeed, it is a knot K in S^3 which we use to construct X_P.

Consider a knot K in S^3, and let m denote a meridional circle to K. Let M_K be the 3-manifold obtained by performing 0-framed surgery on K. Then m can also be viewed as a circle in M_K. In $M_K \times S^1$ we have the smooth torus $T_m = m \times S^1$ of self-intersection 0. Since a neighborhood of m has a canonical framing in M_K, a neighborhood of the torus T_m in $M_K \times S^1$ has a canonical identification with $T_m \times D^2$. Let X_K denote the fiber sum

$$X_K = X\#_{T=T_m}(M_K \times S^1) = [X \setminus (T \times D^2)] \cup [(M_K \times S^1) \setminus (T_m \times D^2)]$$

where $T \times D^2$ is a tubular neighborhood of the torus T in the manifold X (with $\pi_1(X) = \pi_1(X \setminus T) = 0$). The two pieces are glued together so as to preserve the homology class $[\mathrm{pt} \times \partial D^2]$. This latter condition does not, in general, completely determine the diffeomorphism type of X_K (*cf.* [G]). We take X_K to be any manifold constructed in this fashion. Because M_K has the homology of $S^2 \times S^1$ with the class of m generating H_1, the complement $(M_K \times S^1) \setminus (T \times D^2)$ has the homology of $T^2 \times D^2$. Thus X_K has the same homology (and intersection pairing) as X. Furthermore, the class of m normally generates $\pi_1(M_K)$; so $\pi_1(M_K \times S^1)$ is normally generated by the image of $\pi_1(T)$. Since $\pi_1(X \setminus F) = 1$, it follows from Van Kampen's Theorem that X_K is simply connected. Thus X_K is homotopy equivalent to X. It is conceptually helpful to note that X_K is obtained from X by removing a neighborhood of a torus and replacing it with the complement of the knot K in S^3 crossed with S^1. Also, in order to define Seiberg-Witten invariants, the oriented 4-manifold X must also be equipped with an orientation of $H^2_+(X;\mathbf{R})$. The manifold X_K inherits an orientation as well as an orientation of $H^2_+(X_K;\mathbf{R})$ from X.

Let $[T]$ be the class in $H_2(X_K;\mathbf{Z})$ induced by the torus T in X, and let $t = \exp(2[T])$. Our first main theorem, from which Theorem 1.1 is an immediate corollary, is:

Theorem 1.5. *If the torus T is c-embedded, then the Seiberg-Witten invariant of X_K is*

$$SW_{X_K} = SW_X \cdot \Delta_K(t).$$

Let $E(1)$ be the rational elliptic surface with elliptic fiber F. If T is a smoothly embedded self-intersection 0 torus in X, then $E(1)\#_{F=T}X_K = (E(1)\#_{F=T}X)\#_{T=T_m}(M_K \times S^1)$, and in $E(1)\#_{F=T}X$, the torus $T = F$ is c-embedded. Thus we have a slightly more general result:

Corollary 1.6. *Let X be any simply connected smooth 4-manifold with $b^+ > 1$. Suppose that X contains a smoothly embedded torus T of self-intersection 0 with $\pi_1(X \setminus T) = 1$ and representing a nontrivial homology class $[T]$. Then*

$$SW_{E(1)\#_{F=T}X_K} = SW_{E(1)\#_{F=T}X} \cdot \Delta_K(t).$$

More can be said if K is a fibered knot. Consider the normalized Alexander polynomial $\mathrm{A}_K(t) = t^d\Delta_K(t)$, where d is the degree of $\Delta_K(t)$. If K is a fibered knot in S^3 with a punctured genus g surface as fiber, then M_K fibers over the circle with a closed genus g surface Σ_g as fiber. Thus $M_K \times S^1$ fibers over $S^1 \times S^1$ with Σ_g as fiber and with $T_m = m \times S^1$ as section. It is a theorem of Thurston [Th] that such a 4-manifold has a symplectic structure with symplectic section T_m. Thus, if X is a symplectic $4-$manifold with a symplectically embedded torus with self-intersection 0, then $X_K = X\#_{T=T_m}(M_K \times S^1)$ is symplectic since it can be constructed as a symplectic fiber sum [G]. As a corollary to Theorem 1.5 and the theorems of Taubes relating the Seiberg-Witten and Gromov invariants of a symplectic $4-$manifold [T3, T5] we have:

Corollary 1.7. *Let X be a symplectic 4-manifold with $b^+ > 1$ containing a symplectic c-embedded torus T. If K is a fibered knot, then X_K is a symplectic 4-manifold whose Gromov invariant is*

$$Gr_{X_K} = Gr_X \cdot A_K(\tau)$$

where $\tau = \exp([T])$.

Proof. The homology $H_2(M_K \times S^1)$ is generated by the classes of the symplectic curves T_m and Σ_g; so the canonical class of $M_K \times S^1$ has the form $\kappa_{M_K\times S^1} = a[T_m] + b[\Sigma_g]$. Applying the adjunction formula (and using $[T_m]^2 = [\Sigma_g]^2 = 0$ and $[T_m]\cdot[\Sigma_g] = 1$) gives $b = 0$ and $a = 2g-2$. But note that the degree of $\Delta_K(\tau) = a_0 + \sum_{n=1}^{d} a_n(\tau^n + \tau^{-n})$ is $d = g$. Hence $\kappa_{M_K\times S^1} = (2d-2)[T_m]$. This means that the canonical class of the symplectic structure on X_K is $\kappa_{X_K} = \kappa_X + \kappa_{M_K\times S^1} + 2[T] = \kappa_X + 2d[T]$. Taubes' theorem now implies that for any $\alpha \in H_2(X_K)$, the coefficient of $\exp(\alpha)$ in Gr_{X_K} is:

$$\begin{aligned}\mathrm{Gr}_{X_K}(\alpha) &= \mathrm{SW}_{X_K}(2\alpha - \kappa_{X_K}) = \mathrm{SW}_{X_K}(2\alpha - \kappa_X - 2d[T]) \\ &= \sum_{n=-d}^{d} a_n\,\mathrm{SW}_X(2\alpha - \kappa_X - 2(d+n)[T]) = \sum_{n=-d}^{d} a_n \mathrm{Gr}_X(\alpha - (d+n)[T]),\end{aligned}$$

and this is the coefficient of $\exp(\alpha)$ in $\mathrm{Gr}_X \cdot \mathrm{A}_K(\tau)$. □

Of course, if X is simply connected and $\pi_1(X\setminus T) = 1$, then X_K is homeomorphic to X. This implies Corollary 1.2. As corollary of the initial work of Taubes [T1, T2] on the Seiberg-Witten invariants of symplectic manifolds and the adjunction inequality, we have the following corollary which also implies Corollary 1.3.

Corollary 1.8. *If $\Delta_K(t)$ is not monic, then X_K does not admit a symplectic structure. Furthermore, if X contains a surface Σ_g of genus g disjoint from T with $0 \neq [\Sigma_g] \in H_2(X;\mathbf{Z})$ and with $[\Sigma_g]^2 < 2-2g$ if $g > 0$ or $[\Sigma_g]^2 < 0$ if $g = 0$, then X_K with the opposite orientation does not admit a symplectic structure.*

Proof. Suppose that X_K admits a symplectic structure with symplectic form ω and canonical class κ. Taubes has shown that $\mathrm{SW}_{X_K}(\kappa) = \pm 1$ and that if k is any other basic class then $|k\cdot\omega| < \kappa\cdot\omega$.

Since $\mathrm{SW}_{X_K} = \mathrm{SW}_X \cdot \Delta_K(t)$ and since $\Delta_K(t) = a_0 + \sum_{n=1}^{d} a_n(t^n + t^{-n})$ is a polynomial in one variable $t = \exp(2[T])$, any nontrivial basic class, and in particular, the canonical class, is of the form $\kappa = \alpha + 2n[T]$ where $\mathrm{SW}_X(\alpha) \neq 0$ and $|n| \leq d$. Let m be the maximum integer satisfying $\mathrm{SW}_X(\alpha + m[T]) \neq 0$. Note that $m \geq 0$. Set $\beta = \alpha + m[T]$. Because of the maximality of m, we have $\mathrm{SW}_{X_K}(\beta + 2d[T]) = a_d \cdot \mathrm{SW}_X(\beta) \neq 0$.

Replacing $[T]$ with $-[T]$ allows us to assume that $[T]\cdot\omega \geq 0$. First assume that $[T]\cdot\omega > 0$. Then

$$(\beta + 2d[T])\cdot\omega = \kappa\cdot\omega + (m + 2(d-n))[T]\cdot\omega \geq \kappa\cdot\omega$$

because $m \geq 0$ and $d - n \geq 0$, and equality occurs only if $m = 0$ and $n = d$. But a strict inequality contradicts Taubes' theorem, thus $\alpha = \beta$ and $n = d$; so $\kappa = \beta + 2d[T]$. This means that $\pm 1 = \mathrm{SW}_{X_K}(\kappa) = a_d \cdot \mathrm{SW}_X(\beta)$; so $a_d = \pm 1$, *i.e.* Δ_K is monic.

If $[T]\cdot\omega = 0$,

$$(\beta + 2d[T])\cdot\omega = (\alpha + 2n[T])\cdot\omega = \kappa\cdot\omega,$$

which means that $\kappa = \beta + 2d[T]$, and again we see that $a_d = \pm 1$.

Finally, if any manifold Y contains a homologically nontrivial surface Σ_g of genus g with $[\Sigma_g]^2 > 2g - 2$, then, if $g > 0$, it follows from the adjunction inequality [KM, MST] that the Seiberg-Witten invariants of Y vanish, and hence Y does not admit a symplectic structure [T1]. If $g = 0$ and $[\Sigma_g]^2 \geq 0$ one can also show that the Seiberg-Witten invariants of Y must vanish [FS2, Ko2]. □

The authors are unaware of any simply connected smooth oriented $4-$manifold X with $b^+ > 1$ and $SW_X \neq 0$ which does not contain either a sphere with self-intersection -2 or a torus with self-intersection -1.

The techniques used in proving Theorem 1.5 generalize to the more general setting of links. Let $L = \{K_1, \dots, K_n\}$ be an $(n \geq 2)$-component ordered link in S^3 and suppose that X_j, $j = 1, \dots, n$, are simply connected smooth 4-manifolds with $b^+ \geq 1$ and each containing a smoothly embedded torus T_j of self-intersection 0 with $\pi_1(X_j \setminus T_j) = 1$ and representing a nontrivial homology class $[T_j]$. If (ℓ_j, m_j) denotes the standard longitude-meridian pair for the knot K_j, let

$$\alpha_L : \pi_1(S^3 \setminus L) \to \mathbf{Z}$$

denote the homomorphism characterized by the property that $\alpha_L(m_j) = 1$ for each $j = 1, \dots, n$. Now, mimicking the knot case above, let M_L be the 3-manifold obtained by performing $\alpha_L(\ell_j)$ surgery on each component K_j of L. (The surgery curves form the boundary of a Seifert surface for the link.) Then, in $M_L \times S^1$ we have smooth tori $T_{m_j} = m_j \times S^1$ of self-intersection 0 and we can construct the $n-$fold fiber sum

$$X(X_1, \dots X_n; L) = (M_L \times S^1) \#_{T_j = T_{m_j}} \coprod_{j=1}^{n} X_j$$

Here, the fiber sum is performed using the natural framings $T_{m_j} \times D^2$ of the neighborhoods of $T_{m_j} = m_j \times S^1$ in $M_L \times S^1$ and the neighborhoods $T_j \times D^2$ in each X_j and glued together so as to preserve the homology classes $[\mathrm{pt} \times \partial D^2]$. As in the knot case, it follows from Van Kampen's Theorem that $X(X_1, \dots X_n; L)$ is simply connected. Furthermore, the signature and Euler characteristic (*i.e.* the rational homotopy type) of $X(X_1, \dots X_n; L)$ depend only on the rational homotopy type of the X_i and the number of components in the link L. In the special case that all the X_j are the same manifold X we denote the resulting construction by X_L. It is conceptually helpful to note that $X(X_1, \dots X_n; L)$ is obtained from the disjoint union of the X_j by removing a neighborhood of the tori T_j and replacing them with the complement of the link L in S^3 crossed with S^1.

If $\Delta_L(t_1, \dots, t_n)$ denotes the symmetric multivariable Alexander polynomial of the $n \geq 2$ component link L, and $E(1)$ denotes the rational elliptic surface, our second theorem is:

Theorem 1.9. *The Seiberg-Witten invariant of $E(1)_L$ is*

$$SW_{E(1)_L} = \Delta_L(t_1, \dots, t_n)$$

where $t_j = \exp(2[T_j])$.

As a corollary we shall show:

Corollary 1.10. *The Seiberg-Witten invariant of $X(X_1, \dots X_n; L)$ is*

$$SW_{X(X_1, \dots X_n; L)} = \Delta_L(t_1, \dots, t_n) \cdot \prod_{j=1}^{n} SW_{E(1) \#_{F=T_j} X_j}$$

where $t_j = \exp(2[T_j])$.

As before, the work of Taubes [T3, T5] implies that if L is a fibered link, and each (X_j, T_j) is a symplectic pair, then $X(X_1, \dots X_n; L)$ is a symplectic 4-manifold whose Gromov invariant is

$$\mathrm{Gr}_{X(X_1, \dots X_n; L)} = \mathrm{A}_K(t_1, \dots, t_n) \cdot \prod_{j=1}^{n} \mathrm{Gr}(E(1) \#_{F=T_j} X_j).$$

Two words of caution are in order here. First, there may be relations in $X(X_1, \dots X_n; L)$ among the homology classes $[T_j]$ represented by the tori $T_j = T_{m_j}$. These relations are determined by the linking matrix of the link L. In particular, if all the linking numbers are zero, then the $[T_j]$ are linearly independent. At the other extreme, if the n-component link is obtained from the Hopf link by pushing off one component $(n-2)$ times, then all the $[T_j]$ are equal. Second, the ordering of the components of the link can affect these relations.

If L is a two component link with odd linking number, then $E(1)_L$ is a homotopy $K3$-surface and there are many interesting new polynomials which are not products of A-polynomials (*cf.* [Hi]) that can occur in this way as the Seiberg-Witten invariants of a homotopy $K3$-surface.

The first examples of nonsymplectic simply connected irreducible smooth 4-manifolds were constructed by Z. Szabo [S1]. These manifolds $X(k)$ $(k \in \mathbf{Z}, k \neq 0, \pm 1)$ can be shown to be diffeomorphic to $E(1)_{W(k)}$, where $W(k)$ is the 2-component k-twisted Whitehead link (see Figure 1) with Alexander polynomial $k(t_1^{1/2} - t_1^{-1/2})(t_2^{1/2} - t_2^{-1/2})$. By Theorem 1.9 (and by the computation first given in [S1])

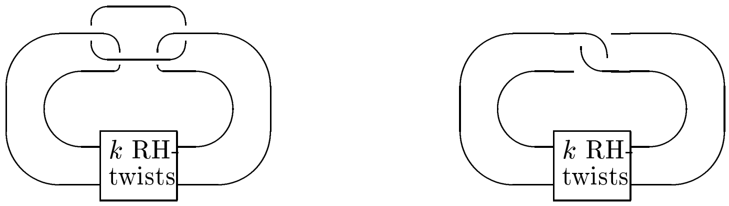
$$\mathrm{SW}_{X(k)} = k(t_1^{1/2} - t_1^{-1/2})(t_2^{1/2} - t_2^{-1/2}).$$

Thus by Taubes' Theorem [T1], $X(k)$ does not admit a symplectic structure with either orientation (since $X(k)$ contains spheres with self-intersection -2). Note also that for $k = \pm 1$ the $k-$twisted Whitehead link is fibered; so $X(\pm 1)$ is, in fact, symplectic.

The first examples of nonsymplectic homotopy $K3$-surfaces were constructed by the authors. These manifolds $Y(k)$ can be shown to be diffeomorphic to $K3_{T(k)}$ where $T(k)$ is the k-twist knot (see Figure 1) with Alexander polynomial $kt - (2k+1) + kt^{-1}$. By Theorem 1.5

$$\mathrm{SW}_{Y(k)} = kt - (2k+1) + kt^{-1}$$

and, by Corollary 1.8, if $k \neq 0, \pm 1$, $Y(k)$ does not admit a symplectic structure with either orientation. Again, for $k = \pm 1$ the $k-$twist knot is fibered; so $Y(\pm 1)$ is symplectic.



$W(k) = k$-twisted Whitehead link $\qquad$ $T(k) = k$-twist knot

Figure 1

Our next task is to prove Theorem 1.5 and Theorem 1.9. The proof of Theorem 1.5 is constructive and gives an algorithm which relates the Seiberg-Witten invariants of X_K with those of X by performing a series of topological log transforms on nullhomologous tori in X_K which reduce it to X. This turns out to be the same algorithm used to compute the Alexander polynomial $\Delta_K(t)$. This proof relies upon important analytical work of Morgan, Mrowka, and Szabo [MMS] (*cf.* [S1, S2]) and Taubes [T6] regarding the effect on the Seiberg-Witten invariants of removing neighborhoods of tori and sewing in manifolds with nonnegative scalar curvature. These we present in Section 2 in the form of gluing theorems. The proof of Theorem 1.9 can take one of two routes. The first is to utilize the algorithm provided by Conway [C] to compute the Alexander polynomial (more precisely the potential function) of a link. The proof then proceeds in a (tedious) manner, similar to the proof of Theorem 1.5. However, we choose to present a more direct proof by showing that the Seiberg-Witten invariants for the manifolds $E(1)_L$ satisfy the axioms for the Alexander polynomial of a link as provided by Turaev [T]. The advantage of this proof, aside from its brevity, is that it isolates the required gluing properties of the Seiberg-Witten invariants and perhaps lays the foundation for determining the axioms for an appropriate gauge theory which may expose the other, more sophisticated, knot and link invariants. In the final section we discuss examples with $b^+ = 1$ which are given by our construction.

It was Meng and Taubes [MT] who first discovered the relationship between Seiberg-Witten type invariants and the Alexander polynomial. In [MT] they defined a 3-manifold invariant by dimension-reducing the Seiberg-Witten invariants, and they showed that this 3-manifold invariant was related to the Milnor torsion. We fell upon Theorems 1.5 and 1.9 by attempting to understand our above mentioned constructions of nonsymplectic homotopy $K3$-surfaces.

We end this introduction with three items. First, we conjecture that if K and K' are two distinct knots (or n-component links) then X_K is diffeomorphic to $X_{K'}$ if and only if K is isotopic to K'. Second, we wish to thank Jim Bryan, Bob Gompf, Elly Ionel, Dieter Kotschick, Wladek Lorek, Dusa McDuff, Terry Lawson, Tom Parker, and Cliff Taubes for useful conversations. Finally, we wish to make it clear that the contributions of the present paper are of a purely topological nature. The gauge theoretic input to our theorems is due to Morgan, Mrowka, and Szabo and to Taubes.

2. Background for the proofs of Theorems 1.5 and 1.9

In this section we shall survey the recent gluing theorems of Morgan, Mrowka, and Szabo [MMS] (*cf.* [S1, S2]) and Taubes [T6] (*cf.* [MT]) which are used in the proof of Theorem 1.5. Also we shall review some of the work of W. Brakes and J. Hoste on 'sewn-up link exteriors' which will be used in our constructions.

The context for the first of the gluing results is as follows. We are given a smooth 4-manifold X with $b_X^+ > 1$ and with an embedded torus T which represents a nontrivial homology class $[T]$ of self-intersection 0 in $H_2(X;\mathbf{Z})$. Any Seiberg-Witten basic class $\alpha \in H_2(X;\mathbf{Z})$ (*i.e.* any α with $\mathrm{SW}_X(\alpha) \neq 0$) must be orthogonal to the homology class $[T]$ since the adjunction inequality states that $0 \geq [T]^2 + |\alpha \cdot [T]|$. The *relative Seiberg-Witten invariant* $\mathrm{SW}_{(X;T)}$ is formally defined to be

$$\mathrm{SW}_{(X;T)} = \mathrm{SW}_{X\#_{T=F}E(1)}$$

where $E(1)$ is the rational elliptic surface with smooth elliptic fiber F.

It is an interesting consequence of the gluing theorems of Morgan-Mrowka-Szabo [MMS] and Taubes [T6] that

Theorem 2.1. *Suppose that $b_X^+ > 1$ and that the torus T is c-embedded. Then*

$$SW_{(X;T)} = SW_X \cdot (t^{1/2} - t^{-1/2})$$

where $t = \exp(2[T])$.

Note that $E(1)$ has a metric of positive scalar curvature. A much more general (and difficult to prove) gluing theorem is:

Theorem 2.2 (Morgan, Mrowka, and Szabo [MMS])**.** *In the situation above*

$$SW_{X_1 \#_{T_1 = T_2} X_2} = SW_{(X_1;T_1)} \cdot SW_{(X_2;T_2)}.$$

The fact that Corollary 1.10 follows from Theorem 1.9 is now an easy consequence of the gluing theorems Theorem 2.1 and Theorem 2.2; for note that $X(X_1, \dots, X_n; L) = X(X_2, \dots, X_n; L) \#_{T_{m_1} = T_1} X_1$. Thus Theorem 2.1 implies that

$$\mathrm{SW}_{X(X_1,\dots,X_n;L)} = \mathrm{SW}_{X(E(1),X_2,\dots,X_n;L)} \cdot \mathrm{SW}_{X_1 \#_{T_1=F} E(1)},$$

and continuing inductively completes the argument. This is the only time we shall need to use the general gluing theorem (2.2).

For our proof of Theorem 1.5 we will form an 'internal fiber sum'. For this construction suppose that we have a pair of c-embedded tori T_1, T_2.

In our manifold X with $b_X^+ > 1$ we formally define the relative Seiberg-Witten invariant $\mathrm{SW}_{(X;T_1,T_2)}$ to be

$$\mathrm{SW}_{(X;T_1,T_2)} = \mathrm{SW}_X \cdot (\tau_1 - \tau_1^{-1}) \cdot (\tau_2 - \tau_2^{-1}).$$

where $\tau_j = \exp([T_j])$. We construct the *internal fiber sum* X_{T_1,T_2} by identifying the boundaries of neighborhoods of the T_i, again preserving the homology classes $[\mathrm{pt} \times \partial D^2]$.

The first gluing theorem we need for the proof of Theorem 1.5 is:

Theorem 2.3 (Morgan, Mrowka, and Szabo [MMS], Taubes [T6](*cf.*[MT]))**.** *The internal fiber sum X_{T_1,T_2} has Seiberg-Witten invariant*

$$SW_{X_{T_1,T_2}} = SW_{(X;T_1,T_2)}|_{\tau_1 = \tau_2}.$$

The other gluing result we need concerns generalized log transforms on nullhomologous tori. Let p and q be relatively prime nonzero integers (or $(1,0)$ or $(0,1)$). If T is any embedded self-intersection 0 torus in a 4-manifold Y with tubular neighborhood

$$N = T \times D^2 = S^1 \times S^1 \times D^2,$$

let $\varphi = \varphi_{p,q}$ be the diffeomorphism $S^1 \times S^1 \times \partial D^2 \to \partial N$ given by

$$\varphi(x, y, z) = (x, y^s z^q, y^r z^p), \qquad \det \begin{pmatrix} p & q \\ r & s \end{pmatrix} = -1.$$

The manifold

$$Y(p/q) = (Y \setminus N) \cup_\varphi (S^1 \times S^1 \times D^2)$$

is called the *(generalized) (p/q)-log transform of Y along T*. Our notation and terminology are incomplete since the splitting $T = S^1 \times S^1$ is necessary information. Throughout this paper, whenever a log transform is performed on a torus T, there will be a natural identification $T = S^1 \times S^1$ and we always perform the transform with $\varphi_{p,q}$ in the coordinates $S^1 \times S^1 \times D^2$ as above.

We shall need to study the situation where a log transform is performed on a nullhomologous torus in Y. If T is such a torus, then in $Y(0/1)$ there appears a new 2-dimensional homology class which is represented by the torus

$$T_0 = S^1 \times S^1 \times \mathrm{pt} \subset S^1 \times S^1 \times D^2 \subset (X \setminus N) \cup_\varphi (S^1 \times S^1 \times D^2) = Y(0/1).$$

(The old torus pushed to $\partial(Y \setminus N)$ is now $\varphi(S^1 \times \mathrm{pt} \times \partial D^2)$.) Notice that each homology class $\alpha \in H_2(Y; \mathbf{Z})$ may be viewed as a homology class in each $Y(p/q)$.

Theorem 2.4 (Morgan, Mrowka, and Szabo [MMS], Taubes [T6] (*cf.* [MT])). *Let Y be a smooth 4-manifold with $b^+ \geq 3$, and suppose that Y contains a nullhomologous torus T with tubular neighborhood $N = T \times D^2 = S^1 \times S^1 \times D^2$. Let τ be the homology class of T_0 in $Y(0/1)$. Then for each characteristic homology class $\alpha \in H_2(Y; \mathbf{Z})$,*

$$SW_{Y(p/q)}(\alpha) = pSW_Y(\alpha) + q \sum_{i=-\infty}^{\infty} SW_{Y(0/1)}(\alpha + 2i\tau).$$

The sum in the above formula reduces to a single term in all the situations which are encountered in the proofs of Theorem 1.5 and Theorem 1.9. Specifically,

2.5. *In Theorem 2.4 suppose that there is a torus in $Y(0/1)$ which is disjoint from T and which represents a homology class σ of self-intersection 0 whose intersection number with τ in $Y(0/1)$ is 1. Suppose furthermore that $\sigma \cdot \alpha = 0$ for all $\alpha \in H_2(Y) \subset H_2(Y(0/1))$. Then*

$$SW_{Y(p/q)} = pSW_Y + qSW_{Y(0/1)}.$$

Proof. Note that $H_2(Y(0/1)) = H_2(Y) \oplus H(\sigma, \tau)$ where $H(\sigma, \tau)$ is a hyperbolic pair. If $\alpha \in H_2(Y; \mathbf{Z})$ satisfies $\mathrm{SW}_{Y(0/1)}(\alpha + 2i\tau) \neq 0$, the adjunction inequality implies that

$$0 \geq \sigma^2 + |(\alpha + 2i\tau) \cdot \sigma| = |2i|.$$

Thus $i = 0$, and $\mathrm{SW}_{Y(p/q)}(\alpha) = p\mathrm{SW}_Y(\alpha) + q\mathrm{SW}_{Y(0/1)}(\alpha)$. Since for $p \neq 0$, each $\alpha \in H_2(Y(p/q))$ arises from a class in $H_2(Y)$, the lemma follows. $\square$

We next wish to describe a method for constructing 3-manifolds which was first studied by W. Brakes [B] and extended by J. Hoste [Ho]. Let L be a link in S^3 with two *oriented* components C_1 and C_2. Fix tubular neighborhoods $N_i \cong S^1 \times D^2$ of C_i with $S^1 \times (\text{pt on } \partial D^2)$ a longitude of C_i, *i.e.* nullhomologous in $S^3 \setminus C_i$. For any $A \in GL(2;\mathbf{Z})$ with $\det A = -1$, we the get a 3-manifold

$$s(L;A) = (S^3 \setminus \operatorname{int}(N_1 \cup N_2))/A$$

called a *sewn-up link exterior* by identifying ∂N_1 with ∂N_2 via a diffeomorphism inducing A in homology. For $n \in \mathbf{Z}$, let $A_n = \left(\begin{smallmatrix} -1 & 0 \\ -n & 1 \end{smallmatrix}\right)$. A simple calculation shows that $H_1(s(L;A_n);\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}_{n-2\ell}$ where ℓ is the linking number in S^3 of the two components C_1, C_2, of L. (See [B].) The second summand is generated by the meridian to either component.

J. Hoste [Ho] has given a recipe for producing Kirby calculus diagrams for $s(L;A_n)$. First we review the notion of a 'band sum' on an oriented knot or link L. Consider a portion of L consisting of a pair of strands, oriented in opposite directions, and separated by a band B. We identify B with an embedding of $I^2 = [0,1] \times [0,1]$ in S^3 such that $B \cap L = (\{0\} \times I) \cup -(\{1\} \times I)$. Let K be the (oriented) knot or link obtained by trading the segments $B \cap L = \{0,1\} \times I$ of ∂B for the complementary oriented segments $I \times \{0\}$ and $-(I \times \{1\})$. The process of exchanging L for K in this fashion is called a *band sum*. Associated with the band move are two unknots: U_ν, an unknotted circle which bounds a disk whose interior meets $B = I^2$ in the arc $\{\frac{1}{2}\} \times I$ and is disjoint from L, and U_o, which spans a disk whose interior meets B in $I \times \{\frac{1}{2}\}$ and is disjoint from K. (See Figure 2.)

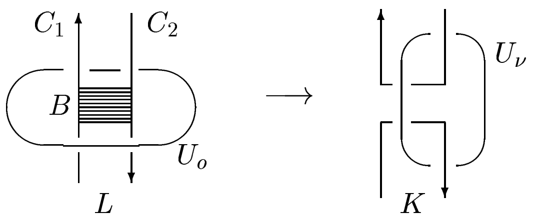


Figure 2

Proposition 2.6 (Hoste [Ho]). *Let $L = C_1 \cup C_2$ be an oriented link in S^3. Consider a portion of L consisting of a pair of strands, one from each component, oriented in opposite directions, and separated by a band B. The band sum of C_1 and C_2 is a knot K, and U_ν links K twice geometrically and 0 times algebraically. The sewn-up link exterior $s(L;A_n)$ is obtained from surgery on the the two component link $K \cup U_\nu$ in S^3 with surgery coefficient 0 on U_ν and $n - 2\ell$ on K, where ℓ is the linking number of C_1 and C_2.*

Next consider a related situation. Let $Z = s(L;A_n)$ where $n = 2\ell$; so Z has $H_1(Z;\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$. Suppose that B is a band in S^3 meeting L as in Proposition 2.6, with the circle U_o, which links L twice geometrically and 0 times algebraically. Then U_o gives rise to a loop, $\bar{U}_o$ in Z. To get a Kirby calculus picture of this situation, apply Hoste's formula. We obtain a two component framed link $K \cup U_\nu$ with 0-framing on each. In S^3, U_o bounds a disk which is disjoint from K and meets a disk spanning U_ν in two points with opposite orientations. Thus U_o bounds a punctured torus in $S^3 \setminus (K \cup U_\nu)$; and so $\bar{U}_o$ is nullhomologous in Z. This means that $\bar{U}_o$ has a naturally defined longitude in Z; so p/q- Dehn surgery on $\bar{U}_o$ is well-defined. Let Z_0 denote the result of 0-surgery on $\bar{U}_o$ in Z.

In Z, let m denote the meridian circle to the sewn-up torus. Then in $Z \times S^1$ we have the torus $T_m = m \times S^1$ of self-intersection 0. Form the fiber sum $X\#_T(Z \times S^1)$ of X with $Z \times S^1$ by identifying T_m with the torus T of X. (In the name of brevity, we shall sometimes make, as in this case, a mild change in our notation for fiber sum.) Similarly we can form the fiber sum $X\#_T(Z_0 \times S^1)$, which is seen to be the result of performing a $(0/1)$-log transform on the torus $\bar{U}_o \times S^1$ in $X\#_T(Z \times S^1)$. We wish to compute the Seiberg-Witten invariant of this 4-manifold.

By cutting open $s(L; A_n)$ along one of the two components of L, we obtain the exterior of the link L in S^3, and if this is done after performing 0-surgery on U_o, we obtain a link exterior in $S^2 \times S^1$. Now perform the corresponding task in $X\#_T(Z_0 \times S^1)$, removing a torus $C_i \times S^1$. We obtain $X\#_T(S^2 \times S^1 \times S^1)$ with a pair of tubular neighborhoods of self-intersection 0 tori removed. Call this manifold $Q(L)$. To reiterate — $Q(L)$ is obtained by performing 0-surgery on U_o in $S^3 \setminus L$, crossing with S^1, and then fiber-summing with X along T_m. The boundary components of $Q(L)$ have a natural framing λ, μ, σ, coming from the longitude and meridian of the link components in S^3 and the S^1 in the last coordinate. We can re-obtain $X\#_T(Z_0 \times S^1)$ by sewing up the boundary 3-dimensional tori of $Q(L)$ using the matrix $A_n \oplus (1)$. Instead, let us fill in each of the boundary components of $Q(L)$ with a copy of $S^1 \times D^2 \times S^1$. This can be done in many ways. We wish to do it so that, using the framings obtained from our copies of $S^1 \times D^2 \times S^1$, we obtain $X\#_T(Z_0 \times S^1)$ by sewing up the boundary of the resultant manifold with a neighborhood of the (new) link $(S^1 \times \mathrm{pt} \times S^1) \cup (S^1 \times \mathrm{pt} \times S^1)$ removed using the matrix $\pm A_0 \oplus (1)$. Using the obvious framing for $S^1 \times D^2 \times S^1$, we claim that this is done by gluing each $S^1 \times D^2 \times S^1$ to a component of $\partial Q(L)$ by a diffeomorphism with matrix $B_\ell \oplus (1)$ where $B_\ell = \begin{pmatrix} 0 & 1 \\ 1 & -\ell \end{pmatrix}$. We shall denote by $W(L)$ the manifold formed using $B_\ell \oplus (1)$ to sew in the neighborhoods of the tori $C_i' \times S^1 = S^1 \times pt \times S^1$.

Let $V(L)$ be the result of filling in the exterior of L in S^3 via the diffeomorphism B_ℓ on each component. This is just the result of $(-\ell)$-framed surgery on each component of L. Now if we sew up the link complement $V(L) \setminus L$ via $-A_0$, we get the result of sewing up $S^3 \setminus L$ using the diffeomorphism $B_\ell(-A_0)B_\ell^{-1} = A_{2\ell}$. Thus $s(L; A_n) =$

$s(L'; -A_0)$ where L' is the link C_1', C_2'. Denote by $V_0(L)$ the result of performing 0-surgery on U_o in $V(L) \setminus L' = S^3 \setminus L$. Then $W(L) = X\#_T(V_0(L) \times S^1)$ and $X\#_T(Z_0 \times S^1) = W(L)_{T_1=T_2}$ where $T_i = C_i' \times S^1$.

Proposition 2.7. *Suppose that T is a c-embedded torus in X. Then with the above notation,*

$$SW_{X\#_T(Z_0\times S^1)} = (t^{1/2} - t^{-1/2})^2 \cdot SW_{X_K}$$

as Laurent polynomials, where K is the band sum of C_1 and C_2 using the band B, and $t = \exp(2[T])$.

Proof. Since the matrix $-A_0 \oplus (1)$ identifies the tori $C_i' \times S^1$ in the boundary components of $Q(L)$, Theorem 2.3 tells us that $\mathrm{SW}_{X\#_T(Z_0\times S^1)} = \mathrm{SW}_{W(L)_{T_1=T_2}}$ is obtained from the relative invariant $\mathrm{SW}_{(W(L);C_1',C_2')}$ by identifying the homology classes in $Q(L)$ represented by the tori $T_i = C_i' \times S^1$.

The gluing diffeomorphism B_ℓ identifies the homology class of a longitude of C_i' with the meridian of C_i in $S^3 \setminus L$. Thus Theorem 2.3 implies that

$$\mathrm{SW}_{X\#_T(Z_0\times S^1)} = \mathrm{SW}_{W(L)} \cdot (\tau - \tau^{-1})^2,$$

where $\tau = \exp([m \times S^1])$.

It remains to identify the manifold $W(L)$ as X_K. By construction, $W(L)$ is obtained from the 3-component link $C_1 \cup C_2 \cup U_o$ in S^3 by performing $(-\ell)$-framed surgery on C_1 and C_2 and 0-framed surgery on U_o, crossing with S^1, and fiber summing to X along T_m and T. The result of framed surgery on the 3-component link is, by sliding C_1 over C_2, seen to be the same as 0-framed surgery on K. Thus $W(L) = X_K$, and the handle slide carries the meridian m to a meridian of K. Letting $t = \exp(2[T]) = \tau^2$, we get the calculation as claimed. □

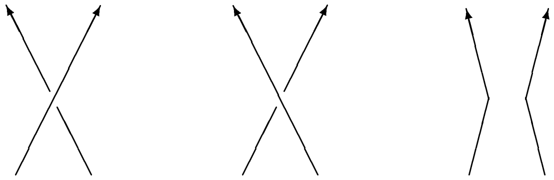
3. The proof of Theorem 1.5

We first recall a standard technique for calculating the (symmetrized) Alexander polynomial of a knot. This uses the skein relation

$$\Delta_{K_+}(t) = \Delta_{K_-}(t) + (t^{1/2} - t^{-1/2}) \cdot \Delta_{K_0}(t) \tag{1}$$

where K_+ is an oriented knot or link, K_- is the result of changing a single oriented positive (right-handed) crossing in K_+ to a negative (left-handed) crossing, and K_0 is the result of resolving the crossing as shown in Figure 3.

Note that if K_+ is a knot, the so is K_-, and K_0 is a 2-component link. If K_+ is a 2-component link, then so is K_-, and K_0 is a knot.

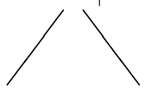


K_+ K_- K_0

Figure 3

The point of using (1) to calculate Δ_K is that K can be simplified to an unknot via a sequence of crossing changes of a projection of the oriented knot or link to the plane. To describe this well-known technique, consider such a projection and choose a basepoint on each component. In the case of a link, order the components. Say that such a projection is *descending*, if starting at the basepoint of the first component and traveling along the component, then from the basepoint of the second component and traveling along it, *etc.*, the first time that each crossing is met, it is met by an overcrossing. Clearly a link with a descending projection is an unlinked collection of unknots. Our goal is to start with a knot K and perform skein moves so as to build a tree starting from K and at each stage adding the bifurcation of Figure 4, where each K_+, K_-, K_0 is a knot or 2-component link, and so that at the bottom of the tree, we obtain only unknots, and split links. Then, because for an unknot U we have $\Delta_U(t) = 1$, and for a split link S (of more than one component) we have $\Delta_S(t) = 0$, we can work backwards using (1) to calculate $\Delta_K(t)$.

K_+



K_- K_0

Figure 4

The recipe for constructing the tree is, in the case of a knot, to change the crossing of the first 'bad' crossing encountered on the traverse described above. In this case, the result of changing the crossing is still a knot, and the result of resolving the crossing is a 2-component link. In the case of a 2-component link, one changes the first 'bad' crossing between the two components which is encountered on the traverse. The result of changing the crossing is still a 2-component link, and the result of resolving the crossing is a knot. In this way we obtain a tree whose top vertex is the given knot K and which branches downward as in the figure above. We shall call this tree a *resolution tree* for the knot K. We claim that the tree can be extended until each bottom vertex represents an unknot or a split link.

For the projection of an oriented, based knot K, let $c(K)$ be the number of crossings and $b(K)$ be the number of bad crossings encountered on a traverse starting at the basepoint. The *complexity* of the projection is defined to be the ordered pair $(c(K), b(K))$. For the projection of an oriented, based 2-component link L, let $c(L)$ be the total number of crossings and let $b(L)$ be the number of bad crossings between the two components. Again the complexity is defined to be $(c(L), b(L))$. Consider a vertex which represents a knot or 2-component link A. Note that $c(A_-) = c(A_+)$, $b(A_-) < b(A_+)$, and $c(A_0) < c(A_+)$. Thus in the lexicographic ordering, $(c(A_-), b(A_-)) < (c(A_+), b(A_+))$ and $(c(A_0), b(A_0)) < (c(A_+), b(A_+))$. Now a knot K_1 with $c(K_1) = 0$ or with $b(K_1) = 0$ is the unknot, and a link L with $c(L) = 0$ is the unlink and with $b(L) = 0$, it is at least a split link. This completes the proof that we can construct the resolution tree as described. (We remark that for the sake of simplicity we have considered only the case where we have changed a positive to a negative crossing in the skein move. Of course, we may as well have to change a negative to a positive crossing in order to lower b at various steps, but this does not change the proof.)

Consider an oriented, based knot K in S^3 and a knot projection of K. We shall use the resolution tree for this projection as a guide for simplifying X_K in a way which leads to a calculation of X_K. Let us consider the first step, say $K = K_+ \to \{K_-,\ K_0\}$. At the crossing of K_- that is in question, there is an unknotted circle U linking K algebraically 0 times, so that the result of $+1$ surgery on U turns K_- into $K_+ = K$. (See Figure 5.) In X_{K_-} we have the nullhomologous torus $S^1 \times U$. (It bounds the product of S^1 with a punctured torus in $S^3 \setminus K_-$.) The fact that $+1$ surgery on U turns K_- into K_+ means that X_{K_+} is the result of a $(1/1)$-log transform on X_{K_-} along $S^1 \times U$. Let $X_{K_-}(0/1)$ denote the result of performing a $(0/1)$-log transform on $S^1 \times U$ in X_{K_-}. We now use (2.4) and (2.5) to compute $\mathrm{SW}(X_K)$. Two tori are central to this calculation. Letting m_U be a meridional circle to U, we get the torus $S^1 \times m_U$ which represents the homology class τ of (2.4). Also, in $X_{K_-}(0/1)$, the boundary of the punctured torus described above is spanned by a disk, and we obtain a torus of self-intersection 0 representing a class σ such that $\sigma \cdot \tau = 1$. Note that $H_2(X_{K_-}(0/1)) = H_2(X_{K_-}) \oplus H(\sigma, \tau)$; so (2.5) applies. Hence $\mathrm{SW}_{X_K} = \mathrm{SW}_{X_{K_-}} + \mathrm{SW}_{X_{K_-}(0/1)}$.

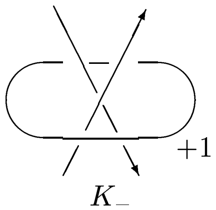


Figure 5

Recall that $X_{K_-}(0/1)$ is obtained by performing 0-framed surgery on both components of the link $K_- \cup U$, crossing with S^1 and fiber-summing with X, using the torus T obtained from a meridian of K_- crossed with S^1. Hoste's recipe, (2.6), allows us to interpret the result of 0-framed surgery on both components of $K_- \cup U$ in S^3 as $s(K_0; A_{2\ell})$ where K_0 is the 2-component link obtained by resolving the crossing under consideration (see Figure 6), and ℓ is the linking number of the two components of K_0.

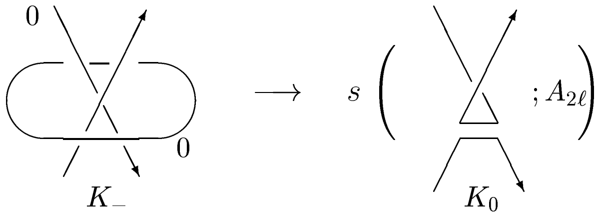


Figure 6

Let $X(s(K_0; A_{2\ell})) = (s(K_0; A_{2\ell}) \times S^1)\#_T X$ where T is the product of S^1 with a meridian to either component of K_0. Because $A_{2\ell}$ sends meridians to meridians, this definition does not depend on the choice of component. Then $X_{K_-}(0/1) \cong X(s(K_0; A_{2\ell}))$; so

$$\mathrm{SW}_{X_{K_+}} = \mathrm{SW}_{X_{K_-}} + \mathrm{SW}_{X(s(K_0;A_{2\ell}))}, \tag{2}$$

mimicking the skein move which gives the second tier of the resolution tree.

Now consider the next stage of the resolution tree and the skein move $L \to \{L_-, L_0\}$ where $K_0 = L = L_+$. This move corresponds to changing a bad crossing involving both components of L. If the bad crossing under consideration is, say, right-handed, then $L = L_+$ can be obtained from $L_- = C_1 \cup C_2$ by $+1$-surgery on an unknotted circle U_o as in Figure 7. This means that $X(s(L; A_{2\ell}))$ is the result of a $(1/1)$-log transform on the torus $S^1 \times U_o$ in $X(s(L_-; A_{2\ell_-}))$ where ℓ_- is the linking number of the components C_1, C_2 of L_- and is determined by the fact that $H_1(s(L_-; A_{2\ell_-}); \mathbf{Z}) = H_1(s(L; A_{2\ell}); \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$.

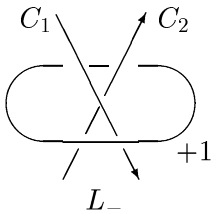


Figure 7

In $X(s(L_-; A_{2\ell_-}))$, the torus $S^1 \times U_o$ is nullhomologous. This is precisely the situation of (2.7) where $Z = s(L_-; A_{2\ell_-})$. We wish to apply (2.5) to this situation. Let $X(s(L_-; A_{2\ell_-}))(0/1)$ denote the 4-manifold obtained by performing a $(0/1)$-log transform on $S^1\times U_o$ in $X(s(L_-;A_{2\ell_-}))$. This is the manifold $X\#_T(Z_0\times S^1)$ of (2.7), and the 3-manifold Z_0 is the result of 0-surgery on U_o in $s(L_-;A_{2\ell_-})$. Let m_{U_o} be a meridional circle to $U_o\subset s(L_-;A_{2\ell_-})$. The torus $m_{U_o}\times S^1$ in $X(s(L_-;A_{2\ell_-}))(0/1)$ is the T_0 mentioned in the statement of (2.4). As we argued in the proof of (2.7), in Z, the loop U_o bounds a punctured torus, and this gets completed to a torus of self-intersection 0 in Z_0. Let σ be its homology class in $X(s(L_-;A_{2\ell_-}))(0/1) = X\#_T(Z_0\times S^1)$. Since the class σ satisfies the hypothesis of (2.5), we have

$$\mathrm{SW}_{X(s(L_+;A_{2\ell}))}=\mathrm{SW}_{X(s(L_-;A_{2\ell_-}))}+\mathrm{SW}_{X\#_T(Z_0\times S^1)}.$$

Applying (2.7), this becomes:

$$\mathrm{SW}_{X(s(L_+;A_{2\ell}))}=\mathrm{SW}_{X(s(L_-;A_{2\ell_-}))}+(t^{1/2}-t^{-1/2})^2\cdot \mathrm{SW}_{X_{L_0}} \tag{3}$$

where L_0 is the result of resolving the crossing of L which is under consideration, and $t=\exp(2[T])$.

In order to see that this process calculates $\Delta_K(t)$, for fixed X, we define a formal Laurent series Θ, which is an invariant of knots and 2-component links. For a knot K, define Θ_K to be the quotient, $\Theta_K=\mathrm{SW}_{X_K}/\mathrm{SW}_X$, and for a 2-component link with linking number ℓ between its components, $\Theta_L=(t^{1/2}-t^{-1/2})^{-1}\cdot\mathrm{SW}_{X(s(L;A_{2\ell}))}/\mathrm{SW}_X$, where as usual $t=\exp(2[T])$. It follows from (2) and (3) that for knots or 2-component links, Θ satisfies the skein relation

$$\Theta_{K_+}=\Theta_{K_-}+(t^{1/2}-t^{-1/2})\cdot\Theta_{K_0}.$$

For a split 2-component link, L, the 3-manifold $s(L;A_{2\ell})$ contains an essential 2-sphere (coming from the 2-sphere in S^3 which splits the link). This means that $X(s(L;A_{2\ell}))$ contains an essential 2-sphere of self-intersection 0, and this implies that $\mathrm{SW}_{X(s(L;A_{2\ell}))}=0$ (see [FS2]). Thus for a split link, $\Theta_L=0$. For the unknot U, the manifold X_U is just $X\#_T(S^2\times T^2)=X$, and so $\Theta_U=1$. Subject to these initial values, the resolution tree and the skein relation (1) determine $\Delta_K(t)$ for any knot K. It follows that Θ_K is a Laurent polynomial in a single variable t, and $\Theta_K(t)=\Delta_K(t)$, completing the proof of Theorem 1.4.

4. The proof of Theorem 1.9

We first review the axioms which determine the Alexander polynomial of a link. The reference for this material is [T]. The fact, proved in [T], that we shall use here is that there is but one map ∇ which assigns to each $n-$component ordered link L in S^3 an element of the field $\mathbf{Q}(t_1,\dots,t_n)$ with the following properties:

1. $\nabla(L)$ is unchanged under ambient isotopy of the link L.
2. If L is the unknot, then $\nabla(L)=1/(t-t^{-1})$.
3. If $n\geq 2$, then $\nabla(L)\in\mathbf{Z}[t_1,t_1^{-1},\dots,t_n,t_n^{-1}]$.
4. The one-variable function $\tilde\nabla(L)(t)=\nabla(L)(t,t,\dots,t)$ is unchanged by a renumbering of the components of L.

5. (Conway Axiom). If L_+, L_-, and L_0 are links coinciding (except possible for the numbering of the components) outside a ball, and inside this ball have the form depicted in Figure 8, then
$$\tilde{\nabla}(L_+) = \tilde{\nabla}(L_-) + (t - t^{-1}) \cdot \tilde{\nabla}(L_0).$$
6. (Doubling Axiom). If the link L' is obtained from the link $L = \{K_1, \ldots, K_n\}$ by replacing the K_j by its $(2,1)-$cable, then
$$\nabla(L')(t_1, \ldots, t_n) = (T + T^{-1}) \cdot \nabla(L)(t_1, \ldots, t_{j-1}, t_j^2, t_{j+1}, \ldots, t_n)$$
where $T = t_j \prod_{i \neq j} t_i^{\ell k(K_j, K_i)}$.

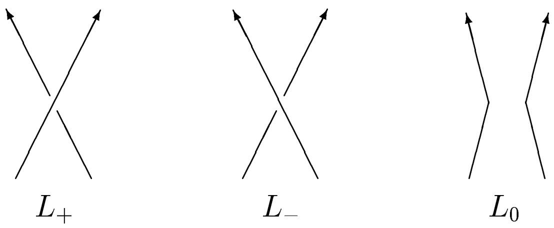


Figure 8

The Alexander polynomial Δ_L in Theorem 1.9 is just
$$\Delta_L(t_1^2, \ldots, t_n^2) \doteq \nabla(L)(t_1, \ldots, t_n)$$
where the symbol $\doteq$ denotes equality up to multiplication by -1 and powers of the variables.

Recall the construction of $E(1)_L$. If $L = \{K_1, \ldots, K_n\}$ is an $(n \geq 2)$-component ordered link in S^3 and (ℓ_j, m_j) denotes the standard longitude-meridian pair for the knot K_j, we let
$$\alpha_L : \pi_1(S^3 \setminus L) \to \mathbf{Z}$$
denote the homomorphism characterized by the property that $\alpha_L(m_j) = 1$ for each $j = 1, \ldots, n$. Define M_L to be the 3-manifold obtained by performing $\alpha_L(\ell_j)$ surgery on each component K_j of L. Then, in $M_L \times S^1$ we use the smooth tori $T_{m_j} = m_j \times S^1$ to construct the $n-$fold fiber sum
$$E(1)_L = (M_L \times S^1) \#_{T_{m_j} = F} \coprod_{j=1}^{n} E(1),$$
the fiber sum being performed using the natural framings $T_{m_j} \times D^2$ of the neighborhoods of $T_{m_j} = m_j \times S^1$ in M_j and the neighborhoods $F \times D^2$ of an elliptic fiber in each copy of $E(1)$ and glued together so as to preserve the homology classes $[\mathrm{pt} \times \partial D^2]$. It is amusing to note that this later condition is unnecessary in this

special situation. For, since $E(1) \setminus F$ has a big diffeomorphism group, we can fiber sum in the $E(1)$ with any gluing map and end up with diffeomorphic 4-manifolds.

Now let ∇ be that function which associates to every ordered n-component link L, the polynomial $\nabla(L)(t_1, \dots, t_n) = \mathrm{SW}_{E(1)_L}(t_1^2, \dots, t_n^2)$, with $t_j = \exp(2[T_{m_j}])$. We show that ∇ satisfies the above stated axioms. However, there is a small obstruction to doing this in a straightforward manner: SW_X is only defined when $b^+ > 1$ and when L has but one component $E(1)_L$ has $b^+ = 1$. Although our Theorem 1.9 is stated only for $n \geq 2$, the axioms insist that we consider 1-component links. We overcome this problem as follows. Given an ordered n-component link L we always fiber sum in the $K3$-surface (rather than $E(1)$) to $M_L \times S^1$ along T_{m_1}. In the case that $n \geq 2$, the resulting manifold, which we temporarily denote by $E(2,1)_L$, has

$$\mathrm{SW}_{E(2,1)_L} = (t_1^{1/2} - t_1^{-1/2}) \cdot \mathrm{SW}_{E(1)_L}$$

We shall complete the proof of Theorem 1.9 by showing that

$$\nabla(L)(t_1, \dots, t_n) = \frac{\mathrm{SW}_{E(2,1)_L}(t_1^2, \dots, t_n^2)}{t_1 - t_1^{-1}}$$

satisfies all the axioms.

Axiom 1 is clear.

For Axiom 2 note that if L is the unknot, then $E(2,1)_L$ is the $K3$-surface, so that $\nabla(L)(t) = \mathrm{SW}_{E(2,1)_L}(t^2)/(t - t^{-1}) = 1/(t - t^{-1})$.

To verify Axiom 3, consider an $(n \geq 2)$-component link L with components $K_1, \dots, K_n$. We need to see that the only possible basic classes of $E(2,1)_L$ are the classes T_{m_i} (which are identified in $E(2,1)_L$ with the fiber classes F_i). A Mayer-Vietoris sequence argument shows that $H_2(E(2,1)_L) \cong \mathrm{im}(\varphi) \oplus G$ where

$$H_2(E(2) \setminus F) \oplus \sum_1^{n-1} H_2(E(1) \setminus F) \oplus H_2((S^3 \setminus L) \times S^1) \xrightarrow{\varphi} H_2(E(2,1)_L) \xrightarrow{\delta} G$$

and G is isomorphic to the kernel of $\sum_1^n H_1(T^3) \to H_1((S^3 \setminus L) \times S^1)$. Now $H_2(E(2) \setminus F) \cong 2E_8 \oplus 2H \oplus 3(0)$ and $H_2(E(1) \setminus F) \cong E_8 \oplus 3(0)$ where H denotes a hyperbolic pair. Each copy of E_8 is represented by eight -2-spheres in the usual configuration, say W, with $\partial W = \Sigma(2,3,5)$, the Poincaré homology sphere. Since W embeds in $E(2)$, whose only basic class is 0, and since $\Sigma(2,3,5)$ has positive scalar curvature, a rudimentary gluing formula implies that each basic class of $E(2,1)_L$ is orthogonal to the image of the E_8 summands. Each of the two hyperbolic pairs H is the homology of a nucleus in $E(2) \setminus F$ and is generated by a torus τ of self-intersection 0 and a sphere σ of self-intersection -2 which intersect at one point. For any basic class κ of $E(2,1)_L$, the adjunction formula implies $0 \geq \tau^2 + |\kappa \cdot \tau|$; so κ is orthogonal to τ. Also, $\tau + \sigma$ is represented by another torus of self-intersection 0; so k is in fact orthogonal to H. Furthermore, each of the summands (0) in $H_2(E(1) \setminus F)$ and $H_2(E(2) \setminus F)$ is represented by a torus in the boundary of the tubular neighborhood of F, and each of these tori is glued to a torus in $(S^3 \setminus L) \times S^1$ in $E(2,1)_L$. It follows that the only possible basic classes lying in the image of φ in fact lie in the image of $H_2((S^3 \setminus L) \times S^1)$. This image is spanned by the classes of the tori T_{m_i}, $i = 1, \dots, n$ and the tori V_j, $j = 1, \dots, n-1$ where V_i is the boundary of a tubular neighborhood of K_i in S^3. The nonzero elements of G determine classes in $E(2,1)_L$ with nonzero Mayer-Vietoris boundary. These are generated by classes γ_i, $i = 1, \dots, n$ and Σ_j, $j = 1, \dots, n-1$. A representative of γ_i is formed as follows. Let S_i denote intersection of a Seifert surface for the *knot* K_i with the link exterior. The intersection of S_i with V_j $(j \neq i)$ consists of $\ell k(K_i, K_j)$ copies of m_j. Each of these is glued to a circle on the fiber F of the corresponding $E(1)$ or $E(2)$, and this circle bounds in the elliptic surface. The same is true for the longitude of the knot K_i. The result represents γ_i. Note that $\gamma_i \cdot F_j = \delta_{ij}$ and $\gamma_i \cdot V_j = 0$ for each $j = 1, \dots, n$. The generators Σ_j are constructed by starting with an arc A_j in $S^3 \setminus L$ from a point on V_n to a point on V_j. The boundary of $A_j \times S^1$ consists of two circles, and each is identified with a circle in $H_2(E(1) \setminus F)$ or $H_2(E(2) \setminus F)$. In the elliptic surfaces, these circles bound vanishing cycles, disks of self-intersection -1. Thus Σ_i is represented by a -2-sphere. We have $\Sigma_j \cdot F_i = 0$ for all i, and $\Sigma_j \cdot V_i = \delta_{ij}$. Suppose we have a basic class

$$\kappa = \sum_{i=1}^{n} a_i F_i + \sum_{j=1}^{n-1} b_j V_j + \sum_{k=1}^{n} c_k \gamma_k + \sum_{\ell=1}^{n-1} d_\ell \Sigma_\ell$$

Since F_i is a torus of self-intersection 0, the adjunction inequality implies that $\kappa \cdot F_i = 0$, *i.e.* that $c_i = 0$. Similarly, V_j is a torus of self-intersection 0; so $0 = \kappa \cdot V_j = d_j$. Also $\Sigma_\ell + V_\ell$ is represented by a torus of self-intersection 0; thus $0 = \kappa \cdot (\Sigma_\ell + V_\ell) = \kappa \cdot \Sigma_\ell = b_\ell$.

Axiom 4 is clear.

Axiom 5 is verified in the spirit of Theorem 1.5. However, we must first construct an auxiliary manifold $\bar{E}(1)_L$ as follows. In $E(2,1)_L$, L an n-component link, let $F_1, \dots, F_n$ be tori with the torus F_1 the elliptic fiber in the $K3$-surface and, for $j \geq 1$, F_j the elliptic fiber in the $(j-1)^{st}$ copy of $E(1)$. (Note that $F_i = T_{m_i}$ in $E(2,1)_L$.) Now perform $(n-1)$ internal fiber sums, identifying F_1 with F_2, a parallel copy of F_2 with F_3, and so on. The homology classes represented by the F_j in $\bar{E}(1)_L$ are all equal, and we denote this homology class by $[F]$. Let $t = \exp(2[F])$. It follows from the gluing formula Theorem 2.3 that

$$\mathrm{SW}_{\bar{E}(1)_L}(t) = \mathrm{SW}_{E(2,1)_L}(t, \dots, t) \cdot (t^{1/2} - t^{-1/2})^{(2n-2)},$$

or by defining $\widetilde{\mathrm{SW}}_{\bar{E}(1)_L}(t) = \mathrm{SW}_{E(2,1)_L}(t, \dots, t)$,

$$\widetilde{\mathrm{SW}}_{\bar{E}(1)_L}(t) = \frac{\mathrm{SW}_{\bar{E}(1)_L}(t)}{(t^{1/2} - t^{-1/2})^{(2n-2)}}.$$

Suppose now that L_+, L_-, and L_0 are links which coincide (except possibly for the numbering of the components) outside a ball, and inside this ball have the form depicted in Figure 8. Furthermore, assume $L_\pm$ has n components. Then, as in the proof of Theorem 1.5, there is a nullhomologous torus T in $\bar{E}(1)_{L_-}$ so that $\bar{E}(1)_{L_+}$ is the result of a (1/1)-log transform on T. By the log transform formula (Theorems 2.4 and 2.5),

$$\mathrm{SW}_{\bar{E}(1)_{L_+}} = \mathrm{SW}_{\bar{E}(1)_{L_-}} + \mathrm{SW}_{\bar{E}(1)_{L_-}(0/1)}.$$

There are two cases. For the first case, the two strands of L_+ are from distinct components K_{i_1}, K_{i_2} of L_+. The manifold $\bar{E}(1)_{L_-}(0/1)$ is obtained as follows: Perform surgeries on the link components $\{K_i\}$ of L_- with surgery coefficient $\alpha_{L_-}(\ell_i)$ on K_i and perform 0-surgery on the unknotted component U which links K_{i_1} and K_{i_2} as shown in Figure 9.

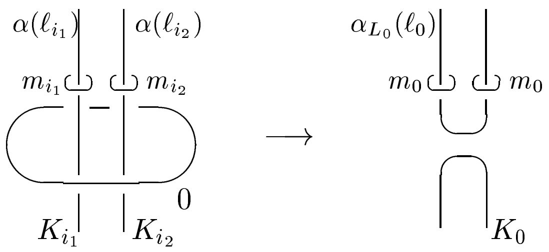


Figure 9

Let M^3 be the resulting 3-manifold. Next form

$$E(2,1)_{L_-}(0/1) = (M^3 \times S^1) \#_{T_{m_1}=F_1} E(2) \#_{T_{m_2}=F_2} E(1) \# \cdots \#_{T_{m_n}=F_n} E(1). \tag{4}$$

Finally, $\bar{E}(1)_{L_-}(0/1)$ is obtained from $E(2,1)_{L_-}(0/1)$ by performing $n-1$ internal fiber sums, as described above. If we slide the handle corresponding to K_{i_2} over the handle corresponding to K_{i_1} then we obtain a new Kirby calculus description of M^3: it is obtained from surgery on the link L_0 with surgery coefficients again given by the $\alpha_{L_0}(\ell_i)$, as in Figure 9. (Note that if the new component is called K_0 then for $j \neq i_1$, i_2, the linking number $\ell k_{L_0}(K_0, K_j) = \ell k_{L_-}(K_{i_1}, K_j) + \ell k_{L_-}(K_{i_2}, K_j)$; so the total linking number for the longitude ℓ_0 is $\alpha_{L_0}(\ell_0) = \alpha_{L_-}(\ell_{i_1}) + \alpha_{L_-}(\ell_{i_2}) - 2\ell k_{L_-}(K_{i_1}, K_{i_2})$ which is exactly the framing which is given to K_0 by the handle slide.)

Now the link L_0 has $n-1$ components. We see that the difference between the constructions for $\bar{E}(1)_{L_-}(0/1)$ and $\bar{E}(1)_{L_0}$ is that an extra copy of $E(1)$ needs to be fiber-summed into $\bar{E}(1)_{L_0}$ and then an extra internal fiber sum needs to be performed on the result, in order to get $\bar{E}(1)_{L_-}(0/1)$. Theorem 2.1 and Theorem 2.3 then imply that $\mathrm{SW}_{\bar{E}(1)_{L_-}(0/1)}(t) = (t^{1/2} - t^{-1/2})^3 \mathrm{SW}_{\bar{E}(1)_{L_0}}(t)$, where one factor $(t^{1/2} - t^{-1/2})$ comes from the extra fiber sum with $E(1)$, and $(t^{1/2} - t^{-1/2})^2$ comes from the extra internal fiber sum. We have:

$$\begin{aligned}
\widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_+}}(t) &= \frac{\mathrm{SW}_{\bar{E}(1)_{L_+}}(t)}{(t^{1/2}-t^{-1/2})^{(2n-2)}}\\
&= \frac{\mathrm{SW}_{\bar{E}(1)_{L_-}}(t)+\mathrm{SW}_{\bar{E}(1)_{L_-}(0/1)}(t)}{(t^{1/2}-t^{-1/2})^{(2n-2)}}\\
&= \widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_-}}(t)+\frac{(t^{1/2}-t^{-1/2})^3\cdot \mathrm{SW}_{\bar{E}(1)_{L_0}}(t)}{(t^{1/2}-t^{-1/2})^{(2n-2)}}\\
&= \widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_-}}(t)+\frac{(t^{1/2}-t^{-1/2})\cdot \mathrm{SW}_{\bar{E}(1)_{L_0}}(t)}{(t^{1/2}-t^{-1/2})^{(2n-4)}}\\
&= \widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_-}}(t)+(t^{1/2}-t^{-1/2})\cdot \widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_0}}(t),
\end{aligned}$$

as desired.

For the second case, the two strands of L_+ are from the same component (say the j^{th}) of L_+; so L_0 has $(n+1)$ components. Then $\bar{E}(1)_{L_-}(0/1)$ is obtained by first performing surgeries on the components $\{K_i\}$ of L_- with surgery coefficient $\alpha_{L_-}(\ell_i)$ on K_i and then performing 0-surgery on an unknotted circle which links K_j twice geometrically and 0-times algebraically as in Figure 10.

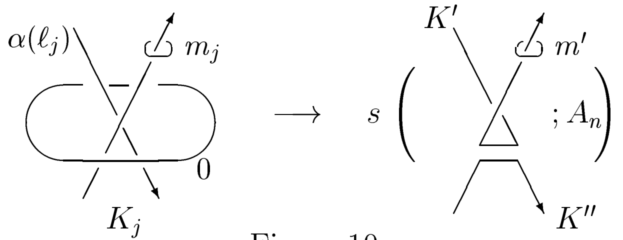


Figure 10

This gives a 3-manifold, M^3. Then we form $E(2,1)_{L_-}(0/1)$ as given by (4), and $\bar{E}(1)_{L_-}(0/1)$ is obtained from this by performing $(n-1)$ internal fiber sums.

Denote the components of L_0 by $K_1,\dots,K_{j-1},K',K'',K_{j+1},\dots,K_n$. Let L_j denote the 2-component link $L_j=\{K',K''\}$. It follows from Hoste's theorem (2.6) that M^3 may be obtained from the sewn-up link exterior $s(L_j;A_n)$ by further surgering the K_i, $i\neq j$ with framing $\alpha_{L_-}(\ell_i)$ (where $n=\alpha_{L_-}(\ell_j)+2\ell k(K',K'')$). Again see Figure 10. Because $\alpha_{L_0}(\ell')+\alpha_{L_0}(\ell'')=\sum_{i\neq j}\ell k(K_i,K_j)+2\ell k(K',K'')=n$, and also $\alpha_{L_-}(\ell_i)=\alpha_{L_0}(\ell_i)$, $i\neq j$, the discussion preceding Proposition 2.7 relates the Seiberg-Witten invariant of $E(2,1)_{L_-}(0/1)$ with the Seiberg-Witten invariant of $E(2,1)_{L_0}$. We need to keep in mind, however, that because L_0 has $n+1$ components, there is an extra fiber sum with $E(1)$ in the construction for $E(2,1)_{L_0}$. Thus,

$$\mathrm{SW}_{E(2,1)_{L_-}(0/1)}=(t_j^{1/2}-t_j^{-1/2})\cdot \mathrm{SW}_{E(2,1)_{L_0}}|_{t'=t''=t_j}$$

Furthermore, $\bar{E}(1)_{L_0}$ has one more internal fiber sum than does $\bar{E}(1)_{L_-}(0/1)$; so

$$\mathrm{SW}_{\bar{E}(1)_{L_-}(0/1)}(t) = \frac{1}{(t^{1/2} - t^{-1/2})} \cdot \mathrm{SW}_{\bar{E}(1)_{L_0}}(t).$$

Thus

$$\begin{aligned}
\widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_+}}(t) &= \frac{\mathrm{SW}_{\bar{E}(1)_{L_+}}(t)}{(t^{1/2} - t^{-1/2})^{(2n-2)}} \\
&= \frac{\mathrm{SW}_{\bar{E}(1)_{L_-}}(t) + \mathrm{SW}_{\bar{E}(1)_{L_-}(0/1)}(t)}{(t^{1/2} - t^{-1/2})^{(2n-2)}} \\
&= \widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_-}}(t) + \frac{\frac{1}{(t^{1/2}-t^{-1/2})} \cdot \mathrm{SW}_{\bar{E}(1)_{L_0}}(t)}{(t^{1/2} - t^{-1/2})^{(2n-2)}} \\
&= \widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_-}}(t) + \frac{(t^{1/2} - t^{-1/2}) \cdot \mathrm{SW}_{\bar{E}(1)_{L_0}}(t)}{(t^{1/2} - t^{-1/2})^{2n}} \\
&= \widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_-}}(t) + (t^{1/2} - t^{-1/2}) \cdot \widetilde{\mathrm{SW}}_{\bar{E}(1)_{L_0}}(t),
\end{aligned}$$

completing the proof of Axiom 5.

Finally, to verify Axiom 6 we first note that $E(2,1)_{L'}$ is obtained from $E(2,1)_L$ by performing an order 2 log transform on the torus $\bar{T}_j = \bar{K}_j \times S^1$ where $\bar{K}_j$ is the core of the surgered K_j in M_L. The homology class of the resulting meridian m_j' is twice that of m_j so that $[T_{m_j'}] = 2[T_{m_j}]$. The log transform formulas of [FS1], then state that

$$\begin{aligned}
&\mathrm{SW}_{E(2,1)_{L'}}(t_1, \dots, t_{j-1}, t_j', t_{j+1}, \dots, t_n) \\
&\qquad = (\bar{t}_j^{\,1/2} + \bar{t}_j^{\,-1/2}) \cdot \mathrm{SW}_{E(2,1)_L}(t_1, \dots, t_{j-1}, t_j^2, t_{j+1}, \dots, t_n).
\end{aligned}$$

The result now follows since

$$[\bar{T}_j] = [T_j] + \sum_{i \neq j} \ell k(K_j, K_i)[T_i].$$

5. Examples with $b^+ = 1$

In this section we shall discuss examples which have $b^+ = 1$. For such manifolds, the Seiberg-Witten invariant depends on a choice of metric and self-dual 2-form as follows. Let X be a simply connected oriented 4-manifold with $b_X^+ = 1$ with a given orientation of $H_+^2(X;\mathbf{R})$ and a given metric g. Since $b_X^+ = 1$, there is a unique g-self-dual harmonic 2-form $\omega_g \in H_+^2(X;\mathbf{R})$ with $\omega_g^2 = 1$ and corresponding to the positive orientation. Fix a characteristic cohomology class $k \in H^2(X;\mathbf{Z})$. Given a pair (A, ψ), where A is a connection in the complex line bundle corresponding to

k and ψ a section of the bundle W^+ of self-dual spinors for the associated *spin*^{c} structure, the perturbed Seiberg-Witten equations are:

$$\begin{aligned} D_A\psi &= 0 \\ F_A^+ &= q(\psi) + i\eta^+ \end{aligned} \tag{5}$$

where F_A^+ is the self-dual part of the curvature of A , D_A is the twisted Dirac operator, η^+, is a self-dual 2-form on X, and q is a quadratic function. Write $\mathrm{SW}_{X,g,\eta^+}(k)$ for the corresponding invariant evaluated on the class k $(= \frac{i}{2\pi}[F_A])$. As the pair (g,η^+) varies, $\mathrm{SW}_{X,g,\eta^+}(k)$ can change only at those pairs (g,η^+) for which there are solutions of (5) with $\psi = 0$. These solutions occur for pairs (g,η^+) satisfying $(2\pi k+\eta^+)\cdot\omega_g = 0$. This last equation defines a codimension 1 subspace ('wall') in $H^2(X;\mathbf{R})$. The point ω_g lives in the double cone $\mathcal{C}_X = \{\alpha \in H^2(X;\mathbf{R}) | \alpha\cdot\alpha > 0\}$, and, if $(2\pi k+\eta^+)\cdot\omega_g \neq 0$ for a generic η^+, $\mathrm{SW}_{X,g,\eta^+}(k)$ is well-defined, and its value depends only on the sign of $(2\pi k+\eta^+)\cdot\omega_g$. A useful lemma, which follows from the Cauchy-Schwarz inequality (see [LL]) is:

Lemma 5.1. *Suppose that α and β are nonzero elements of $H^2(X;\mathbf{R})$ which lie in the closure of the same component of C_X. Then $\alpha\cdot\beta \geq 0$ with equality if and only if $\alpha = \lambda\beta$ for some $\lambda > 0$.*

It follows from this lemma that that $\mathrm{SW}_{X,g,\eta^+}(k)$ depends only on the component of $\mathcal{C}_X$ which contains the g-self-dual projection of $2\pi k + \eta^+$. Furthermore, if the g_i-self-dual projections of $2\pi k + \eta_i^+$ lie in different components of $\mathcal{C}_X$ and are oriented so that $(2\pi k + \eta_1^+)\cdot\omega_{g_1} > 0 > (2\pi k + \eta_0^+)\cdot\omega_{g_0}$, then

$$\mathrm{SW}_{X,g_1,\eta_1^+}(k) - \mathrm{SW}_{X,g_0,\eta_0^+}(k) = (-1)^{\frac{1}{2}\delta(k)+1} \tag{6}$$

where $\delta(k) = \frac{1}{4}(k^2 - (3\mathrm{sign} + 2\mathrm{e})(X))$ is the formal dimension of the moduli spaces. Thus, as the pair (g,η^+) is varied, there are exactly two values of $\mathrm{SW}_{X,g,\eta^+}(k)$. Furthermore, in case $b^- \leq 9$, for any fixed $a \in H_2(X;\mathbf{Z})$ with $\delta(a) = a^2 + b^- - 9 \geq 0$, the self-dual projections of $2\pi a$ all lie in the same component of $\mathcal{C}_X$; so, if $a = k$ is characteristic, then for small enough perturbations, the Seiberg-Witten invariants agree, independent of metric [KM, S2].

Suppose that X contains a smooth essential torus T of self-intersection 0. By Lemma 5.1, the class $[T]$ orients $\mathcal{C}_X$ by declaring $\mathcal{C}_X^+$ to be the component of $\mathcal{C}_X$ which contains classes α with $\alpha\cdot[T] > 0$. Denote the other component by $\mathcal{C}_X^-$ and the corresponding Seiberg-Witten invariants by $\mathrm{SW}_X^{\pm}$; *i.e.* $\mathrm{SW}_X^+(k) = \mathrm{SW}_{X,g,\eta^+}(k)$ where the g-self-dual projection of $2\pi k+\eta^+$ has positive intersection with $[T]$, and we define $\mathrm{SW}_X^-(k)$ similarly. The $[T]^\perp$*-restricted Seiberg-Witten invariants* are defined to be

$$\mathrm{SW}_{X,T}^{\pm} = \sum_{k\cdot[T]=0} \mathrm{SW}_X^{\pm}(k)\ \exp(k).$$

When $k^2 \geq 0$ and $k \cdot [T] = 0$, Lemma 5.1 implies that $k = \lambda[T]$. In particular, if $b_X^- \leq 9$ and $\delta(k) \geq 0$, then $k = \lambda[T]$ if $k \cdot [T] = 0$.

Lemma 5.2. *Let X be a simply connected smooth 4-manifold with $b_X^+ = 1$, and suppose that X contains a smooth homologically nontrivial torus T of self-intersection 0. Let $t = \exp(2[T])$. Then*

$$(t^{1/2} - t^{-1/2}) \cdot SW_{X,T}^+ = (t^{1/2} - t^{-1/2}) \cdot SW_{X,T}^-.$$

Proof. The coefficient of $\exp(k)$ in $(t^{1/2} - t^{-1/2}) \cdot \mathrm{SW}_{X,T}^+$ is

$$c^+(k) = \mathrm{SW}_X^+(k - [T]) - \mathrm{SW}_X^+(k + [T]).$$

Since $k \cdot [T] = 0$, we have $(k - [T])^2 = (k + [T])^2$; so the wall-crossing formula (6) implies that $c^+(k) = \mathrm{SW}_X^-(k - [T]) - \mathrm{SW}_X^-(k + [T])$, the coefficient of $\exp(k)$ in $(t^{1/2} - t^{-1/2}) \cdot \mathrm{SW}_{X,T}^-$. □

For example, consider the case of the rational elliptic surface $E(1)$ and its fiber class $[F]$. For a Kahler metric g on $E(1)$, the Kahler form ω is self-dual, and since F is a complex curve, $\omega \cdot [F] > 0$. So the self-dual projection of $[F]$ is a positive multiple of ω, and we see that the small-perturbation component of $\mathcal{C}_{E(1)}$ is $\mathcal{C}_{E(1)}^+$ for $n[F]$, $n > 0$, and similarly is $\mathcal{C}_{E(1)}^-$ for $n[F]$, $n < 0$. Since $E(1)$ carries a metric of positive scalar curvature, this means $\mathrm{SW}_{E(1)}^+(n[F]) = 0$ for $n > 0$, and $\mathrm{SW}_{E(1)}^-(n[F]) = 0$ for $n < 0$. The wall-crossing formula (6) implies that (up to an overall sign) $\mathrm{SW}_{E(1)}^-((2n+1)[F]) - \mathrm{SW}_{E(1)}^+((2n+1)[F]) = 1$. Hence (up to that same sign)

$$\mathrm{SW}_{E(1)}^-((2n+1)[F]) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

In other words,

$$\mathrm{SW}_{E(1),F}^- = \sum_{n=0}^{\infty} t^{(2n+1)/2}$$

where $t = \exp(2[F])$. Similarly,

$$\mathrm{SW}_{E(1),F}^+ = -\sum_{n=0}^{\infty} t^{-(2n+1)/2}.$$

Notice that in this case, $(t^{1/2} - t^{-1/2}) \cdot \mathrm{SW}_{E(1),F}^+$ and $(t^{1/2} - t^{-1/2}) \cdot \mathrm{SW}_{E(1),F}^-$ both equal -1, as Lemma 5.2 demands.

Let X be a simply connected smooth 4-manifold with $b_X^+ = 1$, and suppose that X contains a c-embedded oriented torus T. Suppose further that $\pi_1(X \setminus T) = 1$. Then for any knot K in S^3 we can form X_K, which is homeomorphic to X. Since

$[T] = [T_m]$ orients $\mathcal{C}_{X_K}$, we have invariants $\mathrm{SW}^{\pm}_X$ and $\mathrm{SW}^{\pm}_{X_K}$. Our discussion of § 3 applies in this case as well, once we have the analogues of the log transform and gluing formulas of § 2. These formulas are also due to Taubes and Morgan, Mrowka, and Szabo, *cf.* [S2].

Theorem (Morgan, Mrowka, and Szabo [MMS], Taubes)**.** *Consider the simply connected 4-manifolds X_1 and X_2 with $b^+_{X_1} = 1$ and $b^+_{X_2} \geq 1$. Suppose that X_1 and X_2 contain c-embedded tori T_1 and T_2. Then*

$$SW_{X_1\#_{T_1=T_2}X_2} = \begin{cases} SW^{\pm}_{X_1,T_1} \cdot SW^{\pm}_{X_2,T_2} \cdot (t^{1/2} - t^{-1/2})^2, & \text{if } b^+_{X_2} = 1 \\ SW^{\pm}_{X_1,T_1} \cdot SW_{X_2} \cdot (t^{1/2} - t^{-1/2})^2, & \text{if } b^+_{X_2} > 1 \end{cases}$$

where $t = \exp(2[T])$ and $[T] = [T_1] = [T_2] \in H_2(X_1\#_{T_1=T_2}X_2)$.

It follows from Lemma 5.2 that these formulas actually make sense. The restriction to $\mathrm{SW}^{\pm}_{X_1,T_1}$ is accounted for by the fact that $b^+_{X_1\#_{T_1=T_2}X_2} \geq 3$, and so any basic class in $H_2(X_1\#_{T_1=T_2}X_2)$ must be orthogonal to $[T]$.

In order to state the internal fiber sum formula, let X be a simply connected 4-manifold with $b^+_X = 1$. Suppose that X contains a pair of c-embedded tori T_1, T_2, and let X_{T_1,T_2} denote the internal fiber sum. Denote by $\mathrm{SW}^{\pm}_{X,T_1,T_2}$ the $([T_1],[T_2])^{\perp}$-restricted Seiberg-Witten invariants of X,

$$\mathrm{SW}^{\pm}_{X,T_1,T_2} = \sum_{\substack{k\cdot[T_i]=0 \\ i=1,2}} \mathrm{SW}^{\pm}_X(k)\ \exp(k).$$

Theorem (Morgan, Mrowka, and Szabo [MMS], Taubes)**.** *Let X be a simply connected 4-manifold with $b^+_X = 1$. Suppose that X contains a pair of c-embedded tori T_1, T_2 representing homology classes $[T_1]$ and $[T_2]$.*

$$SW_{X_{T_1,T_2}} = (SW^{\pm}_{X,T_1,T_2})\big|_{\tau_1=\tau_2} \cdot (t^{1/2} - t^{-1/2})^2$$

where $\tau_i = \exp([T_i])$ and $t = \exp(2[T])$.

Of course we also need the log transform formula:

Theorem (Morgan, Mrowka, and Szabo [MMS], Taubes)**.** *Let Y be a smooth 4-manifold with $b^+ = 1$, and suppose that Y contains a nullhomologous torus T. Let τ be the homology class of T_0 in $Y(0/1)$. For each characteristic homology class $\alpha \in H_2(Y;\mathbf{Z})$ and $p \neq 0$,*

$$SW^{\pm}_{Y(p/q)}(\alpha) = pSW^{\pm}_Y(\alpha) + q\sum_{i=-\infty}^{\infty} SW_{Y(0/1)}(\alpha + 2i\tau).$$

Now the arguments of § 3 go through verbatim to give:

Theorem 5.3. *Let X be a simply connected smooth 4-manifold with $b^+_X = 1$, and suppose that X contains a c-embedded oriented torus T. Suppose further that $\pi_1(X \setminus T) = 1$. For $t = \exp(2[T])$ the $[T]^\perp$-restricted Seiberg-Witten invariants of X_K are*

$$SW^{\pm}_{X_K,T} = SW^{\pm}_{X,T} \cdot \Delta_K(t).$$

As in the $b^+ \geq 3$ case, if X is a symplectic 4-manifold with $b^+ = 1$ containing a symplectic embedded torus T of self-intersection 0 and K is a fibered knot, then X_K is a symplectic 4-manifold. Conversely, if X is symplectic, choose a metric g for X so that the symplectic form ω is self-dual, and let κ_X denote the canonical class. In [T1, T2], Taubes shows that for $r << 0$, we have $\mathrm{SW}_{X,g,r\omega}(-\kappa_X) = \pm 1$ and also that if $\mathrm{SW}_{X,g,r\omega}(k) \neq 0$ then $-\kappa_X \cdot \omega \leq k \cdot \omega$, with equality only when $k = -\kappa_X$. Now let $b^+_X = 1$, and let T be an embedded torus of self-intersection 0, such that $[T] \cdot \omega > 0$. For example, this holds if T is symplectically embedded. Then for any $k \in H_2(X; \mathbf{R})$, we have $(2\pi k^+ + r\omega) \cdot [T] < 0$ for $r << 0$. This means that $\mathrm{SW}_{X,g,r\omega}(k) = \mathrm{SW}^-_X(k)$. Hence:

Proposition 5.4 (Taubes). *Let X be a symplectic 4-manifold with $b^+ = 1$ containing an embedded torus T of self-intersection 0 such that $[T] \cdot \omega > 0$. Then*

$$SW^-_X(-\kappa_X) = \pm 1$$

and if $SW^-_X(k) \neq 0$, then

$$-\kappa_X \cdot \omega \leq k \cdot \omega$$

with equality only when $k = -\kappa_X$. □

Lemma 5.5. *Let X be a simply connected smooth 4-manifold with $b^+_X = 1$, and suppose that X contains a c-embedded oriented torus T and that $\pi_1(X \setminus T) = 1$. Suppose also that $SW_{X\#_{T=F}E(1)} \neq 0$. Then there is a Laurent polynomial S_X such that*

$$SW^{\pm}_{X,T} = S_X \cdot \sum_{n=0}^{\infty} t^{(2n+1)/2} \quad \text{or} \quad S_X \cdot \sum_{n=0}^{\infty} t^{-(2n+1)/2}.$$

Proof. The gluing formula implies that

$$\mathrm{SW}_{X\#_{T=F}E(1)} = \mathrm{SW}^{\pm}_{X,T} \cdot \mathrm{SW}^{\pm}_{E(1),F} \cdot (t^{1/2} - t^{-1/2})^2 = \mathrm{SW}^{\pm}_{X,T} \cdot (t^{1/2} - t^{-1/2})$$

by Lemma 5.2 and the above calculation of $\mathrm{SW}^{\pm}_{E(1),F}$. Because $X\#_{T=F}E(1)$ has $b^+ = 3$, its Seiberg-Witten invariant is a Laurent polynomial, $S_X \neq 0$; so the lemma follows. □

Lemma 5.6. *Suppose that $b^+_X = 1$ and $\alpha \in H^2(X)$, then for any pair (g, η^+) with $\eta^+ \cdot \omega_g \neq 0$ and $|r| >> 0$, we have*

$$|\, SW_{X,g,r\eta^+}(\alpha) \pm SW_{X,g,r\eta^+}(-\alpha)\,| = 1.$$

Proof. Since $F_{\bar{A}} = -F_A$ and $q(\bar{\psi}) = -q(\psi)$, if (A, ψ) is a solution to the Seiberg-Witten equations for α corresponding to $(g, r\eta^+)$, then $(\bar{A}, \bar{\psi})$ is a solution to the equations for $-\alpha$ corresponding to $(g, -r\eta^+)$. Thus

$$\mathrm{SW}_{X,g,-r\eta^+}(-\alpha) = (-1)^{(\mathrm{sign+e})(X)/4}\mathrm{SW}_{X,g,r\eta^+}(\alpha).$$

For $|r| >> 0$, the signs of $(-2\pi\alpha - r\eta^+) \cdot \omega_g$ and $(-2\pi\alpha + r\eta^+) \cdot \omega_g$ are opposite. Thus the wall-crossing formula implies that

$$|\,\mathrm{SW}_{X,g,-r\eta^+}(-\alpha) \pm \mathrm{SW}_{X,g,r\eta^+}(-\alpha)\,| = 1$$

and the lemma follows. □

In the symplectic case, the above lemma is essentially [MS, Prop.2.2].

Corollary 5.7. *Let X be a symplectic 4-manifold with $b^+ = 1$ containing a symplectic c-embedded torus T. Suppose also that $SW_{X\#_{T=F}E(1)} \neq 0$. If Δ_K is not monic, then X_K does not admit a symplectic structure.*

Proof. Write $\Delta_K(t) = a_0 + \sum_{j=1}^{d} a_j(t^j + t^{-j})$ with $a_d \neq 0$. Suppose that X_K admits a symplectic structure with symplectic form ω and canonical class κ. If $[T] \cdot \omega = 0$, it follows from Lemma 5.1 that $[T] = \lambda\omega$ for some $\lambda \neq 0$. Such a symplectic form is clearly nongeneric and we may perturb it so that $[T] \cdot \omega \neq 0$. We may also assume that T is oriented so that $[T] \cdot \omega > 0$. The adjunction inequality of Li and Liu [LL, Thm.E] then implies that $\kappa \cdot [T] \leq 0$.

The hypothesis that $\mathrm{SW}_{X\#_{T=F}E(1)} \neq 0$ and Lemma 5.5 imply that there is a Laurent polynomial $S_X \neq 0$ such that, for $t = \exp(2[T])$, one of

$$\mathrm{SW}^-_{X,T} = S_X \cdot \sum_{n=0}^{\infty} t^{(2n+1)/2} \tag{7}$$

$$\mathrm{SW}^-_{X,T} = S_X \cdot \sum_{n=0}^{\infty} t^{-(2n+1)/2} \tag{8}$$

holds. Let α be any class such that the coefficient of $\exp(\alpha)$ in S_X is nonzero. There are finitely many such classes; so we may list the integers $m_1 < \cdots < m_r$ such that the coefficient $S_X(\alpha + m_i[T])$ of $\exp(\alpha + m_i[T])$ in S_X is nonzero. Theorem 5.3 implies that if the case (7) holds, then

$$\mathrm{SW}^-_{X_K}(\alpha + (m_1 - 2d + 1)[T]) = a_d\, S_X(\alpha + m_1[T]) \neq 0 \tag{9}$$

and if the case (8) holds then

$$\mathrm{SW}^-_{X_K}(\alpha + (m_r + 2d - 1)[T]) = a_d\, S_X(\alpha + m_r[T]) \neq 0. \tag{10}$$

If $\kappa \cdot \omega < 0$ then, by a result of Liu and Ohta and Ono, X_K is the blowup of a rational or ruled surface [MS, Cor.1.4]. Every such surface has a metric of positive scalar curvature; so it follows from the wall-crossing formula that for each α, $\mathrm{SW}^{\pm}_{X_K}(\alpha) = \pm 1$ or 0. Equations (9) and (10) imply in this case that $a_d = \pm 1$, *i.e.* that Δ_K is monic.

Thus we may assume that $\kappa \cdot \omega > 0$. This means that κ and $[T]$ both lie in the same component of the cone $\mathcal{C}_{X_K}$; so Lemma 5.1 implies that $\kappa \cdot [T] \geq 0$. Since we already have the opposite inequality, we must have $\kappa \cdot [T] = 0$. By Lemma 5.6,

$$|\,\mathrm{SW}^-_{X_K}(\kappa + 2m[T]) \pm \mathrm{SW}^-_{X_K}(-\kappa - 2m[T])\,| = 1$$

for each m. However $[T] \cdot \omega > 0$; so for m large, we have $(-\kappa - 2m[T]) \cdot \omega < -\kappa \cdot \omega$. Thus Proposition 5.4 implies that $\mathrm{SW}^-_{X_K}(-\kappa - 2m[T]) = 0$ for $m >> 0$. This means that $\mathrm{SW}^-_{X_K}(\kappa + 2m[T]) = \pm 1$ for $m >> 0$. But $\mathrm{SW}^-_{X_K,T} = \mathrm{SW}^-_{X,T} \cdot \Delta_K(t)$ and $(\kappa + 2m[T]) \cdot [T] = 0$; so the case (7) must hold.

Again by Proposition 5.4, we have $\mathrm{SW}^-_{X_K}(-\kappa) \neq 0$; so we may write $\kappa = -\alpha + 2n[T]$ where $\mathrm{SW}^-_X(\alpha) \neq 0$ and $|n| \leq d$, and again we have $m_1 < \cdots < m_r$ and (9). From (7):

$$\mathrm{SW}^-_X(\alpha) = \sum_{m_i \,\mathrm{odd} < 0} S_X(\alpha + m_i[T]).$$

But $(\alpha + (m_1 - 2d + 1)[T]) \cdot \omega = -\kappa \cdot \omega + (m_1 + 1 - 2(d-n))[T] \cdot \omega \leq -\kappa \cdot \omega$ because $m_1 < 0$ (since $\mathrm{SW}^-_X(\alpha) \neq 0$) and $d - n \geq 0$. Thus Proposition 5.4 implies that $m_1 = -1$ and $n = d$; so $\kappa = \alpha + 2d[T]$. This means that $\pm 1 = \mathrm{SW}^-_{X_K}(-\kappa) = a_d \cdot \mathrm{SW}^-_X(-\alpha)$; so $a_d = \pm 1$, *i.e.* Δ_K is monic. □

As in the $b^+ > 1$ case, if X contains a surface Σ_g of genus g disjoint from T with $0 \neq [\Sigma_g] \in H_2(X; \mathbf{Z})$ and with $[\Sigma_g]^2 < 2 - 2g$ if $g > 0$ or $[\Sigma_g]^2 < 0$ if $g = 0$, then X_K with the opposite orientation does not admit a symplectic structure. These results along with the blowup formula of [FS2] imply:

Corollary 5.8. *For any knot K in S^3 whose Alexander polynomial $\Delta_K(t)$ is not monic, the manifolds $E(1)_K$ admit no symplectic structure with either orientation, even after an arbitrary number of blowups.*

The first examples of this sort were obtained by Szabo [S2], and in fact they are the manifolds $E(1)_{K_k}$ where K_k is the k-twist knot. For these examples (with $T = F$)

$$\mathrm{SW}^-_{E(1)_{K_k},T} = (kt - (2k+1) + kt^{-1}) \cdot \sum_{n=0}^{\infty} t^{(2n+1)/2}$$

$$\mathrm{SW}^+_{E(1)_{K_k},T} = -(kt - (2k+1) + kt^{-1}) \cdot \sum_{n=0}^{\infty} t^{-(2n+1)/2}$$

Szabo computes the 'small perturbation' invariant $\mathrm{SW}^0_{E(1)_{K_k}}$. As is the case for $E(1)$, for $m > 0$, this is $\mathrm{SW}^0_{E(1)_{K_k}}(m[T]) = \mathrm{SW}^+_{E(1)_{K_k}}(m[T])$, and for $m < 0$, it is

$\mathrm{SW}^0_{E(1)_{K_k}}(m[T]) = \mathrm{SW}^-_{E(1)_{K_k}}(m[T])$. Hence

$$\mathrm{SW}^0_{E(1)_{K_k},T} = kt^{-1/2} - kt^{1/2},$$

as Szabo calculates.

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