A FAKE SMOOTH $\mathbf{CP}^2\#\mathbf{RP}^4$

DANIEL RUBERMAN AND RONALD J. STERN

Abstract. We show that the manifold $*\mathbf{RP}^4\# *\mathbf{CP}^2$, which is homotopy equivalent but not homeomorphic to $\mathbf{RP}^4\#\mathbf{CP}^2$, is in fact smoothable.

1. Introduction

In Kirby's problem list [Kir97, Problem 4.82] and in a recent lecture at MSRI, P. Teichner raised the question of the smoothability of a certain non-orientable 4-manifold. In this note, we show that the manifold in question, denoted $*\mathbf{RP}^4\# *\mathbf{CP}^2$, which is homotopy equivalent but not homeomorphic to $\mathbf{RP}^4\#\mathbf{CP}^2$, is in fact smoothable. The smooth model we construct will have the additional property that its universal cover is diffeomorphic to $\mathbf{CP}^2\#\overline{\mathbf{CP}}^2$. To describe the manifold in question, we remind the reader that one of the first consequences of Freedman's simply-connected surgery theory was a construction of a manifold $*\mathbf{CP}^2$, sometimes called CH in honor of Chern, which is homotopy equivalent but not homeomorphic to $\mathbf{CP}^2$. The manifold $*\mathbf{CP}^2$ is not smoothable for classical reasons: it has non-trivial Kirby-Siebenmann invariant $\mathrm{KS} \in \mathbf{Z}_2$. Given any simply-connected *non-spin* manifold M, a similar construction produces a homotopy equivalent '$*$–partner' $*M$ with opposite Kirby-Siebenmann invariant [Teich96]. In 1983, the first author [Rub84] constructed what is in effect the $*$-partner of $\mathbf{RP}^4$. The connected sum $*\mathbf{CP}^2\# *\mathbf{RP}^4$ has trivial KS-invariant and so might expected to be smoothable; on the other hand [HKT94] it is not homeomorphic to $\mathbf{CP}^2\#\mathbf{RP}^4$.

Theorem 1. *The manifold $*\mathbf{CP}^2\# *\mathbf{RP}^4$ has a smooth structure. Moreover, it has a smooth structure such that its universal cover is diffeomorphic to $\mathbf{CP}^2\#\overline{\mathbf{CP}}^2$*

The classification [HKT94] of non–orientable manifolds with $\pi_1 = \mathbf{Z}_2$ implies that such manifolds which have $b_2 > 1$ are smoothable if and only if their Kirby-Siebenmann invariant vanishes. Together with theorem 1 this yields:

Corollary 2. *Let X be a closed non–orientable 4-manifold with $\pi_1(X) = \mathbf{Z}_2$. Then X has a smooth structure if and only if $\mathrm{KS}(X) = 0$.*

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2. Construction of the manifold

The proof of Theorem 1 is constructive; we will find a smooth manifold homeomorphic to $*\mathbf{CP}^2 \# * \mathbf{RP}^4$. The construction uses a homology sphere satisfying the conclusion of the following lemma, whose proof will be given in the next section.

Lemma 2.1. *There is a homology 3–sphere Σ^3 with the following properties.*

(i) *Σ is obtained by ± 1 surgery on a knot K in S^3.*
(ii) *The Rohlin invariant $\mu(\Sigma) = 1 \pmod 2$.*
(iii) *Σ admits a free, orientation preserving involution τ, which is isotopic to the identity.*

Different Σ's could in principle give rise to different smooth structures on $*\mathbf{CP}^2 \# * \mathbf{RP}^4$, but we know of no way to tell them apart. The situation is quite analogous to that for the fake $\mathbf{RP}^4$'s constructed in [FS81].

Proof of Theorem 1. Let Σ be a homology 3–sphere as described in the lemma; choose an orientation on Σ so that it becomes surgery on a knot with coefficient $= +1$. Items *(i)* and *(ii)* are the ingredients in Freedman's construction [Fre82] of $*\mathbf{CP}^2$. That is, let Y be the result of adding a 2–handle to B^4 along K, with framing 1, then $\partial Y = \Sigma$ and

$$*\mathbf{CP}^2 = Y \cup_\Sigma \Delta^4$$

where Δ^4 is a contractible 4-manifold with boundary $-\Sigma$. (The sign of the framing is not really important, for the difference between $*\mathbf{CP}^2$ and $*\overline{\mathbf{CP}}^2$ will disappear when we connect sum with $*\mathbf{RP}^4$.) The non-trivial μ-invariant is readily identified with the Kirby-Siebenmann invariant of $*\mathbf{CP}^2$.

Now items *(ii)* and *(iii)* are exactly the ingredients for the construction of $*\mathbf{RP}^4$ given in [Rub84], i.e.

$$*\mathbf{RP}^4 = \Delta^4/(x \in \Sigma \sim \tau(x)) = (\Sigma/\tau \mathbin{\widetilde{\times}} \mathbf{I}) \cup_\Sigma \Delta^4$$

(The authors of [HKT94] seem to have been unaware of this earlier construction of $*\mathbf{RP}^4$; compare the discussion in [Kir97, Problem 4.74].)

Let X be the smooth manifold obtained as the union of Y and the mapping cylinder of the orbit map of the free involution τ on Σ, i.e.

$$X = Y \cup_\Sigma (\Sigma/\tau \mathbin{\widetilde{\times}} \mathbf{I}) = Y/(x \in \Sigma \sim \tau(x)).$$

Then X is manifestly smooth, and we claim that it is homeomorphic to $*\mathbf{CP}^2 \# * \mathbf{RP}^4$. This seems quite plausible, for the construction amounts to performing a sort of connected sum, where instead of removing disks and gluing, we remove the 'pseudo-disc' Δ^4 and glue up. Unfortunately, we do not know an elementary proof, and must appeal to the homeomorphism classification theorem of [HKT94].

According to that work, the manifold $*\mathbf{CP}^2\# * \mathbf{RP}^4$ is distinguished among non-orientable manifolds with $\pi_1 = \mathbf{Z}_2$ by having $b_2 = 1$, trivial Kirby-Siebenmann invariant, and by a codimension-2 Pin^c Arf-invariant. (The other possible manifolds, up to homeomorphism, with the same homology are $\mathbf{CP}^2\#\mathbf{RP}^4$, $*\mathbf{CP}^2\#\mathbf{RP}^4$, and $\mathbf{CP}^2\# * \mathbf{RP}^4$.) The Arf-invariant, whose value for $*\mathbf{CP}^2\# * \mathbf{RP}^4$ is $\pm 3 \pmod 8$, is that of a surface pulled back from $\mathbf{CP}^N$ via a map $\varphi : X \to \mathbf{CP}^{N+1}$ which classifies c_Φ of the (primitive) Pin^c structure Φ.

A (topological) Spin^c structure on $*\mathbf{CP}^2$ also determines such a map, say φ'; it is easy to see that (in terms of the decomposition of $*\mathbf{CP}^2$ given above) that φ' can be taken to be smooth on Y, and constant on Δ^4. To be more concrete, the dual surface F could be taken as a Seifert surface of K, capped off in the 2-handle. The Arf invariant of F (in $*\mathbf{CP}^2$) is $4 \pmod 8$, as can be seen from this description of F, or by using Rohlin's theorem as in [HKT94].

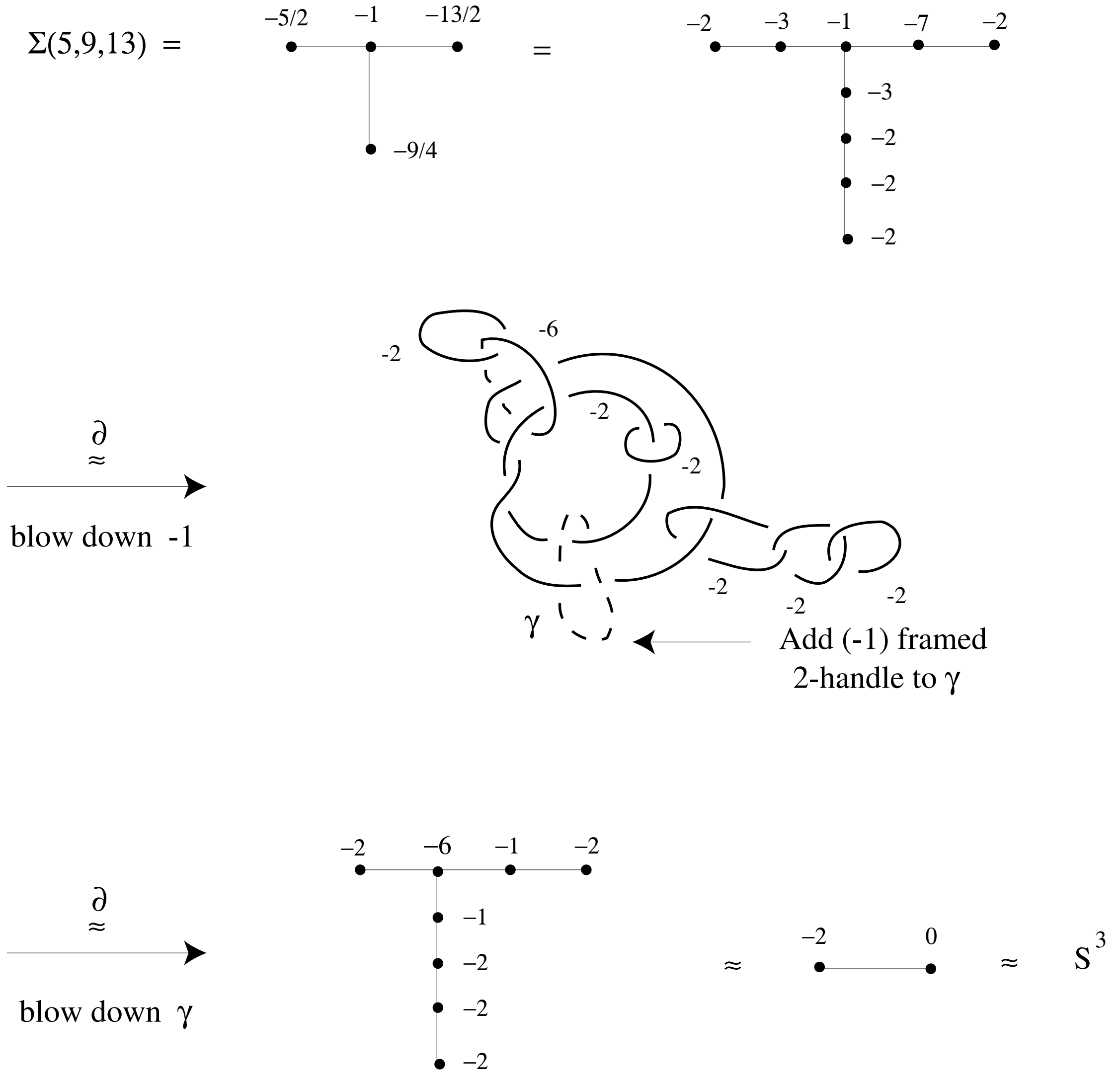
The Pin^c structure on $*\mathbf{RP}^4$ has for its characteristic class the non-trivial class in $H^2(*\mathbf{RP}^4;\mathbf{Z})$. This class is 'dual' to a surface in $*\mathbf{RP}^4$ which again may be assumed to lie in $*\mathbf{RP}^4 - \Delta^4$. By the homotopy invariance of the Arf-invariant for Pin^- structures, $\mathrm{Arf}(*\mathbf{RP}^4) \equiv \mathrm{Arf}(\mathbf{RP}^4) \equiv \pm 1 \pmod 8$. There is a unique Pin^c structure on Σ, so the Pin^c structures on $*\mathbf{RP}^4 - \Delta^4$ and $*\mathbf{CP}^2 - \Delta^4$ glue up to give a Pin^c structure Φ_X on X. The characteristic class c_{Φ_X} is clearly dual to the disjoint union of surfaces lying in the two pieces of X, so the Arf invariant is $\pm 4 \pm 1 \equiv \pm 3 \pmod 8$, just as for $*\mathbf{CP}^2\# * \mathbf{RP}^4$. Since X is smooth, its Kirby-Siebenmann invariant is trivial, and so X is homeomorphic to $*\mathbf{CP}^2\# * \mathbf{RP}^4$.

The additional remark about the universal cover of X being standard may be seen as follows(cf. [FS81]). By the construction of X, its cover $\widetilde{X} \cong Y \cup_\tau \overline{Y} \cong Y \cup \overline{Y}$ since τ is isotopic to the identity. On the other hand, $Y \cup \overline{Y}$ is obtained by adding two 2-handles to B^4, together with a 4-handle. The first is added along K, with framing 1, and the second is added along a meridian of K, with framing 0. (This is a standard argument in handle theory, see for example [Kir89].) It is then easy to unknot K, by repeatedly sliding over the 0-framed handle, resulting in a standard picture of $\mathbf{CP}^2\#\overline{\mathbf{CP}}^2$. □

3. Proof of Lemma 2.1

In this section, we give two examples of homology spheres satisfying the conclusions of Lemma 2.1. Both examples are Brieskorn spheres, i.e. Seifert-fibered homology spheres of the form $\Sigma(p,q,r)$, where p, q, and r are relatively prime odd numbers. The involution τ is nothing more than multiplication by $-1 \in S^1$ in the natural circle action on $\Sigma(p,q,r)$. The condition that the numbers p, q, and r be odd guarantees that τ is free; since -1 is contained in a circle, the involution is isotopic to the identity.

There are many Brieskorn spheres which are integral surgery on a knot–for some examples see [KT90, MM97] or adapt the technique of [CH81]. For most of these constructions one of the indices turns out to be even. One construction is given below, where it is shown that adding a handle (along the curve denoted γ) to the Brieskorn sphere $\Sigma(5,9,13)$ yields S^3. Turning the picture upside down shows that $\Sigma(5,9,13)$ is integral surgery on a knot in S^3. As remarked in the proof of Theorem 1, it doesn't matter whether the coefficient is positive or negative. Again, the μ-invariant is 1 (from the picture just after blowing down the first -1 curve), so this example proves the lemma.



Another construction from the literature which provides Seifert fibered spaces is $rs(p+q)^2+pq$ surgery on the knot denoted $K_{p,q}(r,s)$ in the recent paper [MM97, §9].

Choosing $p = -13$, $q = 23$, $r = 3$, and $s = 1$ gives the homology sphere $\Sigma(3, 13, 23)$ as $+1$ surgery on a hyperbolic knot. Since $\mu(\Sigma(3, 13, 23)) = 1$, this manifold gives an example which yields the proof of Lemma 2.1. This is the only example of a μ-invariant 1 homology sphere constructible by this method found by a moderately long computer search. It is possible to give a Kirby-calculus proof that $\Sigma(3, 13, 23)$ is surgery on a knot similar to the one for $\Sigma(5, 9, 13)$; aficionados of the subject may wish to check if the knot is the same as the one in the knot from the paper [MM97].

References

[CH81] A. Casson and J. Harer, *Some homology lens spaces which bound rational homology balls*, Pacific J. Math. **96** (1981), 23–36.

[Fre82] M. Freedman, *The topology of four–dimensional manifolds*, J. Diff. Geo. **17** (1982), 357–432.

[FS81] R. Fintushel and R. Stern, *An exotic free involution on S^4*, Annals of Math. **113** (1981), 357–365.

[HKT94] I. Hambleton, M. Kreck, and P. Teichner, *Nonorientable 4-manifolds with fundamental group of order 2.*, Trans. A.M.S. **344** (1994), 649–665.

[Kir89] R.C. Kirby, *Topology of 4-manifolds*, Lecture Notes in Math., vol. 1374, Springer-Verlag, 1989.

[Kir97] R.C. Kirby, *Problems in low–dimensional topology*, Geometric Topology (W. Kazez, ed.), American Math. Soc./International Press, Providence, 1997.

[KT90] John Kalliongis and Chichen M. Tsau, *Seifert fibered surgery manifolds of composite knots*, Proc. Amer. Math. Soc. **108** (1990), 1047–1053.

[MM97] Katura Miyazaki and Kimihiko Motegi, *Seifert fibred manifolds and Dehn surgery*, Topology **36** (1997), 579–603.

[Rub84] D. Ruberman, *Equivariant knots of free involutions of S^4*, Top. Appl. **18** (1984), 217–224.

[Teich96] P. Teichner, *On the star-construction for topological 4-manifolds*, Geometric Topology (W. Kazez, ed.), American Math. Soc./International Press, Providence, 1997, 300–312.

Daniel Ruberman, Department of Mathematics, Brandeis University, Waltham, Massachusetts 02254

E-mail address: `ruberman@binah.cc.brandeis.edu`

Ronald J. Stern, Department of Mathematics, University of California, Irvine, California 92697

E-mail address: `rstern@math.uci.edu`