ON TWO-GENERATOR SATELLITE KNOTS

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Abstract. Techniques are introduced which determine the geometric structure of non-simple two-generator 3-manifolds from purely algebraic data. As an application, the satellite knots in the 3-sphere with a two-generator presentation in which at least one generator is represented by a meridian for the knot are classified.

1. Introduction

We consider classical knot and link exteriors as a context in which to introduce techniques which determine the geometric structure of non-simple 2-generator 3-manifolds from purely algebraic data. Here the application of these techniques are to a pair of related conjectures on the Heegaard realization of generators. These are the so called "two generator is tunnel one" and "meridional generator" conjectures.

These two conjectures arise from two of the problems in R. Kirby's famous 1978 list [K]. The first is a problem attributed to L. Moser which asks for a geometric characterization of those knots and links whose fundamental group may be presented using two group elements. The "two generator is tunnel one" conjecture purports to be the solution to this particular problem. This conjecture is usually attributed to M. Scharlemann, who attributes it to A. Casson, who notes it is implicit in a question asked in 1967 by F. Waldhausen [W]. The second is a problem attributed to S. Cappell and J. Shaneson which asks if generators which can be represented (under a suitable choice of basepoint) by a simple closed curve on the boundary (the so-called *meridional* generators) correspond to the bridges of a bridge decomposition of the knot exterior. The meridional generator conjecture asserts an affirmative answer to Cappell and Shanneson's question although considered in the broader context of a generalized notion of bridge decomposition.

Little is known on either of these conjectures. The only published result on the first is due to S. Bleiler [B] where the conjecture is established for cable knots. A single published result exists for the second as well. This is due to M. Boileau and B. Zimmermann [BZ], who use Thurston's orbifold technology to show that a knot

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or link exterior whose group is generated by a pair of meridional elements is in fact two bridge.

The principal result of this paper, Theorem 3.1, classifies those satellites with a two generator presentation in which one of the generators is meridional. Note that this includes all known two generator satellites. The two generator is tunnel one and meridional generator conjecture for these knots follows as a corollary to the classification. It also follows from this classification that a two generator satellite knot out in this class is a counter-example to the two generator is tunnel one conjecture. The key to the classification is an algebraic theorem given in 3.10.

The paper is organized as follows. The first and second sections respectively contain the topological and algebraic backgound necessary for the classification theorem which is stated and proved in section three.

The techniques developed here apply more generally to the classification of non-simple 3-manifolds. The details of this will appear in a subsequent publication.

2. Topological Background

1.1. Begin by recalling B. Clark's construction of a knot whose fundamental group is at most g-generator [C]. Start with a genus g Heegaard splitting of the 3-sphere and consider one of the handlebodies as the union of a single 0-handle and g 1-handles. Form the exterior of a knot by removing the 0-handle and one of the 1-handles. The cores of the remaining 1-handles form a set of *unknotting tunnels* for the knot K. The minimal cardinality of a set of unknotting tunnels for K is called the *tunnel number* of K. It follows that the rank of the fundamental group of a tunnel number n knot is at most $n + 1$. It is a consequence of Dehn's lemma that the rank of the group of a tunnel number one knot is exactly two. Clark's construction can be generalized to form complements of n component links by decomposing the genus g handlebody into a collection of n 0-handles and $g+n-1$ 1-handles and deleting a collection of n 0-handles and n 1-handles. It is occasionally useful to consider this construction in the context of general 3-manifolds.

1.2. Next, recall H. Doll's generalization of Schubert's notion of bridge number [D]. A properly embedded arc α in a handlebody is said to be *trivial* if there exists an arc β in the boundary of the handlebody whose endpoints agree with those of α such that $\alpha \cup \beta$ bounds a disc in the handlebody. A knot in a 3-manifold M has a (g, b)*-decomposition* or alternately, is said to be in *b-bridge position with respect to a Heegaard surface F of genus g* if the knot intersects the closure of each complementary handlebody to F in b trivial arcs. One now defines the *genus g bridge number* for a link L to be the minimal number b for which the link has a (g, b) decomposition with respect to a genus g splitting of the ambient manifold M. Note that Schubert's original definition of bridge number is the above definition applied to S^3 with $g = 0$. Also note that when $b \geq 1$, by stabilizing the Heegaard splitting one can always change

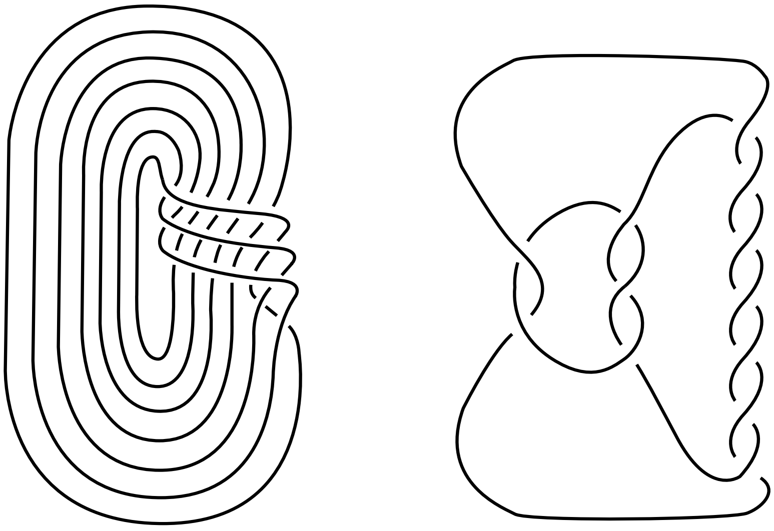

FIGURE 1

bridges to genus, that is, a link with a (g, b) decomposition always has a $(g+1, b-1)$ decomposition. That the converse is false is demonstrated by the $(-2, 3, 7)$ pretzel knot which has no $(0, 2)$ decomposition (i.e. is not two bridge), but as the closure of the 7-string braid $(\sigma_1\sigma_2)^{-1} \cdot (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6)^3$ has a $(1, 1)$ decomposition. See Figure 1. Also note from Figure 1 that as a pretzel knot, the $(-2, 3, 7)$ pretzel also has a $(0, 3)$ decomposition.

For convenience, we refer to a knot with a $(1, 1)$ decomposition as *one bridge with respect to a torus*. Note that such knots are naturally tunnel one, see [G].

1.3. We now develop the terminology that describes how these geometric concepts appear in the algebraic data. An element of the fundamental group of a link exterior is said to be *meridional* if there is a choice of basepoint so that the element is represented by a simple closed curve in the boundary. Note that an element can be meridional and yet not lie in the fundamental group of the boundary of the knot, as exemplified by the two bridge knot and link exteriors. The fundamental group of a two bridge link can be generated by two meridional elements, both of which cannot lie in the fundamental group of the boundary.

A link exterior is said to have a (g, b) *presentation* if its fundamental group has a presentation with $g+b$ generators, b of which are meridional. Note that if $b \geq 1$ that a knot with a (g, b) decomposition has a (g, b) presentation. The converse is:

Meridional Generator Conjecture. *A knot with a (g, b) presentation has a (g, b) decomposition.*

Remark: That a knot with a $(g, 0)$ decomposition need not have a $(g, 0)$ presentation is exemplified by the torus knots. However, as of this writing, no counterexample

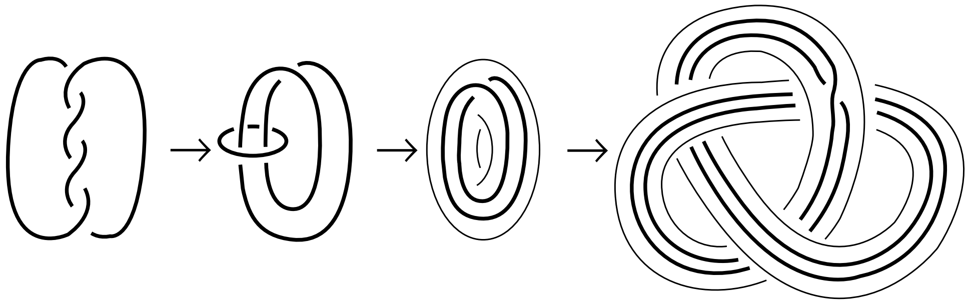

FIGURE 2

is known to the conjecture that a knot with a $(g, 0)$ presentation has a $(g, 0)$ decomposition.

Remark: Cappell and Shaneson's original problem that n-meridional generators implies n-bridge is just the special case $g = 0$.

1.4. The tunnel one satellite knots in S^3 were classified in 1988 by K.Morimoto and H. Sakuma in [MS] and the tunnel one satellite knots in general 3-manifolds were classified in 1995 by J. Neil [N]. Per Morimoto and Sakuma's classification, a satellite knot K is tunnel one when it satisfies the following three conditions. First, the exterior of K must be the union of a simple link exterior X_A and a torus knot exterior X_B. Second, the link exterior X_A must be attached to X_B in a manner that takes a meridian of X_A to the Seifert fibre of X_B. Finally, the link exterior X_A must embed in S^3 as the exterior of a 2-bridge link other than the un- or Hopf links. As can be seen from the construction, all the knots obtained in this manner are one bridge with respect to the torus knot's torus. See Figure 2 where this is illustrated for the Rolfsen-Bailey knot, i.e. the one obtained by attaching the exterior of the two bridge link $\frac{4}{1}$ to the exterior of the left hand trefoil, as described above. It is interesting to note that M. Eudave-Munoz was able to recover this classification by first proving that a tunnel one satellite in S^3 has a $(1, 1)$-decomposition, see [E].

3. 2. ALGEBRAIC BACKGROUND

2.1. Here we review some of the properties of groups needed in future discussion. Let A denote a group. A subgroup S of A is called *proper* if S is nontrivial and there exist elements of A outside of S. A proper subgroup S is said to be *malnormal* in A if the intersection of subgroups aSa^{-1} and S is trivial if and only if a is an element of A outside of S. Thus S intersects each of its nontrivial conjugates trivially. An element $a \in A$ is said to be *central* if a commutes with each element of A. A *torsion* element of A is an element a satisfying the relation $a^n = 1$ for some nonzero integer n.

2.2. The *rank* of a finitely presentable group is the minimal number of generators needed to present the group. A group A is called an *n-generator group* if A is known to have rank n. The groups of rank one are the finite cyclic groups together with the infinite cyclic group. The groups of higher rank form a much more complicated class. Indeed, determining the rank of a group is in general a difficult problem.

2.3. The following concepts are presented in greater detail in Magnus, Karrass, and Solitar [MKS] and are recalled here for the convenience of the reader. Let $A *_C B$ denote the free product with amalgamation of the non-trivial groups A and B along the group C, that is, C is isomorphic to a proper subgroup A_C of A and also isomorphic to a proper subgroup B_C of B. The subgroups A_C and B_C are identified with the group C via the respective isomorphisms. When the context is clear, we simply refer to the subgroup A_C of A by C, and similarly the subgroup B_C of B will be referred to by C. Note that group G contains each of A, B and C as proper subgroups. The groups A and B are called the *free factors* for the group $A *_C B$ and the group C is called the *amalgamated subgroup.*

Any element x of $A *_C B$ may be uniquely expressed in *normal form* as $x = x_1x_2 \cdots x_m c$ where c is an element of C and the x_i are alternately elements of a fixed set T_A of right transversals for the (right) cosets of C in A and a corresponding fixed set T_B of right transversals for the (right) cosets of C in B. The coset C in either of A or B always receives the transversal represented by 1 and the transversal representative 1 never appears in the normal form word for x. If x is expressed in the above normal form, then x is said to have length m, and one writes $\text{length}(x) = m$. If x_1 is an element of A, then x is said to begin in A. If x_m is an element of A, then x is said to end in A. Similarly, if x_i is an element of B for $i = 1$ or $i = m$, then x is said to begin or end respectively in B. If x begins in A and ends in B, then x can be written concretely in the normal form $x = a_1b_2 \cdots b_m c$. If x can be written in this form, then x is said to be a word of length m beginning in A and ending in B, and similarly for the other cases.

Two words x and y written in normal form are multiplied by concatenation. The product is subsequently converted to normal form by moving the elements of C to the right through the word y. If the word $x = x_1 \cdots x_n c$ ends in the same group that the word $y = y_1 \cdots y_m c'$ begins in, then the product $x_n c y_1$ is an element of one of A or B. If $x_n c y_1$ is an element of the complement of C in that factor, then xy is said to have *amalgamation* and $\text{length}(xy) = n+m-1$. If the product $x_n c y_1 = \bar{c}$, an element of C, then the word xy has *cancellation* and the product $x_{n-1}\bar{c}y_2$ is then examined for additional cancellation and/or amalgamation and $\text{length}(xy) \leq m + n - 2$. If x ends in a different group than y begins in, there is no amalgamation or cancellation in xy and $\text{length}(xy) = m + n$.

Now recall the results of [BJ] which will be relevent to the discussion at hand.

2.4. Lemma [BJ, 1.5]. *Suppose that the group $A *_C B$ is generated by n elements g_1, g_2, ... g_n. The set S_A consisting of those transversals of T_A appearing in any of the g_i together with all of the elements of C generates the group A.*

2.5. By symmetry, the group B may be generated by the set S_B consisting of all transversal elements of T_B that appear in at least one of the generators g_i together with the elements of C.

2.6. Theorem [BJ 3.1]. *Let $G = A *_C B$ be the free product with amalgamation of the torsion free group A and the group B. If the amalgamated subgroup C is of rank at least two and malnormal in A and if G has rank two, then either B contains torsion elements or G has a generating pair of one of four types (listed here as in* [BJ, 5.1]*):*

$p_1 : \{bc, ac_0\}$
$p_2 : \{c, a_1b_2c_0\}$
$p_3 : \{bc, a_1b_2c_0\}$
$p_4 : \{bc, a_1b_2a_3c_0\}$

2.7. Proposition [BJ 5.1.1]. *If G has a generating pair of type p_1 and C is abelian, then there is a minimal positive integer m such that the element $(bc)^m$ is an element $\tilde{c}$ of C which is central in B.*

4. The $(1,1)$ Presented Satellite Knots

3.1. Theorem. *The exterior of a $(1,1)$ presented satellite knot in the 3-sphere decomposes as the union along a torus of a simple link exterior X_A and a torus knot exterior X_B. Moreover, the link exterior X_A embeds in the 3-sphere as the complement of a two bridge link and is attached to X_B so that a meridian for this two bridge link is identified with the Seifert fibre of the torus knot exterior.*

Combining 1.4 and 3.1 gives the following corollories.

3.2. Corollary. *A $(1,1)$ presented satellite knot in the 3-sphere is one bridge with respect to a torus, in particular, is tunnel one.*

3.3. Corollary. *A two-generator satellite knot in the 3-sphere with no $(1,1)$ presentation has tunnel number at least two.*

3.4. Let K be a $(1,1)$ presented satellite knot in S^3. If K is a cable knot, then by [B], K is the $(spq \pm 1, s)$ cable on the (p,q) torus knot. Here, the space X_A embeds in S^3 as the complement of the two bridge link corresponding to the even integer $\frac{2s}{1}$. The conclusion of 3.1 follows.

Without loss of generality, the knot K is not cabled. Denote the exterior of K by X_K. The space X_K has a decomposition by essential tori into simple pieces, see [JS] or [J]. One of these simple pieces has ∂X_K as one of its boundary components. Denote this piece by X'_A. Thurston's geometrization for Haken manifolds [T] implies that X'_A is either a Seifert fibred space or its interior admits a hyperbolic metric. If not hyperbolic, the space X'_A contains a non-boundary parallel essential annulus, vertical with respect to the Seifert fibration on X'_A, with both boundary components lying on the boundary of X_K. As K is not a cabled knot, it follows that the knot K is composite, contradicting Norwood's result [N] that two-generator knots are prime. Hence the space X'_A is hyperbolic.

The space X'_A has at least two boundary tori. Let the torus T be a boundary torus that is different from ∂X_K. This torus T separates the space X_K into two pieces. Let X_A be the component of $X_K - T$ containing X'_A and X_B the complementary component. Since T is a separating torus, by the Seifert-Van Kampen theorem, the group $G = \pi_1(X_K)$ splits as the free product $A *_C B$ where A is the group $\pi_1(X_A)$, B is the group $\pi_1(X_B)$ and C is the group of the torus T.

3.5. Proposition. *The group C is a malnormal subgroup of A.*

3.6. (*Proof of 3.5.*) Let g be an element of the group A such that $gcg^{-1} = c'$ for elements $c, c' \in C$. Hence there exists a proper, π_1-injective map f of the annulus $S^1 \times I$ into X_A. Put the image of f into general position with respect to the decomposing tori for X_A inherited from the torus decomposition of X_K. While the intersection of the image of f with one of these tori may be hideous on the torus, the intersection on the annulus may, by standard techniques, be reduced to a family $\mathcal{F}$ of parallel essential simple closed curves on the annulus. The restricition of f to a complementary component of $\mathcal{F}$ which contains a component of $\partial(S^1 \times I)$ is a proper, π_1-injective map of an annulus into the hyperbolic manifold X'_A, hence the image of this restricted f can be isotoped into the torus T. Induction then shows that the entire image f can be isotoped into T and hence that $g \in C$. $\diamondsuit$

3.7. Proposition. *The group $G = \pi_1(X_K)$ has a generating pair of type $\{ac_0, bc\}$.*

3.8. (*Proof of 3.7.*) As the group G is $(1,1)$-presented, the group $G = A *_C B$ has a generating pair $\{g_1, g_2\}$. Without loss of generality, the generator g_1 is meridional and hence it has normal form ac. Thus there are exactly five possibilites for the normal forms of the generating set $\{g_1, g_2\}$. These are:

$p_1 : \{ac, bc\}$
$q_2 : \{ac_0, a_1b_2 \cdots b_{2k}a_{2k+1}c\}$
$q_3 : \{ac_0, a_1b_2 \cdots b_{2k}c\}$
$q_4 : \{ac_0, b_1a_2 \cdots a_{2k}c\}$
$q_5 : \{ac_0, b_1a_2 \cdots a_{2k}b_{2k+1}c\}$

By inverting the second element, we can consider a pair of type q_4 to be of type q_3. But by [BJ 4.4], a pair of type q_3 cannot generate G and by [BJ 4.5] neither can a pair of type q_5. So if $\{g_1, g_2\}$ is not of type p_1, it must be of type q_2. Note that by an argument similar to that in 3.6, the element $a_1^{-1}ac_0a_1$ is never an element of the subgroup C. This is essentially the argument that a hyperbolic manifold contains no essential annuli. Thus conjugating a pair of type q_2 by the element a_1^{-1} produces a generating set of type q_4 or q_5. Thus $\{g_1, g_2\}$ is not of type q_2. The desired result follows. $\diamondsuit$

3.9. From 3.7 and 2.7, the element $\tilde{c} = (bc)^m$ is central in B. It follows from G. Burde and H. Zieschang [BZ] that X_B is the complement of a torus knot and that the element $\tilde{c}$ is a power of the class of the torus knot's regular Seifert fibre. There is a choice of basis $\{x, y\}$ for the group C so that x is the class of a meridian for the torus knot and y coincides with $\tilde{c}$ if $\tilde{c}$ is a primitive element of C and is a primitive root of $\tilde{c}$ otherwise.

3.10. Theorem. *The elements ac_0 and $\tilde{c}$ generate the group A. Hence A has a $(0,2)$ presentation. In particular, the space X_A embeds in S^3 as the exterior of a two bridge link with meridians representing ac_0 and a primitive root of $\tilde{c}$.*

3.11. Lemma. *The elements ac_0 and $\tilde{c}$ generate a subgroup of A containing an element of the form x^q for some nonzero integer q.*

3.12. (*Proof of 3.11.*) Assume not. For any integer p, an element of form x^p is an element of C outside of the cyclic subgroup generated by $\tilde{c}$, and as an element of G can be expressed as a word w in the generating pair $\{ac_0, bc\}$. The word w can be written in the following form:

$$w = (bc)^{p_1}(ac_0)^{p_2}(bc)^{p_3}\cdots(ac_0)^{p_{2j}}(bc)^{p_{2j+1}}.$$

The integers p_1 and p_{2j+1} are allowed to be zero, but all other integers p_i are nonzero. As x^p is an element of C the word w admits amalgamations and cancellations that reduce its length to zero. We now begin a series of rewritings of the word w. Note that if p_{2i+1} is a multiple of the integer m, then we may replace the phrase $(bc)^{p_{2i+1}}$ by a power of $\tilde{c}$. Do this throughout the word w and combine maximal length phrases within w that are words in the elements ac_0 and $\tilde{c}$. The word w now has the following appearance:

$$w = (bc)^{p_1}w_2(bc)^{p_3}\cdots w_{2l}(bc)^{p_{2l+1}}$$

Each word w_i is a word in the elements ac_0 and $\tilde{c} = (bc)^m$. With the exception that p_1 and p_{2l+1} are allowed to be zero, no power p_{2i+1} is a multiple of the integer m, else it is combined into a neighboring w_{2i} or w_{2i+2}. If w has form $(bc)^0w_2$, then we have established 3.11. Hence we assume that each phrase w_{2i}, if not in $A - C$ is a power of $\tilde{c}$. Since $\tilde{c}$ and bc commute, all pwers of bc my be moved right through the word w. If w_{2i} is a power of $\tilde{c}$, any occurence of the phrase $(bc)^{p_{2i-1}}w_{2i}(bc)^{p_{2i+1}}$ may be rewritten as $w_{2i}(bc)^{(p_{2i-1}+p_{2i+1})}$, and then reindexed as necessary. In this way, the word w has been rewritten in the form:

$$w = (bc)^{p_1}w_2(bc)^{p_3}\cdots w_{2n}(bc)^{p_{2n+1}}$$

Each word w_i is a word in $A-C$. With the exception that p_1 is allowed to be zero and p_{2l+1} could be an arbitrary integer, no other power p_{2i+1} is a multiple of the integer m. Notice that w has now been written in a form admitting no further cancellation or amalgamation. Thus if w is to have length zero, it must have form $(bc)^p$ where p is a multiple of m or $(bc)^0 w_2$. But either of these options produce a power of $\tilde{c}$, a contradiction. $\diamondsuit$

3.13. (*Proof of 3.10.*) By 3.11, the elements ac_0 and $\tilde{c}$ generate an element of form $x^p \in C$. If $|p| = 1$, then by 2.4, the elements ac_0 and $\tilde{c}$ generate the entire group A. Assume that ac_0 and $\tilde{c}$ do not generate the element x and let k be the minimal positive integer for which the elements ac_0 and $\tilde{c}$ generate the element x^k. It follows from the argument in 3.12 that since the pair $\{ac_0, bc,\}$ generate the group G, that the elements bc and x^k must generate an element of form x^j where $1 \leq j \leq k-1$. Thus, Theorem 3.10 follows from the proposition below.

3.14. Proposition. *The elements bc and x^k cannot generate an element of form x^j where $1 \leq j \leq k-1$.*

3.15. (*Proof of 3.14.*) As noted in 3.9, B is the group of a torus knot. Hence the group B decomposes as the free product with amalgamation of form $E *_D F$, where each of the groups D, E and F are infinite cyclic. Geometrically, this corresponds to the familiar picture of constructing a torus knot exterior by glueing together two solid tori along an annulus in the boundary of each. Each infinite cyclic group E and F corresponds to the group of a solid torus. Let $h : B \longrightarrow \mathbf{Z}_p * \mathbf{Z}_q$ be the homomorphism defined by quotienting the group B by the infinite cyclic subgroup generated by the class of the regular Seifert fibre. By abusive notation we both e_i and $h(ei)$ will be denoted by the symbol e_i. Similarly, both f_i and $h(fi)$ will be denoted by the symbol f_i, the context making the meaning clear. Note for later use that as p and q are the indices of the exceptional fibres in the Seifert fibration of a torus knot exterior, they are relatively prime. Thus one of these integers must be odd and so one of $\mathbf{Z}_p$ or $\mathbf{Z}_q$ has no element of order two.

As the meridian for the torus knot, the element x has normal form $e_1 f_2 d \in E *_D F$, and so the element $h(x)$ has normal form $e_1 f_2$ in the free product $\mathbf{Z}_p * \mathbf{Z}_q$. Note that the normal form for the element x^k is then $(e_1 f_2)^k$, in particular has length $2k$. Since $(bc)^m = \tilde{c}$, the element $h(bc)$ lies in either a free factor of $\mathbf{Z}_p * \mathbf{Z}_q$ or a conjugate thereof. Each of these cases is examined in turn.

3.15.1. Suppose the element $h(bc)$ lies in a free factor of $\mathbf{Z}_p * \mathbf{Z}_q$. Without loss of generality, $h(bc) = e'$. Let w be a word in the elements $h(bc)$ and $h(x^k)$. Then we may assume that w appears as follows:

$w = e'^{p_1} h(x^k)^{p_2} e'^{p_3} \cdots h(x^k)^{p_{2j}} e'^{p_{2j+1}}$.

With the possible exception of p_1 and p_{2j+1}, all integers p_i are nonzero and such that no power of e' is trivial. If w is a nontrivial power of the element e', note that length(w) in $\mathbf{Z}_p * \mathbf{Z}_q$ is one. If w is a power of $h(x^k)$, say $h(x^k)^p$, then length$(w) = |2kp|$. We now assume that w has neither of these forms and compute the length of w. Before computing lengths, some preliminaries are required:

Given a fixed word w as above, associate a series of integers to w as in [BJ]. In particular, let P denote the associated sequence $\{p_2, p_4, \cdots p_{2k}\}$. The integer σ denotes the number of times that the sequence P changes algebraic sign. For example, if w is the word $e_1^8 h(x^k)^4 e_1^{-2} h(x^k)^{227} e_1 h(x^k)^{-88} e_1 h(x^k)$, then the sequence P is $\{4, 227, -88, 1\}$ and the associated integer σ is 2. Note that the relation $0 \leq \sigma \leq j-1$ holds. Further, for each integer $2i+1$ satisfying $0 \leq i \leq j$, let ϵ_{2i+1} denote the length of the phrase $e_1^{p_{2i+1}}$. Also note that with the possible exception of p_1 and p_{2j+1} all these integers are one. Without amalgamation or cancellation, the "uncancelled" length of the word w is given by:

$$\sum_{i=1}^{i=j} |p_{2i}|2k + \sum_{i=0}^{i=j} \epsilon_{2i+1} \geq 2kj + (j-1) + \epsilon_1 + \epsilon_{2j+1}$$

In what follows, assume that the integer k is at least two. Note that as the sequence P proceeds from p_{2i} to $p_{2(i+1)}$, the phrase $h(x^k)^{p_{2i}} e'^{p}_{2i+1} h(x^k)^{p_{2i+2}}$ may admit cancellation or amalgamation as outlined below:

· If both p_{2i} and $p_{2(i+1)}$ are positive, the phrase $h(x^k)e'^{p}_{2i+1}h(x^k) = (e_1 f_2)^k e'^{p}_{2i+1} (e_1 f_2)^k$ may have cancellations and amalgamation. By 3.15, there are at most two cancellations and a single amalgamation yielding a maximal length reduction of five.

· If both p_{2i} and $p_{2(i+1)}$ are negative, then again by 3.15, the phrase

$$h(x^k)^{-1} e'^{p}_{2i+1} h(x^k)^{-1} = (f_1 e_2)^{-k} e'^{p}_{2i+1} (f_1 e_2)^{-k}$$

has at most two cancellations and a single amalgamation yielding a maximal length reduction of five.

· If p_{2i} is positive and $p_{2(i+1)}$ negative, then writing out the phrase $h(x^k)e'^{p}_{2i+1}h(x^k)^{-1} = (e_1 f_2)^k e'^{p}_{2i+1} (e_1 f_2)^{-k}$ shows there is no cancellation or amalgamation.

· If p_{2i} is negative and $p_{2(i+1)}$ positive, then since the group $\mathbf{Z}_p$ is abelian, the phrase $h(x^k)^{-1} e'^{p}_{2i+1} h(x^k) = (e_1 f_2)^{-k} e'^{p}_{2i+1} (e_1 f_2)^k$ has a single amalgamation that reduces the length by exactly two.

Now upper bounds for the length of a word w are computed which depend on the variable σ:

3.15.1.a. The integer σ is even. Here length$(w) \geq 2kj + (j-1) + \epsilon_1 + \epsilon_{2j+1} - \sigma - 5(j-1-\sigma) \geq 2kj + (j-1) - \sigma - 5(j-1-\sigma) - 1 = 2kj - 4j + 4\sigma + 3 \geq 2j(k-2) + 3$.

3.15.1.b. The integer σ is odd and there are more occurences of the phrase

$$h(x^k)e'^{p}_{2i+1}h(x^k)^{-1}$$

than of the phrase $h(x^k)^{-1} e'^{p}_{2i+1} h(x^k)$. Here length$(w) \geq 2kj + (j-1) + \epsilon_1 + \epsilon_{2j+1} - (\sigma-1) - 5(j-1-\sigma) \geq 2kj + (j-1) - 2 - (\sigma-1) - 5(j-1-\sigma) = 2kj - 4j + 4\sigma + 3 \geq 2j(k-2) + 3$.

3.15.1.c. The integer σ is odd and there are more occurences of the phrase
$h(x^k)^{-1} e'^{p}_{2i+1} h(x^k)$
than of the phrase $h(x^k)e'^{p}_{2i+1}h(x^k)^{-1}$. Here length$(w) \geq 2kj + (j-1) + \epsilon_1 + \epsilon_{2j+1} - (\sigma-1) - 2 - 5(j-1-\sigma) \geq 2kj + (j-1) - (\sigma+1) - 5(j-1-\sigma) = 2kj - 4j + 4\sigma + 3 \geq 2j(k-2) + 3$.

Note that in all cases the elements e' and $h(x^k)$ cannot generate any elements of length less than $2k-1$ other than those elements of form e'^p. For instance, in 3.15.1.a or in 3.15.1.b, then the length of the word w is at least $2j(k-2)+3$. Note if j = 0, the resulting word is of form e'^p. So $j \geq 1$ and if k is at least three, then the inequality $2j(k-2)+3 < 2k-1$ implies that $j < 1$, a contradicition. But if $k = 2$, then the inequality reduces to $3 < 3$, another contraction. The statement of 3.14 follows.

Now if $h(bc) = f'$, repeating the above argument, replacing x^k by x^{-k} and e' by f' yields the statement of 3.14.

3.15.2. The element $h(bc)$ lies in a conjugate of a free factor, i.e the element $h(bc)$ is a conjugate of an element in a free factor of $\mathbf{Z}_p * \mathbf{Z}_q$.

Without loss of generality suppose $h(bc) = ge'g^{-1}$, where the element g has one of the four normal forms listed below:

i: $g = e'_1 \cdots e'_n$.
ii: $g = e'_1 \cdots f'_n$.
iii: $g = f'_1 \cdots e'_n$.
iv: $g = f'_1 \cdots f'_n$.

Note that in cases i and iii, the length of the element $h(bc)$ is $2n-1$ and in cases ii and iv, that the length of $h(bc)$ is $2n+1$.

Let w be a word in the elements $h(bc)$ and $h(x^k)$. Recall that $h(x^k)$ has normal form $(e_1f_2)^k$. The normal form for the element $h(x^k)$ can be written out and depends upon which of the four normal forms above the element g has. Since $\mathbf{Z}_p$ is an abelian group, we may assume for the purposes of working with $h(bc)$ that g has normal form of type ii or type iv. If not the trivial element, the element $h(bc)^p$ has length at least three. By conjugating, if necessary, we may assume that the element e'_1 beginning the word g in case ii is not the inverse of the element e_1 that begins the word $(e_1f_2)^k$. Similarly, we may assume that the element f'_1 beginning the word g in case iv is not the inverse of the element f_2, ending the word $(e_1f_2)^k$. The word w has the following form:

$w = h(bc)^{p_1}(e_1f_2)^{p_2 \cdot k} \cdots (e_1f_2)^{p_{2j} \cdot k}h(bc)^{p_{2j+1}}$.

With the possible exception of p_1 and p_{2j+1}, all integers p_i are nonzero and no power of $h(bc)$ is trivial. For each of cases ii and iv, the phrase $h(bc)^p(e_1f_2)^{q \cdot k}$ is examined for cancellation and or amalgamation:

3.15.2.1. If p is a positive integer, writing out the phrase using $(ge'g^{-1})^p$ instead of $h(bc)^p$ in case ii shows that there is an amalgamation in the elements ${e'_1}^{-1}e_1$, and no cancellation since the elements e'_1 and e_1 are not inverses. In case iv, there is no cancellation or amalgamation of any kind.

If p is a negative integer, there is no amalgamation or cancellation in case ii. In case iv, there is amalgamation, but no cancellation since the elements f'_1 and f_2 are not inverse.

3.15.2.2. For each of the cases, writing the phrase $(e_1f_2)^k h(bc)^p(e_1f_2)^k$ shows that there is a single amalgamation that reduces the uncancelled length by one.

3.15.2.3. For each of the cases, writing the phrase $((e_1f_2)^k)^{-1}h(bc)^p((e_1f_2)^k)^{-1}$ shows that there is a single amalgamation that reduces the uncancelled length by one.

3.15.2.4. Writing the phrase $(e_1f_2)^k h(bc)^p((e_1f_2)^k)^{-1}$ shows that in case ii there is no cancellation or amalgamation. In case iv, there are two amalgamations that reduce the uncancelled length by two.

3.15.2.5. Writing the phrase $((e_1f_2)^k)^{-1}h(bc)^p(e_1f_2)^k$ shows that in case ii, there are two amalgamations that reduce the uncancelled length by two. In case vi there is no cancellation or amalgamation.

Let w be an arbitary word of form $w = h(bc)^{p_1}(e_1f_2)^{p_2\cdot k}\cdots(e_1f_2)^{p_{2j}\cdot k}h(bc)^{p_{2j+1}}$. For each of the possible forms for g, an upper bound for the length of w will now be computed.

3.15.2.6. Assume that g has type ii. The length of w is computed depending on whether σ is even or odd.

If σ is even, then $\text{length}(w) \geq 2kj+(j-1)(2n+1)-\sigma-(j-1-\sigma) \geq (2n)(j-1)+2kj$.

If σ is odd and there are more occurences of the phrase $(e_1f_2)^k h(bc)^p((e_1f_2)^k)^{-1}$ in w than of the phrase $((e_1f_2)^k)^{-1}h(bc)^p(e_1f_2)^k$, then:
$\text{length}(w) \geq 2kj + (j-1)(2n+1) - (\sigma-1) - (j-1-\sigma) = 2n(j-1) + 2kj + 1$.

If σ is odd and there are more occurences of the phrase $((e_1f_2)^k)^{-1}h(bc)^p(e_1f_2)^k$ in w than of the phrase $(e_1f_2)^k h(bc)^p((e_1f_2)^k)^{-1}$, then:
$\text{length}(w) \geq 2kj + (j-1)(2n+1) - (\sigma-1) - 2 - (j-1-\sigma) = 2kj + 2n(j-1) - 1$.

3.15.2.7. Assume that g has type iv. The length of w is computed depending on whether σ is even or odd.

If σ is even, then $\text{length}(w) \geq 2kj + (j-1)(2n+1) - \sigma - (j-1-\sigma) \geq (2n)(j-1) + 2kj \geq 2kj = 2kj + 2n(j-1)$.

If σ is odd and there are more occurences of the phrase $(e_1f_2)^k h(bc)^p((e_1f_2)^k)^{-1}$ in w than of the phrase $((e_1f_2)^k)^{-1}h(bc)^p(e_1f_2)^k$, then:
$\text{length}(w) \geq 2kj + (j-1)(2n+1) - (\sigma-1) - 2 - (j-1-\sigma) = 2kj + 2n(j-1) - 1$.

If σ is odd and there are more occurences of the phrase $((e_1f_2)^k)^{-1}h(bc)^p(e_1f_2)^k$ in w than of the phrase $(e_1f_2)^k h(bc)^p((e_1f_2)^k)^{-1}$, then:
$\text{length}(w) \geq 2kj + (j-1)(2n+1) - (\sigma-1) - (j-1-\sigma) = 2kj + 2n(j-1) + 1$.

Recall that the integer k is at least two and that the integer n is at least one. Solving each of the following equations:

$2kj + 2n(j-1) - 1 < 2k - 1$
$2kj + 2n(j-1) < 2k - 1$
$2kj + 2n(j-1) + 1 < 2k - 1$

for the variable j, shows that if w is the representative of a word of length less than $2k-1$, then w has form $h(bc)^p$. But by 3.15.2, such an element has odd length. As the length of the element $(e_1f_2)^j$ is the even number $2j$, the elements $h(bc)$ and

$h(x^k) = (e_1 f_2)^k$ cannot generate an element of form $(e_1 f_2)^j$ for $1 \leq j \leq k-1$. Theorem 3.14, and hence Theorem 3.10 follows.

3.16. (*Proof of 3.1.*) Theorem 3.1 now follows from 3.9 and 3.10. $\diamondsuit$

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