ON INVARIANTS AND HOMOLOGY OF SPACES OF KNOTS IN ARBITRARY MANIFOLDS

V. A. VASSILIEV

Abstract. The construction of finite-order knot invariants in $\mathbb{R}^3$, based on resolutions of discriminant sets, can be carried out immediately to the case of knots in an arbitrary three-dimensional manifold M (may be non-orientable) and, moreover, to the calculation of cohomology groups of spaces of knots in arbitrary manifolds of dimension ≥ 3.

Obstructions to the integrability of admissible weight systems to well-defined knot invariants in M are identified as 1-dimensional cohomology classes of certain generalized loop spaces of M. Unlike the case $M = \mathbb{R}^3$, these obstructions can be non-trivial and provide invariants of the manifold M itself.

The corresponding algebraic machinery allows us to obtain on the level of the "abstract nonsense" some of results and problems of the theory, and to extract from others the essential topological (in particular, low-dimensional) part.

Introduction

Finite-order invariants of knots in $\mathbb{R}^3$ appeared in [V2] from a topological study of the discriminant subset of the space of curves in $\mathbb{R}^3$ (i.e., the set of all singular curves).

In [L], [K], finite-order invariants of knots in 3-manifolds satisfying some conditions were considered: in [L] it was done for manifolds with $\pi_1 = \pi_2 = 0$, and in [K] for closed irreducible orientable manifolds. In particular, in [L] it is shown that the theory of finite-order invariants of knots in two-connected manifolds is isomorphic to that for $\mathbb{R}^3$; the main theorem of [K] asserts that for every closed oriented irreducible 3-manifold M and any connected component of the space $C^\infty(S^1, M)$ the group of finite-order invariants of knots from this component contains a subgroup isomorphic to the group of finite-order invariants in $\mathbb{R}^3$. The starting point of these generalizations was the characteristic property of finite-order invariants in $\mathbb{R}^3$, considered in [V2], [BL], [BN]: the triviality of certain "finite differences" of values of these invariants defined with the help of the orientation of the ambient manifold, see § 1.2 below.

We show that all the theory of finite-order invariants and of the cohomology of spaces of knots in $\mathbb{R}^3$, based on the study of discriminants, can be extended almost

Supported in part by NSF grant DMS-9022140, RFBR (project # 95-01-00846a) and INTAS (project # 4373).

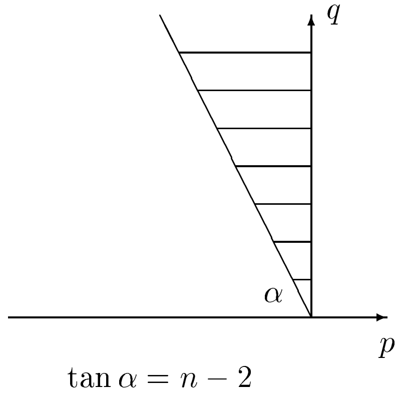


FIGURE 1. Support of the spectral sequence

immediately to the case of arbitrary manifolds of dimension $n \geq 3$ (including non-orientable ones), although the answers generally are not so easy.

The cohomology classes of spaces of knots in M^n come from a spectral sequence with support in the wedge $\{(p,q)|p < 0, p(n-2) + q \geq 0\}$, see fig. 1.

In particular, for $n = 3$ the knot invariants (= 0-dimensional cohomology classes of such a space) are counted by terms $E_\infty^{-i,i}$ of this sequence. In this case all elements of terms $E_0^{-i,i}$ of our spectral sequence, and some elements of $E_0^{-i,i+1}$ appeared in [L] and [K] under the names "singular knot invariants" and "local integrability conditions"; as in [V2], the calculation of further terms of the spectral sequence is nothing but the check whether these singular invariants satisfying these local conditions can be extended to less complicated singular knots or not.

In the case of an arbitrary manifold M^3, groups $E_r^{-j,j+1}$ contain some additional elements, the "global integrability conditions", which can provide non-trivial homological obstructions to this extension, see § 1.5. Our spectral sequence allows us to write these obstructions explicitly and, moreover, to be sure that if for some initial data they vanish, then these data can be extended to a well-defined knot invariant. Also the use of spectral sequences allows us to prove the main theorem of [L] and similar more general comparison theorems just by the methods of "abstract nonsense", and to avoid the technical difficulties overcome in [L], [K] by the methods from the Yablokova work.

The principles of the paper can be formulated in the following five statements.

1. The "manifold" part of invariants also is interesting.

For $n = 3$ the cohomology classes coming from the "principal diagonal" $\{E_\infty^{-i,i}\}$ of the spectral sequence are, generally speaking, the invariants not of knots in M^3 but of both the knots and the manifold M^3; they are exactly the "singular knot invariants" in terminology of [L], [K]. As was pointed out in [L], they take values on pairs of the form {a knot in M; a path in the space of continuous maps $S^1 \to M^3$ connecting this knot to a distinguished knot in its homotopy class (and considered up to homotopy)}.

More generally, for M of arbitrary dimension n let $\Omega_f M$ be the space of smooth maps $S^1 \to M$ and $\Sigma \subset \Omega_f M$ the set of all maps having selfintersections or singular points, so that knot invariants are elements of the group $H^0(\Omega_f M \setminus \Sigma)$. Then our spectral sequence converges to a subgroup of the relative cohomology group $H^*(\Omega_f M, \Omega_f M \setminus \Sigma)$ (if $n > 3$ then to all this group:

$$E_r^{p,q} \to H^{p+q+1}(\Omega_f M, \Omega_f M \setminus \Sigma)). \tag{1}$$

In particular, for $n = 3$ elements of the limit group $E_\infty^{-i,i}$ of this sequence define 1-dimensional cohomology classes of $\Omega_f M$ (and these classes can be nontrivial); and elements, defining zero cohomology classes, can be lifted to well-defined knot invariants.

Restrictions on manifolds, required in [L] and [K], essentially describe some situations, when all or some of these 1-cohomology classes are trivial. In these cases the invariants, provided by the above spectral sequence, coincide with these from [L], [K].

However these cohomology classes are an interesting characteristic of the manifold M and probably should not be considered separately from knot invariants. Moreover, the spectral sequence itself (especially its higher differentials) is a strong invariant of M. By analogy with the main result of [BL], I wonder whether the Jones–Witten–Reshetikhin–Turaev invariants of M can be derived from these ones.

2. One can calculate also higher-dimensional cohomology of spaces of knots, in particular in manifolds of higher dimensions.

The simplest such class is the 1-dimensional cohomology class of order 1 (i.e. coming from the cell $E^{-1,2}$) of the space of knots in $\mathbb{R}^3$. It takes nontrivial value on a 1-cycle in the space of unknots in $\mathbb{R}^3$, in particular proves that this space is not simple-connected, see § 1.8.

On the other hand, for knots in M^n, $n > 3$, we have no problems with the convergence of the spectral sequence, see (1). An attractive (for me) problem is to study explicitly these spectral sequences for simple-connected 4-dimensional manifolds.

3. There is an essential theory of finite-order invariants in non-orientable manifolds.

The orientability of M^3, used in [K] and [L] for the transversal orientation of the discriminant set, is unneseccary. First of all, the entire theory can be carried out without changes to the case of cohomologies and spectral sequences with coefficients in $\mathbb{Z}_2$, when such a coorientation is useless. Moreover, as we shall see in § 1.6, even for non-orientable manifolds our construction can give nontrivial invariants with integer (or real) values. In this case the basic "axiomatic" definition of order i invariants should be replaced by a more general one, as well as the notion of the index of order i, which any invariant defines at a singular knot with i self-intersections.

4. Resolutions of singularities of the discriminant are useful.

To show that the intersection (or linking) number with a subvariety is well defined, one usually tries to prove that this variety is regular (up to a set of codimension 2) and (co)orientable, or (if this is wrong) that it is possible to orient all its smooth pieces in such a way that these orientations will be compatible close to singular points, cf. [A1], [V1], [A3], [L].

Often such considerations can be simplified very much. Indeed, it is sufficient to show that our variety is the image of a regular orientable manifold under a proper map, and to define our indices in terms of the direct image of its fundamental cycle. The representation of varieties as such images is provided by the techniques of singularity resolutions. If our variety is a stratum of the discriminant set (say, the closure a class of singular knots), then there is a "tautological" construction of such resolutions.

5. Spectral sequences also are useful.

Many comparison theorems of topology and algebraic geometry can be proved in the following standard way. Instead of comparing homology groups of two objects, we compare spectral sequences converging to these groups. The convergence process (and the resulting groups) can be very complicated, but we do not need to consider it: it is easier to prove that the initial terms of these spectral sequences are naturally isomorphic, and isomorphisms of their final terms and these groups will follow automatically. Similar considerations often prove that one of these groups is "greater" than the other.

For instance, our spectral sequences are functorial with respect to the inclusion of manifolds, and hence the coincidence theorem from [L] is an immediate corollary of the (very easy) comparison of initial terms of corresponding spectral sequences for M^3 and $\mathbb{R}^3$, induced by any embedding $\mathbb{R}^3 \to M^3$, see [V4] and § 1.7 below. These considerations allow to prove immediately some existence theorems for invariants, or at least to reduce problems of this kind to essential low-dimensional problems concerning initial terms E_1 of these sequences (which are more standard than the study of the "integration process").

Moreover, our sequences are evidently functorial with respect to any submersions (in particular coverings) of manifolds of the same dimension. Another attractive problem: which part of this functoriality survives for arbitrary smooth maps.

In section 1 we outline main features of the spectral sequence and describe in elementary terms its applications to knot invariants; in section 2 we give the exact constructions and technical details. The "invariant–theoretical" part of section 1 can be reformulated and proved in terms not referring to the theory of spectral sequences, however we preserve there the standard notation of this theory.

Also in § 1 I represent four persons: a non-integrable singular knot invariant (see § 1.4), an invariant of singular knots with ≥ 2 crossings, which cannot be extended to invariants of knots with one crossing (§ 1.5), a non-trivial integer first order invariant in a non-orientable manifold (§ 1.6.1), and an invariant of order one, proving the nontriviality of the Whitehead link (§ 1.3).

I thank very much Efstratia Kalfagianni and Xiao-Song Lin for sending me preprints of their works, and Yasha Eliashberg, Sergei Matveev and Kolya Mishachev for related conversations.

Especially I thank MSRI, where the main part of the work was done, for hospitality and excellent conditions.

1. Elementary theory and main results

1.1. **How to overcome the infinitedimensionality.** Let M be a smooth n-dimensional manifold, $n \geq 3$, $\Omega_f M$ the space of all smooth maps $S^1 \to M$, and $\Sigma \subset \Omega_f M$ the set of maps having self-intersections or singular points, so that knots are the elements of $\Omega_f M \setminus \Sigma$.

As in [V2], [V3], we use a sort of the Alexander duality in the space $\Omega_f M$. To justify this duality in the infinitedimensional space, we need to consider a family of finitedimensional approximations to this space. For this, let us embed M into some space $\mathbb{R}^N$ as a regular (may be not closed) submanifold; let U be some open tubular neighborhood of M in $\mathbb{R}^N$, and $\tau : U \to M$ the corresponding C^∞-smooth projection with open discs for the fibers.

For any finitedimensional affine subspace Γ of the space of smooth maps $S^1 \to \mathbb{R}^N$ denote by Γ_U its subset consisting of maps, whose images belong to U. For the approximating subsets of the loop space $\Omega_f M$ we will use the sets of maps of the form $\tau \circ f$, $f \in \Gamma_U$.

For any such space Γ we construct in § 2 a homological spectral sequence

$$E^r_{p,q}(\Gamma) \to \bar{H}_{p+q}(\Sigma \cap \Gamma_U)$$

(where $\bar{H}_*$ denotes the Borel–Moore homology, i.e. the homology of the one-point compactification reduced modulo the added point) in exactly the same way as it was done in [V2] in the special case $M = \mathbb{R}^3 = \mathbb{R}^N = U$. Using the formal change of indices

$$E_r^{p,q} \equiv E^r_{-p,\dim\Gamma-q-1} \tag{2}$$

and the Poincaré–Lefschetz duality

$$\bar{H}_j(\Sigma \cap \Gamma_U) \simeq H^{\dim\Gamma-j}(\Gamma_U, \Gamma_U \setminus \Sigma),$$

we convert this sequence to a cohomological spectral sequence

$$E_r^{p,q}(\Gamma) \to H^{p+q+1}(\Gamma_U, \Gamma_U \setminus \Sigma).$$

If Γ satisfies some genericity conditions (see § 2.2), then the support of its term E_1 (and hence of all subsequent terms) belongs to the wedge shown in fig. 1.

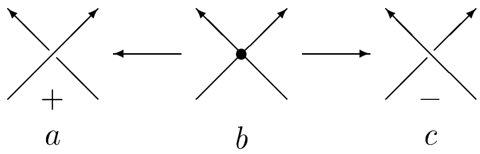


FIGURE 2. Local resolutions of a selfintersection point

1.2. **Stabilization of spectral sequences.** For any natural m, if $\Gamma \subset \Gamma'$ are two subspaces satisfying these genericity conditions, and the dimension of Γ is sufficiently large with respect to m, then for any $p \in [-m, 0]$ and any q the natural homomorphism $E_r^{p,q}(\Gamma') \to E_r^{p,q}(\Gamma)$, $r \geq 1$, is well defined, see § 2.3 below. This homomorphism is compatible with all subsequent differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ of the spectral sequence, and its limit version (for $r = \infty$) is compatible with the map of cohomology groups $H^*(\Gamma'_U, \Gamma'_U \setminus \Sigma) \to H^*(\Gamma_U, \Gamma_U \setminus \Sigma)$ induced by the identical embedding. Thus the limit spectral sequence

$$E_r^{p,q} \equiv \lim \text{ ind } E_r^{p,q}(\Gamma)$$

is well defined. For $n \geq 4$ this spectral sequence converges exactly to the group $H^*(\Omega_f M, \Omega_f M \setminus \Sigma)$. For $n = 3$ the similar statement is not proved (and probably is wrong, at least for sufficiently complicated M) because the sequence has infinitely many nontrivial terms $E_1^{p,q}$ on any line $\{p + q = const \geq 0\}$.

For the most popular case, when $n = 3$, $p + q = 0$ and M is orientable, the group $E_1^{p,q}$ is described in subsection 1.3. The group of finite-order knot invariants, to which these groups $E_r^{-i,i}$ stabilize, can be characterized in the following standard way.

Consider any immersion $\phi : S^1 \to M$, having a transverse selfintersection point $\phi(x) = \phi(y)$, $x \neq y \in S^1$. This selfintersection can be removed in two locally different ways, see fig. 2. Using the orientation of M we can call one of these local perturbations *positive* and the other *negative*, see § 4.3.1 in [V2]; opposite orientations of M define different signs of local perturbations. The index of a knot invariant at the singular knot ϕ is defined as its value at the positive perturbation minus that at the negative one.

Similarly, if ϕ has exactly j different transverse selfintersection points, then there are 2^j different simultaneous perturbations of all these points, moving our immersion ϕ to non-singular knots. Any knot invariant associates the *index of order j* to such a singular knot ϕ: it is equal to the sum of values of the invariant over all perturbations, the number of negative local moves in which is even, minus the sum of similar perturbations with an odd number of negative local moves.

DEFINITION-PROPOSITION. An invariant of knots in an orientable M^3 is *of order i* if any of following equivalent conditions is satisfied:

1) it has filtration $\leq i$ in our spectral sequence (i.e., comes from some of its cells $E_\infty^{-l,l}$ with $l \leq i$);
2) (see [L], [K]) all its indices of orders $j \geq i$ at all immersions with j selfintersections are equal to 0.

The fact, that these two conditions are equivalent, follows immediately from the construction of the spectral sequence. In [V2], [V3] the first of similar two conditions was used to define finite-order invariants in $\mathbb{R}^3$, and the second was mentioned as its "geometrical interpretation". In all the subsequent publications this second condition appears as the main definition.

Definition. The local surgery of an immersion $S^1 \to M^3$, connecting two pictures fig. 2a and 2c, is called *positive* (respectively, *negative*), if it replaces the negative resolution of the corresponding singular immersion by the positive one (respectively, positive by negative).

1.3. Description of the term $E_1^{-i,i}$ of the stable spectral sequence for $n = 3$.

Definition (see [V2]). The $[i]$-*configuration* (or chord diagram, see [BN]) is any collection of $2i$ distinct points in S^1 partitioned into i pairs. An $\langle i\rangle$-*configuration* is any collection of $2i-1$ distinct points in S^1 partitioned into $i-2$ pairs and one triple. An i^*-configuration is any collection of $2i-1$ points in S^1 partitioned into $i-1$ pair and one distinguished point $*$. An $\langle\tilde{i}\rangle$-configuration is an $\langle i\rangle$-configuration, in which one point of the triple is distinguished.

For any symbol $\Upsilon = [i], \langle i\rangle, \langle\tilde{i}\rangle$ or i^*, two Υ-configurations are *equivalent* if they can be transformed into one another by an orientation-preserving homeomorphism of S^1. A map $\phi : S^1 \to M$ *respects* the Υ-configuration if it maps all the points of any pair or triple to one point in M and (in the case of i^*-configurations) has zero derivative at the point $*$.

An Υ_M-*route* is any pair of the form {an equivalence class of Υ-configurations in S^1; a homotopy class of maps $S^1 \to M$ respecting configurations of this class}.

We describe the group $E_1^{-i,i}$ in two equivalent ways. The first of them (following [V2]) reflects better the structure of the resolution space and can be generalized to the calculation of higher-dimensional homology classes, see § 2; the second is more standard (and is formulated in terms of 4-term relations etc.)

First description. For any natural i the term $E_1^{-i,i}$ is the kernel of a certain operator $d_0 : E_0^{-i,i} \to \tilde{E}_0^{-i,i+1}$; let us define the elements of this operator. Suppose first that the manifold M is orientable or that the coefficient group G is the field $\mathbb{Z}_2$. Then the group $E_0^{-i,i}$ (respectively, $\tilde{E}_0^{-i,i+1}$) is the space of G-valued functions on the set of $[i]_M$- and $\langle i\rangle_M$-routes (respectively, on the set of $\langle\tilde{i}\rangle_M$- and i^*_M-routes).

The boundary $d_0(\alpha)$ of the generator α, corresponding to an $[i]_M$-route, is the sum of at most $2i$ generators of the group $\tilde{E}_0^{-i,i+1}$, corresponding to some of segments,

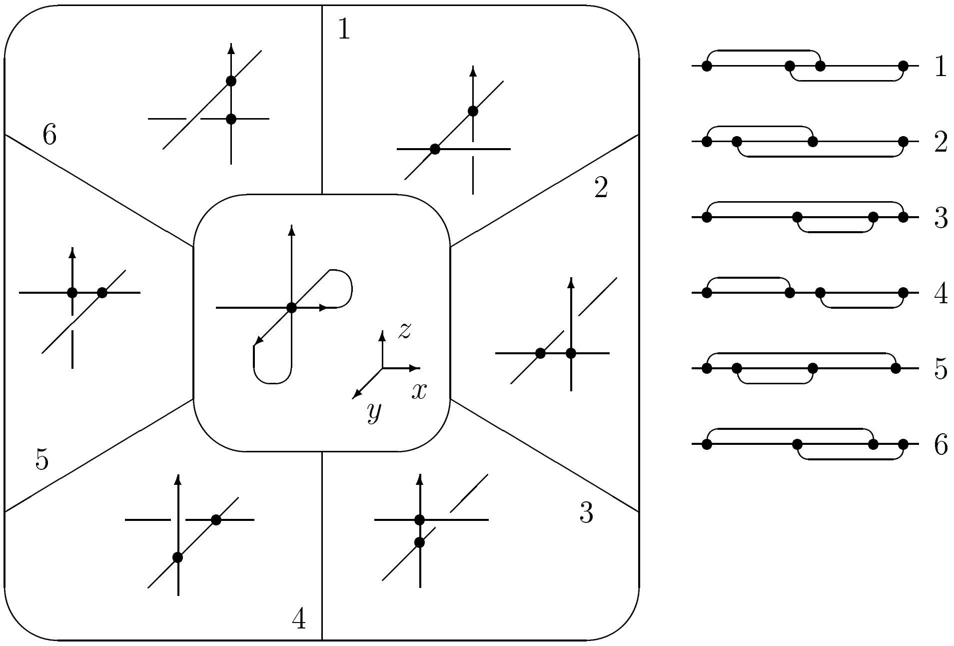
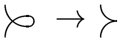


FIGURE 3. Splittings of a triple point

into which this configuration divides the circle. Namely, among these segments there can be "suspicious" ones, which are bounded by the points of one pair of our chord diagram. To such a segment there corresponds a generator in $d_0(\alpha)$ if and only if the loop in M, formed by the image of this segment, is homotopically trivial; this generator is spanned by the i^*_M-route, represented by the map $S^1 \to M$, coinciding with one representing α outside a small neighborhood of this segment and replacing this loop by a cusp at its corner point (which will be the $*$-point of the i^*_M-route):  . To any non-suspicious segment of our chord diagram, there corresponds the generator in $\tilde{E}_0^{-i,i+1}$, equal to the $\langle\tilde{i}\rangle_M$-route, obtained from α by a degeneration, contracting this segment to a point (which will be the distinguished point of the triple of the configuration, cf. [V2]). For instance, the curve with a triple point, shown in the center of fig. 3, is obtained by such a degeneration from any of 6 singular knots around it. (However, there is a subtlety here. Suppose that our $[i]_M$-route has self-equivalences, i.e. any representing it $[i]$-configuration can be transposed into itself by a homeomorphism of S^1 inducing a non-trivial cyclic permutation of its $2i$ vertices in such a way that the composition with this homeomorphism preserves the homotopy class of maps from our $[i]_M$-route. Of course, if we contract segments of S^1 transposed into one another by such a symmetry, then we obtain one and the same $\langle i\rangle_M$- or i^*_M-route; in this case we should count it only once for any class of equivalent segments.)

The differential of an $\langle i\rangle_M$-generator is defined exactly in the same way as in [V2], [V3], i.e. as the sum of three $\langle\tilde{i}\rangle_M$-generators coinciding with it geometrically and taken with appropriate signs (or only one such generator if this $\langle i\rangle_M$-route has a symmetry of order 3).

An equivalent and more standard description of the term $E_1^{-i,i}$ is as follows (cf. [BL], [BN], [L], [K] etc.)

The group $E_1^{-i,i}$ is isomorphic to the space of G-valued functions on the set of (equivalence classes of) $[i]_M$-routes satisfying the following conditions.

1. *Trivial condition.* For any $[i]_M$-route, having suspicious segments such that the corresponding loop in M is contractible, the value of this function should be equal to 0.

2. *Four-term relation.* Take any $\langle i\rangle_M$-route and realize it by a *generic* singular knot, i.e. by an immersion $S^1 \to M$ having $i-2$ transverse double points and one triple point, tangent vectors at which are linearly independent. This singular knot can be slightly perturbed in 6 different ways, so that the triple point splits into two double points, and the $\langle i\rangle$-configuration respected by our knot splits into a $[i]$-configuration, see fig. 3 (= fig. 15 in [V2]). These 6 perturbations can be divided in a natural way into three ordered pairs, numbered by $\langle\tilde{i}\rangle$-configurations, coinciding geometrically with the $\langle i\rangle$-configuration; in fig. 3 these pairs are formed by perturbations 1 and 4, 2 and 5, 6 and 3. For any such pair we take the value of our function on its first member minus the value on the second. The *four-term relation* claims that all these three differences of our six terms should coincide (so that their common value is a characteristic of the central singular knot; in the previous description of $E_1^{-i,i}$ this is the value of our function on the corresponding $\langle i\rangle_M$-route).

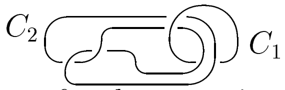
Finally, in the case of nonorientable M (and $G \neq \mathbb{Z}_2$) the group $E_1^{-i,i}$ consists of G-valued functions on the set of $[i]_M$-routes, satisfying all the same relations with small modifications and additionally taking zero value on all $[i]_M$-routes such that the corresponding strata of the discriminant do not satisfy a certain orientability condition, see § 1.6.3 below. In the simplest case of $[1]_M$-routes this condition coincides with the standard coorientability of the corresponding stratum of the discriminant variety.

Examples. 0. Invariants of order 0 are any functions $\pi_0(\Omega_f M) \to G$.

1. The unique [1]-configuration is the pair of points in the circle. Any $[1]_M$-route $\alpha \in E_0^{-1,1}$ is defined by an immersed circle in M with one selfintersection point. Its differential $d_0(\alpha) \in \tilde{E}_0^{-1,2}$ is nontrivial if and only if one of two loops, formed by this circle, defines a zero class in $\pi_1(M)$.

Proposition 1. *For any oriented 3-dimensional manifold M, the group of order 1 elements of the group $H^1(\Omega_f M, \Omega_f M \setminus \Sigma)$ is a free Abelian group whose generators are in the obvious one-to-one correspondence with the unordered pairs $(\alpha, \beta) \equiv (\beta, \alpha) \subset$*

$\pi_1(M)$ such that $\alpha \neq 0 \neq \beta$, and considered up to simultaneous conjugations: $(\alpha, \beta) \sim (\alpha', \beta')$ if there is $\lambda \in \pi_1(M)$ such that $\alpha' = \lambda^{-1}\alpha\lambda, \beta' = \lambda^{-1}\beta\lambda$. $\square$

So, in this theory the non-triviality of the Whitehead link 

in $\mathbb{R}^3$ can be proved already by an invariant of order 1 (and not of order 3, as in the usual theory of finite-order invariants). Indeed, when we try to deform this link to the trivial one, one of two circles (say C_1) can be considered as unmoved one, hence the triviality of the link is equivalent to the triviality of the knot C_2 in the manifold $\mathbb{R}^3 \setminus C_1$. This knot can be obtained from a circle C_0, unknotted and unlinked with C_1, by a deformation, along which it selfintersects only once, and both loops arising in the instant of selfintersection define nonzero elements of the group $\pi_1(\mathbb{R}^3 \setminus C_1)$. By Proposition 3 below the corresponding element of $H^1(\Omega_f M, \Omega_f M \setminus \Sigma)$ belongs to the kernel of the obvious map of this group to $H^1(\Omega_f M)$, and hence defines a knot invariant. The values of this invariant on knots C_0 and C_2 differ by ± 1.

Remark. In the calculation of higher differentials d_r, $r \geq 1$, of the spectral sequence some new generators of groups $E_*^{-i,i+1}$ can arise, namely, certain 1-dimensional cohomology classes of spaces of maps $S^1 \to M$ respecting our $[i]$-configurations, see § 2.4. They can provide some extra obstructions to the integration process, see § 1.5.

1.4. **Example of a nontrivial finite-order class in $H^1(\Omega_f M)$ and some obstructions to the existence of such classes.** Let $M = S^2 \times S^1$. We construct a 1-parametric family of immersed circles in M. All these circles consist of two segments, the first of which is the same for all circles of the family: it starts at the north pole of the distinguished sphere $S^2 \times \{0\}$, finishes at the south pole of the same sphere, has no self-intersections, and the cyclic coordinate of the factor S^1 grows on it monotonically from 0 to 4π. The second segments of these embedded circles will be just all the meridians in this sphere, joining the south pole back to the north one; they (and hence also the entire corresponding embedded circles in M) are parametrized by the equator circle in S^2. There is exactly one selfintersecting circle in this family.

All curves of our family are piece-wise smooth with only two breakpoints at the poles; it easy to improve them slightly at these points in such a way that they become smooth but do not get extra selfintersections. The $[1]_M$-configuration, arising at the unique instant of self-intersection, defines an element of $H^1(\Omega_f M, \Omega_f M \setminus \Sigma)$: indeed, both loops formed by it are homotopically nontrivial in M. The value of this element on our loop in $\Omega_f M$ is equal to ± 1, thus it defines also a nontrivial element in $H^1(\Omega_f M)$.

Problem. To construct a similar example inside the trivial component of $\Omega_f M$ (consisting of contractible loops); may be with a more complicated M. Proposition 3 below shows in particular that it is impossible if M is closed and $\pi_2(M) = 0$.

Proposition 2. *Let M be an arbitrary 3-manifold, and C a connected component of $\Omega_f M$. Suppose that the group $H_1(C,\mathbb{R})$ contains a basis consisting of loops $S^1_\lambda \to \Omega_f M$ (or, which is the same, of maps $S^1 \times S^1_\lambda \to M$) such that for any $\lambda \in S^1_\lambda$ the restriction of this map on any circle $S^1 \times \{\lambda\}$ is an embedding. Then all the elements of $H^1(\Omega_f M)$, coming from our spectral sequence, take zero values on the elements of $H_1(C)$ and hence all elements of groups $E_\infty^{-i,i}$ of this sequence define invariants of knots from this component.* $\square$

For the "trivial" component C_0 of $\Omega_f M$ (i.e. the component of the trivial knot) the condition of the previous proposition follows from the following more standard one.

Proposition 3. *Suppose that the group $\pi_2(M)$ is generated by spheroids $S^2 \to M$ homotopic to embedded spheres. Then the condition of the previous proposition for the component C_0 is satisfied, so that all the elements of $H^1(\Omega_f M)$, coming from our spectral sequence, are trivial in restriction to $H_1(C_0)$.*

Proof. Let us choose a small contractible unknotted parametrized circle $\circ \subset M$ for the basepoint in $\Omega_f M$. Given a spheroid $\varepsilon : S^2 \to M$, denote by τ_ε the toroid $S^1 \times S^1_\lambda \to M$, which almost everywhere coincides with the trivial loop τ_0 in $\Omega_f M$ (sending all of S^1_λ to the point $\circ$) and only in a small closed disc $\delta \in S^1 \times S^1_\lambda \setminus (S^1 \times *)$ replacing τ_0 by such a map that the union of maps τ_ε and τ_0 on two copies of δ, glued on their boundaries, is a spheroid homotopic to ε.

Any loop in $\Omega_f M$, i.e. a continuous map $S^1 \times S^1_\lambda \to M$, is homotopic to a sequence of loops, the first of which is the family of small unknotted embedded circles, moving along a path in M, and all other are toroids τ_{ε_j}, where ε_j are some basis elements of $\pi_2(M)$. Thus we need only to check the conditions of Proposition 2 for any toroid $\tau = \tau_\varepsilon$, where the spheroid ε is homotopic to an embedding.

By definition, there is a small neighborhood I of the distinguished point $* \in S^1_\lambda$ such that for $\lambda \in I$ the corresponding loops $\tau(S^1 \times \{\lambda\})$ coincide with $\circ$. Denote by $D = D_N \cup D_S$ the union of two discs in S^2 bounded by small polar circles, and consider the spheroid $s : S^2 \to M$, coinciding in $S^2 \setminus D$ with τ after the standard identification

$$S^1 \times (S^1_\lambda \setminus I) \sim S^2 \setminus D, \tag{3}$$

and in D coinciding with two small embedded discs contracting the circle $\circ$ in M.

Let $\theta : [0,1] \times S^2 \to M$ be a homotopy of this spheroid, moving it to an embedding of S^2. Since for a generic map $S^2 \to M$ the set of singular points is finite, we can assume that during this homotopy the restrictions of all maps $\theta(t,\cdot)$ on the polar zone D are immersions (and thus also embeddings since this zone can be chosen arbitrarily small). Then consider the toroid $S^1 \times S^1_\lambda \to M$, whose restriction on the cylinder $S^1 \times (S^1_\lambda \setminus I)$ coincides via the identification (3) with the spheroid $\theta(1,\cdot)$, and the image of the cylinder $S^1 \times I$ coincides with the union of traces $\theta([0,1] \times \partial D)$ of polar circles during the homotopy (glued on their common circle $\theta(0\times\partial D_N)\equiv\theta(0\times\partial D_S)$). This toroid satisfies conditions of Proposition 2. $\square$

Corollary. There exists a non-trivial theory of finite-order invariants in reducible 3-manifolds, cf. [K]. For instance, $S^2\times S^1$ is reducible but satisfies conditions of Proposition 3.

I am sure that experts in the low-dimensional topology can prove much stronger statements also proving the triviality of $H^1(\Omega_f M)$-classes of this kind, cf. [K].

1.5. **Integration of elements of $E_1^{-i,i}$ to knot invariants.** In this subsection we assume that M is a three-dimensional oriented manifold. Given an element γ of the corresponding group $E_1^{-i,i}$, can it be continued to a knot invariant of order i? Similarly to [V2], [BL], this question can be reduced to a sequence of systems of linear equations, whose unknowns correspond to topological types of singular knots in M with $< i$ selfintersections, cf. [L], [K]. In this subsection we reformulate this condition back in terms of spectral sequences: γ should belong to the subgroup $E_{i+1}^{-i,i}\subset E_1^{-i,i}$, see [V2]. This allows us to write explicitly the group of additional obstructions to this continuation, arising from the topology of the manifold M, see formula (4) below.

1.5.1. *Actuality table.* Similarly to [V2], any knot invariant of a finite order i can be encoded by an *actuality table*, having $i+1$ levels $0,1,2,\dots,i$. The j-th level consists of cells, corresponding to all $[j]_M$-routes.

In any such cell we draw a generic immersion $S^1\to M$ representing this $[j]_M$-route (i.e. having exactly j transverse selfintersection points and no other singularities). This immersion is an accessoir of the table itself and does not depend on the invariant. To describe the invariant, we write in the cell a number (or, more generally, the element of the coefficient group G), namely, the index of j-th order of the corresponding immersion, see § 1.2.

The calculation of the value of an invariant consists in the same inductive process as in [V2], [V3]. Namely, we join our knot by a generic path in $\Omega_f M$ with the *distinguished* knot from the same homotopy class in $\Omega_f M$ (i.e., to the knot drawn in the corresponding cell of the 0-th level of the actuality table). Such a path has only finitely many intersection points with the variety Σ at its points, corresponding to immersions $S^1\to M$ with one transverse self-intersection.

The value of the invariant at our knot is equal to its value at the distinguished one plus the sum of 1-st order indices of these immersions, taken with coefficients ±1, equal to signs of corresponding local surgeries, see the last definition in § 1.2. To calculate these indices, we join these immersions to distinguished ones (given in the first level of the table) by arbitrary generic paths in Σ and count all the points of the set of transversal self-intersection of Σ, which we meet along these paths, etc. This process stops on the level i, because by the characteristic property (see § 1.2) any invariant of order i defines the same indices of order i for all generic immersions from the same $[i]_M$-route. (In particular, we do not need to draw pictures in the actuality table at the top level.)

If we do not fix values of the invariant at the 0-th level of the table, then we get not an invariant, but just an element of the group $H^1(\Omega_f M, \Omega_f M \setminus \Sigma)$.

Our spectral sequence (more precisely, its restriction on cells $E_r^{-j,j}, E_r^{-j,j+1}$, responsible for the 0-dimensional cohomology of the space of knots) is a method of calculating all actuality tables such that this algorithm works and does not lead to a contradiction.

It starts with any element $\gamma \in E_1^{-i,i}$, see § 1.3. This element defines an upper (i-th) level of the actuality table: into any cell, corresponding to an $[i]_M$-route, we put the value of the function γ at this route. Then, exactly as in [V2], we fill in the table from top to bottom. We can fill in all levels $i-1, i-2, \dots, i-r+1$ if and only if γ belongs to the subgroup $E_r^{-i,i}$ of $E_1^{-i,i}$. Unlike [V2], this subgroup can be proper, see § 1.5.4 below.

1.5.2. *The short spectral sequence.* Exactly as in [V2], we factorize our general spectral sequence (described in § 2) through some elements which surely do not contribute to the calculation of 0-dimensional cohomology of the space of knots in M, and obtain the *short spectral sequence* $E_r^{p,q}$, $r \geq 0$, whose non-trivial groups lie only on two half-lines with $p < 0$ and $p+q = 0$ or $p+q = 1$, and coincide with the corresponding groups of the main spectral sequence on the first of these lines. In the rest of the present section we deal only with this short spectral sequence.

The groups $E_0^{-i,i}$ are already described in § 1.3, let us describe $E_0^{-i,i+1}$.

For any $[i]_M$-route I denote by $\{I\}$ the space of its realizations, i.e., of pairs of the form {an $[i]$-configuration of the corresponding equivalence class; a map $S^1 \to M$ respecting this configuration}.

The group $E_0^{-i,i+1}$ is the direct sum of the group $\tilde{E}_0^{-i,i+1}$, described in § 1.3, and the group

$$\breve{E}_0^{-i,i+1} \equiv \prod_I H^1(\{I\}, G), \tag{4}$$

multiplication over all $[i]_M$-routes I.

1.5.3. *The differential* $d^1 : E_1^{-i,i} \to E_1^{-i+1,i}$. Suppose that we have an element γ of $E_1^{-i,i}$, i.e. a G-valued function on the set of $[i]_M$-routes, satisfying the basic relations described in § 1.3. To extend it to a knot invariant, we need to calculate all its higher differentials $d^1(\gamma) \in E_1^{-i+1,i}$, $d^2(\gamma) \in E_2^{-i+2,i-1}, \dots, d^{i-1}(\gamma) \in E_{i-1}^{-1,2}$, $d^i(\gamma) \in E_i^{0,1}$, and to prove that all these differentials are trivial; if we prove all these conditions but the last one, then we extend our element γ only to an element of the group $H^1(\Omega_f M, \Omega_f M \setminus \Sigma)$.

In this subsubsection we describe explicitly the first of these conditions.

The map d^1 splits into the sum of two operators $\breve{d}^1 : E_1^{-i,i} \to \breve{E}_1^{-i+1,i}$ and $\tilde{d}^1 : E_1^{-i,i} \to \tilde{E}_1^{-i+1,i}$, where $\breve{E}_1^{-i+1,i} \equiv \breve{E}_0^{-i+1,i}$, $\tilde{E}_1^{-i+1,i} \equiv \tilde{E}_0^{-i+1,i}/d^0(E_0^{-i+1,i-1})$, and we need to check both conditions $\breve{d}^1(\gamma) = 0$, $\tilde{d}^1(\gamma) = 0$. Let us describe these conditions.

Consider any $[i-1]_M$-route I, and any 1-cycle l in the manifold $\{I\}$. We can realize it by a smooth generic path, only finitely many times intersecting the set of immersions, having i transverse self-intersections, i.e. defining points of $[i]_M$-routes. At any such point we take the value of our function γ on this $[i]_M$-route, and multiply it by the coefficient 1 or -1 depending on the direction in which we traverse this stratum (i.e. on the sign of the corresponding local surgery close to the additional i-th crossing point, see § 1.2). The sum of all such values taken with these coefficients depends only on the class of the path l in the group $H_1(\{I\}, G)$. We denote this sum by $\langle \breve{d}^1(\gamma), l\rangle$ and thus define the element $\breve{d}^1(\gamma) \in H^1(\bigcup_I \{I\}; G)$.

If this element is non-trivial, then γ is not equal to the i-th index of any invariant of order i.

Now suppose that $\breve{d}^1(\gamma) = 0$. Then we can define a locally constant function A_{i-1} on the regular part of the manifold $\{I\}$ (i.e. on its part corresponding to immersions with exactly $i-1$ selfintersections) in such a way that the difference of its values on two sides of any hypersurface, corresponding to any $[i]_M$-route, coincides with the value of γ at this route. Such a locally constant function A_{i-1} is defined by γ almost uniquely, only up to addition of functions, which are constant on any manifold $\{I\}$ (i.e., up to elements of the group $E_0^{-i+1,i-1}$).

The second condition $\tilde{d}^1(\gamma) = 0$, which this function A_{i-1} should satisfy, consists of following two subconditions:

a) given any generic immersion with $i-3$ transverse double crossings and one triple point, the values of A_{i-1} at all 6 local moves of the triple point, decomposing it into a pair of double points, satisfy the 4-term relations, see fig. 3;

b) given any generic map $S^1 \to M$ with $i-2$ transverse self-intersections and one cusp point, the value of A_{i-1} at its local move, replacing the cusp by a self-intersection point, is equal to 0.

It is sufficient to check this second condition close to one generic point of any $\langle i-1\rangle_M$-route or $(i-1)^*$-route: if it is satisfied for some choice of such points (and the first homological condition $\breve{d}^1(\gamma) = 0$ also holds), then it will be satisfied automatically at all other points of the same $\langle i-1\rangle_M$-route or $(i-1)^*$-route.

This condition can be easily identified as the triviality of a certain element $d^1(\gamma)$ of the quotient group $\tilde{E}_1^{-i+1,i} \equiv E_0^{-i+1,i}/d^0(E_0^{-i+1,i-1})$.

Suppose that $d^1(\gamma) = 0$, i.e. there exists a locally constant function A_{i-1} satisfying all above conditions. Then we say that γ belongs to the group $E_2^{-i,i}$, and are able to fill in the $(i-1)$-th level of the actuality table: in any cell, containing a generic point of the $[i-1]_M$-route, we put the number (or element of G) equal to the value of the function A_{i-1} at this point.

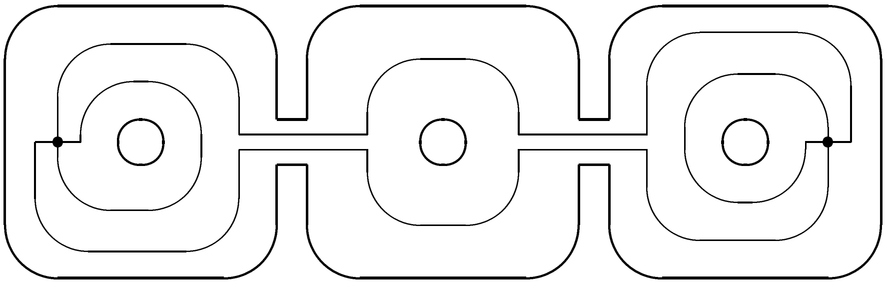


FIGURE 4. A non-integrable singular knot.

The choice of this function A_{i-1} is not unique (if exists): it is defined by γ up to addition of arbitrary elements of the group $E_1^{-i+1,i-1}$.

1.5.4. *An example of non-degenerating spectral sequence.* Let M be the connected sum of three copies of $S^2 \times S^1$. A planar outline of M is shown in fig. 4 by the domain with three holes, bounded by thick curves. Consider the loop $S^1 \to M$ with two self-intersections, shown in fig. 4 by the thin line. Its chord diagram is *trivial*, i.e., consists of two non-crossing chords.

DEFINITION. Two $[2]_M$-routes with non-crossing chords are *neighbors*, if there exists an immersion $S^1 \to M$ with unique generic triple point such that some two of three its perturbations, shown in fig. 3 and respecting trivial chord diagrams, belong to these $[2]_M$-routes. Two $[2]_M$-routes are *related*, if there exists a chain of $[2]_M$-routes, joining them, any two neighboring members of which are neighbors. A $[2]_M$-route is *marginal* if one of two its suspicious loops (see § 1.3) is contractible (so that any element of the group $E_1^{-2,2}$ should take zero value on the corresponding $[2]_M$-route).

LEMMA. *Among relatives of the $[2]_M$-route, represented by the curve from fig. 4, there are no marginals.*

Indeed, the subgroup in $H_1(M)$, generated by cycles, lying in an immersed circle, is the same for all its relatives. For the initial curve from fig. 4 this subgroup is of rank 3, and for any marginal $[2]_M$-route of rank at most 2. □

Now define the $\mathbb{Z}$-valued function γ on the set of all $[2]_M$-routes, which takes value 1 at all relatives of the route represented by the curve from fig. 4, and zero value at all other $[2]_M$-routes. This function satisfies both conditions from § 1.3, and hence belongs to the subgroup $E_1^{-2,2} \subset E_0^{-2,2}$.

However $\check{d}^1(\gamma) \neq 0$. Indeed, consider the horizontal segment in the picture of our curve in the rightmost copy of $S^2 \times S^1$, joining two points of the sphere $S^2 \times \{0\}$. We can suppose that these points are poles of this sphere, and the segment is its distinguished meridian. Consider the family of curves C_α, $\alpha \in [0, 2\pi]$, coinciding with the one from fig. 4 everywhere outside this segment and replacing it by all other meridians; the parameter α of this family is the cyclic coordinate α of the equator in S^2.

This family defines a 1-cycle in the corresponding $[1]_M$-route, and its intersection index with the cycle γ obviously is equal to ± 1, in particular γ defines a non-zero class in the group H^1 of this $[1]_M$-route. Thus $\breve{d}^1(\gamma) \neq 0$.

1.5.5. *Differentials* d^2, d^3 *etc.* Now suppose that we already have calculated (and defined) all differentials $d^s : E_s^{-j,j} \to E_s^{-j+s,j-s}$, $s < r$, for any $j \leq i$; let γ be an element of the subgroup $E_r^{-i,i} \subset E_1^{-i,i}$. This means in particular that we can fill in all levels $i-1, i-2, \dots, i-r+1$ of the actuality table with the upper level γ. The indices of this table are then determined by γ up to addition of similar tables corresponding to all elements of groups $E_{r-1}^{-i+1,i-1}, E_{r-2}^{-i+2,i-2}, \dots, E_1^{-i+r-1,i-r+1}$.

Moreover, in this case we can define similar indices (of corresponding orders) for *all* immersions $S^1 \to M$, having exactly $i, i-1, i-2, \dots, i-r+1$ transverse self-intersection points. Then for any manifold $\{I\}$, where I is a $[j]_M$-route with $j \geq i-r+1$, these indices form a locally constant function on its open submanifold, consisting of immersions with exactly j transverse self-intersections. Let us fix such a collection of locally constant functions A_j and the actuality table, representing them, in which only levels $i, i-1, \dots, i-r+1$ are filled in. We call this partially completed table the *tentative actuality table.*

This collection of locally constant functions (or, equivalently, the representing them tentative actuality table) is called 1-*integrable,* if there exists a locally constant function on the union of regular subsets of all manifolds $\{I\}$ for all $[i-r]_M$-routes I, such that a) the difference of its values at two sides of any hypersurface in $\{I\}$, consisting of generic immersions with $i-r+1$ self-intersections, is equal to the value at the corresponding piece of this hypersurface of its index A_{i-r+1}, encoded in the existing part of table, and b) close to any generic immersion, respecting an $\langle i-r\rangle$-configuration (respectively, $(i-r)^*$-configuration), all corresponding 4-term relations (respectively, the trivial relation) are satisfied.

Exactly as in § 1.5.3, the obstruction to the existence of such a function is just an element of the group $E_1^{-i+r,i-r+1}$. Denote by $D^r(E_r^{-i,i})$ the subgroup of this group, generated by all such obstructions over all elements $\gamma \in E_r^{-i,i}$ and all their admissible extensions to collections of locally constant functions A_j, $j = i-1, \dots, i-r+1$.

The condition $\gamma \in E_{r+1}^{-i,i}$ means that this obstruction is trivial for at least one choice of the tentative actuality table with upper level equal to γ. To check this condition, we need to calculate this obstruction for an arbitrary such tentative table, and check whether it belongs to the subgroup generated by similar subgroups

$$d^1(E_1^{-i+r-1,i-r+1}), D^2(E_2^{-i+r-2,i-r+2}), \dots, D^{r-1}(E_{r-1}^{-i+1,i-1}). \tag{5}$$

Here is one more way to say the same: we denote by $E_r^{-i+r,i-r+1}$ the quotient group of $E_1^{-i+r,i-r+1}$ by the subgroup generated by all subgroups (5), denote by $d^r(\gamma)$ the class of our obstruction in this quotient group, and check whether it is trivial or not.

If yes, then we a) change the levels $i-1, \dots, i-r+1$ of our tentative actuality table by the elements of an arbitrary 1-integrable one with the same leading term γ, and b) fill in the $(i-r)$-th level of this new table, putting in any cell, corresponding to a $[i-r]_M$-route, the value, which any locally constant function A_{i-r} on this route, satisfying the above integration conditions (and defined by this 1-integrable tentative table), takes at the immersion depicted in this cell.

Thus the inductive step of the calculation (and definition) of our spectral sequence is completed.

REMARK. We could replace the group (4) by certain its subgroup. Namely, for an $[i]_M$-route I denote by $h(I,G)$ the subgroup in $H_1(\{I\},G)$ spanned by such loops in $\{I\}$, all whose points are generic immersions with exactly i selfintersections. Then the group $\check{E}_0^{-i,i+1}$ could be redefined as $\prod_I(\text{Ann } h(I,G)) \subset \prod_I H^1(\{I\},G)$. Indeed, all our additional obstructions to integrability take zero values on elements of $h(I,G)$.

1.6. **The case of non-orientable manifolds.** If the three-dimensional manifold M is non-orientable, then many natural strata of (the resolution of) the discriminant turn out to be non-(co)orientable, and hence can participate only in the construction of (mod 2)-invariants. However, if M has a sufficiently complicated fundamental group, then many strata are still orientable and define integer invariants and homology classes.

1.6.1. *First example.* Consider the main stratum of the discriminant $\Sigma \subset \Omega_f M$, corresponding to some $[1]_M$-route (i.e. an irreducible component of the set of maps $S^1 \to M$ with a transverse self-intersection). Let ϕ be a generic representative of this stratum, and α and β the classes of two corresponding loops in the group $\pi_1(M)$ with basepoint at the self-intersection point.

PROPOSITION 4. *This stratum of the discriminant is non-orientable (i.e. its intersection with any sufficiently large approximating space Γ_U is) if and only if there exists an element $\lambda \in \pi_1(M)$ such that*

a) λ destroys the orientation of M: $\langle w_1(M), \lambda\rangle \neq 0$;

b) the conjugation operator $T_\lambda : \pi_1(M) \to \pi_1(M)$, $T_\lambda(\cdot) = \lambda^{-1}(\cdot)\lambda$, either preserves both elements α and β, or permutes them. $\square$

Consider the connected sum $M^2 = K\#K$ of two copies of the Klein bottle, and an immersed curve in it with unique self-intersection on the "neck" of the connected sum, such that any of two obtained loops lies in its own summand of the connected sum and defines in it a basic loop destroying its orientation. Using the obvious identification $M^2 = M^2 \times \{0\}$, we will consider this curve as a loop in the manifold $M^3 = M^2 \times S^1$. It follows from the van Kampen theorem, that an element $\lambda \in \pi_1(M^3)$, satisfying the conditions of the previous proposition, does not exist. Hence our stratum defines a class in the integer cohomology group $H^1(\Omega_f M, \Omega_f M \setminus \Sigma)$.

It is easy to prove that the condition of Proposition 2 is satisfied for the containing this curve component of $\Omega_f M$, hence this class defines a knot invariant. There are infinitely many knots distinguished by this invariant. Namely, let ν be the "neck" cylinder connecting two summands K of M^2. Everywhere outside $\nu \times S^1$ our knots coincide with our immersed circle in $M^2 \times \{0\}$, and in $\nu \times S^1$ they coincide with 2-string braids going from one boundary component to the other and twisted arbitrarily many times.

1.6.2. *Coding order i invariants in non-orientable 3-manifolds and computation of their values.* Consider any immersion $\phi : S^1 \to M^3$ with i distinguished transverse double crossings. As usual, we can resolve all these crossings in 2^i locally different ways so that they become nonsingular knots. These 2^i resolutions can be obviously partitioned into two groups in such a way that any two neighboring resolutions (i.e., two resolutions, obtained from one another by one local surgery of fig. 2) belong to different groups.

Definition. The *supercoorientation* (or simply *s-orientation*) of the containing ϕ $[i]_M$-route I is the simultaneous choice of one of these two groups close to all points of the manifold $\{I\}$, depending continuously of these points.

For singular knots in oriented manifolds these s-orientations are defined canonically, see § 1.2. The problem of deciding whether an $[i]_M$-route in a non-orientable manifold satisfies this condition or not, is an independent problem, related in particular with its symmetry properties. If $\pi_1(M)$ is sufficiently complicated and the loops in M, lying in the representing this $[i]_M$-route curve, are "sufficiently independent" in (the set of conjugacy classes of) $\pi_1(M)$, then this route should be s-orientable.

Definition. The *global stratum* (or simply *stratum*) of Σ corresponding to an $[j]_M$-route is an irreducible component of the set of maps $S^1 \to M$, respecting this route and having transverse crossings at all its j double points. (This transversality condition is not very restrictive, indeed, the set of maps not satisfying it has codimension 2 in the set of all maps respecting this route.) Such a stratum contains an open smooth subset, consisting of maps, having no extra singular or multiple points; its path-components are called *small strata* or *pieces* of the global stratum.

Any knot invariant can be extended in the almost standard way (cf. § 1.2) to a function on immersions with arbitrarily many double transverse crossings and no other singularities. More precisely, this extension takes values on pairs of the form {such an immersion, a choice of the local s-orientation of its route}. It is equal to the sum of values of our invariant over all resolutions from the chosen group minus the similar sum over the remaining group; in particular it changes the sign if we change the s-orientation. The invariant is of order i if and only if its extension to all singular immersions with $> i$ crossings is equal to 0.

Any invariant of order i is encoded by the actuality table, having $i+1$ levels $0, 1, \dots, i$. The cells of this table on level $j \leq i$ are numbered by $[j]_M$-routes.

In any such cell we draw an immersion ϕ, representing this $[j]_M$-route I, having exactly j transverse crossings, and supplied with a certain s-orientation of the corresponding small stratum of I (these data do not depend on the invariant). To specify the invariant, we write in this cell a number: the index of order j of our invariant at this (s-oriented) singular immersion, cf. [V2], [V3].

Remark. We need to fill in the cells even for not s-orientable $[j]_M$-routes, however if already the corresponding small stratum is not s-coorientable, then the corresponding index will be equal to zero.

The calculation of values of invariants on knots consists essentially in the same inductive process as in § 4.3 of [V2] (see also §1.5.1 above), only with following modifications.

Suppose that we go along a smooth path in our stratum, consisting of immersions with j crossings, and at some instant traverse the stratum of $(j+1)$-crossed curves (i.e., at that instant our curve has the $(j+1)$-th selfintersection point). We need to compare three s-orientations: these of our $[j]_M$-route at some its point a_+ before traversing, after it (at the point a_- of the same global stratum), and the s-orientation of the $[j+1]_M$-route at the very point a_0 of traversing. Set $\alpha = 1$ if two first s-orientations are compatible in obvious way (i.e. the same perturbations of first j crossings will belong to the chosen groups independently on what is happening close to the $(j+1)$-st one). Otherwise set $\alpha = -1$.

Set $\beta = 1$ if the perturbations from the chosen group close to the $[j]$-stratum at the point a_+ belong also to the group chosen in correspondence with the s-orientation of the $[j+1]_M$-route at the point a_0. Otherwise set $\beta = -1$.

Finally, the value of the index (of order j) of our invariant at the point a_+ of the $[j]_M$-route is equal to that at the point a_-, taken with the coefficient α, plus the value of the index of order $j+1$ at the point a_0 of the $[j+1]_M$-route, taken with the coefficient β.

1.6.3. *Basic relations for non-orientable manifolds.* The basic relations defining the group $E_1^{-i,i}$ of the spectral sequence (see § 1.3) should be slightly modified in the case of non-orientable M.

First, our G-valued function on the set of $[i]_M$-routes should vanish on all non-s-orientable routes. The trivial relation stays unchanged, and in the 4-term relation our 6 perturbations should be taken with certain signs, depending on their s-orientations. To define them we need the following notion of the s-orientability of $\langle i \rangle_M$-routes.

Consider a *generic* map ϕ realizing some $\langle i\rangle_M$-route. In its small neighborhood the pair $(\Gamma_U, \Sigma\cap\Gamma_U)$ is diffeomorphic to the direct product of the $(\dim\Gamma - i - 1)$-dimensional real space and the pair ($\mathbb{R}^{i+1}$, the union of coordinate hyperplanes in $\mathbb{R}^{i+1}$). In particular in this neighborhood the discriminant locally separates the space of knots into 2^{i+1} octants.

Let us divide all these octants into two groups in such a way that any two neighboring octants belong to different groups.

Definition. The (local) s-orientation of the $\langle i\rangle_M$-route, containing ϕ, is a choice of one of these two groups.

The *global* s-orientation of the $\langle i\rangle_M$-route is its simultaneous local s-orientation at all its generic points, depending continuously of these points and compatible in the obvious way close to generic points, respecting $\langle i+1\rangle$-configurations. (I.e., if we move along a path in the $\langle i\rangle_M$-route and traverse the set of maps, having an additional selfintersection point, not participating in the definition of the $\langle i\rangle_M$-route, then local s-orientations, defined in the terms of resolutions of multiple points, participating in this definition, should not remark this traversing.)

The s-orientation (local or global) of an $\langle\tilde{i}\rangle_M$-route is just the s-orientation of the corresponding $\langle i\rangle_M$-route.

The s-orientation of a i^*_M-route $\{I\}$ at its point, having $i-1$ transverse self-intersections, is any local s-orientation of the $[i-1]_M$-route, obtained from I by forgetting about its singular point.

Let us fix any local s-orientation of our $\langle i\rangle_M$-route at its generic point ϕ. The union of all $[i]_M$-routes is represented close to ϕ by 6 locally different components, see fig. 3. Supply any of these components with a sign, equal to 1 or -1 depending on whether the restriction of this s-orientation of the $\langle i\rangle_M$-route at ϕ onto the set of 2^i local resolutions of singular knots from this component coincides with the own s-orientation of the corresponding $[i]_M$-route or not.

Then three **sums** of values of our G-valued function on perturbations 1 and 4 (respectively, 2 and 5, respectively, 3 and 6, see fig. 3), taken with these coefficients, should coincide.

1.6.4. *Higher obstructions to the integration.* Given an element $\gamma \in E_1^{-i,i}$ (i.e., a function on the set of s-oriented $[i]_M$-routes, satisfying the relations from § 1.6.3), its integration to an order i knot invariant with upper level γ (in particular obstructions to the existence of such an invariant) can be formulated in terms of the short spectral sequence, generalizing that from § 1.5.

Again, its non-trivial groups $E_r^{p,q}$ lie on only two lines $p+q=0$ and $p+q=1$, i.e. are of the form $E_r^{-i,i}$ or $E_r^{-i,i+1}$, $i \geq 0$.

Its group $E_0^{-i,i}$ is the space of G-valued functions on the space of s-oriented $[i]_M$- and $\langle i\rangle_M$-routes, taking opposite values on any route supplied with opposite s-orientations.

The group $E_0^{-i,i+1}$ is the sum of two groups $\tilde{E}_0^{-i,i+1}$ and $\breve{E}_0^{-i,i+1}$. The first of them is generated by all s-oriented $\langle\tilde{i}\rangle_M$- and i^*-routes. The second is defined by

$$\breve{E}_0^{-i,i+1} \equiv \prod_I H^1(\{I\}, sG), \tag{6}$$

summation over *all* $[i]_M$-routes I, where sG is the local system of groups, locally isomorphic to G and such that the monodromy over a loop in $\{I\}$, destroying the s-orientation, acts in the fibre as multiplication by -1.

The operator d^0 acts from $E_0^{-i,i}$ to the first summand $\tilde{E}_0^{-i,i+1}$, and its kernel is (naturally isomorphic to) the group $E_1^{-i,i}$ described in § 1.6.3. The construction of forthcoming operators d^r essentially repeats that from § 1.5.

Remark. All considerations and events from sections 1.6.2—1.6.4 are valid if M is orientable and coincide then with their standard versions, see [V2], [BN].

1.7. **Functoriality of spectral sequences.** If M' is a submanifold of M (of the same dimension), then there appears the natural homomorphism of our spectral sequences, $E_r^{p,q}(M) \to E_r^{p,q}(M')$. Indeed, the space of maps $S^1 \to M'$ is an open subset in the space of maps $S^1 \to M$. (In the framework of finitedimensional approximations from § 1.1, for the approximating set of the space $\Omega_f M'$ we can take the subset in Γ_U, consisting of maps, whose images belong to $\tau^{-1}(M')$.) This embedding induces the restriction homomorphism from the Borel–Moore homology group of the discriminant set of the former space to that for the latter one. This homomorphism can be extended naturally to spaces of resolutions of these discriminants and to any terms of their natural filtrations, thus inducing a homomorphism of spectral sequences, see § 2.3. The explicit form of these spectral sequences implies the following theorem.

For any $[i]_M$-route I denote by $\{I_{M'}\}$ the subspace in $\{I\}$, formed by maps, whose images belong to M'.

Theorem 1. *Let $M' \subset M$ be two three-dimensional oriented manifolds, and suppose that the identical embedding $M' \to M$ induces*

a) an isomorphism $\pi_1(M') \to \pi_1(M)$, and

b) for any $[i]_M$-route I an epimorphism (respectively, isomorphism) $H_1(\{I_{M'}\}) \to H_1(\{I\})$.

Then the group of finite-order invariants of knots (or, more generally, d-component links with any fixed d) in the manifold M' is canonically isomorphic to a quotient group of the similar group of invariants of knots or links in M (respectively, to all this group).

Indeed, our condition a) implies that for any i our embedding induces the natural isomorphism $E_1^{-i,i}(M) \to E_1^{-i,i}(M')$, and condition b) implies that all the forthcoming maps $E_r^{-i,i}(M) \to E_r^{-i,i}(M')$, $r > 1$, are epimorphic (respectively, isomorphic), so that the limit homology map $H^1(\Omega_f M, \Omega_f M \setminus \Sigma) \to H^1(\Omega_f M', \Omega_f M' \setminus \Sigma)$ also is epimorphic (respectively, isomorphic). Moreover, condition b), applied to the $[0]_M$-routes (i.e. path-components of spaces $\Omega_f M$ and $\Omega_f M'$) implies that the kernel of the map $H^1(\Omega_f M', \Omega_f M' \setminus \Sigma) \to H^1(\Omega_f M')$ is a quotient group of the similar kernel for M. $\square$

The isomorphism theorem from [L] follows immediately from this one, see [V4]. Indeed, we can take $M' = \mathbb{R}^3$, then our conditions a) and b) will be satisfied for all 2-connected 3-manifolds M.

Here is a slightly more general statement.

THEOREM 1$'$. *Let $M' \subset M$ be two three-dimensional oriented manifolds, such that*

a) for any $[i]_M$-route I the space $\{I_{M'}\}$ consists of at most one path-component (respectively, of exactly one), and

b) for any I such that $\{I_{M'}\}$ is non-empty, the map $H_1(\{I_{M'}\}) \to H_1(\{I\})$, induced by the identical embedding, is epimorphic (respectively, isomorphic).

Then the graded group of finite-order invariants of knots in M' is naturally isomorphic to a quotient group of the similar group of invariants of knots in M (respectively, to entire this group).

Indeed, condition a) ensures that the map $E_1^{-i,i}(M) \to E_1^{-i,i}(M')$ is epimorphic for all i; the rest of the proof is the same as for Theorem 1.

1.8. **First-order cohomology classes of knots in $\mathbb{R}^n$.** It is well-known that there are no first-order knot invariants in $\mathbb{R}^3$, see [V2]. However, the subgroup $F^*_{1,\mathbb{Z}_2} \subset H^*(\Omega_f \mathbb{R}^3 \setminus \Sigma, \mathbb{Z}_2)$ of all $\mathbb{Z}_2$-valued first-order cohomology classes is non-trivial: it has exactly two non-trivial components $F^1_{1,\mathbb{Z}_2} \simeq F^2_{1,\mathbb{Z}_2} \simeq \mathbb{Z}_2$.

The generator of the first of them can be defined as the linking number with the cycle in Σ, formed by all maps $\phi : S^1 \to \mathbb{R}^3$, gluing together some two *opposite* points of S^1; the generator of the group $F^2_{1,\mathbb{Z}_2}$ is just the square of this one.

More generally, the following statement holds.

THEOREM 2. *For any $n \geq 3$, the subgroup $F^*_{1,\mathbb{Z}_2} \subset H^*(\Omega_f \mathbb{R}^n \setminus \Sigma, \mathbb{Z}_2)$ of first-order cohomology classes of the space of knots in $\mathbb{R}^n$ contains exactly two non-trivial components $F^{n-2}_{1,\mathbb{Z}_2} \sim F^{n-1}_{1,\mathbb{Z}_2} \sim \mathbb{Z}_2$. The generator of the first of them is equal to the linking number with the set of maps gluing together some two opposite points of the circle. The generator of the second can be realized by the linking number with a similar variety, where these two opposite points are fixed, say are equal to 0 and π, and in the case of odd n is equal to the Bockshtein of the first generator.*

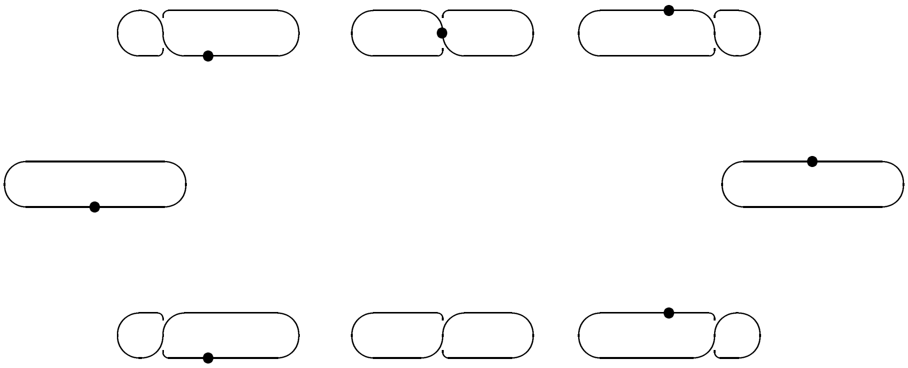


FIGURE 5. Non-trivial 1-cycle in the space of unknots

If n is even, then both these cohomology classes give rise to integer cohomology classes, i.e. $F^{n-2}_{1,\mathbb{Z}} \sim F^{n-1}_{1,\mathbb{Z}} \sim \mathbb{Z}$, and there are no other non-trivial integer cohomology groups $F^d_{1,\mathbb{Z}}$, $d \neq n-2, n-1$. $\square$

These cocycles in $\Omega_f\mathbb{R}^3 \setminus \Sigma$ are non-trivial already in restriction to the component of unknots. Indeed, consider the standard embedded circle in $\mathbb{R}^3$ and rotate it by all angles $\alpha \in [0, 2\pi]$ around any of its diameters. Then we obtain a nontrivial element of $H_1(\Omega_f\mathbb{R}^3 \setminus \Sigma, \mathbb{Z}_2)$, which takes non-zero value on the generator of the group $F^1_{1,\mathbb{Z}_2}$.

Indeed, let us realize this 1-cycle by the (obviously homotopic to it) family of unknots shown in fig. 5. Then span it by a disc in $\Omega_f\mathbb{R}^3$, swept out by the 1-parametric family of segments, connecting in the shortest way any two unknots of our family, placed in this picture one over the other, so that along any such segment the projection to $\mathbb{R}^2$ remains the same. It is obvious that the intersection number of this disc with the above-mentioned subvariety in Σ, generating the group $F^1_{1,\mathbb{Z}_2}$, is non-trivial (mod 2).

On the other hand, it is easy to prove that our 1-cycle in the space of unknots is homotopic there to the cycle, consisting of embeddings with the one and the same image, which are obtained one from the another by shifts of the cyclic parameter α.

Consider the space of naturally parametrized great circles in a sphere $S^2 \subset \mathbb{R}^3$. This space is obviously homeomorphic to $SO(3) \sim \mathbb{R}P^3$. The restriction on it of our generator of $F^1_{1,\mathbb{Z}_2}$ coincides with the generator of its $\mathbb{Z}_2$-cohomology ring, hence also its square (generating $F^2_{1,\mathbb{Z}_2}$) is nontrivial in restriction to this space.

2. CONSTRUCTION OF THE SPECTRAL SEQUENCE(S)

2.1. **Resolution spaces.** Denote by Ψ the space of all unordered pairs of points in S^1 (may be coinciding): $\Psi = S^1 \times S^1/\{\alpha \times \beta = \beta \times \alpha\}$. It is easy to see that Ψ is diffeomorphic to the closed Möbius band.

Let $\Upsilon : \Psi \to \mathbb{R}^\kappa$ be a generic embedding of Ψ into the space of a huge (may be finite) dimension. For any map $\phi : S^1 \to M$ of the class Γ_U, consider all the points $(\alpha, \beta) \in \Psi$ such that either $\alpha = \beta$ and $\phi(\alpha) = \phi(\beta)$, or $\alpha = \beta$ and $\phi' = 0$ at the point α. If Γ is not very degenerate, then the number of such points for any $\phi \in \Sigma \cap \Gamma_U$ is estimated from above by an uniform number (depending on Γ); we suppose that the dimension of $\mathbb{R}^\kappa$ is sufficiently large with respect to this number.

Consider all images $\Upsilon(\alpha, \beta) \in \mathbb{R}^\kappa$ of all such points for this ϕ. If κ is sufficiently large and the embedding Υ is generic, then all these points are vertices of a certain simplex in $\mathbb{R}^\kappa$; denote this simplex by $\Delta(\phi)$. Define the space $\sigma(\Gamma)$ as the subset in $\Gamma_U \times \mathbb{R}^\kappa$ swept out by all simplices of the form $\phi \times \Delta(\phi)$ over all $\phi \in \Sigma \cap \Gamma_U$.

Proposition 5 (cf. [V3]). *The map $\sigma(\Gamma) \to \Sigma \cap \Gamma_U$, defined by the obvious projection $\Gamma_U \times \mathbb{R}^\kappa \to \Gamma_U$, is proper, and the induced map*

$$\bar{H}_*(\sigma(\Gamma)) \simeq \bar{H}_*(\Sigma \cap \Gamma_U) \tag{7}$$

is an isomorphism. □

2.2. **Filtration and stratification of the resolution set.** The spaces $\sigma(\Gamma)$ have nice structures which allow (in principle) to calculate groups (7), namely, the filtration $F_1 \subset F_2 \subset \cdots$ (by the "complexities" of underlying singularities in Σ) and a decomposition of terms $F_i \setminus F_{i-1}$ of this filtration in correspondence with a certain classification of these singularities. These structures have two useful properties:

1) cohomology groups, associated with these objects, are functorial with respect to embeddings $\Gamma \subset \Gamma'$, see § 2.3 below;

2) these groups converge (in some weak sence) to cohomology groups of more or less standard topological spaces like the space of continuous maps of a given graph to M, see § 2.4.

The construction of these structures is based on the following classification of singular knots.

Definitions (cf. [V2], [V3]). Let $A = \{a_1, \dots, a_{\#A}\}$ be an arbitrary finite unordered collection of naturals, all whose members a_l are not less than 2; let b be an nonnegative integer. Denote by $|A|$ the sum of all numbers a_l. An *A-configuration* is any family of $|A|$ pairwise distinct points in S^1 partitioned into $\#A$ groups of cardinalities $a_1, \dots, a_{\#A}$ respectively. An *(A, b)-configuration* is a pair consisting of an A-configuration and an additional family of b pairwise distinct points in S^1 (some of which can coincide with points of the A-configuration). The map $\phi : S^1 \to M$ *respects* the (A, b)-configuration if it sends all points of any of its groups of cardinalities $a_1, \dots, a_{\#A}$ into one point in M, and $\phi' = 0$ at all points of its b-part. Two (A, b)-configurations are *equivalent* if they can be transformed into one another by an orientation-preserving homeomorphism of S^1.

For instance, the $[i]$-, $\langle i \rangle$- and i^*-configurations from § 1.3 are respectively the (A, b)-configurations with $A = (2, \dots, 2)$ (i twos), $b = 0$; $A = (3, 2, \dots, 2)$ ($i - 2$

twos), $b = 0$, and $A = (2, \dots, 2)$ ($i-1$ twos), $b = 1$ with the last point different from $2i-2$ points forming the A-part.

The *complexity* of an (A, b)-configuration is the number $|A| - \#A + b$, so that the codimension in $\Omega_f M$ of the set of respecting it maps is equal to $\dim M$ times this number.

For any equivalence class $\mathbf{J}$ of (A, b)-configurations, $\delta(\mathbf{J})$ is the dimension of this class, i.e. the number of geometrically distinct points in any configuration J of this class.

An (A, b, M)-*configuration* is any pair, consisting of an (A, b)-configuration in S^1 and some $\#A + b$ points $m_1, \dots, m_{\#A+b}$ in M. A map $\phi : S^1 \to M$ *respects* such a configuration, if it sends any group of a_l points, participating in the definition of the A-part of the configuration, into the point m_l, sends any point v_j, participating in the definition of the b-part, into $m_{\#A+j}$, and $\phi'(v_j) = 0$ for any such point v_j. The configuration is *acceptable*, if it can be respected by at least one map (this means that if some of points v_j coincide with the points of the A-part, then the corresponding points m_* also coincide).

Definition. An affine finite-dimensional subspace Γ of the space of smooth maps $S^1 \to \mathbb{R}^N$ is (M, d)-*nondegenerate*, if

a) for any acceptable (A, b, M)-configuration of complexity $\leq d$, the set of maps $\phi \in \Gamma_U$, respecting this configuration, is a smooth submanifold in Γ_U, and differentials of all $n(|A| + 2b)$ conditions distinguishing this manifold (i.e. of conditions $\phi(x_1) = \dots = \phi(x_{a_1}) = m_1, \dots, \phi(v_1) = m_{\#A+1}, \phi'(v_1) = 0, \dots$) are linearly independent at any its point;

b) for any equivalence class $\mathbf{J}$ of (A, b)-configurations of arbitrary complexity, the codimension in Γ_U of the set of maps, respecting some configurations of this class, is not less than $n(|A| - \#A + b) - \delta(\mathbf{J})$.

Proposition 6. *For any d, (M, d)-nondegenerate spaces exist and are dense in the space of all affine subspaces of sufficiently large dimension in $C^\infty(S^1, \mathbb{R}^N)$.*

Indeed, for any natural D and any (A, b, M)-configuration, the set of D-dimensional subspaces, not satisfying condition a) at this configuration, is a subvariety in the space of all D-dimensional subspaces in $\Omega_f \mathbb{R}^N$. The codimension of this subvariety grows to infinity together with D, in particular for large D becomes greater than the dimension of the space of all (A, b, M)-configurations with given A and b. Condition b) follows from the Thom transversality theorem, cf. [V3]. $\Box$

Definition. For any (A, b)-configuration J, the simplex $\Delta(J) \subset \mathbb{R}^\kappa$ is defined as the simplex $\Delta(\phi)$ for any generic $\phi \in \Omega_f M$ respecting J (i.e. having no extra singularities).

The number of vertices of this simplex is equal to $\sum_{j=1}^{\#A} \binom{a_j}{2} + b$.

Definition. Given any equivalence class $\mathbf{J}$ of (A, b)-configurations, the corresponding $\mathbf{J}$-*block* $B(\mathbf{J}, \Gamma)$ is the union of all points $(\phi, \zeta) \in \sigma(\Gamma) \subset \Gamma_U \times \mathbb{R}^\kappa$ such that for some (A, b)-configuration $J \in \mathbf{J}$

a) ϕ respects J, and

b) ζ belongs to the simplex $\Delta(J)$.

The term F_i of the *main filtration* of $\sigma(\Gamma)$ is defined as the union of all $\mathbf{J}$-blocks over all $\mathbf{J}$ of complexity $\leq i$.

By definition, $B(\mathbf{J}, \Gamma)$ consists of simplices $\zeta \times \Delta(J) \sim \Delta(J)$, $J \in \mathbf{J}$. Any such simplex lies in F_i (where i is the complexity of $\mathbf{J}$), but some of its points belong to F_{i-1}. These points constitute several faces of $\Delta(J)$. Namely, any face of $\Delta(J)$ is characterized by the collection of its vertices, i.e. by a collection of $\#A$ graphs with $a_1, \dots, a_{\#A}$ vertices respectively, and a choice of some of b "singular" points of the configuration. The faces lying in F_{i-1} are exactly those that either one of corresponding graphs is not connected, or at least one of b points is missed in this choice, cf. [V2].

Proposition 7 (see e.g. [V3]). *For any (A, b)-configuration J of complexity i, the group $\bar{H}_*(\Delta(J) \setminus F_{i-1})$ is trivial in all dimensions other than $\sum_{j=1}^{\#A}(a_j - 1) + b - 1 \equiv i - 1$, and in dimension $i-1$ it is isomorphic to*

$$\otimes_{j=1}^{\#A} \mathbb{Z}^{(a_j - 1)!} \tag{8}$$

(in particular to $\mathbb{Z}$ if all a_j are equal to 2).

For $\mathbf{J}$ of complexity i denote by $\tilde{B}(\mathbf{J}, \Gamma)$ the "pure part" $B(\mathbf{J}, \Gamma) \setminus F_{i-1}$ of the $\mathbf{J}$-block $B(\mathbf{J}, \Gamma)$. By the construction, it is the space of a fiber bundle, whose base is the space of all pairs of the form $\{$an (A, b)-configuration $J \in \mathbf{J}$, a map $\phi \in \Gamma_U$ respecting $J\}$, and the fiber over such a point is the set of interior points of faces of the simplex $\Delta(J)$, not belonging to F_{i-1}, so that the Borel–Moore homology group of the fiber is described by Proposition 7.

Denote by $\beta(\mathbf{J}, \Gamma)$ the base of this fiber bundle, and by $\beta(\mathbf{J})$ the space of similar pairs $\{J, \phi\}$ over all $J \in \mathbf{J}$ and all $\phi \in \Omega_f M$ respecting J (and not only over such $\phi \in \Gamma_U$).

Proposition 8. *1. For any equivalence class $\mathbf{J}$ of (A, b)-configurations of complexity d and any homology class $\xi \in H_*(\beta(\mathbf{J}))$ (with coefficients in any local system of groups) there exists a (M, d)-nondegenerate space Γ such that ξ can be realized by a cycle belonging to $\beta(\mathbf{J}, \Gamma)$.*

2. If two such cycles in $\beta(\mathbf{J}, \Gamma)$ define the same homology class in $\beta(\mathbf{J})$, then there exists a (M, d)-nondegenerate space Γ', containing Γ, such that these cycles are homological already in $\beta(\mathbf{J}, \Gamma')$.

Proof. This follows from the Weierstrass approximation theorem: it is sufficient to take weakly moved spaces of maps $S^1 \to \mathbb{R}^N$ given by trigonometric polynomials of sufficiently large degrees. $\square$

DEFINITION. A sequence of finitedimensional affine subspaces $\Gamma^1 \subset \Gamma^2 \subset \cdots$ in $\Omega_f \mathbb{R}^N$ is *exhausting* if

a) for any d almost all its terms Γ^j (i.e. all except may be for finitely many) are (M, d)-nondegenerate;

b) for any $\mathbf{J}$ and any class $\xi \in H^*(\beta(\mathbf{J}))$, the condition 1) of the previous proposition is satisfied for almost all Γ^j;

c) for any term Γ^j of this sequence and any two cycles $\xi, \zeta \in H_*(\beta(\mathbf{J}, \Gamma^j))$, defining the same element of $H_*(\beta(\mathbf{J}))$, these cycles are homological in almost all spaces $\beta(\mathbf{J}, \Gamma^k)$, $k \geq j$.

Proposition 8 implies that such sequences exist.

2.3. **Stabilization of spectral sequences.** For any subspace $\Gamma_U \subset \Omega_f \mathbb{R}^N$, consider the homological spectral sequence $E^r_{p,q}(\Gamma)$, converging to the group $\bar{H}_*(\sigma(\Gamma)) \equiv \bar{H}_*(\Sigma \cap \Gamma_U)$ and defined by the main filtration of $\sigma(\Gamma)$, described in the previous subsection. By definition, $E^1_{p,q}(\Gamma) \simeq \bar{H}_{p+q}(F_p(\sigma(\Gamma)) \setminus F_{p-1}(\sigma(\Gamma))$.

Using the formal inversion (2), we convert it to the *cohomological* spectral sequence $E_r^{p,q}(\Gamma) \to H^{p+q+1}(\Gamma_U, \Gamma_U \setminus \Sigma)$.

For any d, denote by ${}_dE_r^{p,q}(\Gamma)$ the truncated spectral sequence, obtained from the previous one by replacing by 0 all terms ${}_dE_1^{p,q}$ with $p < -d$. It converges to the Borel–Moore homology group of $F_d(\sigma(\Gamma))$: ${}_dE_r^{p,q}(\Gamma) \to \bar{H}_{\dim \Gamma - p - q - 1}(F_d(\sigma(\Gamma)))$.

Let $\Gamma \subset \Gamma'$ be two (M, d)-nondegenerate subspaces in $\Omega_f \mathbb{R}^N$. Then there is a natural homomorphism

$$ {}_dE_r^{p,q}(\Gamma') \to {}_dE_r^{p,q}(\Gamma). \tag{9} $$

Indeed, by the definition of (M, d)-nondegeneracy $F_d(\sigma(\Gamma))$ admits in $F_d(\sigma(\Gamma'))$ a tubular neighborhood, homeomorphic to the direct product of $F_d(\sigma(\Gamma))$ and an open $(\dim \Gamma' - \dim \Gamma)$-dimensional disc, in such a way that this homeomorphism preserves natural filtrations of both spaces.

The homomorphism (9) is defined as the composition of the restriction on this neighborhood and the Künneth formula in it.

The *stable spectral sequence* $E_r^{p,q}$ is defined by $E_r^{p,q} \equiv \lim \operatorname{ind} {}_dE_r^{p,q}(\Gamma^j)$ over such homomorphisms for any exhausting sequence of approximating spaces Γ^j.

It is easy to see that it does not depend on the choice of this sequence of spaces and converges to some subgroup in $H^*(\Omega_f M, \Omega_f M \setminus \Sigma)$.

PROPOSITION 9. *The support of the stable sequence $E_r^{p,q}$ (i.e. the set of such p, q that $E_1^{p,q} \neq 0$) belongs to the wedge from fig. 1: $p < 0, q + (n-2)p \geq 0$.*

Moreover, the same is true for any non-stable spectral sequence $E_r^{p,q}(\Gamma)$ with any (M,d)-nondegenerate Γ.

Proof. By Proposition 7, for any equivalence class $\mathbf{J}$ of (A,b)-configurations of complexity i and any element Γ^j of our exhausting sequence, the contribution of the block $\tilde{B}(\mathbf{J},\Gamma^j) \subset \sigma(\Gamma^j)$ into the group $E_1^{-i,q} \simeq \bar{H}_{\dim\Gamma-q+i-1}(F_d \setminus F_{d-1})$ can be nontrivial only if $\dim\Gamma - q + i - 1 \leq \dim\beta(\mathbf{J},\Gamma) + \sum_{k=1}^{\#A}(a_k-1) + b - 1 \leq \dim\Gamma - ni + 2i + i - 1$; the summand $2i$ in the last expression is the upper estimate for the dimension $\delta(J)$ of the space of (A,b)-configurations of the class $\mathbf{J}$. This implies the first statement of the proposition. The second statement follows in the same way from condition b) of the definition of (M,d)-nondegenerate spaces. $\square$

In the next subsection we show that such stable sequences are not too wild and huge.

2.4. **On the calculation of the term $E_1^{p,q}$ of the stable spectral sequence.** Let us fix a certain natural i.

By the definition of spectral sequences,

$$E_1^{-i,q}(\Gamma) \simeq \bar{H}_{\dim\Gamma+i-q-1}(F_i \setminus F_{i-1}), \tag{10}$$

where $F_i \equiv F_i(\sigma(\Gamma))$. The space $F_i \setminus F_{i-1}$ splits into pure $\mathbf{J}$-blocks $\tilde{B}(\mathbf{J},\Gamma)$ with $\mathbf{J}$ of complexity i. As in [V2], [V3], we introduce the *auxiliary filtration* in it, defining its term Φ_α as the union of all blocks $\tilde{B}(\mathbf{J},\Gamma)$ such that $\delta(\mathbf{J}) \leq \alpha$, i.e. the configuration $\mathbf{J}$ consists of $\leq \alpha$ geometrically distinct points. Let $\mathcal{E}^\rho_{\mu,\nu}(\Gamma,i)$ be the spectral sequence, converging to the group $\bar{H}_*(F_i \setminus F_{i-1})$ and generated by this filtration. Its term $\mathcal{E}^1_{\mu,\nu}$ is the direct sum of groups $\bar{H}_{\mu+\nu}(\tilde{B}(\mathbf{J},\Gamma))$ over all $\mathbf{J}$ of complexity i and $\delta(\mathbf{J}) = \mu$.

The stabilization of this spectral sequence over growing Γ is defined in the same way as for the main spectral sequence and allows us to define the stable cohomological spectral sequence $\mathcal{E}_\rho^{a,b}(i) \equiv \lim\operatorname{ind} \mathcal{E}^\rho_{-a,\dim\Gamma^k-b-1}(\Gamma,i)$ over any exhausting sequence $\{\Gamma^k\}$.

PROPOSITION 10. *The stable auxiliary spectral sequence converges to the term E_1 of the stable main spectral sequence. Namely, its group $\oplus_{a+b=t}\mathcal{E}_\infty^{a,b}(i)$ is adjoined to the group $E_1^{-i,i+t}$.* $\square$

On the other hand, the stable member $\mathcal{E}_1^{a,b}(i)$ of this sequence can be expressed in terms of cohomology groups of spaces $\beta(\mathbf{J})$. Indeed,

a) this term splits into the direct sum of certain homology groups, associated with all $\mathbf{J}$ with complexity i and $\delta(\mathbf{J}) = -a$;

b) namely, for any such $\mathbf{J}$ the corresponding summand is the stabilization of groups $\bar{H}_{\dim\Gamma^k-a-b-1}(B(\mathbf{J},\Gamma^k))$;

c) since $B(\mathbf{J},\Gamma^k)$ is a fiber bundle with base $\beta(\mathbf{J},\Gamma^k)$ and fiber described in Proposition 7, these stable homology groups are isomorphic to the stabilization of $(\dim\Gamma^k -$

$a-b-i$)-dimensional Borel–Moore homology groups of $\beta(\mathbf{J}, \Gamma^k)$ with coefficients in a certain local system Ξ with fiber (8);

d) by the definition of (M,d)-nondegeneracy, all spaces $\beta(\mathbf{J}, \Gamma^k)$ with sufficiently large k are smooth $(\dim \Gamma^k - a - n\cdot i)$-dimensional manifolds, therefore by the Poincaré duality theorem previous homology groups are isomorphic to groups

$$H^{b-(n-1)i}(\beta(\mathbf{J}, \Gamma^k), \Xi^* \otimes Or(\mathbf{J}, k)),$$

where $Or(\mathbf{J}, k)$ is the orientation sheaf of the manifold $\beta(\mathbf{J}, \Gamma^k)$;

e) these sheaves $Or(\mathbf{J}, k)$ stabilize to a common sheaf $Or(\mathbf{J})$ on $\beta(\mathbf{J})$ (i.e., they are preserved by all inclusions $\beta(\mathbf{J}, \Gamma^k) \to \beta(\mathbf{J}, \Gamma^l)$ with sufficiently large $k < l$);

f) and finally we obtain from Proposition 8 that the summand in $\mathcal{E}_1^{a,b}(i)$, corresponding to the class $\mathbf{J}$, is isomorphic to

$$H^{b-(n-1)i}(\beta(\mathbf{J}), \Xi^* \otimes Or(\mathbf{J})). \tag{11}$$

Certainly, it splits into the direct product of similar groups over all path-components of $\beta(\mathbf{J})$, i.e., over the $\mathbf{J}_M$-*routes*, cf. § 1.3.

Example (cf. § 1). Let $n=3$. To calculate the groups $E_r^{-i,i}$, we need to consider only the groups $\mathcal{E}_\infty^{a,b}$ with $a+b=0$ or $a+b=1$. First suppose that $a+b=0$. Since $a \geq -2i$, we have $b \leq 2i$, thus the group (11) can be nontrivial only if $b = 2i$, i.e., $a=-2i, \mu=2i$, and $\mathbf{J}$ is a class of $[i]$-configurations. It is easy to calculate that for such $\mathbf{J}$ the sheaf $\Xi^* \otimes Or(\mathbf{J})$ is isomorphic to $\mathbb{Z}$ if M is orientable, and coincides with the sheaf $s\mathbb{Z}$ from formula (6) otherwise.

Thus the unique group $\mathcal{E}_1^{a,b}$ with $a+b=0$ is the group $\mathcal{E}_1^{-2i,2i} \equiv H^0(\beta(\mathbf{J}), s\mathbb{Z})$, formally generated by s-orientable $[i]_M$-routes.

Similarly, the only two non-trivial groups $\mathcal{E}_1^{a,b}$ with $a+b=1$ are the group $\mathcal{E}_1^{-2i,2i+1}$, which is just the group (6) (or (4) if M is orientable) and the group $\mathcal{E}_1^{-2i+1,2i}$, which is the direct sum of certain 0-dimensional homology groups of $\langle i\rangle_M$- or i^*-routes. In the case of $\langle i\rangle_M$-routes, the coefficient sheaf Ξ^* is locally isomorphic to $\mathbb{Z}^2$ and consists of $\mathbb{Z}$-linear combinations of corresponding $\langle\tilde{i}\rangle_M$-routes factorized through the diagonal, consisting of such combinations with coinciding coefficients.

Concluding remark. All the above considerations can be word-for-word carried out to the case of d-component links in M with any fixed d: in the basic construction instead of the space of maps of one circle to M (or to $\mathbb{R}^N$) we need only to consider the space of similar maps of the disjoint union of d circles, cf. [S]

References

[A1] V. I. Arnold, *On a characteristic class entering into conditions of quantization,* Funct. Anal. and its Appl., 1:1, 1967, 1–15.

[A2] V. I. Arnold, *On some topological invariants of algebraic functions,* Trudy Moskov. Mat. Obshch. 1970, 21, 27–46. Engl. transl.: Transact. Moscow Math. Soc., 1970, 21, 30–52.

[A3] V. I. Arnold, *Plane curves, their invariants, perestroikas and classifications,* Adv. in Sov. Math., AMS, Providence RI, vol.21, 1994, 33–91.

[BL] J. Birman, X.-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math., 1993, 111, 225–270.

[BN] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology, 1995, 34:2, 423–472.

[G] V. Goryunov, *Finite order invariants of framed knots in a solid torus and in Arnold's J^+-theory of plane curves*, Preprint, Univ. of Liverpool, 1995.

[K] E. Kalfagianni, *Finite type invariants for knots in 3-manifolds*, preprint.

[L] X.-S. Lin, *Finite type link invariants of 3-manifolds*, Topology 33:1, 1994, 45–71.

[M] A. B. Merkov, *On classification of ornaments,* Adv. in Soviet Math., vol. 21, Providence RI, 1994, 199–211.

[S] T. Stanford, *Finite-type invariants of knots, links, and graphs*, Topology, 1996, 35:4, 1027–1050.

[V1] V. A. Vassiliev, *Lagrange and Legendre Characteristic Classes, 2-d edition,* Gordon and Breach Publ., 1993.

[V2] V. A. Vassiliev, *Cohomology of knot spaces,* Advances in Soviet Math., Vol. 1, Providence RI, 1990, 23–69.

[V3] V. A. Vassiliev, *Complements of Discriminants of Smooth Maps: Topology and Applications*, revised ed., AMS, Transl. of Math. Monogr., vol. 98, Providence RI, 1994.

[V4] V. A. Vassiliev, *Topology of discriminants and their complements*, Proc. of the Intern. Congress of Math., Zürich 1994; Birkhäuser 1995, 209–226.

[V5] V. A. Vassiliev, *Invariants of ornaments,* Advances in Sov. Math., Vol. 21, AMS, Providence RI, 1994, 225–262.

[V6] V. A. Vassiliev, *Topology of complements of discriminants,* Moscow, Phasis, 1997 (to appear, in Russian).

[W] M. Weiss, *Calculus of embeddings,* Bull. AMS, 33:2, 1996, 177–187.

Steklov Math. Inst., Gubkina st. 8, 117966 Moscow, RUSSIA
Current address: MSRI, 1000 Centennial Drive, Berkeley, CA 94720, USA
E-mail address: `vassil@vassil.mccme.rssi.ru`