A COMBINATORIAL PROOF OF A RESULT OF HETYEI AND REINER ON FOATA-STREHL TYPE PERMUTATION TREES

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Abstract. We give a combinatorial proof of the known [4] result that there are exactly $n!/3$ permutations of length n in the minmax tree representation of which the ith node is a leaf. We also prove the new result that the number of n-permutations in which this node has one child is $n!/3$ as well, implying that the same holds for those in which this node has two children.

1. Background and definitions

In [4] the authors gave a new proof for a theorem of Purtill [5] by introducing a new $\mathbb{Z}_2^{n-1}$-action on the symmetric group S_n. This had the flavor of a $\mathbb{Z}_2^n$-action defined and succesfully used by Foata and Strehl [2] [3], but it proved to have a more symmetric structure. In order to define their new group action, Hetyei and Reiner studied a binary tree representation of the permutation $p = (p_1, p_2, \cdots, p_n)$ called *minimum-maximum trees* or, in the rest of this paper, *minmax* trees. These are defined as follows.

Definition 1. *Let p be a permutation of length n and let $p = u\,m\,v$ where m is the leftmost of the minimum and maximum letters of p, u is the subword preceding m and v is the subword following m. The* minmax tree *T_p^m has m as its root. The right subtree of T_p^m is obtained by applying the definition recursively to v. Similarly, the left subtree of T_p^m is obtained by applying the definition recursively to u.*

So every node of T_p^m which is not a leaf is either a minimum node or a maximum node. Useful information can be read off these trees, for example, it can be shown that p is an André-permutation if and only if all these nodes are minimum nodes.

The minmax tree of the permutation $p = 3\,6\,7\,1\,5\,2\,10\,4\,9\,8$ is shown on Figure 1.

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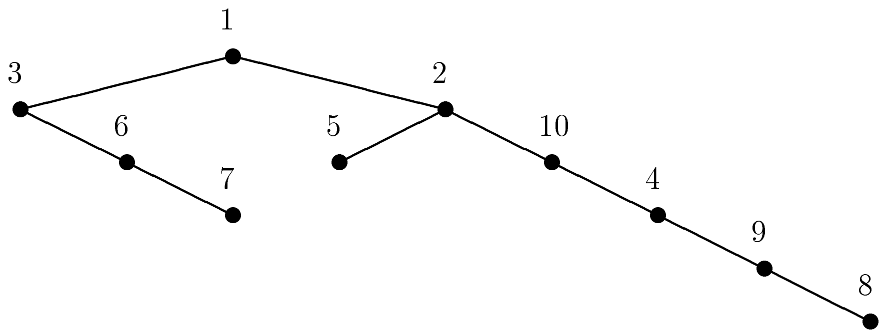

Figure 1: T_p^m for $p = 3\,6\,7\,1\,5\,2\,10\,4\,9\,8$.

If $a = a_1, a_2, \cdots, a_k$ is a subword of the n-permutation p, then the *pattern* of a is the unique k-permutation for which the pairwise comparisons between the ith and jth entries are the same as those for a. For example, the pattern of the subword $3\,6\,7\,1$ is $2\,3\,4\,1$, while the paattern of the subword $4\,9\,8$ is $1\,3\,2$.

The group action defined by Hetyei and Reiner is as follows: Let $1 \leq i \leq n$. Then $\psi_i(T_p^m)$ is obtained from T_p^m by changing only the subtree whose root is the entry p_i. If p_i was a minimum node, then we take the *maximal* entry among all entries of this subtree, put it into the place of p_i (so to the root of this subtree), and write the other entries of the subtree to the nodes of the subtree so that their *pattern* is the same as it was before the operation. Similarly, if p_i was a maximum node, then we put the *minimal* among the entries of its subtree to the place of p_i, then we write the other entries in the subtree to the nodes so that their pattern is unchanged.

Figure 2 shows the image of our previuos example p, under the operator ψ_7.

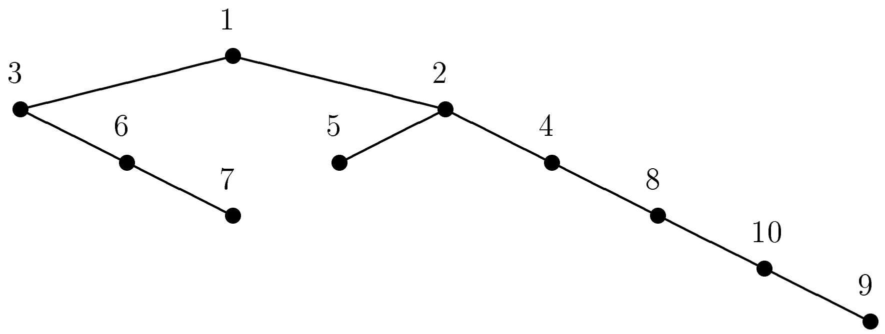

Figure 2: $\psi_7(T_p^m)$ for $p = 3\,6\,7\,1\,5\,2\,10\,4\,9\,8$.

It is clear that $\psi_i(T_p^m) = T_p^m$ if and only if p_i is a leaf in T_p^m. Hetyei and Reiner proved that any for any $i \leq n-2$, p_i is a leaf in exactly $n!/3$ permutations of length n (in the rest of this paper, n-permutations), thus ψ_i fixes $n!/3$ n-permutations. This is a little surprising: it is not even intuitively obvius why this number should be independent from i, and indeed, in the original group action of Foata and Strehl, it was not. Hetyei and Reiner used exponential generating functions and partial differential equations to get this result, however they noted that such a simple formula would deserve a combinatorial proof.

Present paper provides such a proof. We also prove the new result that the number of permutations in which p_i has 1 child as well as the number of n-permutations in which p_i has two children is $n!/3$, too, whenever $2 \leq i \leq n-2$. Thus this group action has even more symmetries, which supports the inventors' claim that it could be used in analysing random permutations.

2. The Proof of the Theorem

First we make some simple observations which will be useful later. For any $i \leq n-1$, either p_i is an ancestor of p_{i+1}, or p_{i+1} is an ancestor of p_i, for otherwise these two entries would have a common ancestor, which would have an index *between* i and $i+1$, a contradiction. So in particular, no permutation's minmax tree can have both p_i and p_{i+1} as leaves. This allows the following definition.

Definition 2. *Let $1 \leq i \leq n-2$. Then the* ith local extremum *of a permutation p is the entry which is closest to the root of the minmax tree of p among p_i, p_{i+1} and p_{i+2}. This entry will be denoted e_i.*

Being closest to the root means having the shortest path to the root in this definition. Note that e_i always exists: either p_{i+1} is an ancestor of both of its neighbors and $e_i = p_{i+1}$, or only one of its neighbors, say p_i is an ancestor of p_{i+1} and p_{i+2} is a descendant of p_{i+1}, implying $e_i = p_i$, or both of its neighbors are ancestors of p_{i+1}, in which case, by the tree-property, one must be an ancestor of the other.

Lemma 1. *The number of n-permutations whose minmax tree contains p_1 as a leaf is $n!/3$.*

Proof. By symmetry, we can suppose that the entry 1 of our permutation precedes the entry n. Indeed, the complement of p (the permutation whose ith entry is $n+1-p_i$) has an isomorphic minmax tree to T_p^m.

First we consider the case when the entry 1 is among the first three elements, so in particular, $e_1 = 1$. This gives rise to three subcases:

- If $p_1 = 1$, then p_1 is the root of the minmax tree.
- If $p_2 = 1$, then p_2 is the root, and p_1 is a leaf.
- If $p_3 = 1$, then p_3 is the root, its left subtree has p_1 and p_2 as nodes, and among these, by definition, p_2 is a leaf, and p_1 is not.

Clearly, these subcases are equally likely to occur, so each of them occurs with probability $1/3$.

Now suppose that the entry of the permutation is not among the first 3 elements. This entry is the root of the minmax tree of the permutation, and its left subtree has at least three nodes. Let these nodes be $b_1 < b_2 < \cdots < b_k$. Repeat the previous argument for this subtree, with b_1 playing the role of 1 and get that whenever b_1 is among the leftmost three elements of the permutation, p_1 is a leaf with probability 1/3.

Iterate this algorithm as long as it is necessary. It will eventually stop because we either get a left subtree of size three, or a subtree whose minimal entry is among the first three ones. In every subcase, p_1 is a leaf with probability 1/3, completing the proof of the lemma. □

We will use the same methods to prove the general theorem.

Theorem 1. *The number of n-permutations containing p_i as a leaf is $n!/3$, when $1 \leq i \leq n-2$. If $i = n-1$, then p_i is never a leaf, whereas if $i = n$, then p_i is always a leaf.*

Proof. Clearly, p_n is always a leaf because it cannot be the leftmost in any comparison, thus it cannot have descendants. Similarly, p_n is always the child of p_{n-1}, thus p_{n-1} is never a leaf.

Now let $1 \leq i \leq n-2$. Again, we can suppose that 1 precedes n in our permutations. First we consider the case when $1 \in \{p_i, p_{i+1}, p_{i+2}\}$.

- If $p_i = 1$, then p_i is the root of the minmax tree.
- If $p_{i+1} = 1$, then p_{i+1} is the root, so p_i is the rightmost element of its left subtree, and as such, it is necessarily a leaf.
- If $p_{i+2} = 1$, then p_{i+2} is the root, p_i is the next-to-last element of the root's left subtree, and as such, it is always an internal node (having the leaf p_{i+1} for its only child).

Again each of these subcases occurs with probability 1/3.

If $1 \notin \{p_i, p_{i+1}, p_{i+2}\}$, then we can proceed as we did in proof of Lemma 1: look for the entry 1 of the permutation, then only consider the subtree which contains the positions $i, i+1$ and $i+2$. If 1 was not in any of these positions, then all the three of them are in the same subtree. Iterating this algorithm we eventually reach a subtree where we can apply the above method. The structure of the other subtrees do not influence whether p_i is a leaf or not, so p_i is a leaf with probability 1/3. □

Corollary 1. *Let $n \geq 4$ and let $d_{i,j}$ be the number of n-permutations whose entry p_i has exactly j children. Then $d_{i,j} = n!/3$ for $2 \leq i \leq n-2$ and $j \in \{0, 1, 2\}$.*

Proof. For $j = 0$, this is just the statement of the Theorem. For $j = 2$, note that a node p_{i+1} has two children if and only if it is the local extremum e_i, and this happens exactly in the second subcase of the proof of the theorem, thus its probability is 1/3. Similarly, for $j = 1$, p_{i+1} must be the last element of the left subtree of the local extremum e_i, thus $e_i = p_{i+2}$ must hold, which is just the third subcase. □

As p_1 cannot have two children, and p_{n-1} cannot be a leaf, we also have $d_{1,1} = 2n!/3$ and $d_{n-1,1} = 2n!/3$. This latter is true since p_{n-1} has two children if and only if p_{n-2} is a leaf.

We note that there exists a natural bijection between the set of minmax trees and that of min1-min2 trees (in these trees the root is the leftmost of the minimal and the second minimal element) and thus our results hold for the min1-min2 trees as well.

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