ON A COMPUTER RECOGNITION OF 3-MANIFOLDS

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Abstract. We describe theoretical backgrounds for a computer program that recognizes all closed orientable 3-manifolds up to complexity 8. The program can treat also not necessarily closed 3-manifolds of bigger complexities, but here some unrecognizable (by the program) 3-manifolds may occur.

Introduction

Let M be an orientable 3-manifold such that ∂M is either empty or consists of tori. Then, modulo W. Thurston geometrization conjecture [Scott 1983], M can be decomposed in a unique way into graph-manifolds and hyperbolic pieces. The classification of graph-manifolds is well-known [Waldhausen 1967], and a list of cusped hyperbolic manifolds up to complexity 7 is contained in [Hildebrand, Weeks 1989]. If we possess an information how the pieces are glued together, we can get an explicit description of M as a sum of geometric pieces. Usually such a presentation is sufficient for understanding the intrinsic structure of M; it allows one to label M with a *name* that distinguishes it from all other manifolds.

We describe theoretical backgrounds and a general scheme of a computer algorithm that realizes in part the procedure. Particularly, for all closed orientable manifolds up to complexity 8 (all of them are graph-manifolds, see [Matveev 1990]) the algorithm gives an exact answer.

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1. Special and almost special spines

Let Δ denote the underlying space of the one-dimensional skeleton of the 3-simplex, i.e. the polyhedron homeomorphic to a circle with three radii.

Definition. A compact polyhedron P is called a *fake surface* if the link of each of its points is homeomorphic to one of the following one-dimensional polyhedra:

(1) a circle; (2) a circle with a diameter; (3) a circle with three radii (i.e. Δ).

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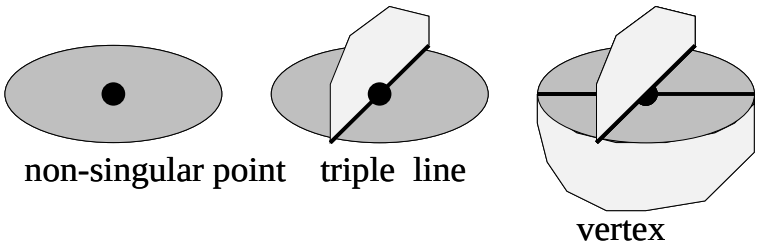

FIGURE 1.

Typical neighborhoods of points of a fake surface are shown in Fig. 1, where these points are shown as fat dots.

The union of singular points (i.e. vertices and triple lines) of a fake surface P is called its *singular graph* (and is denoted by SP). Connected components of $P \setminus SP$ are called *2-components* of P. Each 2-component is a 2-manifold without boundary.

Definition. A compact polyhedron P is called *almost special* if it can be embedded in a fake surface.

There is a close relation between fake surfaces and almost special polyhedra. For example, the wedge of any fake surface and any graph is an almost special polyhedron. The example is very typical, since any almost special polyhedron can be collapsed onto a polyhedron of the form $F \cup G$, where F is a collection of disjoint fake surfaces, G is a graph, and $F \cap G$ is a finite set of non-vertex points in F.

Definition. A fake surface P is said to be a *special polyhedron* if it contains at least one vertex and if all its 2-components are 2-cells.

Note that there are finitely many special polyhedra with a given number of vertices.

Definition. A subpolyhedron $P \subset \text{Int } M$ of a compact 3-manifold M with a non-empty boundary is said to be its *spine* if M collapses to P or, equivalently, if $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$.

The spine is called almost special or special if it is a polyhedron of the corresponding type.

We always assume that an almost special spine can not be collapsed onto a proper subpolyhedron.

By a spine of a closed manifold M we mean a spine of the punctured M, i.e. $M \setminus \text{Int } D^3$. It is known that any compact 3-manifold possesses an almost special (even a special) spine. Moreover, one can easily construct a special spine of M starting from practically any presentation of M [Matveev 1990]. Special spines possess an important property that favorably distinguishes them from fake surfaces and almost special spines: a 3-manifold can be uniquely recovered from its special spine. Note that special spines can be considered as combinatorial objects and admit presentations in computer's memory as strings of integer numbers that show how 2-cells are attached to singular graphs of spines. To present a manifold by its almost special

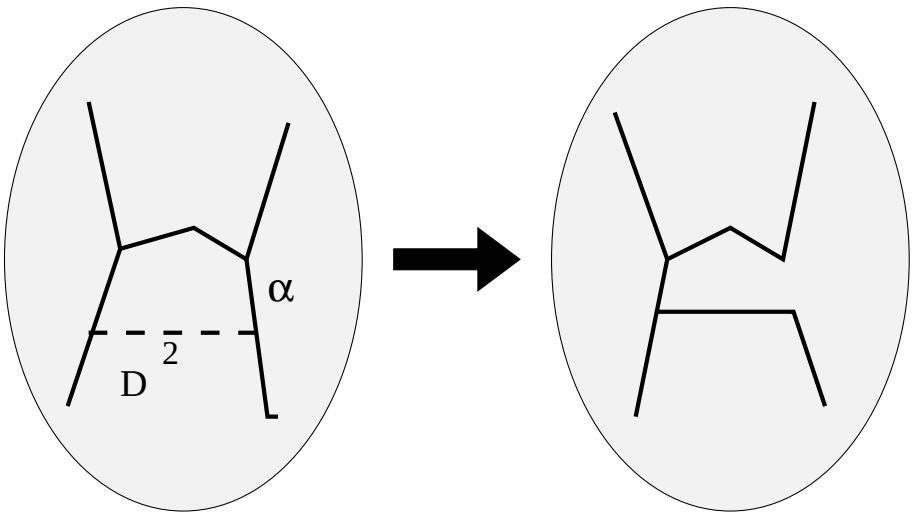

FIGURE 2.

spine, an additional information is needed about the way how the spine should be thickened to the 3-manifold.

2. Simplifying moves on spines

In what follows we will consider compact orientable 3-manifolds whose boundaries consist of spheres and tori.

We introduce five types of moves on almost special spines. Each of them does not increase the number of vertices of the spine and quite often decreases it. We call them *simplifying moves*. Any spine admits only finitely many simplifying moves. The moves transform not only spines, but may also transform the corresponding manifolds. Therefore, one should keep in memory an additional information sufficient for recovering the original manifolds from the new ones.

Let P be an almost special spine of a 3-manifold M.

Move 1. (*Disc replacement*). Let P be a fake surface. Suppose D^2 is a disc in M such that $D^2 \cap P = \partial D^2$ and the curve ∂D^2 is in general position in P. Then D^2 cuts of $M \setminus P$ a ball B^3. Let $\alpha \neq D^2$ be a 2-component of the fake surface $P \cup D^2$ such that α separates B^3 from $M \setminus B^3$. Removing from α an open 2-disc and collapsing the resulting polyhedron until possible, we get another almost special spine P_1 of M, see Fig. 2.

We require that Move 1 should not increase the number $v(P)$ of vertices of P, i. e. $v(P_1) \leq v(P)$. We will say that the move is *monotone* if $v(P_1) < v(P)$, and *horizontal* if $v(P_1) = v(P)$. The number $v(P \cup D^2) - v(P)$ is called *the degree* of the move.

Remark. It is easy to see that any special spine admits only finitely many different disc replacement moves of a given degree n.

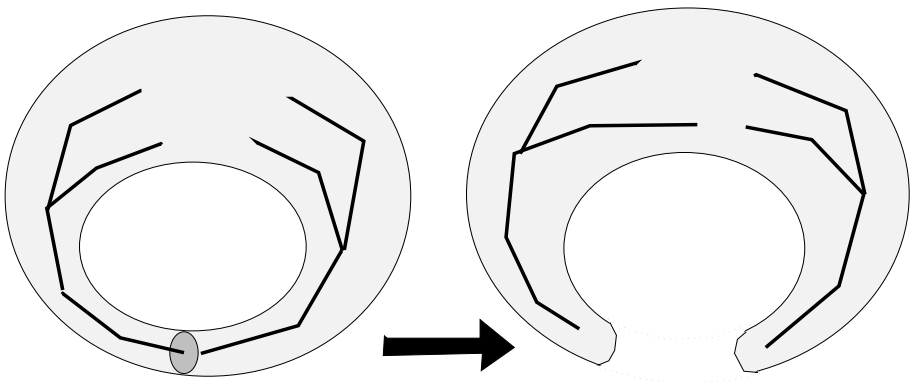

FIGURE 3.

In what follows we will consider disc replacement moves of degree ≤ 4.

Move 2. (*Cutting a 2-component along a non-trivial circle*). Suppose a 2-component $\alpha \subset P$ contains an orientation preserving simple closed curve l such that l determines a non-trivial element of $\pi_1(M)$. Then ∂M contains at least one torus, and there is an incompressible proper annulus $A \subset M$ such that $A \cap P = l$. Cutting P along l and collapsing the resulting polyhedron until possible, we get an almost special polyhedron $P_1 \subset M$. Denote by M_1 a regular neighborhood of P_1 in M. The manifold M_1 is obtained from M by cutting along A. It is easy to see that the complement of M_1 in M is homeomorphic to $Y^3 = N^2 \times S^1$, where N^2 is a disc with two holes. Therefore, $M = M_1 \cup Y^3$, where $M_1 \cap Y^3 = \partial M_1 \cap \partial Y^3$ consists of one or two tori depending on whether or not the boundary circles of A lie in different tori in ∂M.

Moves 1 and 2 are basic ones. Applying them, we may obtain almost special spines with one-dimensional parts as well as spines of 3-manifolds with several spherical boundary components. To simplify them, we use additional moves.

Move 3. (*Cutting free arcs*). Suppose P contains an arc l such that no 2-dimensional sheets are attached to l. Then we replace P by $P_1 = P - \text{Int } l$. To describe the corresponding transformation of M, denote by D^2 a proper disc in M such that D^2 intersects l transversely at exactly one point. Then a regular neighborhood M_1 of P_1 in M is obtained from M by cutting along D^2. In other words, M can be obtained from the new manifold M_1 by attaching index 1 handle, see Fig. 3. There are three cases.

A) $M = (M_1 + D^3)\#D^2 \times S^1$, if ∂D^2 does not separate ∂M;

B) $M = (M_1 + D^3)\#S^2 \times S^1$ if D^2 does not separate M, but ∂D^2 separates ∂M;

C) $M = M_1'\#(M_1'' + D^3)$ if D^2 separates M, where M_1', M_1'' are the connected components of M_1 and "$+D^3$" means that we fill up by a 3-ball a spherical component of the boundary.

Move 4. (*Delicate piercing*). Suppose ∂M consists of at least two components, and at least one of them is spherical. Then one can find a proper arc $l \subset M$ such that

$A = l \cap P$ is a non-singular point of P and l joints a spherical component of ∂M with another one. Removing from P an open disc neighborhood of A and collapsing the resulting polyhedron until possible, we get an almost special spine P_1 of $M_1 = M + D^3$. We call the piercing delicate since it induces a very mild modification of M.

Move 5. (*Rough piercing*). Let M be closed, and let α be a 2-component of P such that the boundary curve of α contains the maximal number of vertices. Removing α from P and collapsing the resulting polyhedron until possible, we get an almost special spine P_1 of a new manifold $M_1 \subset M$ such that ∂M_1 is a torus. Clearly, $M \setminus \mathrm{Int} M_1$ is a solid torus, i. e. M is obtained from M_1 by a Dehn filling.

3. Experimental results

We recall the notion of complexity of 3-manifolds that is naturally related to practically all the known methods of presenting manifolds and adequately describes complexity of manifolds in the informal sense of the expression [Matveev 1990].

Definition. The *complexity* $c(M)$ of a compact 3-manifold M equals k if M possesses an almost special spine with k vertices and admits no almost special spines with a smaller number of vertices.

The complexity possesses the following properties:

1). For any integer k there exists only a finite number of distinct closed irreducible orientable 3-manifolds of complexity k;
2). The complexity of the connected sum of 3-manifolds is equal to the sum of their complexities;
3). Let M_F be obtained from a 3-manifold M by cutting along a proper incompressible surface $F \subset M$. Then $c(M_F) \leq c(M)$.

Remark. Using moves 1, 3 and 4, one can easily prove that any minimal almost special spine of a closed orientable irreducible 3-manifold M with $c(M) > 0$ is a special one. There are exactly three closed orientable irreducible 3-manifolds of complexity 0: S^3, RP^3, and $L(3,1)$. Their minimal almost special spines (a point, RP^2, and a fake surface without vertices) are not special.

Recall that a 3-manifold M is a *graph-manifold* if it can be obtained by pasting together several copies of $D^2 \times S^1$ and $Y^3 = N^2 \times S^1$ (where N^2 is a disc with two holes) along some homeomorphisms of their boundaries.

Theorem 1. *All closed orientable 3-manifolds of complexity* ≤ 8 *are graph-manifolds.*

Theorem 1 was initially proved by a computer. Let us describe the main steps of the program.

Step 1. The computer enumerates all the special polyhedra with ≤ 8 vertices (there are finitely many of them);

Step 2. The computer selects spines of closed orientable 3-manifolds;

STEP 3. Then it tries to apply to each spine degree 4 disc replacement moves that strictly decrease the number of vertices. If such a move is possible, then the corresponding 3-manifold M is not interesting for either it has a smaller complexity (and we had met it earlier), or it is a connected sum of closed manifolds of smaller complexities. Otherwise, we go to the next step.

STEP 4. The computer applies Move 5 (rough piercing) and simplifies the new spine by Moves 1 – 4. Note that Move 5 is allowed only if the manifold is closed or has a spherical boundary, and it produces a manifold with a torus boundary.

The main observation resulting from the computer experiment is that if we start with a special spine of a closed orientable manifold M with ≤ 8 vertices, then after Moves 1 – 5 we always get a collection of points presenting spines of 3-dimensional spheres. It means that M is a graph manifold.

Later, a purely theoretical proof of Theorem 1 was found [Matveev 1990]. Note that Theorem 1 is exact in the following sense: there exist closed orientable 3-manifolds of complexity 9 that are hyperbolic and therefore are not graph-manifolds. The volume of one of them is equal to 0.94...; it is the smallest known value for volumes of closed orientable hyperbolic 3-manifolds. [Fomenko, Matveev 1988; Hildebrand, Weeks, 1989].

Theorem 2. *If a special spine of a closed orientable 3-manifold M contains < 8 vertices and is not minimal, then it admits a degree ≤ 4 monotone disc replacement move. Any two minimal special spines of M are related by degree ≤ 4 horizontal disc replacement moves.*

Theorem 2 had been verified by a computer program.

4. CONJECTURES

The following conjectures have been motivated by above-stated experimental results as well as by other observations.

Conjecture 1. *If a special spine of a compact 3-manifold is not minimal, then the number of its vertices can be decreased by degree ≤ 4 disc replacement moves.*

If the conjecture is true, then one can get a simple algorithm for recognition of the unknot. Let us apply to a spine of the knot space degree ≤ 4 disc replacement moves until possible. The knot is trivial if and only if we get a circle. In the same way one can get a simple algorithm for recognition of the 3-sphere.

Most probably, the conjecture is not true, but it is true "in general". In other words, the above algorithms would give the circle or the point for almost all spines of the solid torus or the ball, respectively. It means that we have good practical procedures for recognizing the unknot and the sphere.

Conjecture 2. *If a special spine of a closed graph-manifold is minimal, then any Move 5 (rough piercing) transforms it into a special spine of a graph-manifold.*

The conjecture is true for all graph-manifolds up to complexity 8. It allows one to reduce the recognition problem for closed graph-manifolds to that for manifolds with boundaries.

Note that if the boundary ∂M of an irreducible, boundary irreducible graph-manifold M is not empty, then M contains an essential annulus. Conjectures 1, 2 and the following theorem show that Moves 1 – 5 are "almost sufficient" for recognizing graph-manifolds with boundaries and, more generally, for decomposing 3-manifolds into geometric pieces.

Theorem 3. *If a compact 3-manifold M contains an essential annulus, then its minimal almost special spine is not special.*

Remark. If an almost special spine is not special, then it contains either a 2-component not homeomorphic to the 2-disc or an one-dimensional part. Hence we can apply either Move 2 or Move 3.

Before proving Theorem 3, let us recall some notions of normal surface theory [Haken 1961]. Let ξ be a handle decomposition of a 3-manifold M with non-empty boundary. It consists of index 0, 1, and 2 handles called *balls, beams,* and *plates,* respectively. Connected components in the intersection of balls and beams are called *islands*, connected components in the intersection of balls and plates are called *bridges*. The boundaries of balls meet ∂M along *lakes*. Any normal surface $F \subset M$ should intersect balls, beams and plates in a very specific way (see [Haken 1961]). Particularly, the intersection of F with balls should consist of *elementary* discs. The boundary curve ∂E of each elementary disc E should satisfy the following conditions:

(1) The intersection of ∂E with any bridge and any lake consists of no more than one segment;
(2) If l is an arc in the intersection of ∂E with a lake L then the endpoints of l should lie in different connected components of the intersection of L with islands;
(3) If a lake and a bridge are adjacent then ∂E intersects no more than one of them.

We will say that an elementary disc E has the *type* (m, n) if the circle ∂E intersects m bridges and n lakes.

Any special spine P of M generates a handle decomposition ξ_P of M. Balls, beams, and plates of the decomposition correspond to vertices, edges and 2-components of P, respectively. The boundary of each ball contains exactly four islands, and any two of them are joined by exactly one bridge, see Fig. 4.

It is not hard to see that any elementary disc for $\xi(P)$ has one of the following types: $(4,0), (3,0), (2,1), (1,2), (0,2), (0,3), (0,4)$. Each type determines the corresponding elementary disc in a unique way (up to homeomorphisms of the ball taking islands to islands, bridges to bridges, and lakes to lakes), except the type (0,3) that determines two elementary discs.

For any beam $D^2 \times I$ (with $D^2 \times \{0\}$ and $D^2 \times \{1\}$ being islands), the disc $D^2 \times \{1/2\}$ is called the *transverse disc* of the beam.

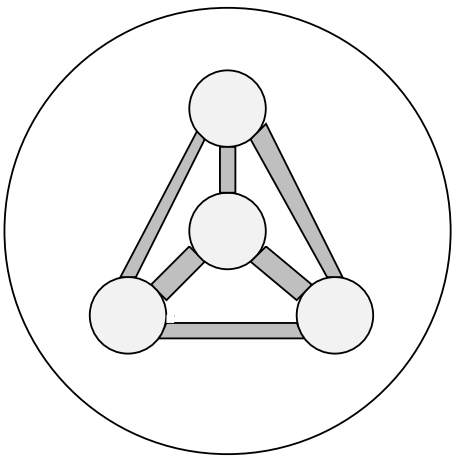

FIGURE 4.

Definition. Let A be a proper annulus in a 3-manifold M with a special spine P such that A is normal with respect to ξ_P. We say that A has *a tail* if the intersection of A with the transverse disc of a beam contains a proper arc l such that the endpoints of l lie in the same circle of ∂A. The arc l cuts off A a disc D_l. We will refer to D_l as to a *tail* of A.

Lemma 1. *If the generated by a special spine P of a 3-manifold M handle decomposition ξ_P contains a normal annulus A with a tail D_l, then P is not minimal.*

Proof. Denote by M_{D_l} the 3-manifold obtained from M by cutting along D_l. Evidently, M_{D_l} is homeomorphic with M. The tail decomposes the balls of ξ_P into balls, plates into plates, and beams into beams except the beam B_0 containing l. Coherently collapsing new balls, beams, and plates onto 2-dimensional subsets, we get an almost special spine P' of M_{D_l} . Since each ball of ξ_P contains no more than one vertex of P', we have $v(P') \leq v(P)$, where $v(P)$ denotes the number of vertices. Note that P' has a free edge arising from cutting and collapsing the beam B_0, see Fig. 5. After collapsing P' through this free edge, we get an almost special spine of M_{D_l} with a fewer number of vertices.

Proof of Theorem 3. Let P be a special spine of a 3-manifold M with an essential annulus. Replace the annulus by an annulus A that is normal with respect to the generated by P handle decomposition ξ_P of M. If A has a tail, then we apply Lemma 1 to find a simpler spine of M. Assume that A has no tails. Since each elementary disc of type $(0,3)$ or $(0,4)$ in A would determine at least one tail, only types $(4,0),(3,0),(2,1),(1,2),(0,2)$ for elementary discs in A are possible. Moreover, if E is an elementary disc of type $(1,2)$ or $(0,2)$, then two arcs in $\partial E \cap \partial M$ must lie in different components of ∂A.

Let us cut now M along A such that one component S of ∂A is preserved. In other words, we remove from M the subset $S^1 \times (0,1] \times I$, where $A = S^1 \times [0,1]$ and

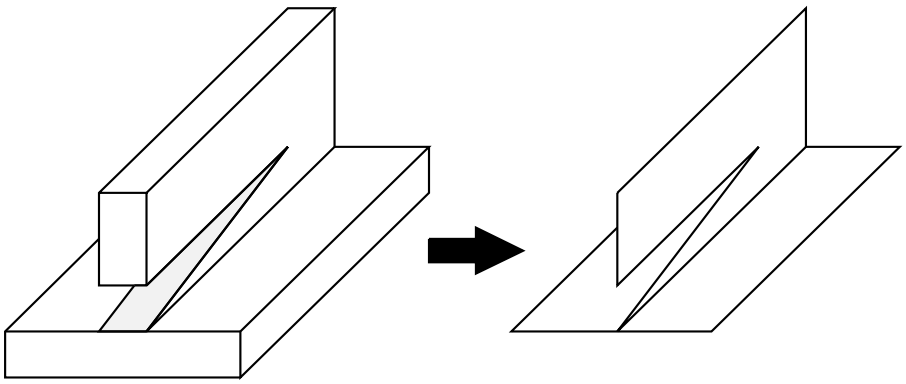

FIGURE 5.

$S^1 \times [0,1] \times I$ is a thin regular neighborhood of A in M. As above, coherently collapsing the new balls, beams, and plates onto 2-dimensional subsets, we get an almost special spine P' of M with $v(P') \leq v(P)$. Moreover, if at least one elementary disc of type $(1,2)$ is present, then $v(P') < v(P)$. It is because each type $(1,2)$ elementary disc in the intersection of A with a ball of ξ_P annihilates the corresponding vertex of P, see Fig. 6.

We conclude the proof with the following remark: if there are no type $(1,2)$ elementary discs in A, then all elementary discs in A have type $(0,2)$. In this case the core circle of A can be shifted into a 2-component of P. Since all 2-components of P are 2-cells, it implies that the core circle is contractible, which contradicts the assumption that A is incompressible.

5. The algorithm

Let M be a compact 3-manifold such that ∂M is either empty or consists of tori. Our goal is to decompose M into geometric pieces, particularly, to determine whether or not M is a graph-manifold.

STEP 1. Construct a special spine P of M;

STEP 2. Apply to P Moves 1 – 4 until possible. In the case of Move 2 when $Y^3 \times S^1$ is cut off we store the information how $Y^3 \times S^1$ is attached to the remaining part of M. It can be done by selecting a meridian-longitude pair on boundary tori and controlling their behavior under further moves.

STEP 3. Assume that we have started with a non-closed manifold M. Then by Theorem 3 (and modulo Conjecture 1), M is a graph-manifold if and only if after Step 2 we get a collection of points. We use the stored information to present M as a union of Seifert manifolds with explicitly given parameters and gluing matrices.

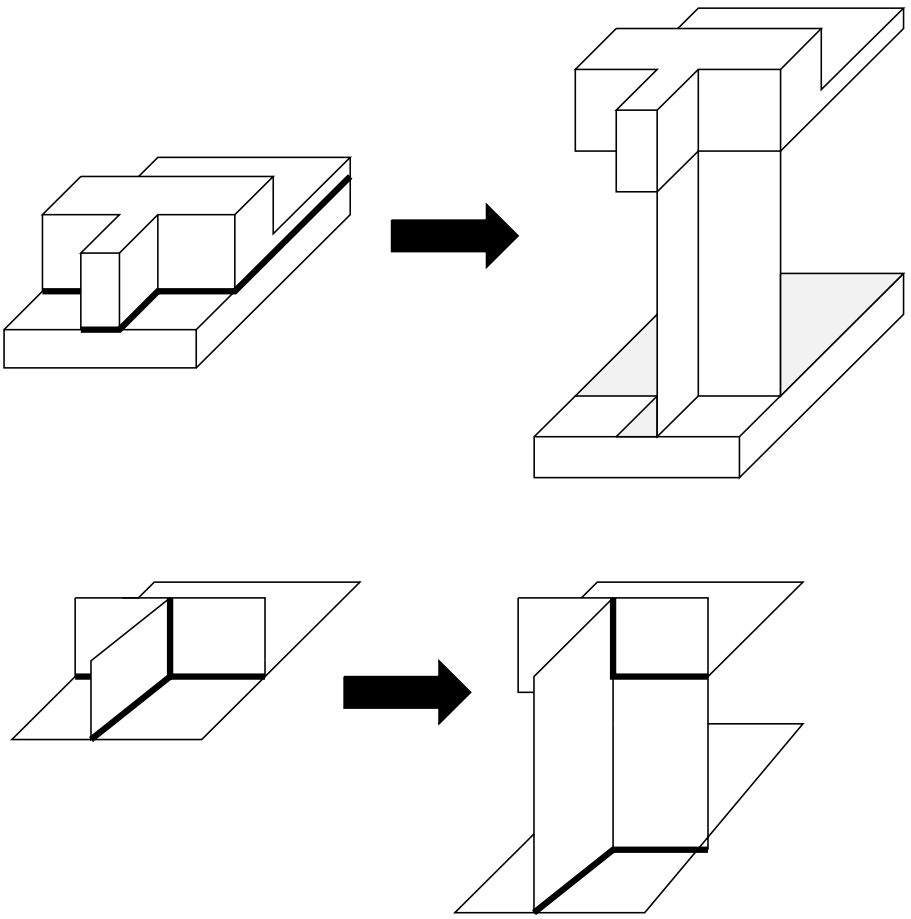

FIGURE 6.

STEP 4. In general, we get a collection of special spines of irreducible, boundary irreducible manifolds. Then we apply Move 5 to those of them that are spines of closed manifolds.

STEP 5. Iterate Steps 2 – 4 until possible.

If we have started with a graph-manifold M, then (modulo Conjectures 1 and 2) we get an explicit presentation of M as a connected sum of unions of Seifert manifolds with known parameters and known gluing matrices. In general, we may get unknown pieces that should be tested for hyperbolicity by comparing with J. Weeks table.

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