A SUSPENSION LEMMA FOR BOUNDED POSETS

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ABSTRACT. Let P and Q be bounded posets. In this note, a lemma is introduced that provides a set of sufficient conditions for the proper part of P being homotopy equivalent to the suspension of the proper part of Q. An application of this lemma is a unified proof of the sphericity of the higher Bruhat orders under both inclusion order (which is a known result by ZIEGLER) and single step inclusion order (which was not known so far).

1. INTRODUCTION

One way to draw conclusions about the homotopy type of a poset P is to consider an order-preserving map f from P to another poset Q the homotopy type of which is known. If one can show that f carries a homotopy equivalence the problem is solved. If P and Q are bounded then one is rather interested in the homotopy type of the proper part $\overline{P}$ of P; to take advantage of the map f, it is then usually crucial that $f : P \to Q$ restricts to a map of the proper parts $\overline{f} : \overline{P} \to \overline{Q}$. However, even if this is not the case, the map $f : P \to Q$ may be exploited to determine the homotopy type of $\overline{P}$: in this note we present a set of sufficient conditions on $f : P \to Q$ that guarantees that $\overline{P}$ is homotopy equivalent to the *suspension* of $\overline{Q}$ (Suspension Lemma).

We apply the Suspension Lemma to show that the higher Bruhat orders by MANIN & SCHECHTMAN [5] (a certain generalization of the weak Bruhat order on the symmetric group) are spherical, no matter whether we order by *inclusion* or by *single step inclusion*.

The Suspension Lemma has been applied again in [2] to uniformly prove the sphericity of the two (possibly different) *higher Stasheff-Tamari orders* [3] on the set of triangulations of a cyclic polytope.

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2. THE LEMMA

In this section we state and prove the Suspension Lemma.

Lemma 2.1. *Let P, Q be bounded posets with $\hat{0}_Q \neq \hat{1}_Q$. Assume there exist a dissection of P into green elements* $\mathrm{green}(P)$ *and red elements* $\mathrm{red}(P)$*, as well as order-preserving maps*

$$f : P \to Q \quad \textit{and} \quad i, j : Q \to P$$

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with the following properties:

(i) *The green elements form an order ideal in P.*
(ii) *The maps $f \circ i$ and $f \circ j$ are the identity on Q.*
(iii) *The image of i is green, the image of j is red.*
(iv) *For every $p \in P$ we have $(i \circ f)(p) \leq p \leq (j \circ f)(p)$.*
(v) *The fiber $f^{-1}(\hat{0}_Q)$ is red except for $\hat{0}_P$, the fiber $f^{-1}(\hat{1}_Q)$ is green except for $\hat{1}_P$.*

Then the proper part $\overline{P}$ of P is homotopy equivalent to the suspension of the proper part $\overline{Q}$ of Q.

Proof. Define

$$g : \begin{cases} \overline{P} & \to & \overline{Q \times \{\hat{0}, \hat{1}\}}, \\ p & \mapsto & \begin{cases} (f(p), \hat{0}) & \text{if } p \text{ is green,} \\ (f(p), \hat{1}) & \text{if } p \text{ is red;} \end{cases} \end{cases}$$

and

$$h : \begin{cases} \overline{Q \times \{\hat{0}, \hat{1}\}} & \to & \overline{P}, \\ (q, \hat{0}) & \mapsto & i(q), \\ (q, \hat{1}) & \mapsto & j(q). \end{cases}$$

The assumptions guarantee that the above maps are well-defined and order-preserving. We claim that $h \circ g$ is homotopic to the identity on P. In order to prove this, consider the following carrier on the order complex $\Delta(\overline{P})$ of $\overline{P}$.

$$C : \begin{cases} \Delta(\overline{P}) & \to & 2^{\Delta(\overline{P})}, \\ \sigma & \mapsto & \Delta\big(P_{\geq (i \circ f)(\min \sigma)} \cap P_{\leq (j \circ f)(\max \sigma)} \cap \overline{P}\big). \end{cases}$$

We claim that $C(\sigma)$ is contractible for all $\sigma \in \Delta(\overline{P})$. To this end, let σ be a chain in $\overline{P}$. If $\min \sigma$ were contained in $f^{-1}(\hat{0}_Q)$ and $\max \sigma$ were contained in $f^{-1}(\hat{1}_Q)$ then—because of (v)—the chain σ would have a red minimal and a green maximal element; contradiction to (i). Because $f \circ i$ and $f \circ j$ are the identity on Q, the maps i and j are in particular injective. Hence, at least one of the elements $(i \circ f)(\min \sigma)$ and $(j \circ f)(\max \sigma)$ is contained in $\overline{P}$. Therefore, $C(\sigma)$ is a cone for all $\sigma \in \Delta(\overline{P})$, thus contractible.

We further claim that the identity on P and $(h \circ g)$ are both carried by C. To see this, consider a chain σ in $\overline{P}$ and an element p in σ. Since

$$(i \circ f)(\min \sigma) \overset{\text{(iv)}}{\leq} \min \sigma \leq p \leq \max \sigma \overset{\text{(iv)}}{\leq} (j \circ f)(\max \sigma),$$

the identity on P is carried by C. Because

$$(i \circ f)(\min \sigma) \leq (h \circ g)(\min \sigma) \leq (h \circ g)(p) \leq (h \circ g)(\max \sigma) \leq (j \circ f)(\max \sigma),$$

also $(h \circ g)$ is carried by C.

Thus, the identity on $\overline{P}$ and $(h \circ g)$ are homotopic by the Carrier Lemma [1, Lemma 10.1]. Together with the fact that $(g \circ h)$ is the identity on Q, this proves that $\overline{P}$ is homotopy equivalent to $\overline{Q \times \{\hat{0}, \hat{1}\}}$.

Finally, the poset

$$\overline{Q \times \{\hat{0}, \hat{1}\}} = \left(\overline{Q} \times \{\hat{0}, \hat{1}\}\right) \cup \left\{(\hat{0}_Q, \hat{1}), (\hat{1}_Q, \hat{0})\right\},$$

where

$$(\hat{0}_Q, \hat{1}) < \overline{Q} \times \hat{1} \quad \text{and} \quad (\hat{1}_Q, \hat{0}) > \overline{Q} \times \hat{0},$$

is homeomorphic to the suspension of $\overline{Q}$ by elementary computation rules for products and suspension of topological spaces. Therefore, $\overline{P}$ is homotopy equivalent to the suspension of $\overline{Q}$, as desired. □

3. An Application to Higher Bruhat Orders

In the following we present a proof for the sphericity of the higher Bruhat orders $\mathcal{B}(n,k)$ with respect to *single step inclusion order*. Higher Bruhat orders were defined by Manin & Schechtman [5] as a generalization of the weak Bruhat order of the symmetric group. They were further studied by Kapranov & Voevodski [4] and Ziegler [6]. For basic facts see these references.

For any $(k+2)$-subset P of $[n]$ the set of all its $(k+1)$-subsets is called a $(k+1)$*-packet*. By abuse of notation, we denote this $(k+1)$-packet again by P. A subset U of $\binom{[n]}{k+1}$ is *consistent* if for any $(k+1)$-packet P the intersection $U \cap P$ is empty, all of P, or a beginning or ending segment in the lexicographic ordering of P. For two consistent sets $U, U' \subseteq \binom{[n]}{k+1}$ the *single step inclusion order* is defined by $U \leq U'$ if there is a sequence $U = U_0, \ldots, U_m = U'$ of consistent sets with $\#(U_i \backslash U_{i-1}) = 1$ for $i = 1, \ldots, m$.

The higher Bruhat order $\mathcal{B}(n,k)$ is the set of all consistent subsets of $\binom{[n]}{k+1}$, partially ordered by single step inclusion. In contrast to this, $\mathcal{B}_{\subseteq}(n,k)$ is the set of all consistent subsets of $\binom{[n]}{k+1}$ partially ordered by ordinary inclusion of sets (*inclusion order*). Ziegler [6] has shown that these partial orders do not coincide in general.

While sphericity for the inclusion order was already established in [6], the topological type of the single step inclusion order remained an open problem. We solve this problem in the following theorem, the proof of which works equally fine for $\mathcal{B}_{\subseteq}(n,k)$.

Theorem 3.1. *The proper part of the higher Bruhat order $\mathcal{B}(n,k)$ has the homotopy type of an $(n-k-2)$-sphere.*

Proof. We prove the theorem by induction on $n-k$. For $n = k+1$ the higher Bruhat orders are isomorphic to the poset $\{\hat{0}, \hat{1}\}$. Therefore, $\overline{\mathcal{B}(k+1,k)}$ is the empty set, i.e., it has the homotopy type of a (-1)-sphere.

We show that for $n > k+1$ the conditions of the Suspension Lemma are satisfied for

$$\begin{aligned}
P &= \mathcal{B}(n,k), \\
Q &= \mathcal{B}(n-1,k), \\
\operatorname{green}(\mathcal{B}(n,k)) &= \left\{ U \subseteq \tbinom{[n]}{k+1} \,:\, \{n-k,\ldots,n\} \notin U \right\}, \\
\operatorname{red}(\mathcal{B}(n,k)) &= \left\{ U \subseteq \tbinom{[n]}{k+1} \,:\, \{n-k,\ldots,n\} \in U \right\}, \\
f &: \begin{cases} \mathcal{B}(n,k) & \to \quad \mathcal{B}(n-1,k), \\ \qquad U & \mapsto \quad U \backslash n := \{ I \in U \,:\, n \notin I \}; \end{cases} \\
i &: \begin{cases} \mathcal{B}(n-1,k) & \to \quad \mathcal{B}(n,k), \\ \qquad V & \mapsto \quad V; \end{cases} \\
j &: \begin{cases} \mathcal{B}(n-1,k) & \to \quad \mathcal{B}(n,k), \\ \qquad V & \mapsto \quad V \cup \left\{ I \in \tbinom{[n]}{k+1} \,:\, n \in I \right\}. \end{cases}
\end{aligned}$$

Assumptions (i), (ii), and (iii) are obvious by the definitions. For the inclusion order also (iv) is obvious.

To prove (iv) for the single step inclusion order, we proceed as follows. Let $U \in \mathcal{B}(n,k)$ be a consistent set. We show in the sequel that U can be obtained from $(i \circ f)(U) = U \backslash n$ by adding one element at a time without getting inconsistent. Then $(i \circ f)(U) \leq U$, and we are done. (The statement about j follows by taking complements.)

Let α an *admissible permutation* of $\binom{[n-1]}{k}$ corresponding to $U \backslash n$. That is, the restriction of α to a k-packet P is the lexicographic order on P if P is contained in $U \backslash n$; it is the reverse lexicographic order on P otherwise (see [6]). We now build up U from $U \backslash n$ by adding the elements I' of $\{ I \in U \,:\, n \in I \}$ in the order in which the elements $I' \backslash n$ appear in α. Consistency at every step follows by construction and the fact that α is admissible. This completes the proof of (iv).

To see (v), assume, for the sake of contradiction, that there is a non-empty consistent set $U \in \mathcal{B}(n,k)$ with

$$U \backslash n = \varnothing \quad \text{and} \quad \{n-k,\ldots,n\} \notin U.$$

In the following we show that every non-empty consistent set, in particular U, contains at least one interval. We call $j \in [n] \backslash I$ an *internal gap* of I if $\min I < j < \max I$. Note that the subsets of $[n]$ without internal gaps are exactly the intervals. Assume $I \in U$ has $c > 0$ internal gaps. Let j be one of them. Consider the $(k+1)$-packet $P := I \cup \{j\}$. Since U is consistent, $P \backslash \min I$ or $P \backslash \max I$ is in U as well. Both of them have at most $c-1$ internal gaps. By induction we conclude that U contains at least one interval.

Since $U \backslash n = \varnothing$, every element in U contains n. The only interval in $\binom{[n]}{k+1}$ containing n, however, is $\{n-k,\ldots,n\}$. Hence, $\{n-k,\ldots,n\} \in U$; contradiction.

Thus, $\hat{0} = \varnothing$ is the only green element in $f^{-1}(\hat{0}) = f^{-1}(\varnothing)$. The second statement in (v) is again achieved by taking complements.

Therefore, the assumptions of the Suspension Lemma are satisfied, and $\overline{\mathcal{B}(n,k)}$ is homotopy equivalent to the suspension of $\overline{\mathcal{B}(n-1,k)}$. This proves the theorem by induction on $n-k$. □

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