ON SUBDIVISION POSETS OF CYCLIC POLYTOPES

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ABSTRACT. There are two related poset structures, the higher Stasheff-Tamari orders, on the set of all triangulations of the cyclic d polytope with n vertices. In this paper it is shown that both of them have the homotopy type of a sphere of dimension $n-d-3$.

Moreover, we resolve positively a new special case of the *Generalized Baues Problem*: The Baues poset of all polytopal decompositions of a cyclic polytope of dimension $d \leq 3$ has the homotopy type of a sphere of dimension $n-d-2$.

1. INTRODUCTION

This paper continues the investigation of certain posets of triangulations of cyclic polytopes, the *higher Stasheff Tamari posets*, initiated in [4] and continued in [5].

The first higher Stasheff Tamari poset is the poset $\mathcal{S}_1(n,d)$ of all triangulations of the cyclic d-polytope with n vertices $C(n,d)$, partially ordered by increasing bistellar operations; the second higher Stasheff Tamari poset is the poset $\mathcal{S}_2(n,d)$ of all triangulations of $C(n,d)$, partially ordered by the height of their characteristic sections in $C(n,d+1)$ (see [4], [5]).

Our first main result is the following.

Theorem 1.1. (i) *For all $n > d+1$ the proper part $\overline{\mathcal{S}_1(n,d)}$ of $\mathcal{S}_1(n,d)$ is homotopy equivalent to a sphere of dimension $n-d-3$.*

(ii) *For all $n > d+1$ the proper part $\overline{\mathcal{S}_2(n,d)}$ of $\mathcal{S}_2(n,d)$ is homotopy equivalent to a sphere of dimension $n-d-3$.*

In [4], it was proved for $d \leq 3$ that the poset structures $\mathcal{S}_1(n,d)$ and $\mathcal{S}_2(n,d)$ coincide. It was also shown that the poset $\mathcal{S}_2(n,d)$ is a lattice for $d \leq 3$. If $d = 2$ this is the well-known *Tamari lattice* on triangulations of a convex n-gon. We will use this lattice structure to resolve in the affirmative a special case of the *Generalized Baues Problem* of Billera, Kapranov, and Sturmfels (see [1], [6], [8]).

Theorem 1.2. *For cyclic polytopes $C(n,d)$ of dimension $d \leq 3$, the refinement ordering on the set of polytopal subdivisions gives a poset which is homotopy equivalent to a $(n-d-2)$-sphere.*

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We will prove Theorem 1.1 by induction on $n-d$, showing that the poset $\overline{\mathcal{S}_1(n,d)}$ (resp. $\overline{\mathcal{S}_2(n,d)}$) is homotopy equivalent to the suspension of $\overline{\mathcal{S}_1(n-1,d)}$ (resp. $\overline{\mathcal{S}_2(n-1,d)}$).

The proof of Theorem 1.2 is via a reduction to the poset $\overline{\mathcal{S}_2(n,d)}$, by showing that the poset of polytopal subdivisions of $C(n,d)$ is homotopy equivalent to the suspension of $\overline{\mathcal{S}_2(n,d)}$. We will make use of a lemma (Lemma 6.5) about the homotopy type of non-contractible intervals in a poset which we think is of interest in its own right.

This paper is structured as follows: in Section 2 we recall some notation and basic facts about simplicial complexes, posets, and cyclic polytopes. In Section 3 we prove Theorem 1.1. Sections 4 and 5 provide the necessary details. In Section 6 we prove Theorem 1.2, the special case of the Generalized Baues Problem. Section 7 discusses some of the remaining open problems in the area of triangulations of cyclic polytopes.

2. Notation and Basic Facts

In this section we will introduce our notation and discuss some basic facts that have appeared previously.

Let $[n] := \{1,2,\dots,n\}$. We regard the *cyclic d-polytope with n vertices* as the convex hull of points on the moment curve

$$C(n,d) = \mathrm{conv}\{(i,i^2,\dots,i^d) \in \mathbb{R}^d : i \in [n]\}$$

Since we are dealing with the combinatorial structure of all triangulations of cyclic polytopes we may choose these special coordinates without any loss of generality. We will often refer to the i^{th} vertex $(i,i^2,\dots,i^d)$ of $C(n,d)$ as simply i.

The *canonical projection* $p = p_{n,d}$ from $C(n,d+1)$ onto $C(n,d)$ is given by deletion of the x_{d+1}-coordinate. Facets of $C(n,d)$ that can be seen from a point in $\mathbb{R}^{d+1}$ with a very large (negative) x_{d+1}-coordinate are called *upper (lower) facets*.

Two simplices are said to be *admissible* if they intersect in a common (possibly empty) face of each. A *triangulation* of a polytope P is a set of simplices with vertices in the vertex set of P such that

- the union of the simplices equals P,
- every face of a simplex in the triangulation is itself in the triangulation, and
- any two simplices are admissible.

Triangulations are often identified with their sets of inclusion-maximal faces. Simplices are usually identified with their vertex sets.

To test intersections of simplices S_1 and S_2 we will use the concept of *zig-zag-paths* based on the alternating oriented matroid property of cyclic polytopes (see [5]). We construct a table with n columns, corresponding to the labels $1,\dots,n$, and two rows, corresponding to the simplices S_1 and S_2. In row i, column j, there is a star $*$ if and only if $j \in S_i$. An *(S_1,S_2)-zig-zag-path of length k* is a set of k stars in the columns $s_1 < \cdots < s_k$ such that $s_1,s_3,s_5,\dots$ are in S_1 and $s_2,s_4,s_6,\dots$ are in S_2, or vice versa. The simplices

	1	2	3	4	5	6	7	8	9
S_1	$*$		$*$			$*$		$*$	$*$
S_2		$*$		$*$		$*$	$*$	$*$	

(a)

	1	2	3	4	5	6	7	8	9
S_1	$*$		$*$			$*$		$*$	$*$
S_2		$*$	$*$			$*$	$*$	$*$	

(b)

FIGURE 1. Zig-zag-paths: (a) S_1 and S_2 are non-admissible in dimensions 5 or less, hence S_1 and S_2 cannot be in a triangulation of, e. g., $C(9,4)$ at the same time; (b) S_1 and S_2 are admissible in dimensions 3 or greater, therefore S_1 and S_2 may be in a triangulation of $C(9,4)$.

	1	2	3	4	5	6	7	8	9
S_1	$*$	e	e	e	$*$	$*$	e	$*$	$*$
S_2	o	o	$*$	$*$	o	o	$*$	$*$	$*$

(a)

	$\cancel{1}$	2	3	$\cancel{4}$	$\cancel{5}$	6	7	8	9
S_1		e	$*$			$*$	e	$*$	$*$
S_2		$*$	$*$			$*$	o	o	$*$

(b)

FIGURE 2. Gaps: (a) S_1 is a lower facet of $C(9,4)$, S_2 is an upper facet of $C(9,4)$; (b) neither S_1 nor S_2 are facets of $C(9,4)$, but S_1 is a lower facet and S_2 an upper facet of the cyclic subpolytope $C(\{2,3,6,7,8,9\},4)$. Therefore, S_1 is lower than S_2 in $C(9,5)$.

S_1 and S_2 are admissible in dimension d if and only if there is no (S_1,S_2)-zig-zag-path of length $d+2$ (see Figure 1).

Any subset $V \subseteq [n]$ gives rise to a cyclic subpolytope $C(V,d)$, namely the convex hull of the subset V. For a cyclic subpolytope $C(V,d)$ and a d-subset F of V we call a label $i \in V \setminus F$ an *even (odd) gap* of F (in V) if the number of labels $j \in V$ with $j > i$ is even (odd). Then we know that the set of *lower (upper) facets* of $C(V,d)$ is the set of all $F \in \binom{V}{d}$ containing only even (odd) gaps [5]. This applies, in particular, to simplices so that we can talk about the upper and lower facets of a d-simplex in $C(n,d)$. For a visualization, we use the same table as for the zig-zag-paths and fill an e (resp. o) for an even (resp. odd) gap into the corresponding field (see Figure 2).

Let T be a triangulation of $C(n,d)$ and $\tilde{S}$ be a $(d+1)$-simplex in $C(n,d+1)$ all of whose lower facets lie in T. An *increasing bistellar operation* or *increasing flip* in T at $\tilde{S}$ is an operation that replaces in T the lower facets of $\tilde{S}$ by the upper facets of $\tilde{S}$. The result, it is clear, is a new triangulation of $C(n,d)$. The transitive closure of this operation defines the *first higher Stasheff-Tamari poset* $\mathcal{S}_1(n,d)$. We write $T <_1 T'$ to indicate that T is less than T' in $\mathcal{S}_1(n,d)$.

The *characteristic section* of a triangulation T of $C(n,d)$ is the unique piecewise linear map (with respect to the simplicial complex T) from $C(n,d)$ to $C(n,d+1)$ that is inverted by the canonical projection p and has the property that it sends the i^{th} vertex of $C(n,d)$ to the i^{th} vertex of $C(n,d+1)$. We identify a triangulation T with its characteristic section

$T : C(n,d) \to C(n,d+1)$ and with its image $T(C(n,d))$ in $C(n,d+1)$. The *second higher Stasheff-Tamari poset* $\mathcal{S}_2(n,d)$ is the set of all triangulations of $C(n,d)$ partially ordered by the height of characteristic sections. That is, $T \leq_2 T'$ if and only if $T(x)_{d+1} \leq T'(x)_{d+1}$ for all $x \in C(n,d)$, where here v_{d+1} denotes the $(d+1)^{st}$ coordinate of the vector v in $\mathbb{R}^{d+1}$. We then say that T is weakly lower than T'. If $T(x)_{d+1} \leq T'(x)_{d+1}$ holds for all x in the (geometric) intersection of a simplex $S \in T$ and a simplex $S' \in T'$ we say that S is weakly lower than S'. We write $T <_2 T'$ to denote that T is less than T' in $\mathcal{S}_2(n,d)$.

The unique minimal element in $\mathcal{S}_1(n,d)$ respectively $\mathcal{S}_2(n,d)$ (which is the set of lower facets of $C(n,d+1)$) is denoted by $\hat{0}_{n,d}$. Similarly, the unique maximal element (which is the set of upper facets of $C(n,d+1)$) is denoted by $\hat{1}_{n,d}$. The d-simplices in $C(n,d)$ are partially ordered by the following relation: $S \prec S'$ if and only if $S \cap S'$ is a lower facet of S' and an upper facet of S (see [5]).

We will make use of some standard constructions on simplicial complexes. Let Δ be a simplicial complex on the ground set X. That is, Δ is a collection of subsets of X that is closed under containment. If $S \subseteq X$ define the *link of S in Δ* to be the complex

$$\mathrm{lk}_\Delta(S) := \{R \backslash S : R \in \Delta, S \subseteq R\};$$

the *star of S in Δ* is the complex

$$\mathrm{st}_\Delta(S) := \{R \in \Delta : S \subseteq R\};$$

and the *deletion of S in Δ* is the complex

$$\mathrm{del}_\Delta(S) := \{R \in \Delta : S \not\subseteq R\}.$$

If there is another complex Δ' on a ground set Y disjoint from X we will define the *combinatorial join of Δ and Δ'* to be the complex on the ground set $X \cup Y$

$$\Delta * \Delta' := \left\{ S \cup S' : S \in \Delta, S' \in \Delta' \right\}.$$

If T, T' are the sets of inclusion maximal faces of Δ, Δ' then the above formulas yield the sets of inclusion maximal faces of the link, the star, the deletion, and the join, respectively.

Given an i-simplex σ spanned by some $(i+1)$-subset (also denoted σ) of vertices of $C(n,d)$, there is also a unique linear section $\sigma : \sigma \to C(n,d+1)$ of p having the property that it sends each vertex i of σ to the vertex labelled i of $C(n,d+1)$. Say that σ *submerged* by the triangulation T of $C(n,d)$ if

$$\sigma(x)_{d+1} \leq T(x)_{d+1}$$

for every point x in σ. For a triangulation T of $C(n,d)$ let its i^{th} *submersion set* $\mathrm{sub}_i(T)$ be the set of i-simplices submerged by T.

When we refer to the topology or homotopy type of a poset P, we will always mean the topology of the *geometric realization* of its *order complex*, i. e., $|\Delta(P)|$ [3, §9]. If P is a poset with bottom and top elements $\hat{0}, \hat{1}$, then its *proper part* $\overline{P}$ is simply the subposet $P \backslash \{\hat{0}, \hat{1}\}$.

We recall the following facts from [4] and [5] which will be crucial for our main results:

Theorem 2.1. [5, Theorem 1.1] *The first higher Stasheff Tamari poset $\mathcal{S}_1(n,d)$ is bounded.*

Theorem 2.2. [5, Theorem 4.2(iii), Proposition 5.14(iii)] *The following map is well-defined and order-preserving:*

$$f : \begin{cases} \mathcal{S}_1(n,d) & \rightarrow & \mathcal{S}_1(n-1,d), \\ T & \mapsto & \mathrm{del}_T(n) \cup \big(\mathrm{del}_{\mathrm{lk}_T(n)}(n-1) * \{n-1\}\big). \end{cases}$$

Proposition 2.3. [4, Proposition 2.15] *For any two triangulations T_1, T_2 of $C(n,d)$, we have $T_1 \leq T_2$ in $\mathcal{S}_2(n,d)$ if and only if*

$$\mathrm{sub}_{\lceil \frac{d}{2} \rceil}(T_1) \subseteq \mathrm{sub}_{\lceil \frac{d}{2} \rceil}(T_2).$$

Proposition 2.4. [4, Propositions 3.2, 4.1] *Membership in $\lceil \frac{d}{2} \rceil$- submersion sets for $d = 2,3$ has the following characterization.*

For T a triangulation of $C(n,2)$ and $e = \{i,j\}$ an edge inside $C(n,2)$, we have that $e \in sub_1(T)$ if and only if there does not exist an edge $e' = \{k,l\}$ of T with $k < i < l < j$.

For T a triangulation of $C(n,3)$ and $t = \{i,j,k\}$ a triangle inside $C(n,3)$, we have that $t \in sub_2(T)$ if and only if there does not exist an edge $\{x,y\}$ of T with $i < x < j < y < z$.

Theorem 2.5. [4, Theorems 3.6, 4.9] *For $d \leq 3$, the higher Stasheff-Tamari poset $\mathcal{S}_2(n,d)$ is a lattice, i. e., any subset of its elements has a meet (greatest lower bound) and a join (least upper bound).*

Theorem 2.6. [4, Theorems 3.9, 4.11] *For $d \leq 3$, the proper part $\overline{\mathcal{S}_2(n,d)}$ of the higher Stasheff-Tamari poset has the homotopy type of an $(n-d-3)$-sphere.*

3. The Homotopy Types of $\mathcal{S}_1(n,d)$ and $\mathcal{S}_2(n,d)$

In this section, Theorem 1.1 will be proven by induction on $n-d$, using the Suspension Lemma 3.1 below to show that the proper part of $\mathcal{S}(n,d)$ is homotopy equivalent to the suspension of the proper part of $\mathcal{S}(n-1,d)$, where $\mathcal{S}(n,d)$ can be either $\mathcal{S}_1(n,d)$ or $\mathcal{S}_2(n,d)$. (A more detailed proof of the Suspension Lemma can be found in [7].)

Lemma 3.1 (Suspension Lemma). *Let P,Q be bounded posets with $\hat{0}_Q \neq \hat{1}_Q$. Assume there exist a dissection of P into green elements* green(P) *and red elements* red(P)*, as well as order-preserving maps*

$$f : P \to Q \quad \text{and} \quad i,j : Q \to P$$

with the following properties:

(i) *The green elements form an order ideal in P.*
(ii) *The maps $f \circ i$ and $f \circ j$ are the identity on Q.*
(iii) *The image of i is green, the image of j is red.*
(iv) *For every $p \in P$ we have $(i \circ f)(p) \leq p \leq (j \circ f)(p)$.*
(v) *The fiber $f^{-1}(\hat{0}_Q)$ is red except for $\hat{0}_P$, the fiber $f^{-1}(\hat{1}_Q)$ is green except for $\hat{1}_P$.*

Then the proper part $\overline{P}$ of P is homotopy equivalent to the suspension of the proper part $\overline{Q}$ of Q.

Sketch of proof. Define

$$g : \begin{cases} \overline{P} & \to \quad \overline{Q \times \{\hat{0}, \hat{1}\}}, \\ p & \mapsto \quad \begin{cases} (f(p), \hat{0}) & \text{if } p \text{ is green,} \\ (f(p), \hat{1}) & \text{if } p \text{ is red;} \end{cases} \end{cases} \tag{1}$$

and

$$h : \begin{cases} \overline{Q \times \{\hat{0}, \hat{1}\}} & \to \quad \overline{P}, \\ (q, \hat{0}) & \mapsto \quad i(q), \\ (q, \hat{1}) & \mapsto \quad j(q). \end{cases} \tag{2}$$

The assumptions guarantee that the above maps are well-defined and order-preserving. Observe that $g \circ h$ is the identity map on $\overline{Q \times \{\hat{0}, \hat{1}\}}$ and that $\overline{Q \times \{\hat{0}, \hat{1}\}}$ is homeomorphic to the suspension of $\overline{Q}$. It is easy to show that both $h \circ g$ and the identity map on $\overline{P}$ are carried by the following contractible carrier on the order complex $\Delta(\overline{P})$ of $\overline{P}$.

$$C : \begin{cases} \Delta(\overline{P}) & \to \quad 2^{\Delta(\overline{P})}, \\ \sigma & \mapsto \quad \Delta\big(P_{\geq (i \circ f)(\min \sigma)} \cap P_{\leq (j \circ f)(\max \sigma)} \cap \overline{P}\big). \end{cases}$$

Thus, by the Carrier Lemma [3], the map $h \circ g$ is homotopic to the identity on $\overline{P}$, and g and h are homotopy inverses to each other. □

We now prove that the assumptions of the Suspension Lemma are satisfied by the following set of data.

$$\begin{aligned}
P &= \mathcal{S}(n,d),\\
Q &= \mathcal{S}(n-1,d),\\
\operatorname{green}(\mathcal{S}(n,d)) &= \begin{cases} \{\, T \in \mathcal{S}(n,d) \,:\, \{n-d,\dots,n\} \notin T \,\} & \text{for } d \text{ even},\\ \{\, T \in \mathcal{S}(n,d) \,:\, \{n-d,\dots,n\} \in T \,\} & \text{for } d \text{ odd}; \end{cases}\\
\operatorname{red}(\mathcal{S}(n,d)) &= \begin{cases} \{\, T \in \mathcal{S}(n,d) \,:\, \{n-d,\dots,n\} \in T \,\} & \text{for } d \text{ even},\\ \{\, T \in \mathcal{S}(n,d) \,:\, \{n-d,\dots,n\} \notin T \,\} & \text{for } d \text{ odd}; \end{cases}\\
f &: \begin{cases} \mathcal{S}(n,d) & \to \quad \mathcal{S}(n-1,d),\\ \qquad T & \mapsto \quad \operatorname{del}_T(n) \cup \operatorname{del}_{\operatorname{lk}_T(n)}(n-1) * \{n-1\}; \end{cases}\\
i &: \begin{cases} \mathcal{S}(n-1,d) & \to \quad \mathcal{S}(n,d),\\ \qquad T & \mapsto \quad T \cup \operatorname{st}_{\hat{0}_{n,d}}(n); \end{cases}\\
j &: \begin{cases} \mathcal{S}(n-1,d) & \to \quad \mathcal{S}(n,d),\\ \qquad T & \mapsto \quad \operatorname{del}_T(n-1)\\ & \qquad \cup \operatorname{lk}_T(n-1) * \{n\}\\ & \qquad \cup \operatorname{st}_{\hat{1}_{n,d}}(\{n-1,n\}). \end{cases}
\end{aligned}$$

Theorem 2.1 shows that $\mathcal{S}(n,d)$ is bounded. Moreover, by Theorem 2.2 we know that $f(T)$ is a triangulation of $C(n-1,d)$ for all triangulations T of $C(n,d)$. The geometric description of f is as follows: starting with the triangulation T of $C(n,d)$, if one slides the vertex n along the moment curve until it coincides with the vertex $n-1$, then certain d-simplices of T will degenerate. Removing these degenerate simplices and renaming all occurrences of n by $n-1$ yields the triangulation $f(T)$.

The constructions of i and j can be described geometrically as follows: The cyclic polytope $C(n-1,d)$ can be embedded into the cyclic polytope $C(n,d)$ in many different ways. For example, there is an embedding that sends vertex k of $C(n-1,d)$ to vertex k in $C(n,d)$ for all $1 \le k \le n-1$. There is another embedding which sends vertex k to vertex k for all $1 \le k < n-1$ and vertex $n-1$ of $C(n-1,d)$ to vertex n of $C(n,d)$.

The map i uses the first embedding of $C(n-1,d)$ into $C(n,d)$ to embed a triangulation T of $C(n-1,d)$ into $C(n,d)$. This leads to a partial triangulation of $C(n,d)$. Since the "new" vertex n in $C(n,d)$ "sees" a convex polytope from outside, the only possibility to complete that partial triangulation is to join every facet of T that is "visible" by n to n. It is an easy calculation using Gale's Evenness Criterion [10, Theorem 0.7] that the given formula for i describes exactly that.

The map j uses the second embedding of $C(n-1,d)$ into $C(n,d)$ for embedding a triangulation T of $C(n-1,d)$ into $C(n,d)$. Again, the "new" vertex $n-1$ "sees" certain facets of a cyclic polytope with $n-1$ vertices. Given a triangulation of $C(n-1,d)$ that is embedded into $C(n,d)$ in this fashion, the only way to complete it to a triangulation of $C(n,d)$ is to join $n-1$ with the visible facets of the embedded $C(n-1,d)$. Gale's Evenness Criterion again allows us to obtain the formula for j. This proves that i and j are well-defined.

In the following we outline the proof of Theorem 1.1 by verifying the assumptions of the Suspension Lemma. Whenever the details are more involved we give a reference to a Lemma in Section 4 or 5, respectively.

If $n > d+2$ then $\hat{0}_{n-1,d} \neq \hat{1}_{n-1,d}$. From Lemma 4.1 (resp. 5.1) we get that all maps are order-preserving. From Lemma 4.2 (resp. 5.2) we know that no green element can be above a red one. By construction, $f \circ i$ and $f \circ j$ are both the identity on $\mathcal{S}(n,d)$. Since whether or not $\{n-d,\dots,n\}$ is contained in $i(T)$ (resp. $j(T)$) does not depend on T, it can easily be seen that the image of i is green and that the image of j is red. From Lemma 4.3 (resp. 5.3) it follows that the preimages of any $T \in \mathcal{S}(n,d)$ under f are bounded by $i(T)$ and $j(T)$. Finally, Lemma 4.4 (resp. 5.4) imply that $\hat{0}_{n,d}$ is the only green element in $f^{-1}(\hat{0}_{n-1,d})$ and that $\hat{0}_{n,d}$ is the only red element in $f^{-1}(\hat{1}_{n-1,d})$.

The proof of Theorem 1.1 then follows from the well-known fact that $C(d+2,d)$ has exactly two triangulations (i. e., its proper part is the empty set which is a (-1)-sphere) and induction on the codimension $n-d$ using the Suspension Lemma 3.1.

4. Lemmas on $\mathcal{S}_1(n,d)$

We first formulate a lemma that we are using to establish the comparability of elements in $\mathcal{S}_1(n,d)$.

Lemma 4.0. *Let T and T' be triangulations of $C(n,d)$. T is less than or equal to T' in $\mathcal{S}_1(n,d)$ if and only if there is a triangulation of the region between the characteristic sections of T and T' in $C(n,d+1)$.*

In other words, $T \leq_1 T'$ if and only if there is a set $\tilde{T}$ of $(d+1)$-simplices such that the following hold:

(i) *Every pair of $(d+1)$-simplices in T are admissible.*
(ii) *For every lower facet S of a $(d+1)$-simplex in $\tilde{T}$ either there is another $(d+1)$-simplex in $\tilde{T}$ containing S, or S is in T.*
(iii) *For every upper facet S of a $(d+1)$-simplex in $\tilde{T}$ either there is another $(d+1)$-simplex in $\tilde{T}$ containing S, or S is in T'.*
(iv) *Every d-simplex in $T \backslash T'$ is a lower facet of some $(d+1)$-simplex in $\tilde{T}$.*
(v) *Every d-simplex in $T' \backslash T$ is an upper facet of some $(d+1)$-simplex in $\tilde{T}$.*
(vi) *Every d-simplex in $T \backslash T' \cup T' \backslash T$ is a facet of at most one $(d+1)$-simplex in $\tilde{T}$.*

If the above assumptions are met we say "$\tilde{T}$ connects T and T'."

Proof. Given a set of $(d+1)$-simplices $\tilde{T}$ as in the assumption we get a sequence of increasing flips from T to T' by sorting the simplices of $\tilde{T}$ by any linear extension of "$\prec$," as was shown in [5]. On the other hand, every set of $(d+1)$-simplices corresponding to a sequence of increasing flips from T to T' has the properties listed above. $\square$

We now prove a sequence of lemmas that allows us to apply the Suspension Lemma in the case of $\mathcal{S}_1(n,d)$. Throughout this section it is always assumed that $n > d+2$.

Lemma 4.1. *The following maps are order-preserving.*

$$f : \begin{cases} \mathcal{S}_1(n,d) & \to & \mathcal{S}_1(n-1,d), \\ T & \mapsto & \operatorname{del}_T(n) \cup \operatorname{del}_{\operatorname{lk}_T(n)}(n-1) * \{n-1\}; \end{cases}$$
$$i : \begin{cases} \mathcal{S}_1(n-1,d) & \to & \mathcal{S}_1(n,d), \\ T & \mapsto & T \cup \operatorname{st}_{\hat{0}_{n,d}}(n); \end{cases}$$
$$j : \begin{cases} \mathcal{S}_1(n-1,d) & \to & \mathcal{S}_1(n,d), \\ T & \mapsto & \operatorname{lk}_T(n-1) * \{n\} \cup \operatorname{st}_{\hat{1}_{n,d}}(\{n-1,n\}). \end{cases}$$

Proof. The assertion for f is contained in Theorem 2.2. To prove the claims about i and j, observe that any increasing flip $\tilde{S}$ in $T \in \mathcal{S}_1(n-1,d)$ gives rise to an increasing flip $\tilde{S}$ in $i(T)$ and an increasing flip $\tilde{S} \backslash \{n-1\} \cup \{n\}$ in $j(T)$. This completes the proof of the lemma. □

Lemma 4.2. *Let $T <_1 T' \in \mathcal{S}_1(n,d)$ and $S_0 := \{n-d,\ldots,n\}$.*

(i) *If d is even and S_0 is in T then S_0 is also in T'.*
(ii) *If d is odd and S_0 is in T' then S_0 is also in T.*

Proof. The claim follows from the observation that for even d the simplex S_0 is an upper facet of $C(n,d+1)$, whereas for odd d it is a lower facet of $C(n,d+1)$. □

Lemma 4.3. *For all $T \in \mathcal{S}_1(n,d)$ we have $i(f(T)) \leq_1 T \leq_1 j(f(T))$.*

Proof. We start with a geometric description of the flip sequences that are going to establish the claim. Think of the action of $i \circ f$ as sliding vertex n of a triangulation T of $C(n,d)$ continuously to $n-1$ along the edge $\{n-1,n\}$ and then adding a collection of lower facets of $C(n,d+1)$ to the result. If one imagines this process taking place in $C(n,d+1)$ then one observes that the characteristic section of T slides to the characteristic section of $i(f(T))$. Every d-simplex S in T that contains n but not $n-1$ slides exactly across the $(d+1)$-simplex $S \cup \{n-1\}$ (see Figure 3). As the characteristic section T slides, these simplices S are the only ones whose paths sweep out $(d+1)$-dimensional simplices. This yields a set of $(d+1)$-simplices as in the assumptions of Lemma 4.0.

On the other hand, one may regard the action of $j \circ f$ as sliding vertex $n-1$ of T continuously to n along the edge $\{n-1,n\}$ and then adding a bunch of upper facets fo $C(n,d+1)$ to the result. However, the slide—considered in $C(n,d+1)$—moves the characteristic section of T to the characteristic section of $j(f(T))$. Again, the "tracks" of certain d-simplices provide the connecting set of $(d+1)$-simplices.

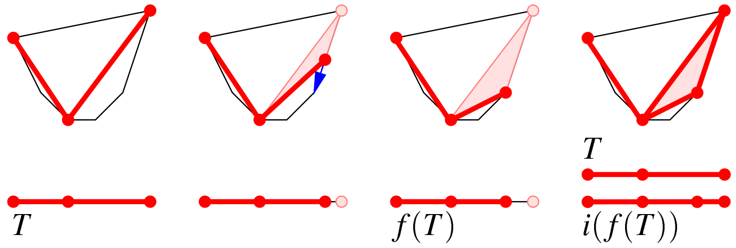

FIGURE 3. The characteristic section of T slides to the characteristic section of $f(T)$. The simplices containing n and not containing $n-1$ are sweeping out the increasing flips from $i(f(T))$ to T.

In the following, we give a combinatorial proof of this idea. For $T \in \mathcal{S}_1(n,d)$ define the following abbreviations.

$$A(T) := \{S \in T \,:\, n \in S, n-1 \notin S\},$$
$$B(T) := \{S \in T \,:\, n \notin S, n-1 \in S\}.$$

We prove that $i(f(T)) \leq_1 T$ for an arbitrary $T \in \mathcal{S}_1(n,d)$. Consider the following set of $(d+1)$-simplices in $C(n,d+1)$.

$$\tilde{A}(T) := \{S \cup \{n-1\} \,:\, S \in A(T)\}.$$

We claim that $\tilde{A}(T)$ connects $i(f(T))$ and T. To verify this claim, we check properties (i)-(vi) from Lemma 4.0 in Steps (i)-(vi) below.

Step (i): All pairs of $(d+1)$-simplices in $\tilde{A}(T)$ are admissible in $C(n,d+1)$ because any zig-zag-path of length $d+3$ can be transformed into a zig-zag-path of length $(d+2)$ by deleting n; deleting n from a simplex in $\tilde{A}(T)$, however gives a simplex in $f(T)$; all of these are clearly admissible in $C(n-1,d)$.

Step (ii): We now show that every lower facet S of a $(d+1)$-simplex $\tilde{S}$ in $\tilde{A}(T)$ is either in $i(f(T))$ or there is another $(d+1)$-simplex $\tilde{S}$ in $\tilde{A}(T)$ containing S.

To this end, let S be an arbitrary lower facet of a $(d+1)$-simplex $\tilde{S}$ in $\tilde{T}$. Hence, $\tilde{S} \backslash S$ is an even gap of S in $\tilde{S}$.

CASE 1: If $\tilde{S} \backslash S = n$ then S is contained in $f(T)$, in particular it is contained in $i(f(T))$.

CASE 2: If $\tilde{S} \backslash S = s < n-1$ then $F := S \backslash \{n-1\}$ is a $(d-1)$-simplex in T.

If F is a facet of $C(n,d)$ then it is an upper facet of $C(n,d)$ because $n-1$ is an odd gap in F. Then $S = F \cup \{n-1\}$ was already a lower facet of $C(n,d+1)$ containing n and $n-1$. However, all these lower facets of $C(n,d+1)$ are in $i(f(T))$ by construction, and thus $S \in i(f(T))$.

If F is not a facet of $C(n,d)$ then there is another simplex $S' \in T$ with $S' \neq S$ and $F \subset S'$. Since $n-1 \notin S'$ we have that $\tilde{S}' := S' \cup \{n-1\} \in \tilde{A}(T)$ with $\tilde{S}' \neq \tilde{S}$ and $S \subset \tilde{S}'$.

Step (iii): Next, we show that every upper facet S of a $(d+1)$-simplex $\tilde{S}$ in $\tilde{A}(T)$ is either in T or there is another $(d+1)$-simplex $\tilde{S}$ in $\tilde{A}(T)$ containing S.

To see this, let S be an arbitrary upper facet of a $(d+1)$-simplex $\tilde{S}$ in $\tilde{T}$. Hence, $\tilde{S}\backslash S$ is an odd gap of S in $\tilde{S}$.

CASE 1: If $\tilde{S}\backslash S = n-1$ then S is contained in T by the definition of $\tilde{A}(T)$.

CASE 2: If $\tilde{S}\backslash S = s < n-1$ then $F := S\backslash\{n-1\}$ is a $(d-1)$-simplex in T.

We show now that F is not a facet of $C(n,d)$: Because s is an odd gap of S in $\tilde{S}$ and $n-1 > s$ is an additional gap of F larger than s we conclude that s is an even gap of F in $\tilde{S}$. However, $n-1$ is clearly an odd gap of F in $\tilde{S}$ because $n \in F$. Thus, F contains an even and an odd gap, and is therefore not a facet of $\tilde{S}$. Consequently, it cannot be a facet of $C(n,d)$.

Hence, there is another simplex $S' \in T$ with $S' \neq S$ and $F \subset S'$. Since $n-1 \notin S'$ we have that $\tilde{S}' := S' \cup \{n-1\} \in \tilde{A}(T)$ with $\tilde{S}' \neq \tilde{S}$ and $S \subset \tilde{S}'$.

Step (iv): We now prove that every d-simplex in $i(f(T))\backslash T$ is a lower facet of some $(d+1)$-simplex $\tilde{S}$ in $\tilde{A}(T)$.

Let S be a d-simplex in $i(f(T))$ but not in T. There are two types of d-simplices in $i(f(T))\backslash T$: simplices of the form $S = S'\backslash\{n\} \cup \{n-1\}$ with $S' \in A(T)$ (case 1), and lower facets of $C(n,d+1)$ containing n and $n-1$ (case 2) .

CASE 1: If S is of the form $S = S'\backslash\{n\} \cup \{n-1\}$ with $S' \in A(T)$ then $\tilde{S} := S \cup \{n\}$ is in $\tilde{A}(T)$, and n is clearly an even gap of S in $\tilde{S}$. Thus, S is a lower facet of the simplex $\tilde{S} \in \tilde{A}(T)$.

CASE 2: If S is a lower facet of $C(n,d+1)$ containing n and $n-1$ then all gaps of S are even. Hence, all gaps of $F := S\backslash\{n-1\}$ are odd. Thus, F is an upper facet of $C(n,d)$. This leads to the existence of a d-simplex S' in T containing F. Since n is in F we know that n is also in S'. If $n-1 \in S'$ then $S = S' \in T$; contradiction to $S \in i(f(T))\backslash T$. Therefore, S' is in $A(T)$ and, consequently, $\tilde{S} := S' \cup \{n-1\}$ is a $(d+1)$-simplex in $\tilde{A}(T)$. Moreover, $S = F \cup \{n-1\}$ is a facet of $\tilde{S}$ because $\tilde{S}$ contains $n-1$. Additionally, S is — by the assumption of this case — a lower facet of $C(n,d+1)$, so it must be a lower facet of $\tilde{S}$.

Step (v): We now prove that every d-simplex in $T\backslash i(f(T))$ is an upper facet of some $(d+1)$-simplex $\tilde{S}$ in $\tilde{A}(T)$.

Let S be a d-simplex in T but not in $i(f(T))$. Then S is, in particular, not contained in $f(T)$. There are two types of d-simplices in $T\backslash f(T)$: simplices from $A(T)$ (case 1), and simplices containing both $n-1$ and n (case 2).

CASE 1: Assume S is in $A(T)$. Then $\tilde{S} := S \cup \{n-1\}$ is in $\tilde{A}(T)$. Since $n-1$ is an odd gap of S in $\tilde{S}$ we conclude that S is an upper facet of the $(d+1)$-simplex $\tilde{S}$ in $\tilde{A}(T)$.

CASE 2: If both $n-1$ and n are in S then S cannot be a lower facet of $C(n,d+1)$, because all lower facets of $C(n,d+1)$ containing both n and $n-1$ are in $i(f(T))$ by construction. Assume, for the sake of contradiction, that $F := S\backslash\{n-1\}$ is a facet of $C(n,d)$. Then either all gaps of F are even or all gaps of F are odd. Since $n \in F$ we know that $n-1$ is an odd gap of F, thus all gaps of F must be odd. However, then all gaps of $S = F \cup \{n-1\}$ are even; contradiction to the fact that S is not a lower facet of $C(n,d+1)$. We conclude that F is not a facet of $C(n,d)$. Thus, there is another simplex $S' \neq S$ in T containing F. Moreover, because $n-1 \notin S'$ but $n \in S'$, we have $S' \in A(T)$, and, consequently, $\tilde{S} := S' \cup \{n-1\}$ is in $\tilde{A}(T)$. Because $n-1$ is an odd gap of S' in $\tilde{S}$ we know that S' is an upper facet of $\tilde{S}$. Moreover, since S' and S are both in T they are admissible in $C(n,d)$. That means that S is also an upper facet of $\tilde{S}$. (A lower and an upper facet of a $(d+1)$-simplex in $C(n,d+1)$ are never admissible in $C(n,d)$.)

Step (vi): Finally we prove that every simplex in $T \backslash i(f(T)) \cup i(f(T)) \backslash T$ is a facet of at most one $(d+1)$-simplex in $\tilde{A}(T)$.

CASE 1: S is a d-simplex in $T \backslash i(f(T))$. If S is in $A(T)$ then $\tilde{S} = S \cup \{n-1\}$ is the only $(d+1)$-simplex in $\tilde{A}(T)$ containing S because membership in $\tilde{A}(T)$ requires the containment of $n-1$. If both n and $n-1$ are in S then we proceed as follows. Assume, for the sake of contradiction, that there are two distinct $(d+1)$-simplices $\tilde{S}$ and $\tilde{S}'$ in $\tilde{A}(T)$ containing S. Then S is a lower facet of one of them, say $\tilde{S}$ and an upper facet of the other one, say $\tilde{S}'$. In other words, $s := \tilde{S} \backslash \tilde{S}' < n-1$ is an odd gap of $\tilde{S}'$ and $s' := \tilde{S}' \backslash \tilde{S} < n-1$ is an even gap of $\tilde{S}$ in $\tilde{S} \cup \tilde{S}'$. By construction of $\tilde{A}(T)$, we know that $\tilde{S} = R \cup \{n-1\}$ and $\tilde{S}' = R' \cup \{n-1\}$ for some $R, R' \in A(T)$. In particular, R and R' are in T, thus admissible in $C(n,d)$. However, $R = \tilde{S} \backslash \{n-1\}$ and $R' = \tilde{S}' \backslash \{n-1\}$. Therefore, s is an even gap of R' and s' is an odd gap of R in $R \cup R'$. But that means, R' is a lower and R is an upper facet of the $(d+1)$-simplex $R \cup R'$; contradiction to the fact that R and R' are admissible in $C(n,d)$.

CASE 2: S is a d-simplex in $i(f(T)) \backslash T$. If S is of the form $S' \backslash \{n\} \cup \{n-1\}$ for some $S' \in A(T)$ then $S \cup \{n\}$ is the only $(d+1)$-simplex in $\tilde{A}(T)$ containing S because membership in $\tilde{A}(T)$ requires the containment of n. On the other hand, if S is a lower facet of $C(n,d+1)$ then there cannot be two distinct $(d+1)$-simplices which both contain S and are admissible in $C(n,d+1)$.

Steps 1 to 6 prove that the assumptions of Lemma 4.0 are satisfied, thus $\tilde{A}(T)$ connects $i(f(T))$ and T, proving $i(f(T)) \leq_1 T$.

Analogously, the set

$$\tilde{B}(T) := \left\{ \tilde{S} \cup \{n\} \, : \, S \in B(T) \right\}.$$

connects T and $j(f(T))$, proving $T \leq_1 j(f(T))$. We omit the details verifying this, which are similarly tedious. □

Lemma 4.4. *Let T be in $\mathcal{S}_1(n,d)$ and $S_0 := \{n-d, \ldots, n\}$.*

(i) *Let d be even, $f(T) = \hat{0}_{n-1,d}$, and $S_0 \notin T$. Then $T = \hat{0}_{n,d}$.*
(ii) *Let d be even, $f(T) = \hat{1}_{n-1,d}$, and $S_0 \in T$. Then $T = \hat{1}_{n,d}$.*
(iii) *Let d be odd, $f(T) = \hat{0}_{n-1,d}$, and $S_0 \in T$. Then $T = \hat{0}_{n,d}$.*
(iv) *Let d be odd, $f(T) = \hat{1}_{n-1,d}$, and $S_0 \notin T$. Then $T = \hat{1}_{n,d}$.*

Proof. For the proof of (i), let $T \in \mathcal{S}_1(n,d)$ for even d with $f(T) = \hat{0}_{n-1,d}$. Assume that $T \neq \hat{0}_{n,d}$. Recall that any such element T in $\mathcal{S}_1(n,d)$ can be connected to $\hat{0}_{n,d}$ by a sequence of decreasing flips (see Theorem 2.1). The map f is order-preserving (see Theorem 2.2); thus every element in such a sequence is mapped by f to $\hat{0}_{n-1,d}$. Because of Lemma 4.2, we may therefore assume that T differs from $\hat{0}_{n,d}$ by exactly one increasing flip corresponding to a $(d+1)$-simplex $\tilde{S}$. The simplex $\tilde{S}$ must contain both $n-1$ and n because otherwise it would give rise to a (non-trivial) flip from $\hat{0}_{n-1,d}$ to $f(T)$ in contradiction to $f(T) = \hat{0}_{n-1,d}$. The only flip in $\hat{0}_{n,d}$ containing $n-1$ and n corresponds, however, to the $(d+1)$-simplex $\tilde{S} = \{1, n-d, \dots, n\}$. The fact that S_0 is an upper facet of $\tilde{S}$, thus is contained in the triangulation resulting from this flip, gives a contradiction. Thus, claim (i) is proved.

The proofs of the remaining statements are analogous with

$$\tilde{S} = \begin{cases} \{n-d-1, n-d, \dots, n\} & \text{decreasing flip in } \hat{1}_{n,d} \text{ for (ii)}, \\ \{n-d-1, n-d, \dots, n\} & \text{increasing flip in } \hat{0}_{n,d} \text{ for (iii)}, \\ \{1, n-d, \dots, n\} & \text{decreasing flip in } \hat{1}_{n,d} \text{ for (iv)}. \end{cases}$$

□

5. Lemmas for $\mathcal{S}_2(n,d)$

This section is devoted to proving an analogous set of lemmas to the ones in the previous section, in order to guarantee the assumptions of the Suspension Lemma for $\mathcal{S}_2(n,d)$. Again, in the following $n > d+2$.

Lemma 5.1. *The following maps are order-preserving.*

$$f : \begin{cases} \mathcal{S}_2(n,d) & \to & \mathcal{S}_2(n-1,d), \\ T & \mapsto & T \backslash n := \mathrm{del}_T(n) \cup \mathrm{del}_{\mathrm{lk}_T(n)}(n-1) * \{n-1\}; \end{cases}$$

$$i : \begin{cases} \mathcal{S}_2(n-1,d) & \to & \mathcal{S}_2(n,d), \\ T & \mapsto & T \cup \mathrm{st}_{\hat{0}_{n,d}}(n); \end{cases}$$

$$j : \begin{cases} \mathcal{S}_2(n-1,d) & \to & \mathcal{S}_2(n,d), \\ T & \mapsto & \mathrm{lk}_T(n-1) * \{n\} \cup \mathrm{st}_{\hat{1}_{n,d}}(\{n-1,n\}). \end{cases}$$

Proof. That i and j are order-preserving is easily seen by considering the following facts: both maps embed a triangulation of $C(n-1,d)$ into $C(n,d)$; i copies the original triangulation, j renames $n-1$ to n. This does not change any height relations of piecewise linear sections to each other. Then both maps add a set of simplices which does not depend upon T. These are consequently at the same height for all triangulations. Thus, all height relations are maintained.

We now prove the assertion concerning f. We use the fact that the map $f : \mathcal{S}(n,d) \to \mathcal{S}(n-1,d)$ has the following geometric interpretation: given a triangulation T of $C(n,d)$, imagine a homotopy that "slides" the vertex n down the moment curve toward the vertex $n-1$, so that at $t=0$ one has the triangulation $T(0) = T$ of the original cyclic polytope $C(n,d)$, and at $t=1$ some of the simplices of $T(1)$ (namely those containing both $n-1$ and n) have become degenerate (volume zero). If one eliminates these degenerate simplices from $T(1)$ and relabels the vertex n by $n-1$ in the remaining simplices, one obtains the triangulation $f(T)$ of $C(n-1,d)$.

To prove that f is order-preserving, assume $T \leq_2 T'$, and we will show that $f(T) \leq_2 f(T')$. Fix a point $x \in C(n-1,d)$, and for $0 \leq t \leq 1$, let $T(t)(x)_{d+1}, T'(t)(x)_{d+1}$ be the $(d+1)^{st}$-coordinates of the image of x under the parametrized characteristic sections $T(t), T'(t) : C(n,d) \to C(n,d+1)$. Since $T \leq_2 T'$, we have

$$T'(t)(x)_{d+1} - T(t)(x)_{d+1} \geq 0 \text{ for } 0 \leq t < 1.$$

However $T'(t)(x)_{d+1} - T(t)(x)_{d+1}$ is clearly a continuous function of t, so the same inequality holds for $t = 1$. Hence

$$f(T)(x)_{d+1} = T(1)(x)_{d+1} \leq T'(1)(x)_{d+1} = f(T')(x)_{d+1}$$

which shows that $f(T) \leq_2 f(T')$

□

Lemma 5.2. *Let $T < T' \in \mathcal{S}_2(n,d)$ and $S_0 := (n-d,\ldots,n)$.*

(i) *If d is even and S_0 is in T then S_0 is also in T'.*
(ii) *If d is odd and S_0 is in T' then S_0 is also in T.*

Proof. The assertion follows from exactly the same argument as given in the proof of Lemma 4.2. □

Lemma 5.3. *For all $T \in \mathcal{S}_2(n,d)$ we have $i(f(T)) \leq_2 T \leq_2 j(f(T))$.*

Proof. This follows from Lemma 4.3 and the fact that $T \leq_1 T'$ always implies $T \leq_2 T'$ (see [4]). □

Lemma 5.4. *Let T be in $\mathcal{S}_2(n,d)$ and $S_0 := \{n-d,\ldots,n\}$.*

(i) *Let d be even, $f(T) = \hat{0}_{n-1,d}$, and $S_0 \notin T$. Then $T = \hat{0}_{n,d}$.*
(ii) *Let d be even, $f(T) = \hat{1}_{n-1,d}$, and $S_0 \in T$. Then $T = \hat{1}_{n,d}$.*
(iii) *Let d be odd, $f(T) = \hat{0}_{n-1,d}$, and $S_0 \in T$. Then $T = \hat{0}_{n,d}$.*
(iv) *Let d be odd, $f(T) = \hat{1}_{n-1,d}$, and $S_0 \notin T$. Then $T = \hat{1}_{n,d}$.*

Proof. This statement is independent of the partial order $\mathcal{S}_1(n,d)$ or $\mathcal{S}_2(n,d)$ under consideration. Thus the proof of Lemma 4.4 is valid here as well. □

6. THE GENERALIZED BAUES PROBLEM FOR $C(n,d)$ WITH $d \leq 3$

The goal of this section is to prove a new special case of the generalized Baues problem, but we must first recall the definition of the Baues poset Baues$(C(n,d))$. A *polytopal decomposition* δ of $C(n,d)$ is a collection $\{V_\alpha\}$ of vertex subsets $V_\alpha \subseteq [n]$ satisfying

- For all α, $|V_\alpha| \geq d+1$.
- Any two cyclic subpolytopes $C(V_\alpha,d), C(V_\beta,d)$ intersect in a common face (possibly empty).

- The union of the cyclic subpolytopes $C(V_\alpha, d)$ covers $C(n,d)$, i. e.,

$$\bigcup_\alpha C(V_\alpha, d) = C(n,d)$$

Say that a polytopal decomposition is *proper* if it is not the trivial decomposition $\{[n]\}$.

The *Baues poset* $\mathrm{Baues}(C(n,d))$ is the set of all proper polytopal decompositions ordered by refinement, i. e., $\delta = \{V_\alpha\} \leq \delta' = \{V_{\alpha'}\}$ if for every $V_\alpha \in \delta$ there exists a $V_{\alpha'} \in \delta'$ with $V_\alpha \subseteq V_{\alpha'}$. One can check that this agrees with the poset considered in the Generalized Baues Problem [1] for the case of subdivisions of a cyclic polytope. Theorem 1.2 now reads as follows.

Theorem 6.1. *For $d \leq 3$ the poset* $\mathrm{Baues}(C(n,d))$ *is homotopy equivalent to a sphere of dimension $n-d-2$.*

As was said in the introduction, our method will be to show that $\mathrm{Baues}(C(n,d))$ is homotopy equivalent to the suspension $\mathrm{susp}(\overline{\mathcal{S}_2(n,d)})$. We begin by defining a map ϕ from $\mathrm{Baues}(C(n,d))$ to intervals in $\mathcal{S}_2(n,d)$. An element δ of $\mathrm{Baues}(C(n,d))$ is a polytopal subdivision of $C(n,d)$, so let $\phi(\delta)$ be the set of all triangulations of $C(n,d)$ which refine it.

Lemma 6.2. *For any δ in* $\mathrm{Baues}(C(n,d))$*,*

- *the set $\phi(\delta)$ is a non-empty interval in $\mathcal{S}_2(n,d)$.*
- *$\phi(\delta)$ is* not *the improper interval consisting of all $\mathcal{S}_2(n,d)$.*
- *$\delta \leq \delta'$ in* $\mathrm{Baues}(C(n,d))$ *implies $\phi(\delta) \subseteq \phi(\delta')$.*
- *ϕ is injective, i. e., $\phi(\delta) = \phi(\delta')$ implies $\delta = \delta'$.*

Proof. Since δ is a polytopal subdivision of $C(n,d)$, and subsets V of the vertices of $C(n,d)$ span cyclic subpolytopes $C(V,d)$, we know that δ gives a decomposition

$$C(n,d) = \bigcup_\alpha C(V_\alpha, d)$$

for some vertex sets V_α in which the $C(V_\alpha,d)$ all have disjoint interiors. If we let $\hat{0}_\alpha$, $\hat{1}_\alpha$ denote the bottom and top triangulations of $C(V_\alpha,d)$, then one can form two triangulations T and T' respectively, by refining δ according to $\hat{0}_\alpha$ and $\hat{1}_\alpha$ respectively on each subpolytope $C(V_\alpha,d)$. It is then clear from the definition of $\mathcal{S}_2(n,d)$ that $\phi(\delta) = [T,T']$. This proves the first assertion of the lemma.

To prove the second assertion, note that since δ is a non-trivial polytopal subdivision of $C(n,d)$, it must use at least one $(d-1)$-simplex σ spanned by the vertices of $C(n,d)$ which lies interior to $C(n,d)$, and therefore this simplex σ would lie in every triangulation in $\phi(\delta)$. If $\phi(\delta)$ were all of $\mathcal{S}_2(n,d)$, then in particular this would imply that the bottom and top triangulations $\hat{0}, \hat{1}$ have this simplex σ in common. But one can easily check from the explicit description of the triangulations $\hat{0}, \hat{1}$ given in [4] or [5] that they have no interior $(d-1)$-simplices in common.

To see the third assertion, note $\delta \leq \delta'$ means that δ refines δ' as a polytopal subdivision, so any triangulation T which refines δ will also refine δ', and hence $\phi(\delta) \subseteq \phi(\delta')$.

To see the fourth assertion, it suffices to show that δ is completely determined by $\phi(\delta)$, in the sense that the set of $(d-1)$-simplices of δ is the intersection of all the sets of $(d-1)$-simplices of its triangulation refinements. Certainly the $(d-1)$-simplices of δ are contained in this intersection. This intersection cannot be larger because for each α, (using the notation of the first paragraph), the two triangulations $\hat{0}_\alpha, \hat{1}_\alpha$ share no common $(d-1)$-simplices interior to $C(V_\alpha, d)$. □

We next recall and introduce some notions about lattices. Given a lattice L with bottom and top elements $\hat{0}, \hat{1}$, an element of L which covers $\hat{0}$ (resp. is covered by $\hat{1}$) is called an *atom (coatom)*, resp. The lattice L is *atomic* (resp. *coatomic*) if the join of all the atoms is $\hat{1}$ (resp. the meet of all the coatoms is $\hat{0}$). Any interval $[x,y]$ in a lattice is a lattice itself, and will be called atomic or coatomic if it satisfies the previous conditions. An interval $[x,y]$ will be called *proper* if it is not the whole lattice $L = [\hat{0}, \hat{1}]$. Recall that the *proper part* of L is the subposet $\overline{L} := L \setminus \{\hat{0}, \hat{1}\}$.

We now define three *interval posets* as certain collections of intervals in L ordered by inclusion of intervals:

- $\mathrm{Int}(L)$ — all non-empty intervals in L,
- $\overline{\mathrm{Int}}(L)$ — all non-empty, proper intervals in L,
- $\overline{\mathrm{Int}}_{atomic}(L)$ — all non-empty, proper, atomic intervals in L.

Similarly one can define $\overline{\mathrm{Int}}_{coatomic}(L)$.

In [9] it was shown that $\mathrm{Int}(L)$ is canonically homeomorphic to L, and that $\overline{\mathrm{Int}}(L)$ is canonically homeomorphic to $\mathrm{susp}(\overline{L})$, i. e., the suspension of the proper part of L. One can view Lemma 6.4 below as asserting an analogous statement, up to homotopy, for $\overline{\mathrm{Int}}_{atomic}(L)$.

We recall (Theorem 2.5) that for $d \leq 3$ $\mathcal{S}_2(n,d)$ is a lattice, and note that Lemma 6.2 shows that ϕ defines an injective, order-preserving map $\mathrm{Baues}(C(n,d)) \to \overline{\mathrm{Int}}(\mathcal{S}_2(n,d))$.

Lemma 6.3. *For $d \leq 3$, the image of $\phi : \mathrm{Baues}(C(n,d)) \to \overline{\mathrm{Int}}\mathcal{S}_2(n,d)$ is exactly*

$$\overline{\mathrm{Int}}_{coatomic}(\mathcal{S}_2(n,d)).$$

Proof. To see that $\phi(\delta)$ is always a coatomic interval in $\mathcal{S}_2(n,d)$, we use the notation from the proof of Lemma 6.2, and note the following isomorphism of posets:

$$\phi(\delta) = [T, T'] \cong \prod_\alpha [\hat{0}_\alpha, \hat{1}_\alpha].$$

Since each interval $[\hat{0}_\alpha, \hat{1}_\alpha]$ is isomorphic to $\mathcal{S}_2(n',d)$ for some $n' < n$, the coatomicity of $\phi(\delta)$ would follow if we knew that $\mathcal{S}_2(n,d)$ is a coatomic lattice for $d \leq 3$. But if $\mathcal{S}_2(n,d)$ were *not* coatomic then its proper part $\overline{\mathcal{S}_2(n,d)}$ would be contractible (see, e. g.,[3, Theorem 10.14]), contradicting Theorem 2.6 above.

It remains then to show that every coatomic interval in $\mathcal{S}_2(n,d)$ is of the form $\phi(\delta)$ for some δ in $\mathrm{Baues}(C(n,d))$. For $d = 1$, this is trivial since the cyclic polytope $C(n,1)$ is simply a line segment with $n-2$ interior subdivision points. Triangulations of $C(n,1)$ are specified by their subset of interior vertices and $\mathcal{S}_2(n,d)$ is a Boolean algebra $\mathcal{B}_{n-2}$, so that every interval is coatomic, and it is easy to see that every interval is $\phi(\delta)$ for some δ in $\mathrm{Baues}(C(n,d))$.

For $d = 2,3$ the fact that every coatomic interval in $\mathcal{S}_2(n,d)$ is of the form $\phi(\delta)$ requires some argument. Assume we have such a coatomic interval $[T,T']$, and we will show how to construct its preimage δ. Form a graph G whose vertices are the d-simplices σ in the triangulation T', and whose edges correspond to a pair of d-simplices σ,σ' which share a $(d-1)$-simplex τ that is *not* a simplex in T. Let $\{G_\alpha\}$ be the various connected components of G, and define V_α to be the set of all vertices of $C(n,d)$ which lie in a simplex of G_α. We wish to prove two claims about these graphs:

- If σ,σ' are simplices of T' which correspond to an edge of G, then their union is a cyclic subpolytope $C(d+2,d)$ which supports a bistellar operation corresponding to a covering relation between T' and some coatom of the interval $[T,T']$.
- For each α, the connected component G_α is a path, and the set of d-simplices σ corresponding to G_α gives exactly the maximal simplices of the top triangulation $\hat{1}_\alpha$ of the cyclic subpolytope $C(V_\alpha,d)$.

Assuming these two claims for the moment, we show how to finish the proof. The second claim implies that the decomposition $C(n,d) = \bigcup_\alpha C(V_\alpha,d)$ defines a polytopal subdivision δ. Furthermore, as in the first paragraph of this proof, we know that $\phi(\delta)$ is equal to some coatomic interval $[T_\delta,T'_\delta]$, where T,T' refine δ and the restriction to $C(V_\alpha,d)$ of T,T' looks like $\hat{0}_\alpha$, $\hat{1}_\alpha$ respectively. By the second claim, this means that $T' = T'_\delta$. By both claims together, every coatom of the interval $[T_\delta,T'_\delta]$ is also a coatom of $[T,T']$ (i. e., all of the former coatoms lie above T), and hence by coatomicity of $[T,T']$ we must have $T = T_\delta$. Therefore $[T,T'] = \phi(\delta)$ as desired.

To show the first claim, assume σ,σ' are simplices of T' which correspond to an edge of G, so there intersection is a $(d-1)$-simplex τ which is not in T. Assume for the sake of contradiction that the union $\sigma\cup\sigma'$ does *not* support a bistellar operation as asserted in the claim. Then every coatom T'' of $[T,T']$ will have τ in its *submersion set* $\mathrm{sub}_{\lceil \frac{d}{2}\rceil}(T'')$ (see Proposition 2.3). Since the meet operation in $\mathcal{S}_2(n,d)$ corresponds to intersection of submersion sets, coatomicity of $[T,T']$ implies that $\mathrm{sub}_{\lceil \frac{d}{2}\rceil}(T)$ would also contain τ. But then the fact that τ is not a $(d-1)$-simplex of T would imply that

- if $d=2$ then $\tau = \{i,j\}$ must be "foiled" by some other $\tau' = \{k,l\}$ in $\mathrm{sub}_{\lceil \frac{d}{2}\rceil}(T)$ which satisfies $i<k<j<l$ (see Proposition 2.4).
- if $d=3$ then $\tau = \{i,j,k\}$ must be "foiled" by one of its edges, say $\{i,j\}$, *intertwining* another triple $\tau' = \{x,y,z\}$ in $\mathrm{sub}_{\lceil \frac{d}{2}\rceil}(T)$ in the sense that $x<i<y<j<z$ (see Proposition 2.4)

However in both of these cases, τ' would also lie in $\mathrm{sub}_{\lceil \frac{d}{2}\rceil}(T')$ since $T<T'$ in $\mathcal{S}_2(n,d)$, and hence would "foil" τ from being a $(d-1)$-simplex of T'. Contradiction.

To show the second claim, note that the first claim implies very stringent requirements on what σ, σ' can look like whenever they correspond to an edge in G:

- if $d = 2$, $\sigma = \{i, j, l\}, \sigma' = \{j, k, l\}$ for some $i < j < k < l$, and
- if $d = 3$, $\sigma = \{i, j, k, m\}, \sigma' = \{i, k, l, m\}$ for some $i < j < k < l < m$.

It is easy to check that these requirements, combined with the fact that a $(d-1)$-simplex τ can lie in at most two d-simplices of T', implies that the degree of any vertex in a connected component G_α can be at most 2. In fact, G_α is constrained to look like the following path of d-simplices:

- for $d = 2$,
$$\{v_1 v_2 v_r\}, \{v_2 v_3 v_r\}, \{v_3 v_4 v_r\}, \ldots, \{v_{r-2} v_{r-1} v_r\}$$
- for $d = 3$,
$$\{v_1 v_2 v_3 v_r\}, \{v_1 v_3 v_4 v_r\}, \{v_1 v_4 v_5 v_r\}, \ldots, \{v_1 v_{r-2} v_{r-1} v_r\}$$

where $v_1 < \cdots < v_r$ are the vertices V_α of G_α written increasing order. In both cases this description matches exactly the top triangulation $\hat{1}_\alpha$ of $C(V_\alpha, d)$. □

Once the image of ϕ has been established, Theorem 6.1 follows by combining

- Lemma 6.4 below,
- the above-mentioned fact that the proper interval poset $\overline{\text{Int}}(L)$ is homeomorphic to $\text{susp}(\overline{L})$, and
- Theorem 2.6 or Theorem 1.1.

Lemma 6.4. *Let L be any finite lattice. Then $\overline{\text{Int}}_{atomic}(L)$ (or $\overline{\text{Int}}_{coatomic}(L)$) is homotopy equivalent to $\overline{\text{Int}}(L)$.*

Lemma 6.4 follows from a more general lemma, which we think is of independent interest. We are indebted to P. Webb for the statement and proof of this lemma.

Lemma 6.5. *Let P be a poset with $\hat{0}, \hat{1}$. If $\{[x_i, y_i]\}_{i=1}^r$ is any finite collection of intervals with the open intervals (x_i, y_i) contractible for all i, then the inclusion*
$$\overline{\text{Int}}P \backslash \{[x_i, y_i]\}_{i=1}^r \hookrightarrow \overline{\text{Int}}P$$
induces a homotopy equivalence.

Lemma 6.4 then follows from Lemma 6.5 by letting $P = L$ and letting $\{[x_i, y_i]\}_{i=1}^r$ be the non-coatomic intervals of L. These non-coatomic intervals satisfy the hypothesis of the lemma by [3, Theorem 10.14].

Lemma 6.5 follows immediately from the following two sublemmas:

Sublemma 6.6. [2] *In a poset Q, if $\{q_i\}_{i=1}^r$ is a finite subset with $Q_{<q_i}$ contractible for all i, then the inclusion*
$$Q \backslash \{q_i\}_{i=1}^r \hookrightarrow Q$$
induces a homotopy equivalence.

Proof. Re-index the elements $\{q_i\}_{i=1}^r$ in such a way that $q_i > q_j$ in Q implies $i < j$. Then

$$(Q \backslash \{q_1, \dots, q_{i-1}\})_{<q_i} = Q_{<q_i}$$

is contractible for all i, so an application of Quillen's Fiber Lemma [3, Theorem 10.5] proves the homotopy equivalence by induction on i. □

We can apply Sublemma 6.6 with $Q = \overline{\mathrm{Int}}P$ to prove Lemma 6.5 once we have established

Sublemma 6.7. *In a poset P with $\hat{0}, \hat{1}$, if an open interval (x,y) is contractible, then $\left(\overline{\mathrm{Int}}P\right)_{<[x,y]}$ is contractible.*

Proof. Note that

$$\left(\overline{\mathrm{Int}}P\right)_{<[x,y]} = \overline{\mathrm{Int}}[x,y].$$

But $\overline{\mathrm{Int}}[x,y]$ is homeomorphic to the suspension $\mathrm{susp}(x,y)$ by [9], and hence contractible since (x,y) was assumed contractible. □

7. Open Problems

The following are some remarks and remaining open problems about triangulations of cyclic polytopes which we consider interesting.

1. The proof of Theorem 6.1 relied heavily on the fact established in [4] that $\mathcal{S}_2(n,d)$ is a lattice for $d \leq 3$. Unfortunately, computer calculations have shown that $\mathcal{S}_2(9,4)$ and $\mathcal{S}_2(10,5)$ are **not** lattices, rendering this lattice-theoretic method of proof invalid for $d \geq 4$ (and resolving negatively Conjecture 2.13 of [4]). However we would still conjecture the following:

   Conjecture 7.1. *The image of $\phi : \mathrm{Baues}(C(n,d)) \to \overline{\mathrm{Int}}\mathcal{S}_2(n,d)$ is exactly the subposet consisting of those closed intervals in $\mathcal{S}_2(n,d)$ whose open interval is not contractible.*

   As in Section 6, this conjecture would resolve in the affirmative the Baues problem for triangulations of all cyclic polytopes. It is easy to see that one direction in this conjecture is true, namely that any interval in the image of ϕ is isomorphic to a Cartesian product of posets isomorphic to $\mathcal{S}_2(n_\alpha, d)$ for various n_α, and hence has proper part homotopy equivalent to a sphere. Consequently, the above conjecture also has as a corollary the calculation of the homotopy type and Möbius function for all (open) intervals in $\mathcal{S}_2(n,d)$.
2. Do the partial orders $\mathcal{S}_1(n,d), \mathcal{S}_2(n,d)$ coincide?

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