CONTINUITY PRINCIPLE AND EXTENSION PROPERTIES OF MEROMORPHIC MAPPINGS WITH VALUES IN NON KÄHLER MANIFOLDS

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Abstract. In this paper we are proving an analogue of E. Levi Continuity Principle for meromorphic mappings with values in general complex spaces. We also describe the singularities of meromorphic mappings into complex spaces carrying pluriclosed Hermitian metric forms and geometrical obstructions to removing them.

Contents

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0. Introduction

0.1. Continuity principle. Let us start with recalling the "Continuity principle" for meromorphic functions due to E. Levi, see [Lv]. Consider a meromorphic function f, which is defined on a ring domain $\Delta^n \times A(r,1)$. Here $A(r,1) = \{z_{n+1} \in \mathbb{C} : r < |z_{n+1}| < 1\}$ is an annulus in $\mathbb{C}$ and Δ^n is a unit polydisk. Denote by S the set of points $s \in \Delta^n$, such that the restriction $f_s := f\mid_{\{s\}\times A(r,1)}$ is well defined and extends as a meromorphic function of one variable onto the whole disk Δ.

Theorem (E. Levi). *If S is not contained in a countable union of locally closed proper analytic subsets of Δ^n then f meromorphically extends onto the whole polydisk Δ^{n+1}.*

Recall that by a locally closed analytic subset of Δ^n one means an analytic set in some open subset of Δ^n.

In this paper we are interested wether this theorem can be generalised to the case of meromorphic mappings with values in more or less arbitrar complex spaces.

Recall that a meromorphic mapping $f : D \to X$ between normal complex spaces D and X is given by a holomorphic map $f : D \setminus I(f) \to X$ of the complement to an analytic set $I(f)$ of codimension at least two, such that the closure $\bar{\Gamma}_f$ of the graph of f is an analytic set in the product $D \times X$. One requires also that $\bar{\Gamma}_f$ should be proper over D, i.e. restriction $\pi\mid_{\bar{\Gamma}_f} : \bar{\Gamma}_f \to D$ of natural projection $\pi : D \times X \to D$ onto $\bar{\Gamma}_f$ is proper. As $I(f)$ one takes usually the *minimal* analytic subset of D such that f is *holomorphic* on $D \setminus I(f)$ and calles $I(f)$ the set of points of indeterminancy of f.

In the case $X = \mathbb{CP}^1$ this gives exactly a meromorphic function on D, see [Re].

One cannot expect for the direct generalisation of the E. Levi theorem to the mappings into general complex spaces as the following couple of examples show.

1. Let $X = \mathbb{C}^2 \setminus \{0\}/(z \sim 2z)$ be a Hopf surface. And let $\pi : \mathbb{C}^2 \setminus \{0\} \to X$ be a natural projection. Embedd the unit bidisk $i : \Delta^2 \to \mathbb{C}^2$ in a standart way, and define $f = \pi \circ i : \Delta^2 \setminus \{0\} \to X$. Then for all $s \in S = \Delta \setminus \{0\}$ f_s is holomorphic on Δ, but f_0 doesn't extends to zero.
2. In [Hs-1] A. Hirschowitz constructed an example of compact complex surface X (of class VII_0) and holomorphic mapping $\pi : \mathbb{B}^2_* \to X$ from the puctured ball into X, such that for any complex curve $C \ni 0$ the restriction $\pi\mid_{C\setminus\{0\}}$ extends holomorphically onto C, but π cannot be meromorphically extended to origin. If we put $f = \pi \circ i : \Delta^2_* \to X$ as above, we obtain a mapping $f : \Delta \times \Delta_* \to X$ such that for all $s \in \Delta$ f_s is holomorphic on Δ, but f is not a meromorphic mapping of the couple (z_1, z_2).
3. In the example of M. Kato, see [Ka-1] or consrtruction of our Example 1, S can be made equal to $\Delta \cap \{z : |z-1| < 1\}$, and for all $s \in \Delta \setminus S$ f_s doesn't extends holomorphically onto Δ. I.e. the maximal set S of points for which f_s extends onto Δ can be "essentially" smaller then Δ!

Hovewer, let us follow the main steps of the proof of E. Levi theorem.
Step 1 . Localisation. For an integer p denote by S_p the set os thouse $s \in S$ that the extension f_s is well defined and has at most p poles counting with multiplicities. Then one immediately sees that there is a point $z_0 \in \Delta^n$ and a natural p such that for any neighborhood $U \ni z_0$ the set $S_p \cap U$ is not contained in a proper analytic subset of U.
Step 2. "Lewisches Einsatz". Take such z_0 as above and take $\varepsilon_1 > \varepsilon_2 > 0$ small enough to find a neighborhood of $U \ni z_0$ such that f is *holomorphic* in $\bar{U} \times A(1-\varepsilon_1, 1-\varepsilon_2)$. Then f meromorphically extends onto $U \times \Delta$.
Step 3. Globalisation. If $U \neq \Delta^n$ one can (obviously) allwayse find $z_1 \in \partial U \cap \Delta^n$ and p such that the set U_p of $z \in U$ such that f_z has at most p poles is not analytic in any neighborhood of z_1. This shows that the maximal U such that f is meromorphic on $U \times \Delta$ is equal to Δ^n.

Our first observation is that the statement of *Step 2 (i.e. main ingredient of E. Levi Theorem)* remains true for the mappings into an "almost" *arbitrary* complex space instead of $\mathbb{CP}^1$. One only need to express the "boundedness of the number of poles" in other terms. Taking into account that the winding number of $f_s\mid_{\partial\Delta}$ is uniformly bounded, say by N, one easily sees that to say that f_s takes the value ∞ not more then p times is equivalent to say that the area of the image of f_s, counted with multiplicities, is not more then $N\cdot(p+1)$. Here $\mathbb{CP}^1$ is equipped with the usual spherical metric of total area one.

Definition 0.1. *We say that a complex space X is disk-convex in dimension k if for any compact $K \subset\subset X$ there is another compact $\hat{K}$ such that for every meromorphic mapping $f : \bar{\Delta}^k \to X$ with $f(\partial\Delta^k) \subset K$ one has $f(\bar{\Delta}^k) \subset \hat{K}$.*

All compact spaces are of course disk-convex. More generally all k- convex spaces are disk-convex in dimension k.

Put $A^k(r,1) = \{z \in \mathbb{C}^k : r < \|z\| < 1\}$ and $A^k_s(r,1) := \{s\} \times A^k(r,1)$ for $s \in \Delta^n$. Let $f : \Delta^n \times A^k(r,1) \to X$ be a holomorphic mapping into a normal complex space X. Denote by S the set of points $s \in \Delta^n$ such that the restriction $f_s := f\mid_{A^k_s(r,1)}$ extends meromorphically onto the polydisk $\Delta^k_s := \{s\} \times \Delta^k$.

Theorem 1 *(Continuity principle). Let $f : \Delta^n \times A^k(r,1) \to X$ be a holomorphic mapping into a normal, disk-convex in dimension k complex space X. Suppose that there is a constant $C_0 < \infty$ and a compact $K \subset X$ such that for s in some subset $S \subset \Delta^n$, which is thick at origin:*

(a) the restriction $f_s := f\mid_{A^k_s(r,1)}$ is well defined and extends meromorphically onto the polydisk $\Delta^k_s := \{s\} \times \Delta^k$, and $\mathrm{Vol}(\Gamma_{f_s}) \le C_0$ for all $s \in S$;

(b) $f(\Delta^n \times A^k(r,1)) \subset K$ and $f_s(\Delta^k) \subset K$ for all $s \in S$.
Then:

1. If $n = 1$ then there is a neighborhood $U \ni 0$ in Δ such that f extends meromorphically onto $U \times \Delta^k$.

2. If $n \geq 2$ and X has bounded cycle geometry in dimension k, then again there is a neighborhood $U \ni 0$ in Δ^n and a meromorphic extension of f onto $U \times \Delta^k$.

Here Γ_{f_s} denotes the graph of f_s and the volumes are taken with respect to some Hermitian metric h on X and a standart Euclidean metric on $\mathbb{C}^k$. The condition of finitness clearly doesn't depends on the particular choice of the metrics. As usually saying that a set $S \subset \Delta^n$ is thick at the point z_0 we mean that for any neighborhood $U \ni z_0$ $S \cap U$ is not contained in a proper analytic subset of U.

Denote by $\mathcal{B}_k(X)$ the Barlet space of compact analytic cycles of dimension k in X. This is an analytic space, which has not more then countable number of components.

Definition 0.2. *We say that a complex space X has bounded cycle geometry in dimension k if all connected components of the Barlet space $\mathcal{B}_k(X)$ are compact.*

In other words X has bounde cycle geometry in dimension k if all *irreducible* components of $\mathcal{B}_k(X)$ are compact and all connected components of $\mathcal{B}_k(X)$ are just the finite unions of irreducible ones. Note again that the property to have bounded cycle geometry doesn't depend on the choice of Hermitian metric.

There are two points to discuus here. The first one is the boundedness of cycle geometry condition in the case $n \geq 2$ on the contrary to the case $n = 1$. The second - local character of the extension obtained, on the contrary to the classical case $X = \mathbb{CP}^1$, i.e. what are the obstructions to the globalisation (Step 3)?

Cocerning the first point we shall construct a following example, showing that if $n \geq 2$ the condition of boundedness of cycle geometry cannot be removed. Denote by z_1, z_2, z_3, z_0 the coordinates in $\mathbb{C}^4 = \mathbb{C}^3_z \times \mathbb{C}^1_{z_0}$, $z = (z_1, z_2, z_3)$.

Example 1. *There is a compact complex 4-dimensional manifold X and a holomorphic mapping $f : \Delta^3 \times A^1(r,1) \to X$ such that:*

(1) for any $s \in S = \{(z_1, z_2, z_3) \in \Delta^3 : |z_1|^2 > |z_2|^2 + |z_3|^2\}$ the restriction $f_s = f\mid_{A^1_s(r,1)}$ holomorphically extends onto Δ_s;

(2) for any $t > 1$ there is a constant $C_t < \infty$ such that for all $s \in S_t = \{(z_1, z_2, z_3) \in \Delta^3 : |z_1|^2 > t \cdot (|z_2|^2 + |z_3|^2)\}$ one has $\mathrm{area}(\Gamma_{f_s}) \leq C_t$;

(3) but for all $z \in \Delta^3 \setminus \bar{S} = \{(z_1, z_2, z_3) \in \Delta^3 : |z_1|^2 < |z_2|^2 + |z_3|^2\}$ every point of the innner circle of the annulai $A^1_z(r,1) := \{z_0 \in \Delta_z : 1 > |z_0|^2 > |z_2|^2 + |z_3|^2 - |z_1|^2\}$ consists of essentially singular points of $f_z : A^1_z(r,1) \to X$, here $r^2 = |z_2|^2 + |z_3|^2 - |z_1|^2$.

In particular this f doesn't extend meromorphically to any open set of the type $U \times \Delta$, where U is a neighborhood of the origin in Δ^3. The cycle geometry of this X is not bounded in dimension one. If one wishes to have an example of such type with $n = 2$, one can take a restriction of f onto $\{z_3 = 0\}$.

Our idea for the proof of the Continuity principle is rufly the following. First, adapting the Barlet construction to our (noncompact) case, one shows that there is a finite dimensional normal complex space $\mathcal{C}_f$, parametrising (analytically) the cycles Z in $\Delta^{n+k} \times X$ which:

1) project onto $\{z\}\times\Delta^k$ for some z in the neighborhood of zero in Δ^n;
2) $Z\cap(\Delta^n\times A^k(r,1)\times X)\subset\Gamma_f$.

From the condition on the thickness of S in the Theorem 1 one gets that $\dim\mathcal{C}_f=n$. Take now the universal space $\mathcal{Z}\to\mathcal{C}_f$ and the evaluation map $F:\mathcal{Z}\to\Delta^{n+k}\times X$. For a neighborhood $U\ni 0$ denote $\mathcal{Z}_U=(\pi\circ F)^{-1}(U)$. Here $\pi:\Delta^{n+k}\times X\to\Delta^n$ is a natural projection. If there exists a neighborhood $U\ni 0$ in Δ^n such that $F\mid_{\mathcal{Z}_U}$: $\mathcal{Z}_U\to U\times\Delta^k\times X$ is proper then $F(\mathcal{Z}_U)$ will be an analytic subset in $U\times\Delta^k\times X$ extending the graph of $f\mid_{U\times A^k(r,1)}$.

For $n=1$ F is allwayse proper, because a nonconstant holomorphic map from the unit disk to the complex space is alwayse proper. For $n\geq 2$ the condition of boundedness of cycle geometry naturally comes out. There is one more case, important in the applications, when F is proper.

Corollary 1. *Let $f:\Delta^n\times A^k(r,1)\to X$ be a holomorphic map into a normal, disk-convex in dimension k complex space X. Suppose that:*

(1) for every $s\in\Delta^n$ outside of thin set, the restriction f_s extends meromorphically onto Δ^k_s;

(2) there is a compact $K\subset\subset X$ such that $f_s(\Delta^k_s)\subset K$ for all s and $f(\Delta^n\times A^k(r,1))\subset K$;

(3) the volumes of the graphs Γ_{f_s} are uniformly bounded in Δ^n, i.e. there exists $C_0<\infty$ s.t. $\mathrm{vol}(\Gamma_{f_s})\leq C_0$ for all s.
Then f meromorphically extend onto Δ^{n+k}.

Note that here one doesn't needs the boundedness of cycle geometry of X. It is worth, probably to point out one case when the boundedness of cycle geometry is satisfied automatically - when $k=\dim X-1$. Really the cycle space of divisors is allwayse compact (provided X is compact).

0.2. Hartogs-type extension theorem and spherical shells. The second point, which should be discussed in concern with the Continuity principle, is the local character of the extension obtained. Namely a mapping into a general space extends only to $U\times\Delta^k$. While in the case $X=\mathbb{CP}^1$ $U=\Delta^n$. And in fact in general the maximal U such that the map can be extended onto $U\times\Delta^k$ can be smaller then Δ^n, as we had see above.

The reason is that the volume of Γ_{f_z} can tend to $+\infty$ when z approach $\partial U\cap\Delta^n$. Take for exapmle the Hopf surface $X=(\mathbb{C}^2\setminus\{0\})/(z\sim 2z)$ and a canonical projection $f:\mathbb{C}^2\setminus\{0\}\to X$. Concider its restriction onto $\Delta^2\setminus\{0\}$. $(1,1)$-form $w=i/2\frac{dz_1\wedge d\bar z_1+dz_2\wedge d\bar z_2}{\|z\|^2}$ defines a canonical metric on X. One easily sees that $\mathrm{area}(\Gamma_{f_s})\sim\log|s|$ when $s\to 0$.

Such type of behavoir means a violation of the Hartogs extension phenomenon for the mapppings into this X. Denote by

$$H^k_n(r):=\{(z',z'')\in\Delta^{n+k}:1-r<\|z''\|<1 or \|z'\|<r\}=$$

$$= \Delta^n \times A^k(1-r,1) \cup \Delta^n(r) \times \Delta^k \tag{0.1}$$

the k-concave Hartogs figure in $\mathbb{C}^{n+k}$.

Definition 0.3. *We say that the meromorphic mappings into the space X have the Hartogs-type extension property in bidimension (n,k) if any meromorphic map $f : H_n^k(r) \to X$ extends meromorphically onto Δ^{n+k}.*

So let us look now for the reasons of falue of Hartogs-type extension of meromorphic mappings with values in (normal) complex spaces. When the image space is Kähler there are no obstructions for the Hartogs-type extension, see [Iv-3]. Here we shall try to propose a general conjecture and state a result in the direction of proving it. Our point of depart will be the following observation:

every compact complex manifold of dimension $k+1$ carries a Hermitian metric form w with $dd^c w^k = 0$.

Really, the condition to carry dd^c-closed strictly positive (k,k)-form for a compact complex manifold is alternative to that of carrying a bidimension $(k+1,k+1)$-current T with $dd^cT \geq 0$ but $\not\equiv 0$. This in the case of $\dim X = k+1$ is a nonconstant plurisubharmonic function, which on compact X doesn't exist.

Let us introduce the class $\mathcal{G}_k$ of normal complex spaces, carrying a nondegenerate positive dd^c-closed strictly positive (k,k)-forms. Note that the sequence $\{\mathcal{G}_k\}$ is rather exaustive: $\mathcal{G}_k$ contains all compact complex manifolds of dimension $k+1$.

Note also that compact spaces from $\mathcal{G}_k$ have bounded cycle geometry in dimension k, see 1.4. We conjecture that meromorphic mappings into the spaces of class $\mathcal{G}_k$ are "almost Hartogs-extendable" in bidimension (n,k) for all $n \geq 1$:

Cojecture. Every meromorphic map $f : H_n^k(r) \to X$, where $X \in \mathcal{G}_k$ and is disk-convex in dimension k, extends to a meromorphic map from $\Delta^{n+k} \setminus A$ to X, where A is an analytic subvariety of Δ^{n+k} (may be empty) of pure codimension $k+1$. Moreover, if $A \neq \emptyset$, then for every sphere $\mathbb{S}^{2k+1}$ embedded into $\Delta^{n+k} \setminus A$ in such a way that $[\mathbb{S}^{2k+1}] \neq 0$ in $H_{2k+1}(\Delta^{n+k} \setminus A, \mathbb{Z})$, $f(\mathbb{S}^{2k+1})$ also is not homologous to zero in X.

In this paper we shall prove this conjecture in the case $k = 1$.

Definition 0.4. *Let us call a Hermitian form w on X plurinegative if $dd^c w \leq 0$.*

The class of normal complex spaces admiting plurinegative Hermitian metrik form we shall denote by $\mathcal{P}_-$.

Theorem 2. *Let $f : H_n^1(r) \to X$ be a meromorphic map into a disk-convex complex space X which admits a plurinegative Hermitian metrik form. Then:*

(1) f extends to a meromorphic map $\hat{f} : \Delta^{n+1} \setminus A \to X$, where A is closed $(n-1)$-polar subset of Δ^{n+1}.

(2) If moreover, w is pluriclosed then A is a analytic subvariety of Δ^{n+1} of pure codimension two (may be empty). If $A \neq \emptyset$ then for every sphere $\mathbb{S}^3$ embedded into

$\Delta^{n+1}\setminus A$ *in such a way that* $[\mathbb{S}^3]\neq 0$ *in* $H_3(\Delta^{n+1}\setminus A,\mathbb{Z})$*, its image* $f(\mathbb{S}^3)$ *also is not homologous to zero in* X*.*

Remarks. 1. One can estimate the number of irreducible components of the singularity set A in this theorem meating a compact subset $P\subset\subset\Delta^{n+1}$. Namely, let a compact $K\subset X$, which contains $\mathrm{cl}[f(P\setminus S)]$, is chosen to be a finite subcomplex of CW- complex X. Choose a point $z'\in\Delta^{n-1}$ such that A intersects $\Delta^2_z:=\{z\}\times\Delta^2$ by discrete set $A_{z'}$. Let $A_{z'}\cap\partial P=\emptyset$. Then

$$|A_{z'}\cap P|\leq|\int_{\partial(P\cap\Delta^2_{z'})}d^cw|\cdot[\inf\{|\int_\gamma d^cw|:\gamma\in H_3(K,\mathbb{Z}),\int_\gamma d^cw\neq 0\}]^{-1}. \quad (0.2)$$

In other words the number of branches of singular set (and moreover, their existence) is bounded by the differential geometry of X. Remark that the subset $\{|\int_\gamma d^cw|:\gamma\in H_3(K,\mathbb{Z}),\int_\gamma d^cw\neq 0\}\subset\mathbb{R}$ is separated from zero, see (2.2.14).
2. Let us call a spherical shell of dimension $k+1$ in complex space X an image Σ of the standard sphere $\mathbb{S}^{2k+1}\subset\mathbb{C}^{k+1}$ under the meromorphic map of some neighborhood of $\mathbb{S}^{2k+1}$ into X, such that Σ is not homologous to zero in X. This notion is close to the notion of the global spherical shell, introduced by Kato, see [Ka-3]. Thus we obtain the following

Corollary 2. *Let X be a disk-convex complex space which possess a pluriclosed Hermitian metric form. Then the following is equivalent:*

(a) X possesses a meromorphic extension property in bidimension $(n,1)$ for all $n\geq 1$,

and thus in all bidimensions (n,k).

(b) X contains no two-dimensional spherical shells.

3. A wide class of complex manifolds without two-dimensional spherical shells is for example a class of such manifolds X for which the Hurewicz homomorphism $\pi_3(X)\to H_3(X,Z)$ vanishes.
4. *Corollary 2* for the case when X is compact complex surface was proved in [Iv-2] using the classification of surfaces.
5. A number of applications will show that characterisation of the obstructions for the extendibility of meromorphic mappings in terms of spherical shells is useful.

Theorem 2 was proved in [Iv-2] under an additional (very restrictive) assumption: *manifold X doesn't containe rational curves.* In this case meromorphic maps into X are just *holomorhpic* . Let us explaine the difference. One can prove, see [Iv-4], that if *holomorphic* mappings into a complex space X are Hartogs extendable in bidimension (n,k), then they are Hartogs extendable in bidimension $(n+1,k)$. Thus, for example, to prove the Hartogs-type extension for holomorphic mappings under the assumption of absence of rational curves, it is sufficient to prove the extendibility of holomorphic mappings with values in X from $H^1_1(r)$ onto the unit bidisk.

Hovewer such reduction fail for the meromorphic maps, i.e. in the presnce of rational curves. In [Iv-4] we had constructed the following example, shoving that increasing of dimension in general for meromorphic mappings doesn't works.

Example 2. *There exists a compact complex three-fold X such that:*

(a) For every domain D in $\mathbb{C}^2$ every meromorphic mapping $f : D \longrightarrow X$ extends to a meromorphic mapping $\hat{f} : \hat{D} \longrightarrow X$. Here $\hat{D}$ stands for the envelope of holomorphy of D.

(b) But there exists a meromorphic mapping $F : B^3 \setminus \{0\} \longrightarrow X$ from punctured threeball into X which does not extend to the origin.

This example shows that for the meromorphic mappings the Hartogs-type extendibility in bidimension $(1,1)$ doesn't imply the extendibility in bidimension $(2,1)$ and, what is more suprusing, it doesn't imply the extendibility in bidimension $(1,2)$! Of course this X doesn't carries a pluruiclosed metric form.

Continuity principle will be applied in the proof of Theorem 2 in the following context. First we shall prove in 2.2 this theorem for n=1. Then in the case $n \geq 2$ first extend the mapping onto the set $\Delta^n \setminus S$, where S is closed (n-1)-polar subset of Δ^{n+1}. Taking an appropriate projections we shall find ourselves in the assumtion of the Corollary 1 with $k = 2$. (In fact we cold also use the C.P.with $k = 1$ together with an observation that spaces carrying a plurinegative metrik-forms have bounded cycle geometry in dimension one.) This will give the result.

In fact extension of the map from the complement of the thin set makes the major diffuculty here. In Kähler case it was done by Y.-T. Siu using his theorem on analyticity of upper level sets of Lelong numbers for the closed positive currents. In our case currents (i.e. preimages of the metric form under the mapping f) are not longer closed, but only pluriclosed, even only plurinegative. For such currents the Theorem of Siu is not longer valid. And this was a motivation for us to develope the approach based on the Continuity principle.

0.3. Kähler case: Griffiths approach revisited. As the title of this article says and as we had just explained above, our object here are meromorphic mappings into *non Kähler* complex manifolds. Hovewer let us make one remark, which is probably interesting especially in the Kähler case. First result on extension of meromorphic mappings is due to P. Griffiths, who proved in [Gr] that every holomorphic mapping from a punctured ball $\mathbb{B}^n_* \subset \mathbb{C}^n$ into a compact Kähler manifold extends meromorphically to origin. He conjectured then that if A is a codimension two analytic subvariety in complex manifold D and G a sufficiently small neighborhood of A then any meromorphic map $f : D \setminus \bar{G} \to X$ into a compact Kähler manifold X extends meromorphically onto the whole D. He also proposed an approach to the proof of this statement in the case when the ampping f is defined on $D \setminus A$. Namely, he proposed to estimate the integrals

$$\int_{K\setminus A}(f^*w)^q, q=1,...,n$$

where K is a relatively compact subdomain in D, and f^*w is a pull back (as a current) of the Kähler form w from X. This whould meen the estimate of the volume of the graph of f in the neighborhood of $A\times X$. So the application of the Bishop extension theorem for analytic sets would give the statement. Indeed, his proof in the case $A=\{point\}$ was exactly this. In fact, as one can see in the proof of the Theorem 2 (and it was known in the Kähler case), for $q=1,2$ one can estimate thouse integrals in the same way as Griffiths did. But then B. Shiffman and Taylor constructed an example of a bedegree (1,1) current T in $\Delta^3\setminus\{line\}$ such that $\int_{\Delta^3\setminus\{line\}}T^3=\infty$, see [Si-3]. As we already had mentioned the codimension two singularities where succefully removed by Y.-T. Siu in [Si-3] using a different approach.

Hovewer our Continuity Principle says exactly that it is enough to estimate just $\int_{K\setminus A}f^*w$! Thus we obtain another proof of the theorem of Siu, which works also in non Kähler case.

There was an attempt in [Sb] to realise the Griffiths approach by means of Fubini theorem. Unfortunatly it containes an unrecouverable gap.

0.4. Applications, generalisations, open questions: meromorphic correspondences, complex Plateau problem, coverings of compact complex manifolds. It is natural to consider also the extension of meromorphic mappings from *singular* spaces. This is equivalent to considering a multivalued meromorphic correspondences from smooth domains, and this to single valued maps into symmetric powers of the image space, see 3.1 for details. However one pays price for such a reductions. In this direction we construct in 3.2 the following

Example 3. *There is a compact complex surface X such that:*
(a) every meromorphic map $f:H^2(r)\to X$ extends meromorphically onto Δ^2, but
(b) there exists a two-valued meromorphic correspondence Z between punctured two-ball $\mathbb{B}^2_*\Subset\mathbb{C}^2$ and X which cannot be extended to origin.

The reason is in fact, that $\mathrm{sym}^2(X)$ containes a two-dimensional spherical shell, while X not. In 3.3 we restate our results for multivalued meromorphic correspondences between (singular) complex spaces.

Results and techniques of this paper can be applied to several questions in complex analysis and geometry. We shall give two such applications.

Recall that a complex Plateau problem for a compact real submanifold M of a complex manifold X consists in finding an analytic chain $A\subset X\setminus M$ with "boundary" M, see 2.5 for details.

Corollary 3. *Let M be a strongly pseudoconvex, maximally complex compact CR-manifold in a disk-convex complex manifold X from class $\mathcal{G}_1$. Suppose that M bounds an abstract smooth Stein domain.*

(a) If $\dim M \geq 5$ then the complex Plateau problem for $M \subset X$ has a solution.

(b) If $\dim M = 3$ then the complex Plateau problem for $M \subset X$ has a solution iff M is

homologous to zero in X.

Remarks. 1. Let $H^2 := \mathbb{C}^2 \setminus \{0\}/z \sim 2z$ be a Hopf surface. Take M to be an image of a standard unit sphere from $\mathbb{C}^2$ under the natural projection $\pi : \mathbb{C}^2 \setminus \{0\} \to H^2$. M is not homologous to zero in H^2 (i.e. it is a spherical shell in H^2!), so a complex Plato problem has no solution for this M.

In the case (a), i.e. when $\dim M \geq 5$ the spherical shells in X are not an obstructions for finding a film with boundary M because we have "enough concavity".

2. Consider a Hopf three-fold $H^3 := \mathbb{C}^3 \setminus \{0\}/(z \sim 2z)$. In this case take a sphere $\mathbb{S}^3$ in a hyperplane $\{z_1 = 0\}$. Its image M under the natural projection will be homologous to zero but will not bound any analytic set in H^3. The reason here is that H^3 doesn't belongs to $\mathcal{G}_1$ but only to $\mathcal{P}_-$.

3. If one doesn't requires strict pseudoconvexity of a "contour" M then counterexamples are known already in $\mathbb{CP}^3$, see [Db].

4. The condition on M to bound an abstract Stein smooth Stein domain is really restrictive in dimension 3, while for bigger dimension one has the Rossi theorem guaranteeing the existence (but in general not smooth) abstract Stein domain with boundary M, see [Rs].

The proof consists in extension of a CR-embedding of M into X onto the Stein domain bounded by this M. We do it along the levels of appropriate plurisubharmonic Morse exhaustion function, see 2.5.

Consider now a domain D in complex manifold Ω, which covers (without ramifications) a compact complex manifold X.

Corollary 4. *Suppose that $X \in \mathcal{G}_1$. Then:*

(a) if $\dim X \geq 3$ then D is equal to a Levi-pseudoconvex domain minus a (possibly empty) variety of pure codimension two;

(b) if X is Kähler then D is Levi-pseudoconvex.

Part (b) was conjectured in [C-H] and was stated without proof in [Iv-3] as a simple corollary of the Hartogs-type extension theorem for mappings into Kähler manifolds. The proof is obtained by applying the extension results to the covering map $D \to X$, see 3.5.

It is interesting to consider the case when Ω is compact and $D = \Omega \setminus E$, where E is removable, i.e. holomorphic functions from $V \setminus E$ extend onto V for the Stein open subsets $V \subset \Omega$. For example if the Haussdorff $(\dim_{\mathbb{R}} \Omega - 2)$-measure of E is zero. Then, provided X carries a pluriclosed (negative) metric, one gets a further information about E, ex. analyticity or pluripolarity. Such examples really occur, see [Ka-2], [La].

In the end of this paper in §4 we give some open questions.

1. Continuity principle.

1.1. Cycle space associated to a meromorphic map. We shall need some notions and results from the theory of cycle spaces developed by D.Barlet, see [Ba]. Partially recalling thouse facts we adapt them to our situation. For the english spelling of the Barlet terminology we refer to [F].

Recall that an analytic cycle of dimension k in complex space Y is a formal sum $Z = \sum_j n_j Z_j$, where $\{Z_j\}$ is a locally finite sequence of analytic subsets (allwayse of pure dimension k) and n_j are positive integers called multiplicities of Z_j. $|Z| := \bigcup_j Z_j$- support of Z. All complex spaces in this paper are reduced, normal and countable at infinity.

With a given meromorphic mapping $f : \Delta^n \times A^k(r,1) \to X$, satifying conditions of the Theorem 1 we shall associate the following space of cycles. Fix some $0 < c < 1$. Consider a set $\mathcal{C}'_{f,C}$ of all analytic cycles Z in $Y := \Delta^{n+k} \times X$ of pure dimension k, such that:

(a) $Z \cap [\Delta^n \times A^k(r,1)] = \Gamma_{f_{z'}} \cap \{z\} \times A^k(r,1) \times X$ for some $z' \in \Delta^n(c)$. This means, in particular, that for this z' $f_{z'}$ extends meromorphically from $A^k_{z'}(r,1)$ onto $\Delta^k_{z'}$.

(b) $\mathrm{vol}(Z) < C$, where C is a some constant, $C > C_0$, C_0 beeing from Theorem 1.

Define $\bar{\mathcal{C}}_{f,C}$ to be a closure of $\mathcal{C}'_{f,C}$ in the usual topology of currents, see below. We shall show that $\mathcal{C}_{f,C} := \{Z \in \bar{\mathcal{C}}_{f,C} : \mathrm{vol}(Z) < C\}$ is an analytic space of finite dimension in the neighborhood of each of its points.

Let Z be an analytic cycle of dimension k in (reduced, normal) complex space Y. In our applications Y will be $\Delta^{n+k} \times X$. By a coordinate chart adapted to Z we shall understand an open neighborhood V in Y such that $V \cap |Z| \neq \emptyset$ together with an isomorphism j of V onto a closed subvariety $\tilde{V}$ in the neighborhood of $\bar{\Delta}^k \times \bar{\Delta}^q$, such that $j^{-1}(\bar{\Delta}^k \times \partial\Delta^q) \cap |Z| = \emptyset$. We shall denote such a chart by (V, j). By an image $j(Z)$ of cycle Z under isomorphism j (or by any other isomorphism) we shall understand the image of underlying analytic set together with multiplicities.

Sometimes we shall, following Barlet, denote: $\Delta^k = U, \Delta^q = B$ and give to the quadriple (V, j, U, B) the name *scale* adapted to Z.

If $\pi : \mathbb{C}^k \times \mathbb{C}^q \to \mathbb{C}^k$ is a natural projection, then $\pi |_{j(Z)}: j(Z) \to \Delta^k$ is a branching covering of degree say d. Number q depends on the embedding dimension of Y (or X in our case). We shall skeep sometimes j in our notations. The branched covering $\pi |_Z := \pi : Z \cap (\Delta^k \times \Delta^q) \to \Delta^k$ defines in a natural way a mapping $\phi : \Delta^k \to \mathrm{Sym}^d(\Delta^q)$ - symmetric power of Δ^q of degree d - as $\phi(z) = (\pi |_Z)^{-1}(z)$. This alloulds represent a cycle $Z \cap \Delta^{k+q}$ with $|Z| \cap (\bar{\Delta}^k \times \partial\Delta^q) = \emptyset$, as a graph of d-valued holomorphic map.

Let S be a normal complex space.

Definition 1.1.1. A holomorphic map $\Phi : S \times \bar{\Delta}^k \to \mathrm{Sym}^d(\Delta^q)$ we shall call an analytic family of k-dimensional subvarities in $\Delta^k \times \Delta^q$ parametrised by S.

We shall need the following Changing of the projection Theorem, due to Barlet. Let (V, j, U, B) be a scale on $\bar{\Delta}^k \times \Delta^q$ adapted to the cycle Z. Let Z be included in an analytic family Φ in $\Delta^k \times \Delta^q$ as above, i.e. there is a holomorphic map $\Phi : S \times \bar{\Delta}^k \to \mathrm{Sym}^d(\Delta^q)$ such that $\Phi(s_0, \cdot)$ represents Z for some $s_0 \in S$. Then for some neighborhood $S_0 \ni s_0$ V is adapted to all $Z_s, s \in S_0$. In particular this defines a map $\Psi : S_0 \times U \to \mathrm{Sym}^{d_1}(B)$.

Theorem (D. Barlet). *Mapping Ψ is holomorphic.*

For the proof see [Ba], Chapitres I and II. Here finite dimensionality and normality of S are essential.

Withough loss of generality we suppose that our mapping f is defined on $\Delta^n \times A^k(r, b)$ with $b > 1$. Now each $Z \in \mathcal{C}_{f,C}$ can be covered by a *finite* number of adapted neighborhoods (V_α, j_α). Their union $\bigcup_\alpha V_\alpha$ we shall denote by W_Z. Taking this covering (V_α, j_α) to be small enough, we can suppose that:

(c) if $V_{\alpha_1} \cap V_{\alpha_2} \neq \emptyset$ then on every irreducible component of the intersection $Z \cap V_{\alpha_1} \cap V_{\alpha_2}$ a point x_1 is fixed such that

(c_1) either there is a polycilinder neighborhood $\Delta_1^k \subset \Delta^k$ of $\pi_1(j_1(x_1))$ such that the chart $V_{12} = j_{\alpha_1}^{-1}(\Delta_1^k \times \Delta^q)$ is adapted to Z and is contained in V_2, here V_{12} is given the same embedding j_{α_1};

(c_2) or this is fulfilled for V_2 instead of V_1.

(d) if $V_\alpha \ni y$ with $p(y) \in \bar{\Delta}^n(c) \times A^k(\frac{r+1}{2}, 1)$ then $p(\bar{V}_\alpha) \subset \bar{\Delta}^n(\frac{c+1}{2}) \times A^k(r, 1)$.

Here we denote by $p : \Delta^{n+k} \times X \to \Delta^{n+k}$ and by $\pi : \Delta^n \times \Delta^k \times X \to \Delta^n$ the natural projections. Case (c_1) can be fulfiled when embedding dimension of V_{α_1} is smaller or equal of that of V_{α_2}, and (c_2) in the opposite case, see [B].

Note that we have allwayse that $p(Z) = \Delta_{z_0}^k$ for some $z_0 \in \Delta^k(c)$ if $Z \in \mathcal{C}_{f,C}$, because Z is a limit of $Z_n \in \mathcal{C}'_{f,C}$.

The subsets W_Z together with the topology of convergence on compact subsets on $\mathrm{Hol}(\Delta^k, \mathrm{Sym}^d(\Delta^q))$ define a (metrizable) topology on our cycle space $\mathcal{C}_{f,C}$.

Each cycle $Z \in \mathcal{C}_{f,C}$ can be viewed as a current of integration and thus $\mathcal{C}_{f,C}$ can be also equiped with the corresponding locally flat topology. Cycles Z_n are converging in this topology to Z if for any smooth (k, k) -form w with compact support in Y

$$\int_{Z_n} w \to \int_Z w.$$

The both topologies are equivalent, see [F]. For us will be important that $\bar{\mathcal{C}}_{f,C}$ is compact. This generalisation of Bishop's theorem is due to Harvey and Shiffman, see [H-S].

Family of cycles Z_s parametrised by normal complex space S we call *analytic* if for any $s_0 \in S$, and any coordinate chart (V, j) adapted to Z_{s_0} the family $Z_s \cap V$ is analytic in V for s in the neighborhood of s_0 in the sence of Definition 1.1.1.

We shall need a criterium for the analyticity of changing the projections map in the case when the parameter space S is of infinite dimension.

Denote by $S_m(\mathbb{C}^q)$ (resp. by $\Lambda^i(\mathbb{C}^q)$) the m-th symmetric (resp. exterior) pover of $\mathbb{C}^q$. Put $F_i := Hom(\Lambda^i(\mathbb{C}^k), \Lambda^i(\mathbb{C}^q))$.

Let (S, s_0) be a germ of Banach analytic space and let

$$\Phi : (S, s_0) \to H(\bar{\Delta}^k, \mathrm{Sym}^d(\Delta^q))$$

be a germ of analytic map.

Fix some $s \in S$ and denote by Z_s the analytic set in $\Delta^k \times \Delta^q$ defined by $\Phi_s := \Phi(s, \cdot) : \Delta^k \to \mathrm{Sym}^d(\Delta^q)$. Let $R(Z_s)$ the locus of ramification of $\pi|_{Z_s} : Z_s \to \Delta^k$. For $z \in \Delta^k \setminus R(Z_s)$ define

$$T^i_m(\Phi_s) : \Delta^k \to F_i \otimes S_m(\mathbb{C}^p) \tag{1.1.1}$$

as

$$T^i_m(\Phi_s)(z) = \Sigma^d_{j=1} \Lambda^i(D\Phi_{s,j}(z)) \otimes \Phi_{s,j}(z)^m. \tag{1.1.2}$$

Here Φ_j, $j = 1, ..., d$ are the local branches of Φ. $T^i_m(\Phi_s)$ extend holomorphically onto the whole polydisk Δ^k, see [B] Proposition 1, Chap.2. This defines a maps

$$T^i_m(\Phi) : S \times \Delta^k \to F_i \otimes S_m(\mathbb{C}^p). \tag{1.1.3}$$

Following Barlet we call the family *isotropic* if thouse maps are analytic for all i, m. The statement of Canging the Projection Theorem remains true for the case of Banach analytic sets S, provided the family Φ is isotropic, see Theorem 4 in [B]. In fact one needs to check analyticity only for $1 \leq m \leq d - 1$. Note also that $F_i = 0$ if $i > \max\{k, q\}$.

Now let $E = (V, j, U, B)$ be a scale on the complex space Y. Let us denote by $\mathrm{Hol}_Y(\bar{U}, \mathrm{sym}^d(B))$ the Banach analytic set of all d-sheeted analytic subsets on $\bar{U} \times B$, which are contained in $j(Y)$. We need to make the tautological family $\mathrm{Hol}_Y(\bar{U}, \mathrm{sym}^d(B)) \times U \to \mathrm{sym}^d(B)$ isotropic. Put $\mathcal{F} := \Sigma_i F_i$ and $\mathcal{S} := \Sigma_m S_m(\mathbb{C}^q)$. Fix some polydisk $U' \subset\subset U$ and call the data $E = (V, j, U, U', B)$ the *double scale* on Y. Consider a continuous map

$$T : \mathrm{Hol}(\bar{U}, \mathrm{sym}^d(B)) \to \mathrm{Hol}(\bar{U}', \mathcal{F} \otimes \mathcal{S})$$

given by

$$h \in \mathrm{Hol}(\bar{U}, \mathrm{sym}^d(B)) \to \Sigma_{i,m} T^i_m(h) \tag{1.1.4}$$

Denote by $\widehat{\mathrm{Hol}}(\bar{U}, \mathrm{sym}^d(B))$ the graph Γ_T of (1.1.4) in $\mathrm{Hol}(\bar{U}, \mathrm{sym}^d(B)) \times \mathrm{Hol}(\bar{U}', \mathcal{F} \otimes \mathcal{S})$.

Let $\widehat{\mathrm{Hol}}_Y(\bar{U}, \mathrm{sym}^d(B))$ be the restriction of Γ_T onto $\mathrm{Hol}_Y(\bar{U}, \mathrm{sym}^d(B))$. Then the set $\widehat{\mathrm{Hol}}_Y(\bar{U}, \mathrm{sym}^d(B))$ is a Banach analytic subset of $\mathrm{Hol}(\bar{U}, \mathrm{sym}^d(B)) \times \mathrm{Hol}(U', \mathcal{F} \otimes \mathcal{S})$, and the tautological family

$$\widehat{\mathrm{Hol}}(\bar{U}, \mathrm{sym}^d(B)) \times U \to \mathrm{sym}^d(B)$$

is (obviously) isotropic! See [B].

Definition 1.1.2. *The family $\mathcal{Z}$ of analytic cycles in an open set $W \subset Y$, parametrised by a Banach analytic set S, is called analytic in the neighborhood of $s_0 \in S$ if for any scale V adapted to Z_{s_0} there is a neighborhood $U \ni s_0$ s.t. $\{\mathcal{Z}_s : s \in U\}$ is isotropic in V.*

1.2. Analyticity of $\mathcal{C}_{f,C}$. Let $f : \Delta^n \times A^k(r,1) \to X$ be our map. Denote by $\mathcal{C}_0$ the subset of $\bar{\mathcal{C}}_{f,C}$ consisting of cycles which are limits of $\{\Gamma_{f_{s_n}}\}$ for $s_n \to 0, s_n \in S$. This is a compact subset (by Bishop's theorem) of the topological space $\mathcal{C}_{f,2C}$. For every cycle $Z \in \mathcal{C}_0$ define its neighborhood W_Z as above. Let $W_{Z_1}, ..., W_{Z_N}$ be a finite covering of $\mathcal{C}_0$. Remark that there is an $\varepsilon_0 > 0$ such that for any $s \in S \cap \Delta^n(\varepsilon_0)$ we have $\Gamma_{f_s} \subset \bigcup_{j=1}^N W_{Z_j}$.

We whant to show now that $\mathcal{C}_{f,2C}$ is an analytic space of finite dimension in the neighborhood of $\mathcal{C}_0$. It is enough to prove this for the W_{Z_1} for example.

Let $W_{Z_1} = \bigcup_{\alpha=1}^{N_1} V_\alpha$. We divide V_α -s into two types.

Type 1. For such V_α as in (d) put

$$H_\alpha := \bigcup_z \{[\Gamma_{f_z} \cap A^k(r,1) \times X] \cap V_\alpha\} \subset \mathrm{Hol}_Y(\Delta^k, \mathrm{Sym}^{d_\alpha}(\Delta^p)). \tag{1.2.1}$$

Union is taken over all $z \in \Delta^n$ for which V_α is adapted to Γ_{f_z}.

Type 2. For all others V_α we put $H_\alpha := \widehat{\mathrm{Hol}}_Y(\bar{\Delta}^k, \mathrm{Sym}^{d_\alpha}(\Delta^p))$.

All H_α are open sets in complex Banach spaces and, for V_α of first type they are of dimension n and smooth.

For every irreducible component of $V_\alpha \cap V_\beta \cap Z_1$ we fix some point $x_{\alpha\beta l}$ on this component (indice l indicates the component), and some chart $V_\alpha \cap V_\beta \supset (V_{\alpha\beta l}, \phi_{\alpha\beta l}) \ni x_{\alpha\beta l}$ adapted to this component as in (c). Put $H_{\alpha\beta l} := \widehat{\mathrm{Hol}}(\Delta^k, \mathrm{Sym}^{d_{\alpha\beta l}}(\Delta^p))$.

Consider a finite products $\Pi_{(\alpha)} H_\alpha$ and $\Pi_{(\alpha\beta l)} H_{\alpha\beta l}$. In the second product we take only triples with $\alpha < \beta$. They are Banach analytic spaces and by Changing the projection Theorem of Barlet, for each pair $\alpha < \beta$ we have two holomorphic mappings $\Phi_{\alpha\beta} : H_\alpha \to \Pi_{(l)} H_{(\alpha\beta l)}$ and $\Psi_{\alpha\beta} : H_\beta \to \Pi_{(l)} H_{\alpha\beta l}$. This defines two holomorphic maps $\Phi, \Psi : \Pi_{(\alpha)} H_\alpha \to \Pi_{\alpha<\beta, l} H_{\alpha\beta l}$. Kernel $\mathcal{A}_1$ of this pair, i.e. thouse $h = \{h_\alpha\}$ that $\Phi(h) = \Psi(h)$, consists exactly from analytic cycles in a neighborhood W_{Z_1} of Z_1. This kernel is a Banach analytic set and moreover, the family $\mathcal{A}_1$ is an analytic family in W_{Z_1}.

Lemma 1.2.1. *$\mathcal{A}_1$ is of finite dimension.*

Proof of the Lemma. Take a smaller covering $\{V_\alpha^{'}, j_\alpha\}$ of Z_1. Namely $V_\alpha^{'} = V_\alpha$ for the V_α of the first type and $V_\alpha^{'} = j_\alpha^{-1}(\Delta_{1-\varepsilon}^k \times \Delta^p)$ for the second. In the same manner we obtain a Banach analytic set $\mathcal{A}_1^{'}$. We have a holomorphic mapping $K : \mathcal{A}_1 \to \mathcal{A}_1^{'}$ defined by the restrictions. Diffe rential $dK \equiv K$ of this map is a compact operator. We also have an in verse map F because of the isotropy of the family $\mathcal{A}_1$, see above. Thus $id - dK \circ dF$ is Fredholm. Because $\mathcal{A}_1^{'} \subset \{h \in \Pi_{(i)}H_i^{'} : (id - K \circ F)(h) = 0\}$ we obtain that $\mathcal{A}_1^{'}$ is an analytic subset in the complex manifold of finite dimension.

Lemma is proved

The same holds for any point in Δ^n instead of zero. Thus $\mathcal{C}_{f,C}$ is finite dimensional analytic space.

1.3. Proof of the Continuity principle. Now we are prepared to prove our Continuity principle. Consider a universal family $\mathcal{Z} := \{Z_a : a \in \mathcal{C}_{f,2C_0}\}$, this time constant C_0 is taken from Theorem 1. This is complex space of finite dimension . We have an evaluation map

$$F : \mathcal{Z} \to \Delta^{n+k} \times X \tag{1.3.1}$$

defined by $Z_a \in \mathcal{Z} \to Z_a \subset \Delta^{n+k} \times X$.

Case $n = 1$. Consider the union $\hat{\mathcal{C}}_0$ of thouse components of $\mathcal{C}_{f,2C}$ which intersect $\mathcal{C}_0$. At least one of thouse components, say $\mathcal{K}$, containes two points s_1 and s_2 s.t. Z_{s_1} projects onto Δ_0^k and Z_{s_2} projects onto Δ_s^k with $s \neq 0$. Consider the restriction $\mathcal{Z}\mid_{\mathcal{K}}$ of the universal space onto $\mathcal{K}$. This is an irreducible complex space of finite dimension. Take points $z_1 \in Z_{s_1}$ and $z_2 \in Z_{s_2}$ and join them by an analytic disk $\phi : \Delta \to \mathcal{Z}\mid_{\mathcal{K}}$, $\phi(0) = z_1, \phi(1/2) = z_2$. Then the composition $\psi = \pi \circ F \circ \phi : \Delta \to \Delta$ is not degenerate because $\psi(0) = 0 \neq s = \psi(1/2)$. Thus ψ is proper and obviously so is the map $F : \mathcal{Z}\mid_{\phi(\Delta)} \to F(\mathcal{Z}\mid_{\phi(\Delta)}) \subset \Delta^{n+k} \times X$. Thus $F : \mathcal{Z}\mid_{\phi(\Delta)})$ is an analytic set in $U \times \Delta^k \times X$ for small enough U extending Γ_f by the reason of dimension.

Case $n \geq 2$. Consider an increasing family of complex spaces $\{\mathcal{C}_{f,C} : C \geq C_0\}$, where C_0 is from the Theorem 1. Note that for $C_1 < C_2$ $\mathcal{C}_{f,C_1}$ is an open subset of $\mathcal{C}_{f,C_2}$. This allouds correctly define an irreducible components of the analytic space $\mathcal{C}_f := \bigcup_{C\geq C_0} \mathcal{C}_{f,C}$.

As above by $\mathcal{C}_0$ denote the compact subset of $\mathcal{C}_f$, which consists from the limits of Γ_{f_s}, $s \to 0, s \in S$, $\mathrm{vol}(\Gamma_{f_s}) \leq C_0$. Clearly only finite number of components of $\mathcal{C}_f$ intersect $\mathcal{C}_0$. Denote them by $\mathcal{K}_1, ..., \mathcal{K}_N$.

Note that if all this components would have zero dimension it would contradict the thickness of S at origin. Take any component, say $\mathcal{K}_1$ of positive dimension. If evaluation map F restricted onto $\mathcal{Z}\mid_{\mathcal{K}_1}$ is proper, then again $F(\mathcal{Z}\mid_{\mathcal{K}_1})$ is an analytic set in the neighborhood $U \times \Delta^k \times X$. If for all $\mathcal{K}_j$ $F(\mathcal{Z}\mid_{\mathcal{K}_j})$ is proper then, because we only have only finitely many this components , we shall have a neighborhood $U \ni 0$ in Δ^n and analytic set $F(\mathcal{Z}\mid_{\bigcup K_j})$ in $U \times \Delta^k \times X$. Would this set be of dimension less then $n+k$, it would contradict the thickness of S at zero. Thus in this case we have again an extension $F(\mathcal{Z}\mid_{\bigcup K_j}$ of Γ_f onto $U\times\Delta^k\times X$.

It remaine to prove that for any $\mathcal{K}_j$ the restriction of F onto $\mathcal{Z}\mid_{\mathcal{K}_j}$ is proper. Suppose now that there is a component, say $\mathcal{K}_1$, such that F restricted to $\mathcal{Z}\mid_{\mathcal{K}_1}$ is not proper. Consider a map $\phi:\mathcal{K}_1\to\Delta^n$ given by $\phi(s)=\pi(F(Z_s))$. If $\phi^{-1}(0)$ is compact, then there are a neighborhoods $W_1\supset\phi^{-1}(0)$ in $\mathcal{K}_1$ and $V_1\ni 0$ in Δ^n such that $\phi\mid_W: W_1\to V_1\subset\Delta^n$ is proper. Thus $F:\mathcal{Z}\mid_{W_1}\to V_1\times\Delta^k\times X$ is proper. If this is the case for all $\mathcal{K}_1,...,\mathcal{K}_N$ then again $F:\mathcal{Z}\mid_{\bigcup W_j}\to\bigcap V_j\times\Delta^k\times X$ is proper and we are done.

So, suppose that $\phi^{-1}(0)$ is not compact in $\mathcal{K}_1$. Each $Z_s, s\in\phi^{-1}(0)$ has a form $\{0\}\times B_s\cup\Gamma_{f_0}$, where B_s is a compact cycle in X. Thus an appropriate connected component of $\phi^{-1}(0)$ parametrizes a noncompact connected and closed subvariety in $\mathcal{B}_k(X)$. This contradicts the bounded cycle geometry condition on X.

Theorem 1 is proved.

Proof of the Corollary 1. In the proof of Corollary 1 we shall need some resuts on the meromorphic families of analytic subsets. They will be used also in the proof of Theorem 2. That's why we had collected them in 2.3. For the proofs of thouse results we refer to [Iv-4].

Denote by ν_j the minimal volume of compact j-dimensional analytic subset in $K\subset\subset X$ - compact from Corollary 1. Put

$$\nu=\min\{vol(A_{k-j}\cdot\nu_j : j=1,...,k\},\tag{1.3.2}$$

where A_{k-j} runs over all $(k-j)$-dimensional analytic subsets of Δ^k, which intersect Δ^k_r. Denote by W the maximal open subset of Δ^n such that f meromorphically extends onto $\Delta^n\times A^k(r,1)\cup W\times\Delta^k$. Put $S=\Delta^n\setminus W$. Let

$$S_l=\{z\in S : vol(\Gamma_{f_z}\leq l\cdot\frac{\nu}{2}\}.\tag{1.3.3}$$

Lemma 2.3.2 tells us that $S_{l+1}\setminus S_l$ are pluripolar and by the Josefson theorem so is S.

Consider a function $v(z)=\mathrm{vol}(\Gamma_{f_z})$. We know that $v\geq 0, v\leq C_0$ and v is lower semicontinuous . The latter follows from Lemma 2.3.1.(4). Let v^* be its upper regularisation, i.e. $v^*=\limsup_{z\to z_0}v(z)$. Then v^* is upper semi-continuous and bounded by C_0. Consider an analytic space

$$\mathcal{C}_{f,2C_0,c}:=\{Z\in\mathcal{C}_{f,2C_0} : \|\pi(Z)\|<c\},\tag{1.3.4}$$

where $0<c<1$ is fixed. Consider the following relatively compact in $\mathcal{C}_{f,2C_0}$ and closed in $\mathcal{C}_{f,2C_0,c}$ set:

$$\mathcal{C}_{f,v^*+\frac{\nu}{4},c}=\{Z\in\mathcal{C}_{f,2C_0,c} : \mathrm{vol}(Z)\leq v^*[\pi(Z)]+\frac{\nu}{4}\},\tag{1.3.5}$$

where, as above $\pi : \Delta^n \times \Delta^k \times X \to \Delta^n$ is a natural projection. Function $\mathrm{vol}(Z)$ is contunious on $\mathcal{C}_{f,2C_0}$. Thus for any $Z_0 \in \bar{\mathcal{C}}_{f,v^*+\frac{\nu}{4},c}$ the set $W_{Z_0,\frac{\nu}{4}} := \{Z : |\mathrm{vol}(Z) - \mathrm{vol}(Z_0)| < \frac{\nu}{4}\}$ is an open neighborhood of Z_0 in $\mathcal{C}_{f,2C_0}$. Closures and neighborhoods are taken in $\mathcal{C}_{f,2C_0,c}$. Put

$$W_{f,v^*,\frac{\nu}{2}} := \bigcup_{Z_0 \in \bar{\mathcal{C}}_{f,v^*+\frac{\nu}{4},c}} W_{Z_0,\frac{\nu}{4}} \tag{1.3.6}$$

and

$$W_{f,v^*,\frac{3\nu}{4}} := \bigcup_{Z_0 \in \bar{\mathcal{C}}_{f,v^*+\frac{\nu}{4},c}} W_{Z_0,\frac{\nu}{2}}. \tag{1.3.7}$$

We are going to show now that $\bar{W}_{f,v^*,\frac{\nu}{2}} = W_{f,v^*,\frac{3\nu}{4}}$.

Find a point $z_0 \in W$ which has a neighborhood, say V, such that for all $z \in V$ $|vol(\Gamma_{f_z}) - vol(\Gamma_{f_{z_0}})| < \frac{\nu}{2}$. Such z_0 exists because W is not pluripolar. Lemma 2.3.2 shows that the analytic spaces $W_{f,v^*,\frac{\nu}{2}}$ and $W_{f,v^*,\frac{3\nu}{4}}$ coincide with $\bigcup_{z\in V}\Gamma_{f_z}$ in the neighborhood of z_0. The same argument shows that the following is true. Take the irreudcible components of analytic spaces $W_{f,v^*,\frac{\nu}{2}}$ and $W_{f,v^*,\frac{3\nu}{4}}$ which contain $\bigcup_{z\in V}\Gamma_{f_z}$. We denote them in the same manner $W_{f,v^*,\frac{\nu}{2}}$ and $W_{f,v^*,\frac{3\nu}{4}}$ respectively. The holomorphis mapping $\pi \circ F : W_{f,v^*,\frac{\nu}{2}} \to \Delta^n$ is surjective. The same for $\pi \circ F : W_{f,v^*,\frac{3\nu}{4}} \to \Delta^n$. There is a pluripolar set $\hat{S} \supset S$ such that $\pi \circ F : \bar{W}_{f,v^*,\frac{\nu}{2}} \setminus (\pi \circ F)^{-1}(\hat{S}) \to \Delta^n \setminus \hat{S}$ is biholomorphic. And the same for $\pi \circ F : W_{f,v^*,\frac{3\nu}{4}} \setminus (\pi\circ F)^{-1}(\hat{S}) \to \Delta^n \setminus \hat{S}$. But this means that $W_{f,v^*,\frac{3\nu}{4}} \setminus W_{f,v^*,\frac{\nu}{2}} \subset (\pi \circ F)^{-1}(\hat{S})$. The latter is a pluripolar subset of an irreducible analytic space, and thus has dense complement. So the statement is proved.

Note now that if $0 < c_1 < c$ then $\bar{W}_{f,v^*,\frac{\nu}{2}} \cap (\pi \circ F)^{-1}(\bar{\Delta}^n_{c_1})$ is compact in $W_{f,v^*,\frac{3\nu}{4}}$. This gives the properness of $\pi \circ F$ restricted to $W_{f,v^*,\frac{3\nu}{4}}$, and thus the properness of F there.

q.e.d.

1.4. Spaces from $\mathcal{G}_k$ have bounded cycle geometry. Let us proove here that spaces from $\mathcal{G}_k$ have bounded cycle geometry in dimension k. In fact somewhat stronger statement is true. Denote by $\mathcal{P}^k_-$ the class of complex spaces which carry a strictly positive (k,k)- forms $\Omega^{k,k}$ which are dd^c-negative, i.e. $dd^c\Omega^{k,k} \leq 0$.

Proposition 1.4.1. *Let $X \in \mathcal{P}^k_-$ be disk-convex in dimension k and let $\mathcal{K}$ be an irreducible component of $\mathcal{B}_k(X)$. Then*

1) $\mathcal{K}$ is compact.

2) If $\Omega^{k,k}$ os some dd^c-negative (k,k)-form on X then $\int_{Z_s}\Omega^{k,k} \equiv const$ for $s \in \mathcal{K}$.

3) X has bounded cycle geometry.

Proof. Let $F : \mathcal{Z}\mid_{\mathcal{K}} \to X$ be the evaluation map, and let $\Omega^{k,k}$ be a sitrictly positive dd^c-negative (k,k)-form on X. Then $\int_{Z_s}\Omega^{k,k}$ measures the volume of Z_s. Let us prove that the function $v(s) = \int_{Z_s}\Omega^{k,k}$ is plurisuperharmonic on $\mathcal{K}$. Take an analytic disk $\phi : \Delta \to \mathcal{K}$. Then by Stokes theorem and reasons of bedegree for any nonnegative test function $\psi \in \mathcal{D}(\Delta)$:

$$< \psi, \Delta\phi^*(v) >= \int_{\Delta}\Delta\psi \cdot \int_{Z_{\phi(s)}}\Omega^{k,k} = \int_{\mathcal{Z}|_{\phi(\Delta)}} dd^c(\pi^*\psi)\wedge\Omega^{k,k} =$$

$$= \int_{\phi(\Delta)}\pi^*\psi\wedge dd^c\Omega^{k,k} \leq 0.$$

Here $\pi : \mathcal{Z}\mid_{\mathcal{K}} \to \mathcal{K}$ is a natural projection. So $\Delta\phi^*(v) \leq 0$ for any analytic disk in $\mathcal{K}$ in the sence of distributions. Thus v is plurisuperharmonic.

Note that by Harvey-Shiffman generalisation of Bishops theorem $v(s) \to \infty$ while $s \to \partial\mathcal{K}$. So by the minimum principle $v \equiv const$ and $\mathcal{K}$ is compact again by Bishop theorem.

2) The same computation shows that $\int_{Z_s}\Omega^{k,k}$ is plurisuperharmonic for any dd^c-negative (k,k)-form. While $\mathcal{K}$ is proved to be compact, we get the statement.

3) Let $\mathcal{R}$ be any connected component of $\mathcal{B}_k(X)$. Write $\mathcal{R} = \bigcup_j\mathcal{K}_j$. From (1) we have that v is constant on $\mathcal{R}$. So if $\{\mathcal{K}_j\}$ is not finite then $\mathcal{R}$ has an accumulation point $s = \lim s_j$, where all s_j belong to different components $\mathcal{K}_j$ of $\mathcal{R}$. This contradicts to the fact that $\mathcal{B}_k(X)$ is a complex space.

q.e.d.

1.5. Construction of Example 1. We start with recalling one example due to M. Kato, see [Ka-1].

In $\mathbb{CP}^3$ with homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ consider a domain $D = \{z \in \mathbb{CP}^3 : |z_0|^2 + |z_1|^2 > |z_2|^2 + |z_3|^2\}$. Group $Sp(1,1)$ naturally acts on $\mathbb{CP}^3$ and preserves D, i.e. $g(D) = D$ for all $g \in Sp(1,1)$. Action of $Sp(1,1)$ is transitive on D and Kato proved, using the result of Vinberg, that there exist a discrete subgroup $\Gamma \subset Sp(1,1)$, acting properly and discontinuously on D, and such that $D/\Gamma = X^3$ is a compact complex manifold, see [Ka-1] for details.

Projective plane $\mathbb{CP}^2 = \{z_3 = 0\}$ intersects D by the complement to the unit ball $[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : |z_2|^2 < |z_0|^2 + |z_1|^2\}$. If $\pi : D \to X^3$ is a natural projection, then its restriction $\pi\mid_{\mathbb{CP}^2\cap D}$: $\mathbb{CP}^2 \setminus \bar{\mathbb{B}}^4 \to X^3$ defines a holomorphic map from the complement to the closed unit ball to X^3, which has singularity at each point of $\partial\mathbb{B}^4$!

Blow up $\mathbb{CP}^4$ at origin of its affine part. Denote by $\mathbb{CP}^4_0$ the resulting manifold. There is a natural holomorphic projection. $p : \mathbb{CP}^4_0 \to \mathbb{CP}^3$-concidered as an exeptional divisor. Γ beeing a group of 4×4 matrices acts naturally on affine part $\mathbb{C}^4$ of $\mathbb{CP}^4$. This action obviously extends onto $\mathbb{CP}^4$ and lifts onto $\mathbb{CP}^4_0$. Moreover the actions of Γ on $\mathbb{CP}^3$ and $\mathbb{CP}^4_0$ are equivariant with respect to the projection p. Put

$\hat{D} := p^{-1}(D)$ and $\hat{X} := \hat{D}/\Gamma$. If take now $f := \{z \in \mathbb{C}^4 : |z_0|^2+|z_1|^2 > |z_2|^2+|z_3|^2\} \to \hat{X}$ to be a restriction of quotient map, we get our example.

2. Hartogs-type extension and spherical shells

2.1. Generalities on pluripotential theory. We start with some well known facts from pluripotential theory . Let D be an open subset of $\mathbb{C}^n$ and S subset of D. Consider the following class of functions:

$$\mathcal{U}(S,D) = \{u \in \mathcal{P}_+(D) : u|_S \geq 1\} \tag{2.1.1}$$

where by $\mathcal{P}_+(D)$ we denote the class of nonnegative plurisuperharmonic functions in D.

Definition 2.1.1. The lower regularization w_* of the function

$$w(\zeta,S,D) = \inf\{u(\zeta) : u \in \mathcal{U}(S,D)\} \tag{2.1.2}$$

is called a $\mathcal{P}$- measure of S in D, i.e.

$$w_*(z,S,D) = \liminf_{\zeta\to z} w(\zeta,S,D) \tag{2.1.3}$$

Note that w_* is plurisuperharmonic in D .

Definition 2.1.2. A point $s_0 \in S$ is called a locally regular point of S if $w^*(s_0, S\cap \Delta^n(s_0,\varepsilon), \Delta^n(s_0,\varepsilon)) = 1$ for all $\varepsilon > 0$.

We shell also say that the set S is locally regular at s_0.

Recall that subset $S \subset D$ is called pluripolar if there exists a plurisubharmonic function u in $D, u \not\equiv -\infty$, such that $u\mid_S \equiv -\infty$. S is complete pluripolar if $S = \{z : u(z) = -\infty\}$. We shall repeatedly use the following statement:

if subset $S \subset D$ (D is now pseudoconvex) is not locally regular at all its points then S

is pluripolar,

see [B-T],[Sd]. We shall repeteadly use the following immediate corollary from the famous Josefson theorem, see [Kl]:

Let Ω be a pseudoconvex set in $\mathbb{C}^n$, and let S_n be a sequence of subsets of Ω such that:

1) S_1 is closed and pluripolar in Ω;

2) $S_{n+1} \subset \Omega\setminus S_n$ is closed in $\Omega\setminus S_n$ and pluripolar;

Then $S := \bigcup_{n=1}^{\infty} S_n$ is pluripolar in Ω.

Denote by $\mathcal{D}^{k,k}(\Omega)$ the space of C^∞-forms of bidegree (k,k) with compact support on complex manifold Ω. $\phi \in \mathcal{D}^{k,k}(\Omega)$ is real if $\bar{\phi} = \phi$. The dual space $\mathcal{D}_{k,k}(\Omega)$ is the space of currents of bidimension (k,k) (bidegree $(n-k,n-k)$, $n = \dim\Omega$). $T \in \mathcal{D}_{k,k}(\Omega)$ is real if $< T,\bar{\phi} >= \overline{< T,\phi >}$ for all $\phi \in \mathcal{D}^{k,k}(\Omega)$.

Definition 2.1.3. Current $T \in \mathcal{D}_{k,k}(\Omega)$ is called positive if for all $\phi_1, ..., \phi_k \in \mathcal{D}^{1,0}(\Omega)$

$$< T, \frac{i}{2}\phi_1 \wedge \bar{\phi}_1 \wedge ... \wedge \frac{i}{2}\phi_k \wedge \bar{\phi}_k >\geq 0.$$

T is negative if $-T$ is positive.

Definition 2.1.4. We shall say that the current $T \in \mathcal{D}_{k,k}(\Omega)$ is pluripositive (-negative) if T is positive and dd^cT is positive (-negative).

T is pluridefinite if it is either pluripositive or plurinegative.

Definition 2.1.5. The current T (not necessarily positive) is pluriclosed if $dd^cT = 0$.

Let $\phi = \Sigma_J \phi_J(x)dx_J \in \mathcal{D}^k(\Omega)$, where Ω open subset of $\mathbb{R}^n$. Euclidean norm of ϕ at x is

$$\|\phi(x)\| = (\Sigma_{|J|=k}|\phi_J(x)|^2)^{1/2}.$$

Further if $T \in \mathcal{D}_k(\Omega)$ and U is open in Ω then the mass of T in U is a number

$$\|T\|(U) = sup\{| < T, \phi > | : \phi \in \mathcal{D}^k(U), \|\phi(x)\| \leq 1, x \in U\}. \tag{2.1.4}$$

Let $K \Subset \Omega$ be closed and $T \in \mathcal{D}_k(\Omega \setminus K)$. We say that T has locally finite mass in the neighborhood of K if for any open relatively compact $U \Subset \Omega$ $\|T\|(U \setminus K) < \infty$. In this case a trivial extension $\tilde{T}$ of T onto Ω is defined in the following way. Take a sequence $u_n \in C^\infty(\Omega)$, $0 \leq u_n \leq 1$, $u_n \equiv 0$ in the neighborhood of K, and $u_n \nearrow \chi_{\Omega\setminus K}$ uniformly on compacts in $\Omega \setminus K$. Here χ_A is a characteristic function of the set A. Then for $\phi \in \mathcal{D}^k(\Omega)$ put

$$< \tilde{T}, \phi >= \lim_{n\to\infty} < u_nT, \phi > \tag{2.1.5}$$

This correctly defines a current on Ω, see [Lg].

If K is complete pluripolar compact in strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and T closed, positive current on $\Omega\setminus K$, then T has locally finite mass in the neighborhood of K, see [Iv-2], *Lemma 2.1* .

Lemma 2.1.1. *(a) Let K be a complete, pluripolar compact in strictly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$ and T be a pluridefinite current of bidegree (1,1) on $\Omega \setminus K$ of locally finite mass in the neighborhood of K, and such that dT has coefficients measures in $\Omega \setminus K$. Then $dd^c\tilde{T}$ has coefficients measures on Ω.*

(b) If K is of Haussdorff dimension zero and $n = 2$ then $\chi_K \cdot dd^c\tilde{T}$ is negative.

Proof. Part (a) of this lemma is proved in [Iv-2], Proposition 2.3 for the currents of bidimension (1,1) (the condition on dT was forgotten there). If T is of bedegree (1,1) then take $T \wedge (dd^c\|z\|^2)^{n-2}$ to get the same conclusion.

(b) Let $\{u_n\}$ be a sequence of smooth p.s.h.functions in Ω, equal to zero in the neighborhood of K, $0 \leq u_n \leq 1$ and such that $u_n \nearrow \chi_{\Omega\setminus K}$ uniformly on compacts in $\Omega \setminus K$, see *Lemma 1.2* from [Sb]. Put $v_n = u_n - 1$.

Let w_e denote the Euclidean volume-form in $\mathbb{C}^2$. Put $\widetilde{dd^cT} = \mu_0 \cdot w_e^n$. Then μ_0 is a measure on Ω. According to part (a) the distribution μ defined from $dd^c\tilde{T} = \mu \cdot w_e^n$ is a measure. Write

$$\mu = \chi_K \cdot \mu + \chi_{\Omega\setminus K} \cdot \mu \tag{2.1.6}$$

Where obviously $\chi_{\Omega\setminus K} = \mu_0$. The measure $\chi_K \cdot \mu$ we denote by μ_1. We shall prove that the measure μ_1 is nonpositive.

Take a ball in $\mathbb{C}^n$ centered at $s_0 \in K$ such that $\partial B \cap K = \emptyset$. One has

$$\mu_1(B \cap K) = -\lim_{k\to\infty}\int_B v_k \cdot \mu w_e^n = -\lim_{k\to\infty} < v_k, dd^c\tilde{T} >=$$

$$= -\lim_{k\to\infty} < dd^c v_k, \tilde{T} > \leq 0, \tag{2.1.7}$$

because $\tilde{T}$ is positive and $dd^c v_k \geq 0$. So for any such ball we have

$$\mu_1(B \cap K) \leq 0 \tag{2.1.8}$$

All that left, is to use the following Vitali-type theorem for a general measures, see [Fd], p.154 . Let D be an open set in $\mathbb{C}^n$ and σ a finite positive Borel measure on D. Let further $\mathcal{B}$ be a family of closed balls of positive radii such, that for any point $x \in D$ the family $\mathcal{B}$ contains the balls of arbitrarily small radii centered at x. Then one can find a countable subfamily $\{B_i\}$ of balls in $\mathcal{B}$ such that

$$\sigma(D \setminus \bigcup_{(i)} B_i) = 0. \tag{2.1.9}$$

Represent our measure μ_1 as a difference $\mu_1 = \mu_1^+ - \mu_1^-$ of two nonnegative measures. Take some relatively compact open subset $D \subset \Omega$. As $\mathcal{B}$ take the family of all balls that $\partial B \cap K = \emptyset$. Because of zero dimensionality of K it is a Vitali-type covering. Let $\{B_i\}$ are such that $\mu_1^+(D \setminus \bigcup_{(i)} B_i) = 0$. Then $\mu_1^+(D) = \mu_1^+(D \setminus \bigcup_{(i)} B_i) + \sum_{(i)} \mu_1^+(B_i) = \sum_{(i)} \mu_1^+(B_i)$. Consequently

$$\mu_1(D) = \mu_1^+(D) - \mu_1^-(D) \leq \mu_1^+(\bigcup_{(i)} B_i) - \mu_1^-(\bigcup_{(i)} B_i) =$$

$$= \sum_i \mu_1^+(B_i) - \sum_i \mu_1^-(B_i) = \sum_i \mu_1(B_i) \leq 0 \tag{2.1.10}$$

by (2.1.8). Thus $\mu_1(D) \leq 0$ for any relatively compact open D in Ω. So the measure μ_1 is negative.

q.e.d.

Remark. As we shall see in the proof of the Theorem 2 (Lemma 2.6.1) the conclusion of the part (b) of Lemma 2.1.1 remains true for arbitrary n and complete, pluripolar closed set of Haussdorff dimension $2n - 4$.

Recall, see [Sb], that a subset $K \subset \Omega$ is called (complete) p-polar if for any $a \in \Omega$ there exists a neighborhood V of a and such coordinates $z_1, ..., z_n$ in V that the sets

$$K_{z_I^0} = K \cap \{z_{i_1} = z_{i_1^0}, ..., z_{i_p} = z_{i_p^0}\} \tag{2.1.11}$$

are (complete) pluripolar in subspaces $V_{z_i^0} := \{z \in V : z_{i_1} = z_{i_1^0}, ..., z_{i_p} = z_{i_p^0}\}$ for almost all $z_I^0 = (z_{i_1}^0, ..., z_{i_p}^0) \in \pi^I(V)$, where I runs over a finite set of multiindices with $|I| = p$, such that $\{(\pi^I)^* w_e\}_I$ generates the space of bedegree (p,p)-forms. Here $\pi^I(z_1, ..., z_n) = (z_{i_1}, ..., z_{i_p})$ stands for the projection onto the space of variables $(z_{i_1}, ..., z_{i_p})$.

2.2. Metrics on complex spaces and proof of Theorem 2 in dimension two. A Hermitian metric form on the complex space X we define in the following way. Let an open covering U_α of X is given together with proper holomorphic injections $\phi_\alpha : U_\alpha \to V_\alpha$ into a domains $V_\alpha \subset \mathbb{C}^{n(\alpha)}$. Let $U_\alpha^{'}$ be the images of U_α. Let $\{w_\alpha\}$ are positive (1,1)-forms on V_α. $\{w_\alpha\}$ defines a Hermitian metric form on X if $(\phi_\alpha \circ \phi_\beta^{-1})^* w_\beta = w_\alpha$ for all α, β. Note that $\phi_\alpha \circ \phi_\beta^{-1}$ is defined in some neighborhood of $\phi_\beta(U_\alpha \cap U_\beta)$ in $\mathbb{C}^{n(\beta)}$. We say that the metric w is Kähler if $dw_\alpha = 0$ on V_α for all α.

Definition 2.2.1. w is pluriclosed (negative) if $dd^c w_\alpha = 0$ ($dd^c w_\alpha \le 0$) in V_α for all α.

Let a meromorphic mapping $f : H^2(r) \to X$ from two-dimensional Hartogs figure into a disk-convex complex space is given. Let w be a plurinegative metric form on X. By W denote the maximal open subset of the unit disk Δ such that f meromorphically extends onto $H_W(r) := W \times \Delta \cup \Delta \times A_{1-r,1}$. Here by $A_{1-r,1}$ we denote the annulus $\{z \in \mathbb{C} : 1 - r < |z| < 1\}$. Let $I(f)$ be the fundamental set of f and by $\hat{f}$ denote the mapping $\hat{f}(z) = (z, f(z))$ into the graph. For $z \in W$ define

$$\mu_t(z) = \text{area}\, \hat{f}(\Delta_z(t)) = \int_{\Delta_z(t)} (dd^c|\lambda|^2 + f\,|^*_{\Delta_z(t)}\, w). \tag{2.2.1}$$

Here $\Delta_z(t) = \{(z, \lambda) : |\lambda| < t\}$. We start with the following simple observation. Denote by $\nu_1 = \nu_1(K)$ the infimum of areas of compact complex curves which are contained in compact $K \subset X$. $\nu_1 > 0$, see *Lemma 2.3.1* below.

Lemma 2.2.1. *Let $f : \Delta \times A_{1-r,1} \to X$ be a holomorphic mapping into a disk-convex complex space X. Suppose that for a sequence of points $\{s_n\} \subset \Delta$, $s_n \to 0$, the following holds:*

(a) $f_s := f\,|_{\{s\} \times A_{1-r,1}}$ extends holomorphically onto $\Delta_s := \{s\} \times \Delta$;

(b) For a compact K in X, which contains $f[(\Delta(1/2) \times A_{1-2/3\cdot r, 1-1/3\cdot r}) \cup \bigcup_{(n)} \{s_n\} \times \Delta_{1-1/3\cdot r}]$, one has

$$|\,\text{area}\, \hat{f}(\Delta_{s_n}(1 - 1/3 \cdot r)) - \text{area}\, \hat{f}(\Delta_0(1 - 1/3 \cdot r)| \le \frac{1}{2} \cdot \nu_1(K). \tag{2.2.2}$$

Then f holomorphically extend onto $V\times\Delta$ for some open $V\ni 0$.

Proof. First of all let us show that $\mathcal{H}-\lim_{n\to\infty}\hat{f}(\bar{\Delta}_{s_n}(1-1/3\cdot r))=\hat{f}(\bar{\Delta}_0(1-1/3\cdot r))$, i.e. the sequence of graphs $\{\hat{f}(\bar{\Delta}_{s_n}(1-1/3\cdot r))\}$ converges in Hausdorff metric to the graph of the limit. If not, there would be a subsequence (still denoted as $\{\hat{f}(\bar{\Delta}_{s_n}(1-1/3\cdot r))\}$), such that $\mathcal{H}-\lim_{n\to\infty}\hat{f}(\bar{\Delta}_{s_n}(1-1/3\cdot r))=\hat{f}(\bar{\Delta}_0(1-1/3\cdot r))\cup\bigcup_{j=1}^{N}C_j$, where $\bigcup_{j=1}^{N}C_j$ is a union of compact curves, see *Lemma 2.3.1* below. Thus by (2.1.2) area $\hat{f}(\bar{\Delta}_{s_n}(1-1/3\cdot r))\geq$ area $\hat{f}(\bar{\Delta}_0(1-1/3\cdot r))+N\cdot\nu_1(K)$. This contradicts (2.2.2).

Take a Stein neighborhood V of $\hat{f}(\bar{\Delta}_0(1-1/3\cdot r))$, see [Si-1]. Then for $\delta>0$ small enough we have that $f(\Delta_\delta\times A_{1-1/3r-\delta,1-1/3r+\delta})\subset V$ and $f(\Delta_{s_n}(1-1/3r))\subset V$ if $s_n\in\Delta_\delta$. From Hartogs theorem for holomorphic functions we get that f extends to a holomorphic map from $\Delta_\delta\times\Delta_{1-1/3r-\delta}$ to V.

q.e.d.

Lemma 2.2.2. *If the metric form w is plurinegative and W is maximal then $\partial W\cap\Delta$ is complete polar in Δ.*

Proof. Take a point $z_0\in\partial W\cap\Delta$. Choose relatively compact in Δ neighborhood U of z_0 and $1-r<t<1$ such that $I(f)\cap\bar{U}\times\partial\Delta(t)=\emptyset$. Denote by $\phi=i\phi^{\alpha\beta}dz_\alpha\wedge d\bar{z}_\beta$ the current $f^*+dd^c\|z\|^2$. The area function from (1.2.1) can be written now as

$$\mu_t(z_1)=i\cdot\int_{|z_2|\leq t}\phi_{22}(z_1,z_2)dz_2\wedge d\bar{z}_2. \tag{2.2.3}$$

The condition that $dd^c\phi$ is negative means that

$$\frac{\partial^2\phi_{11}}{\partial z_2\partial\bar{z}_2}+\frac{\partial^2\phi_{22}}{\partial z_1\partial\bar{z}_1}-\frac{\partial^2\phi_{12}}{\partial z_2\partial\bar{z}_1}-\frac{\partial^2\phi_{21}}{\partial z_1\partial\bar{z}_2}\leq 0 \tag{2.2.4}$$

on $H_W(r)$. Now we can estimate the Laplacian of μ_t:

$$\Delta\mu(z_1)=i\int_{|z_2|\leq t}\frac{\partial^2\phi_{22}}{\partial z_1\partial\bar{z}_1}dz_2\wedge d\bar{z}_2\leq i\int_{|z_2|\leq t}(-\frac{\partial^2\phi_{11}}{\partial z_2\partial\bar{z}_2}+\frac{\partial^2\phi_{12}}{\partial z_2\partial\bar{z}_1}+\frac{\partial^2\phi_{21}}{\partial z_1\partial\bar{z}_2})dz_2\wedge d\bar{z}_2=$$

$$=i\int_{|z_2|=t}\frac{\partial\phi_{11}}{\partial z_2}dz_2+i\int_{|z_2|=t}\frac{\partial\phi_{12}}{\partial\bar{z}_1}d\bar{z}_2-i\int_{|z_2|=t}\frac{\partial\phi_{21}}{\partial z_1}dz_2=\psi(z_1). \tag{2.2.5}$$

Inequality (2.2.5) holds for $z_1\in U\cap W$. But the right hand side ψ is smooth in the whole U. Let Ψ be the smooth solution of $\Delta\Psi=\psi$ in U. Put $\hat{\mu}(z)=\mu_t(z)-\Psi(z)$. Then $\hat{\mu}$ is superharmonic and bounded from below in $U\cap W$, after shrinking of U.

Denote further by E the set of points $z_1\in\partial W\cap U$ such that $\mu_t(z)\to+\infty$ as $z\in W, z\to z_1$. Note that $\hat{\mu}(z)$ also tends to $+\infty$ in this case. For any point $z_\infty\in[\partial W\cap U]\setminus E$ we can find a sequence $\{z_n\}\subset W, z_n\to z_\infty$ such that $\mu_t(z_n)\leq C$. So by *Lemma 2.2.1* $f\mid_{\Delta_{z_\infty}\setminus\Delta_{z_\infty}(1-r)}$ extends onto Δ_{z_∞}. Thus we can define

$$\mu_t(z)=\text{area }\hat{f}(\Delta_z(t))=\int_{\Delta_z(t)}(dd^c|\lambda|^2+f\mid^*_{\Delta_{z_\infty}(t)}w). \tag{2.2.6}$$

Let ν_1 be from *Lemma 2.2.1*. Set $E_j = \{z \in \partial W \cap U : \mu_t(z) \leq \frac{j}{2}\nu_1\}$ for $j = 1, 2, \ldots$. From *Lemma 2.3.1* we see that E_j are closed subsets of $\partial W \cap U$, $E_j \subset E_{j+1}$ and we have that $\partial W \cap U = E \cup \bigcup_{j=1}^{\infty} E_j$.

Further from *Lemma 2.2.1* we see that $E_{j+1} \setminus E_j$ is a discrete subset of $U \setminus E_j$, say $U \setminus E_j = \{a_{ij}\}$. Now put

$$u_1(z) = -\sum_{i,j} c_{ij} \log |z - a_{ji}|. \tag{2.2.7}$$

Here the positive constants c_{ij} are chosen in such a manner that $\sum_{i,j} c_{ij} < +\infty$. Then $u_1(z)$ is superharmonic in U, $u_1(z) \to +\infty$ as $z \to \bigcup_{j=1}^{\infty} E_j$ and $u_1(z) \neq +\infty$ for all $z \in U \cap W$. Now put $u_2(z) = \hat{\mu}(z) + u_1(z)$. Note that u_2 is superharmonic in $W \cap U$ and $u_2(z) \to +\infty$ as $z \to \partial W \cap U$. Define

$$u_n(z) = \min\{n, u_2(z)\} \tag{2.2.8}$$

for $n \geq 3$. Note that u_n are superharmonic in U, because $u_n \equiv n$ in the neighborhood of $\partial W \cap U$. Put now $u(z) = \lim_{n\to\infty} u_n(z)$. Then u is superharmonic in U as a nondecreasing limit of superharmonic functions. Using the fact that $\hat{\mu}$ is finite on W, we obtain that $u(z) = u_2(z) \neq +\infty$ for any $z \in U \cap W$ and $u|_{U\setminus W} \equiv +\infty$, i.e. $\partial W \cap \Delta$ is *complete* polar in Δ. So the *Lemma* is proved.

q.e.d.

In what follows we shall use the fact that the closed set of zero harmonic measure on the plain has zero Hausdorff dimension, see [Gl].

(a) Let us finish the proof of the part (1) of *Theorem 2* in dimension two. Put $S_1 = \Delta \setminus W$ where W is a maximal domain in Δ such that our map f extends meromorphically onto $H_W(r)$. We had proved that S_1 is of harmonic measure zero. In particular S_1 is zero dimensional. For any $\delta > 0$ we can find $0 < \delta_1 < \delta$ such that $\partial\Delta_{1-\delta_1} \cap S_1 = \emptyset$. Now we can change coordinates z_1, z_2 and consider the Hartogs figure $H = \{(z_1, z_2) \in \Delta^2 : 1 - r < |z_2| < 1, |z_1| < 1 \text{ or } |z_2| < 1, 1 - \delta_1 - \varepsilon < |z_1| < 1 - \delta_1 + \varepsilon\}$, where ε is small enough. Applying once more *Lemma 2.2.1* we extend f onto $\Delta \times (\Delta \setminus S_2)$ where S_2 is of harmonic measure zero and obtain the statement of part (1) of *Theorem 2* in the case of dimension two.

(b) Suppose now that our metric form w on X is pluriclosed. Adding to $S := S_1 \times S_2$ the discrete in $\Delta^2 \setminus S$ set of points of indeterminacy of f we can suppose that f is holomorphic on $\Delta^2 \setminus S$. Denote by T the positive (1,1) current (in fact smooth form) f^*w on $\Delta^2 \setminus S$. By *Lemma 3.3* from [Iv-2] we have that T has locally summable coefficients on the whole Δ^2 and $dd^c\tilde{T}$ is a negative measure supported on S.

Lemma 2.2.3. *Suppose that the metric form w is pluriclosed and take a ball $B \subset\subset \Delta^2$ such that $\partial B \cap S = \emptyset$.*

(i) If $f(\partial B)$ is homologous to zero in X then $dd^c\tilde{T} = 0$ on B.

(ii) If $dd^c\tilde{T} = 0$ then f meromorphically extends onto B.

In [Iv-2] *Lemma 4.4*] this statement is proved for the case when $S\cap B=\{0\}$. One can easily check that the same proof holds for the case when $S\cap B$ is closed zero dimensional.

So statement (2) of *Theorem 2* is proved in this case.

Let us prove the Remark 1, stated after the Theorem 2 in Introduction. Take a relatively compact open subset $P\subset\Delta^2$ with smooth boundary and choose a finite subcomplex K of CW-complex X to contain the $\mathrm{cl}[f(\bar P\setminus S)]$. Let $\theta_1,...,\theta_N$ be the generators of $H^3(K,\mathbb{Z})$ and $\psi_1,...,\psi_L$ be the generators of $H_3(K,\mathbb{Z})$. Take a real numbers $r_1,...,r_N$ such that

$$d^cw=r_1\theta_1+...+r_N\theta_N \tag{2.2.9}$$

Take a ball $B\subset\subset\Delta^2$ with $\partial B\cap S=\emptyset$. Then there are integers $z_1,...,z_L$ such that

$$f(\partial B)=z_1\psi_1+...+z_L\psi_L \tag{2.2.10}$$

in $H_3(K,\mathbb{Z})$. For the measure μ defined from $dd^c\tilde T=\mu\cdot(\frac{i}{2})^2dz\wedge d\bar z$ we have that

$$\mu(B\cap S)=\int_B dd^c\tilde T=\int_{\partial B}d^cT. \tag{2.2.11}$$

Using that, we can write

$$\mu(B\cap S)=\int_{f(\partial B)}d^cw=\sum_{k=1}^N\sum_{i=1}^L z_ir_k\int_{\psi_i}\theta_k. \tag{2.2.12}$$

Put $c^{ik}=\int_{\psi_i}\theta_k\in\mathbb{Z}$. Now if we put $\tilde z^k=\sum_{i=1}^L z_ic^{ik}\in\mathbb{Z}$ then

$$\mu(B\cap S)=\sum_{k=1}^N\tilde z^kr_k \tag{2.2.13}$$

So for any such a ball $\mu(B\cap S)$ is a linear combination with integer coefficients of a given reals $r_1,...,r_N$. By induction one easily prove that there is an $\varepsilon=\varepsilon(r_1,..,r_N)>0$ such that each linear combination of $r_1,...,r_N$ with integer coefficients is either zero or greater with modulus then ε. We denote this ε as $\varepsilon(w,K)$.

Now we obtain that

$$|S\cap P|\leq\varepsilon(w,K)^{-1}\cdot|\int_{\partial P}d^cw|. \tag{2.2.14}$$

q.e.d.

2.3. Meromorphic families of analytic sets. When one studies *meromorphic* mappings from domains of dimension more then two, one needs some analog of *Lemma 2.2.1* for "meromorphic polydisks". That will be given in this paragraph, see *Lemmas 2.3.1* and 2.3.2. For the proof of this two Lemmas we refer to [Iv-4].

Fix a complex space X, equipped with some Hermitian metric h. By w_h, or simply by w denote the $(1,1)$-form canonically associated with h. Let Δ^q be a polydisk in $\mathbb{C}^q$ with standard Euclidean metric e. The associated form will be denoted by $w_e = dd^c\|z\|^2 = i/2\sum_{j=1}^q dz_j \wedge d\bar{z}_j$. By $p_1 : \Delta^q \times X \longrightarrow \Delta^q$ and $p_2 : \Delta^q \times X \longrightarrow X$ we denote the projections onto the first and second factors. On the product $\Delta \times X$ we consider the metric form $w = p_1^* w_e + p_2^* w_h$.

Definition 2.3.1. By a meromorphic q-disk in the complex space X we shall understand a meromorphic mapping $\phi : \Delta^q \longrightarrow X$, which is defined in some neighbourhood of the closure $\bar{\Delta}^q$.

It will be convenient for us to consider instead of mappings $\phi : \Delta^q \longrightarrow X$ their graphs Γ_ϕ. By $\hat{\phi} = (z, \phi(z))$ we shall denote the mapping into the graph $\Gamma_\phi \subset \Delta^q \times X$. The volume of the graph Γ_ϕ of the mapping ϕ is given by

$$\mathrm{vol}(\Gamma_\phi) = \int_{\Gamma_\phi} w^q = \int_{\Delta^q} (\phi^* w_h + dd^c\|z\|^2)^q \tag{2.3.1}$$

Here by $\phi^* w_h$ we denote the preimage of w_h under ϕ, i.e. $\phi^* w_h = (p_1)_* p_2^* w_h$.

Recall that the Hausdorff distance between two subsets A and B of the metric space (Y, ρ) is a number $\rho(A, B) = \inf\{\varepsilon : A^\varepsilon \supset B, B^\varepsilon \supset A\}$. Here by A^ε we denote the ε-neighborhood of the set A, i.e. $A^\varepsilon = \{y \in Y : \rho(y, A) < \varepsilon\}$.

Further, let $\{\phi_r\}$ be the sequence of meromorphic mappings of the complex space D into the complex space X.

Definition 2.3.2. We shall say that $\{\phi_r\}$ converge on the compacts in D to the meromorphic mapping $\phi : D \longrightarrow X$, if for every relatively compact open $D_1 \subset\subset D$ the graphs $\Gamma_{\phi_r} \cap (D_1 \times X)$ converge in the Hausdorff metric on $D_1 \times X$ to the graph $\Gamma_\phi \cap (D_1 \times X)$.

First we shall prove the following

Lemma 2.3.1. *Let $\{\phi_r\}$ be a sequence of meromorphic q-disks in complex space X. Suppose that there exists a compact $K \subset X$ and a constant $C < \infty$ such that:*

a) $\phi_r(\Delta^q) \subset K$ for all r;

b) $\mathrm{vol}(\Gamma_{\phi_r}) \leq C$ for all r.

Then there exists a subsequence $\{\phi_{r_j}\}$ and a proper analytic set $A \subset \Delta^q$ such that:

1) the sequence $\{\Gamma_{\phi_{r_j}}\}$ converges in the Hausdorff metric to the analytic subset Γ of $\Delta^q \times X$ of pure dimension q;

2) $\Gamma = \Gamma_\phi \cup \hat{\Gamma}$, *where* Γ_ϕ *is the graph of some meromorphic mapping* $\phi : \Delta^q \longrightarrow X$, *and* $\hat{\Gamma}$ *is a pure* q*-dimensional analytic subset of* $\Delta^q \times X$, *mapped by the projection* p_1 *onto* A;

3) $\phi_{r_j} \longrightarrow \phi$ *on compacts in* $\Delta^q \setminus A$;

4) one has

$$\lim_{j\longrightarrow\infty} \mathrm{vol}(\Gamma_{\phi_{r_j}}) \geq \mathrm{vol}(\Gamma_\phi) + \mathrm{vol}(\hat{\Gamma}). \tag{2.3.2}$$

5) For every $1 \leq p \leq dim X - 1$ *there exists a positive constant* $\nu_p = \nu_p(K, h)$ *such that the volume of every pure* p*-dimensional compact analytic subset of* X *which is contained in* K *is not less then* ν_p.

6) Put $\hat{\Gamma} = \bigcup_{p=0}^{q-1} \Gamma_p$, *where* Γ_p *is a union of all irreducible components of* $\hat{\Gamma}$ *such that* $\dim[p_1(\Gamma_p)] = p$. *Then*

$$\mathrm{vol}_{2q}(\hat{\Gamma}) \geq \sum_{p=0}^{q-1} \mathrm{vol}_{2p}(A_p) \cdot \nu_{q-p} \tag{2.3.3}$$

where $A_p = p_1(\Gamma_p)$.

Let S be a set. By $\Delta^q(b)$ we always denote a polydisk of radii b in $\mathbb{C}^n$ centered at origin. Polydisk is equipped with the usual Euklidean metric from $\mathbb{C}^n$.

Definition 2.3.3. By a family of q-dimensional polydisks in complex space X we shell understand an subset $\mathcal{F} \subset S \times \Delta^q(b) \times X$ such that, for every $s \in S$ the set $\mathcal{F}_s = \mathcal{F} \cap \{s\} \times \Delta^q(b) \times X$ is a graph of a meromorphic mapping of $\Delta^q(b)$ into X.

Suppose further that the set S is equipped with topology and let our space X be equipped with some Hermitian metric h.

Definition 2.3.4. We shall say that the family $\mathcal{F}$ is continuous at point $s_0 \in S$ if $\mathcal{H} - \lim_{s\to s_0} \mathcal{F}_s = \mathcal{F}_{s_0}$.

Here by $\mathcal{H} - \lim_{s\to s_0} \mathcal{F}_s$ we denote the limit of closed subsets of $\mathcal{F}_s$ in the Haussdorff metric on $\Delta^q(b) \times X$. $\mathcal{F}$ is continuous if it is continuous at each point of S. If Ω is open in $\Delta^q(b)$ then the restriction $\mathcal{F}_\Omega$ is naturally defined as $\mathcal{F} \cap (S \times \Omega \times X)$.

When S is a complex space itself, we give the following

Definition 2.3.5. Call the family $\mathcal{F}$ meromorphic if the closure $\hat{\mathcal{F}}$ of the set $\mathcal{F}$ is an analytic subset of $S \times \Delta^q(b) \times X$.

Consider a meromorphic mapping $f : \Delta^p \times \Delta^q(a) \longrightarrow X$ into a complex space X. Let S be some closed subset of Δ^p and $s_0 \in S$ some accumulation point of S. Suppose that for each $s \in S$ the restriction $f_s = f|_{\{s\}} \times \Delta^q(a)$ meromorphically extends onto a q-disk $\Delta^q(b)$. Let, as in Lemma 2.3.1 ν_j denotes the minima of volumes of j-dimensional compact analytic subsets contained in some compact $K \subset X$. Fix some $a < c < b$ and put

$$\nu = \min\{\mathrm{vol}(A_{q-j}) \cdot \nu_j : j = 1, ..., q\}, \tag{2.3.6}$$

where A_{q-j} are running over all $(q-j)$-dimensional analytic subsets of $\Delta^q(b)$, intersecting $\Delta^q(c)$. Here $a < c < b$. Clearly $\nu > 0$. In the following *Lemma* the volumes of graphs over polydisks $\Delta(b)$ are taken.

Lemma 2.3.2. *Suppose that there exists a neighbourhood $U \ni s_0$ in Δ^p such that, for all $s_1, s_2 \in S \cap U$*

$$|\,\mathrm{vol}(\Gamma_{f_{s_1}}) - \mathrm{vol}(\Gamma_{f_{s_2}})| < \nu/2 \tag{2.3.7}$$

Then the family $\{\Gamma_{f_s} : s \in U\}$ is continuous at s_0. If, moreover s_0 is a locally regular point of S then there exists a neighbourhood $V_c \ni s_0$ in Δ^p such, that f meromorphically extends onto $V_c \times \Delta^q(c)$.

2.4. Proof of Theorem 2 in higher dimensions. *Proof.* (1) Let $f : H^1_n(r) \to X$ be our map. For an open subset $W \subset \Delta^n$ denote by

$$H_W(r) = (W \times \Delta) \cup (\Delta^n \times A_{1-r,1})$$

the Hartogs figure *over* W.

Step 1. f extends to a holomorphic map of $\bigcup_{z' \in \Delta^{n-1}_r \setminus R_1} (\Delta^2_{z'} \setminus S_{z'})$ into X, where R_1 is contained in locally finite union of locally closed proper subvarieties of Δ^{n-1}_r and $S_{z'}$ is zerodimensional and pluripolar in $\Delta^2_{z'}$.

Proof of Step 1. For $z' = (z_1, ..., z_{n-1}) \in \Delta^{n-1}_r$ by $H^2_{z'}(r)$ denote the two-dimensional Hartogs domain $\{z'\} \times H^2(r)$ in bidisk $\Delta^2_{z'} = \{z'\} \times \Delta^2 \in \mathbb{C}^{n+1}$. Shrinking $H^1_n(r)$ if necessary, we can suppose that $I(f)$ consists of finitely many irreducible components. Denote by R_1 the set of $z' \in \Delta^{n-1}_r$ such that $\dim[H^2_{z'} \cap I(f)] > 0$. R_1 is clearly contained in finite union of locally closed proper analytic subsets of Δ^{n-1}_r. For $z' \in \Delta^{n-1}_r \setminus R_1$ by 2.1 $f\,|_{H^2_{z'}(r)}$ extends to a holomorphic map $f_{z'} : \Delta^2_{z'} \setminus S_{z'} \to X$, where $S_{z'}$ is zerodimensional and complete pluripolar in $\Delta^2_{z'}$. Also $S_{z'} \supset \Delta^2_{z'} \cap I(f)$.

Take a point $z' \in \Delta^{n-1}_r \setminus R_1$ and a point $z_n \in \Delta \setminus \pi_n(S_{z'})$. Here $\pi_n : \{z'\} \times \Delta \times \Delta \to \{z'\} \times \Delta$ is the projection onto the variable z_n. Take a domain $U \subset\subset \{z'\} \times \Delta \times \{0\}$ which is biholomorphic to the unit disk, doesn't contains points from $\pi_n(S_{z'})$ and contains the points $u := (z', 0, 0)$ and $v := (z', z_n, 0)$. We take also U to intersect $A(r, 1)$. If $\{z'\} \times \{0\}$ is in $\pi_n(S_{z'}$ the take as u some point close to $(z', 0, 0)$ in $\{z'\} \times \Delta$. Find a Stein neighbourhood V of the graph $\Gamma_{f|_{\{z'\} \times \bar{U} \times \Delta}}$. Let $w \in \partial U \cap A(r, 1)$ be some point. We have that $f(\{z', w\} \times \Delta) \subset V$ and $f(\{z'\} \times \partial U \times \Delta) \subset V$. So the usual continuity principle for holomorphic functions gives us the *holomorphic* extension of f to the neighborhood of $\{z'\} \times \bar{U} \times \Delta$ in Δ^{n+1}. Changing little bit a bend of z_{n+1} and repeating the arguments as above we obtain a holomorphic extension of f onto the neighborhood of $\{z'\} \times (\Delta \setminus S_{z'})$ for each $z' \in \Delta^{n-1}_r \setminus R_1$.

Step 2. f extends holomorphically onto $(\Delta_r^{n-1}\times\Delta^2)\setminus R$, where R is a closed subset of $\Delta_r^{n-1}\times\Delta^2$ of Hausdorff codimension 4.

Proof of Step 2. Consider a subset $R_2\subset R_1$ consigning of such $z^{'}\in\Delta_r^{n-1}$ that $\dim[H^2_{z'}\cap I(f)]=2$, i.e. $H^2_{z'}\subset I(f)$. This is a finite union of locally closed subvarieties of Δ_r^{n-1} of complex codimension at least two. Thus $\bigcup_{z'\in R_2}\Delta^2_{z'}$ has Hausdorff codimension at least four.

For $z^{'}\in R_1\setminus R_2=\{z^{'}\in\Delta_r^{n-1}:\dim[H^2_{z'}(r)\cap I(f)]=1\}$ using *Theorem 1* we can extend $f_{z'}$ holomorphically onto $\Delta^2_{z'}$ minus zerodimensional polar set. Repeating the arguments from *Step 1* we can extend f holomorphically to the neighborhood of $\Delta^2_{z'}\setminus C_{z'}$ in $\Delta_r^{n-1}\times\Delta^2$. Here $C_{z'}$ is a complex curve containing one dimensional components of $H^2_{z'}(r)\cap I(f)$.

$\bigcup_{z'\in R_1\setminus R_2}C_{z'}$ has Hausdorff codimension at least four. Thus the proof of *Step 2* will be completed when we put $R=\bigcup_{z'\in R_1\setminus R_2}C_{z'}\cup\bigcup_{z'\in R_2}\Delta^2_{z'}$.

Step 3. We shall state this step in the form of the Lemma.

Lemma 2.6.1. There is a closed (n-1)-polar subset $R_0\subset R$ and a holomorphic extension of f onto $(\Delta_r^{n-1}\times\Delta^2)\setminus R_0$ such that the current $T:=f^*w$ has locally summable coefficients in the neighborhood of R_0. Moreover $dd^c\tilde{T}$ is negative.

Take a point $z_0\in R$ and using the fact that R is of Hausdorff codimension four in $\mathbb{C}^{n+1}$, find a neighbourhood $V\ni z_0$ with a coordinate system $(z_1,...,z_{n+1})$ such that $V=\Delta^{n-1}\times\Delta^2$ is those coordinates and for all $z'\in\Delta^{n-1}$ one has $R\cap\partial\Delta^2_{z'}=0$. By 2.1 the restrictions $f_{z'}$ extend holomorphically onto $\Delta^2_{z'}\setminus R_0(z')$, where $R_0(z')$ are closed complete pluripolar in $\Delta^2_{z'}$ of Hausdorff dimension zero. By the arguments similar to that ones from *Step1* f extends holomorphically to the neighborhood of $V\setminus R_0$, $R_0:=\bigcup_{z'\in\Delta^{n-1}}R_0(z')$.

Consider now a current $T=f^*w$ defined on $(\Delta^{n-1}\times\Delta^2)\setminus R$. Note that T is smooth, positive and $dd^cT\le 0$ there. By the part (b_2) of *Lemma 1.1.3* every restriction $T_{z'}:=T\mid_{\Delta^2_{z'}}\in L^1_{loc}(\Delta^2_{z'})$, $z'\in\Delta^{n-1}$. We shall use the following Oka-type inequality for plurinegative currents, proved in [F-Sb]:

there is a constant C_ρ, such that for any plurinegative current T in Δ^2 one has

$$\|T\|(\Delta^2)+\|dd^cT\|(\Delta^2)\le C_\rho\|T\|(\Delta^2\setminus\bar{\Delta}^2_\rho). \tag{2.4.1}$$

Here $0<\rho<1$.

Use (2.4.1) for the trivial extensions $\tilde{T}_{z'}$ of $T_{z'}$, which are plurinegative by (b) of *Lemma 2.1.1*, to obtain that masses $\|\tilde{T}_{z'}\|(\Delta^2)$ are uniformly bounded on z' on compacts in Δ^{n-1}. On L^1 the mass norm coincides with the L^1-norm. So taking the second factor in $\Delta^{n-1}\times\Delta^2$ with different bends and using Fubini theorem we obtain that $T\in L^1_{loc}(\Delta_r^{n-1}\times\Delta^2)$.

All that left to prove is that $dd^c\tilde{T}$ is negative. It is enough to prove that for any collection L of $(n-1)$ linear functions $\{l_1,...,l_{n-1}\}$ the measure $dd^c\tilde{T}\wedge\frac{i}{2}\partial l_1\wedge\bar{\partial}l_1\wedge...\wedge$

$\frac{i}{2}\partial l_{n-1}\wedge\bar\partial l_{n-1}$ is nonpositive, see [Hm]. Complete those functions to the coordinate system $\{z_1=l_1,...,z_{n-1}=l_{n-1},z_n,z_{n+1}\}$ and note that for almost all $z'\in\Delta^{n-1}$ the set $\Delta^2_{z'}\cap R_0$ is of Hausdorff dimension zero. Thus $\tilde T\mid_{z'}$ is plurinegative for all such z'. Take a nonnegative function $\phi\in\mathcal D(\Delta^{n+1})$. We have that

$$< dd^c(\tilde T\wedge(dd^c\|L\|^2)^{n-1}),\phi >=\int_{\Delta^{n+1}}\tilde T\wedge(dd^c\|z'\|^2)^{n-1}\wedge dd^c\phi =$$

$$=\int_{\Delta^{n-1}}(dd^c\|z'\|^2)^{n-1}\int_{\Delta^2}(\tilde T)_{z'}\wedge dd^c\phi == \int_{\Delta^{n-1}}(dd^c\|z'\|^2)^{n-1}\int_{\Delta^2}\tilde T_{z'}\wedge dd^c\phi\leq$$

$$\leq\int_{\Delta^{n-1}}(dd^c\|z'\|^2)^{n-1}\int_{\Delta^2}dd^c(\tilde T)_{z'}\wedge\phi\leq 0.$$

We had used here Fubini theorem for L^1-functions, the fact that $(\tilde T)_{z'}=\tilde T_{z'}$ for currents from L^1_{loc} which are smooth outside of suitably situated set R_0, and finally the plurinegativity of $\tilde T_{z'}$.

Thus $\tilde T$ is plurinegative. Putting $S=R_0$ we get the statement of (a) for $\Delta^{n-1}_r\times\Delta^2$ instead of Δ^{n+1}. But this obviously implies it for Δ^{n+1}.

(2) Now suppose that the metric form w is pluriclosed. Write $V=B^{n-1}\times B^2$ for some neighborhood of point $a\in S$ such that $\pi\mid_S: S\to B^2$ is proper and $S_{z'}=S\cap B^2_{z'}$ is zerodimensional pluripolar compact in $B^2_{z'}$ for all $z^{'}\in B^{n-1}$.

For every $z'\in B^{n-1}$ let $S^0_{z'}$ be a finite, by *Theorem 1*, set of points $s\in B^2_{z'}$ such that $dd^c\tilde T_{z'}$ has nonzero mass at s. $|S^0_{z'}|$ is uniformly bounded on z' on $\bar B^{n-1}$. The points $s\in S^0_{z'}$ could be also characterized by the condition that for any 3-sphere $S_r(s)\subset B^2_{z'}$ centered at s $\int_{S_r(s)}d^cT_{z'}=C_s<0$. Number C_s doesn't depend on r sufficiently small. From this one immediately gets the closeness of the set $S^0:=\bigcup_{z'\in B^{n-1}}S^0_{z'}$.

Step 4. f extends meromorphically onto $V\setminus S^0$.

Proof of Step 4. Let $b\in B^2_{z'}\setminus S^0_{z'}$. Find a neighborhood $W\cong B^{n-1}\times B^2$ of b such that $W\cap S_0=\emptyset$ and $\pi_2\mid_S: S\cap W\to B^2$ is proper. Here S is a minimal closed subset of V such that $f: V\setminus S\to X$ is holomorphic, see part (1).

First we prove the following:

Lemma 2.6.2. *Suppose that the metric form w on X is pluriclosed and for all $z'\in B^{n-1}$ $f(\partial B^2_{z'})\sim 0$ in X. Then:*

(i) $dd^c\tilde T=0$ in the sense of distributions.

(ii) There is a $(1,0)$-current γ in W, smooth in $W\setminus R_0$, such that $\tilde T=i(\partial\bar\gamma-\bar\partial\gamma)$.

Proof. (i) Let $\tilde T_\varepsilon$ be a smoothing of $\tilde T$ by convolution. Then $\tilde T_\varepsilon$ are plurinegative and $\tilde T_\varepsilon\to T$ in $\mathcal D_{n,n}(W)$. We have that

$$\int_W dd^c\tilde T_\varepsilon\wedge(dd^c\|z'\|^2)^{n-1}=\int_{\partial W}d^c\tilde T_\varepsilon\wedge(dd^c\|z'\|^2)^{n-1}=$$

$$= \int_{\partial B^{n-1}\times B^2} d^cT_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1} + \int_{B^{n-1}\times\partial B^2} d^cT_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1}. \tag{2.6.2}$$

The first integral vanishes by the degree reason. So

$$\|dd^c\tilde{T}_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1}\|(W) = -\int_{B^{n-1}} (dd^c\|z'\|^2)^{n-1}\int_{\partial B^2_{z'}} d^c\tilde{T}_\varepsilon. \tag{2.6.3}$$

Remark now that $\int_{\partial B^2_{z'}} d^c\tilde{T}_\varepsilon \to \int_{\partial B^2_{z'}} d^cT_\varepsilon = \int_{f(\partial B^2_{z'})} d^cw = 0$ because $f(\partial B^2_{z'}) \sim 0$ in X. So the right hand side of (2.6.3) tends to zero as $\varepsilon \searrow 0$. We get that

$$\|dd^c\tilde{T} \wedge (dd^c\|z'\|^2)^{n-1}\|(W) = \lim_{\varepsilon\searrow 0}\|dd^c\tilde{T}_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1}\|(W) = 0. \tag{2.6.4}$$

Taking sufficiently many such coordinate systems we see that $\|dd^c\tilde{T}\|(W) = 0$.

(ii) $\partial\tilde{T}$ is a $\bar\partial$-closed and ∂-closed $(2,1)$-current. So, if $\phi \in \mathcal{D}_{n-2,2}(W)$ is ∂-closed and such that $\bar\partial\phi = \tilde{T}$ then ϕ is smooth on $W \setminus S$ by elliptic regularity of $\bar\partial$. We have now $d\tilde{T} = \partial\tilde{T} + \bar\partial\tilde{T} = \bar\partial\phi + \partial\bar\phi$. Thus $d(\tilde{T} - \phi\bar\phi) = 0$. So $\tilde{T} - \phi - \bar\phi$ is d-closed current of degree two on W. Consider an elliptic system in W:

$$d\gamma = \tilde{T} - \phi - \bar\phi$$

$$d^*\gamma = 0 \tag{2.6.5}$$

Then γ has a solution in W. Indeed, let γ_1 be any solution of the first equation. Find a distribution γ on W with $*d*d\gamma = \Delta\gamma = *d*\gamma_1$ and put $\gamma_2 = \gamma_1 - d\gamma$. γ_2 is smooth on $W \setminus S$ because $\Delta\gamma_2 = d^*d\gamma_2 + dd^*\gamma_2 = d^*(\tilde{T} - \phi + \bar\phi)$ and γ_2 satisfies (3.4.6). Write $\gamma_2 = \gamma^{1,0} + \bar\gamma^{1,0}$. Then we have $\partial\gamma^{1,0} = -\phi$ and $\bar\partial\bar\gamma^{0,1} = -\bar\phi$, so

$$\tilde{T} = d\gamma_2 + \phi + \bar\phi = \partial\bar\gamma^{1,0} + \bar\partial\gamma^{0,1} \tag{2.6.6}$$

with $\gamma^{1,0}$ having needed regularity.

q.e.d.

Lemma 2.6.3. *If $\tilde{T}$ is pluriclosed then the volumes $\Gamma_{f_z} \cap \mathbb{B}^2_z \times X$ are uniformly bounded for $z \in \mathbb{B}^{n-1}_r$.*

Proof. Now let $\gamma^{1,0}$ be as in (2.6.6). Smoothing by convolutions we still have $\tilde{T}_\varepsilon = \partial\bar\gamma^{1,0}_\varepsilon + \bar\partial\gamma^{1,0}_\varepsilon$. Take $0 < r_1 < r_2$ in such a way that $\partial(B^{n-1}_r \times B^2_{r_2}) \cap S \subset \partial B^{n-1}_r \times B^2_{r_2}$ for all $r < r_1$, then:

$$\int_{B^{n-1}_r\times B^2_r\setminus S} T^2 \wedge (dd^c\|z'\|^2)^{n-1} \leq \int_{B^{n-1}_r\times B^2_{r_2}\setminus S} T^2 \wedge (dd^c\|z'\|^2)^{n-1} =$$

$$= \lim_{\varepsilon\searrow 0}\int_{B^{n-1}_r\times B^2_{r_2}\setminus S} \tilde{T}^2_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1} \leq \lim_{\varepsilon\searrow 0}\int_{B^{n-1}_r\times B^2_{r_2}} \tilde{T}^2_\varepsilon \wedge (dd^c\|z'\|^2)^{n-1} =$$

$$= \lim_{\varepsilon \searrow 0} \int_{B_r^{n-1}\times B_{r_2}^2} (\partial\bar\gamma_\varepsilon^{1,0} + \bar\partial\gamma_\varepsilon^{1,0})^2 \wedge (dd^c\|z'\|^2)^{n-1} \leq$$

$$\leq \lim_{\varepsilon \searrow 0} \int_{B_r^{n-1}} (dd^c\|z'\|^2)^{n-1} \int_{B_{r_2}^2} d(\bar\gamma_\varepsilon^{1,0} + \gamma_\varepsilon^{1,0}) \wedge d(\bar\gamma_\varepsilon^{1,0} + \gamma_\varepsilon^{1,0}) =$$

$$= \lim_{\varepsilon \searrow 0} \int_{B_r^{n-1}} (dd^c\|z'\|^2)^{n-1} \int_{B_{r_2}^2} (\bar\gamma_\varepsilon^{1,0} + \gamma_\varepsilon^{1,0}) \wedge d(\bar\gamma_\varepsilon^{1,0} + \gamma_\varepsilon^{1,0}) =$$

$$= \int_{B_r^{n-1}} (dd^c\|z'\|^2)^{n-1} \int_{B_{r_2}^2} (\bar\gamma^{1,0} + \gamma^{1,0}) \wedge d(\bar\gamma^{1,0} + \gamma^{1,0}) \leq c \cdot r^{2(n-1)}. \tag{2.6.7}$$

In the second inequality we had used the positivity of T. In the third the fact that $\bar\partial\bar\gamma_\varepsilon^{1,0}\wedge\partial\gamma_\varepsilon^{1,0}$ is positive and $\bar\partial\bar\gamma_\varepsilon^{1,0}\wedge\bar\partial\bar\gamma_\varepsilon^{1,0} = 0$. Finally $\gamma_\varepsilon^{1,0} \to \gamma^{1,0}$ on $\bar B_r^{n-1}\times\partial B_{r_2}^2$ while $\gamma^{1,0}$ is smooth there. This gives a needed bound for the $\int_{B_r^{n-1}\times B_r^2\setminus S} T^2\wedge(dd^c\|z'\|^2)^{n-1}$.

q.e.d.

Theorem 1 gives us now the proof of *Step 4*.

Step 5. The set S^0 is analytic of pure codimension two.

Proof of Step 5. We had proved that f meromorphically extends onto $V\setminus S^0$, where $V \cong B^{n-1}\times B^2$ and S^0 is a graph of N- valued continuous mapping of B^{n-1} to B^2. N- valued means that $|S^0_{z'}| \leq N$ for every $z' \in \bar B^{n-1}$ and there exist z_0' such $|S^0_{z'}| = N$. Continuous - simply that S^0 is closed. Remark that $\tilde T$ is L^1-current on V with $dd^c\tilde T \leq 0$ and supported on S^0.

Lemma 2.6.4. *Let S^0 be the graph of N- valued continuous mapping of $\bar\Delta^k$ to Δ^l and let R be a closed positive current in Δ^{k+l} of bidimension (k,k) supported on S^0. Then S^0 is a pure k-dimensional analytic variety in Δ^{k+l}.*

Proof. Write $R = R_{K,\bar J}(\frac{i}{2})^k\frac{\partial}{\partial z^K}\wedge\frac{\partial}{\partial\bar z^J}$, where K and J are multiindices of length k. Consider a measures $R_{K,\bar J}$. Disintegrate this measures with respect to the natural projection $\pi : \Delta^k\times\Delta^l \to \Delta^k$, see [Bk], p.58. Denote by $\mu_{K,\bar J} = \pi_*(R_{K,\bar J})$ their direct images. Then disintegration means that one has probability measures $\nu_{K,\bar J,z'}$ on $\Delta^l_{z'} := \{z'\}\times\Delta^l$ with the property that for every continuous function h in Δ^{k+l}

$$< R_{K,\bar J}, h >= \int_{\Delta^k}(\int_{\Delta^l_{z'}} \bar h\mid_{\Delta^l_{z'}} d\nu_{K,\bar J,z'})d\mu_{K,\bar J}, \tag{2.6.8}$$

see [Bk] or [D-M].

Let Ω be the maximal open subset of Δ^k such that the multivalued map s, which is given by its graph S^0 takes exactly N different values (and N is maximal). First we shall prove that $S^0\cap(\Omega\times\Delta^l)$ is analytic.

Let further Ω_1 be some simply connected open subset of Ω. Then $s\mid_{\Omega_1}$ decomposes to N well-defined single valued maps $s^1,...,s^N$. So it is enough to consider the case when s is single valued. Put $s(z') = (s_1(z'),...,s_l(z'))$. Note that in this case $\nu_{K,\bar J,z'} =$

$\delta_{\{z''-s(z')\}}$. We can write for the coefficients of our current R and for $\phi \in C^{\infty}(\Delta^{k+l})$ such that $\pi(\operatorname{supp}\phi) \subset\subset \Delta^k$ that

$$< R_{K,\bar{J}}, \phi >= \int_{\Delta^k} \bar{\phi}(z', s(z'))d\mu_{K,\bar{J}}(z'). \tag{2.6.9}$$

If we choose ϕ not depending on $z'' := (z_{k+1}, ..., z_{k+l})$ then (2.6.9) gives

$$< R_{K,\bar{J}}, \phi >= \int_{\Delta^k} \bar{\phi}(z')d\mu_{K,\bar{J}}(z'). \tag{2.6.10}$$

From the closeness of R we obtain that

$$0 =< R, d[(\frac{i}{2})^k\phi(z')dz_1\wedge ...dz_{p-1}\wedge dz_{p+1}\wedge ...\wedge dz_k\wedge d\bar{z}_J] >=< R_{1...(p-1)(p+1)...k,\bar{J}}, \frac{\partial\phi}{\partial z_p} >=$$

$$= \int_{\Delta^k} \frac{\partial\bar{\phi}}{\partial\bar{z}_p}d\mu_{K,\bar{J}}.$$

So $\mu_{K,\bar{J}}(z_1) = c_{K,\bar{J}}(z_1)\cdot(\frac{i}{2})^k dz'\wedge d\bar{z}'$, where $c_{K,\bar{J}}$ are holomorphic for $K = (1, ..., k)$ and all J. In particular $c_{1...k,\bar{1}...\bar{k}}$ is constant. Now take the $(k-1, k)$- forms $\psi_{q\bar{p}} = \phi(z')\cdot\bar{z}_p\cdot(\frac{1}{2})^k dz_1\wedge ...\wedge dz_{q-1}\wedge dz_{q+1}\wedge ...\wedge dz_k\wedge d\bar{K}$. We have

$$0 =< R, d\psi_{q,\bar{p}} >=< R, \frac{\partial\phi}{\partial z_q}\cdot\bar{z}_p(\frac{i}{2})^k dz^K\wedge d\bar{z}^K >=< R_{K,\bar{K}}, \frac{\partial\phi}{\partial z_q}\bar{z}_{\bar{p}} >=$$

$$= c_{1\bar{1}}\int_{\Delta} \frac{\partial\bar{\phi}}{\partial\bar{z}_q}(z')\cdot s_p(z')(\frac{i}{2})^k dz'\wedge d\bar{z}',$$

i.e. s_p are holomorphic.

Thus we had proved that s is N-valued analytic map of Ω into Δ^l. Considering appropriate discriminants and using Rado's Theorem, we obtain analyticity of s on the whole Δ^k.

If for some sphere S^3 imbedded into $\Delta^{n+1}\setminus S^0$ $f(S^3)$ is homologous to zero in X then one can extend f through one of the branches of S^0 using the same arguments as in the proof of *Step 4.*

q.e.d.

2.5. Complex Plateau problem for three-dimensional contours. We shall apply the results of this paragraph to the complex Plateau problem. Namely, we shall prove part (b) of the *Corollary 3* from *Introduction.*

Recall that complex Plateau problem for a real compact submanifold M of complex manifold X consists in finding a complex analytic subset $A \subset X\setminus M$ such that $\partial[A] = [M]$ in the sense of currents. The necessary condition on M for complex Plateau problem to have a solution is the maximal complexity of M, i.e. M should be a CR-submanifold of X of $\dim_{CR} M = p$, where $\dim_{\mathbb{R}} M = 2p+1$. In the case when X is Stein this is also sufficient, see [H-L]. Already in the case $X = \mathbb{CP}^3$ the maximal complexity of the "contour" M is not sufficient any more, [Dl].

We suppose that M bounds an abstract Stein domain, i.e. there is a complex manifold D with boundary M such that $D \setminus M$ is Stein, and the CR-embedding $f : M \to X$ is given. All that we need to prove is that f extends meromorphically onto D. Clearly we can suppose that f is already holomorphically extended to some neighborhood of M in D.

Proposition 2.5.1. *Let (D, M) be as above and suppose additionally that* $\dim M = 3$.

(a) Then any CR-map $f : M \to X$, where X is a disk-convex complex space admitting

a pluriclosed Hermitian metric form, extends meromorphically onto $D \setminus S$. Here S is a finite subset of D.

(b) If $f(M)$ is homologous to zero in X, or if X doesn't contain spherical shells, then

S is empty.

Proof. Let $\rho : D \to [0, 1]$ be a strictly plurisubharmonic (and thus Morse) exhausting function. Denote by $D^+_\varepsilon = \{z \in D : \rho(z) > \varepsilon\}$. Let $\mathcal{E}$ be the set of such ε that f can be meromorphically extended onto $D^*_\varepsilon \setminus S_\varepsilon$, where S_ε is a discrete set. $\mathcal{E}$ is obviously closed and nonempty. All we need to prove is that $\mathcal{E}$ is open.

Let $\varepsilon_0 = \inf\{\varepsilon \in \mathcal{E}\}$. If ε_0 is a regular value of ρ then the needed result immediately follows from part (2) of *Theorem 2*.

Consider the case of not regular value ε_0 of ρ. Denote by $M_{\varepsilon_0} = \{z : \rho(z) = \varepsilon_0\}$- the critical level set. Fix a critical point $z_0 \in M_{\varepsilon_0}$. All we need to prove is that for any neighborhood W of z_0 the envelope of holomorphy of $W \cap D^+_\varepsilon$ contains some neighborhood of z_0. For convenience we can suppose that $z_0 = 0$ and $\varepsilon_0 = 0$. Write

$$\rho(z) = Q(z) + < z, z > + \bar{Q}(z) + O(\|z\|^3), \tag{2.5.1}$$

where $Q(z)$ is a holomorphic polynomial, $< z, z >$ - Hermitian form - Levi form of ρ. By linear coordinate change we transform $<,>$ to the sum of squares of absolute values. Then by unitary coordinate change we transform Q to the some of squares with real nonnegative coefficients. Now (2.5.1) has a form

$$\rho(z) = \sum_{j=1}^{p} a_j z_j^2 + \sum_{j=1}^{p} a_j \bar{z}_j^2 + \sum_{j=1}^{n} |z_j|^2 + O(\|z\|^3). \tag{2.5.2}$$

In coordinates $z_j = x_j + iy_j$ we rewrite (2.5.2) as follows

$$\rho(z) = 2\sum_{j=1}^{p} a_j(x_j^2 - y_j^2) + \sum_{j=1}^{n}(x_j^2 + y_j^2) + O(\|z\|^3) =$$

$$= \sum_{j=1}^{p}[(1+2a_j)x_j^2 + (1-2a_j)y_j^2] + \sum_{j=p+1}^{n}(x_j^2+y_j^2) + O(\|z\|^3). \tag{2.5.3}$$

Renumerate the coordinates in such a way that $a_j \geq \frac{1}{2}$ for $j = 1,...,q$ and $a_j < 1/2$ for $j = q+1,...,p$. Then

$$\rho(z) \geq \sum_{j=1}^{q}[(2a_j+1)x_j^2 - (2a_j-1)y_j^2] + \delta\cdot\sum_{j=q+1}^{p}|z_j|^2 + O(\|z\|^3) \geq$$

$$\geq \sum_{j=}^{q}[(2a_j-\delta_1+1)x_j^2 - (2a_j+\delta_1-1)y_j^2] + \delta\cdot\sum_{j=q+1}^{p}|z_j|^2 := \rho_1(z), \tag{2.5.4}$$

for some $\delta > 0$ and δ_1 can be chosen arbitrarily small for small $\|z\|$. While obviously $D^+ := \{z \in \mathbb{B}^n : \rho_1(z) > 0\} \subset D^+_{\varepsilon_0}$, all we need is to prove the following

Lemma 2.5.2. The envelope of holomorphy of D^+ contains the origin.

Proof. Consider two cases.

Case 1: $q \leq n-1$. In this case D^+ contains the following Hartogs figure:

$$H := \{z \in \mathbb{B}^n : \sum_{j=1}^{q}[(2a_j-\delta_1+1)x_j^2 - (2a_j+\delta_1-1)y_j^2] > 0, \delta\cdot\sum_{j=q+1}^{n}|z_j|^2 < 1$$

or

$$\sum_{j=1}^{q}(2a_j-\delta_1+1)x_j^2 - (2a_j+\delta_1-1)y_j^2] > -\varepsilon, \delta\cdot\sum_{j=q+1}^{n}|z_j|^2 > \varepsilon\}.$$

The envelope of holomorphy of H obviously contains the origin.

Case 2: $q = n$. In this case

$$D^+ = \{z \in \mathbb{B}^n : \sum_{j=1}^{n}[(2a_j-\delta_1+1)x_j^2 - (2a_j+\delta_1-1)y_j^2] > 0\}. \tag{2.5.5}$$

Put $b_j = 2a_j - \delta_1 + 1$, $c_j = 2a_j + \delta_1 - 1$, $j = 1,...,n$. For small $\delta - 1$, $b_j > c_j$. Write (2.5.5) in the form

$$D^+ = \{z \in \mathbb{B}^n : \sum_{j=1}^{n}b_jx_j^2 > \sum_{j=1}^{n}c_jy_j^2\}. \tag{2.5.6}$$

In the new coordinates $z_j \to \sqrt{b_j}z_j$ (2.5.6) take a form

$$D^+ = \{z \in \mathbb{B}^n : \sum_{j=1}^{n}x_j^2 > \sum_{j=1}^{n}\delta_jy_j^2\}, \tag{2.5.7}$$

where $\delta_j = \frac{c_j}{b_j} < 1$, $j = 1,...,n$. Put $\delta_0 := \max\{\delta_1,...,\delta_n\} < 1$. Then

$$D^+ \supset D_1^+ = \{z \in \mathbb{B}^n : \|x\|^2 > \delta_0\cdot\|y\|^2\}. \tag{2.5.8}$$

The set D_1^+ contains clearly the following complete "tube torus"

$$T = \{x + iy \in \mathbb{C}^n : \|x\| = 1, \|y\| \leq 1/\delta_0\}, \tag{2.5.9}$$

where $1/\delta_0 := \eta > 1$. We shall prove that already the envelope of holomorphy of T contains the origin. For this consider the following continuous family of complex hypersurfaces

$$C_t = \{z \in \mathbb{C}^n : z_1^2 + ... + z_n^2 = t\} \tag{2.5.10}$$

or

$$C_t = \{x + iy \in \mathbb{C}^n : \|x\|^2 - \|y\|^2 = t, (x,y) = 0\}, \tag{2.5.11}$$

where $(x,y) = x_1y_1 + ... + x_ny_n$. Consider the intersections of C_t with a ball of radii $1+\eta^2$:

$$\tilde{C}_t = \{x + iy \in \mathbb{B}^n_{1+\eta^2} : \|x\| - \|y\| = t, (x,y) = 0\}. \tag{2.5.12}$$

This is a continuous family of irreducible analytic hypersurfaces in $\mathbb{B}^n_{1+\eta}$ such that

$$\tilde{C}_{1+\eta^2} = \{x + iy \in \mathbb{B}^n_{1+\eta^2} : |x\|^2 - \|y\|^2 = 1 + \eta^2, (x,y) = 0\} =$$

$$= \{x + iy \in \mathbb{B}^n_{1+\eta^2} : \|x\|^2 + \|y\|^2 = 1 + \eta^2 = \|x\|^2 - \|\|^2, (x,y) = 0\} =$$

$$\{x + iy \in \mathbb{B}^n_{1+\eta^2} : \|x\|^2 = 1 + \eta^2, y = 0\} \subset T,$$

but $\tilde{M}_0 \ni 0$. By continuity principle the envelope of holomorphy of T contains the origin.

q.e.d

End of the proof of Corollary 3b.

So, as in the case of regular value, we can extend our map f meromorphically to the neighborhood of the critical level M_{ε_0} minus discrete set. As a result we obtain the extension $\hat{f}$ of our map onto $\bar{D} \setminus S$ where S is a finite subset of $\bar{D}$ not intersecting $M = \partial D$. If we put $T := f^*w$ then $dd^c\tilde{T}$ is nonpositive measure supported on S. We have

$$\int_S dd^c\tilde{T} = \int_D dd^c\tilde{T} = \int_{\partial D} d^cT = \int_{f(\partial D)} d^cw = \int_M d^cw = 0,$$

if M is homologous to zero in X, or if X doesn't contain spherical shells.

Corollary 3b is proved.

3. Generalisations, applications, open questions

In this last section of the paper we shall give some more applications of the results and (in fact more) techniques used here.

3.1. Multivalued mappings and singular domains of definition. Let D be a domain in complex space Ω and $x_0 \in \partial D$ be a boundary point. D is said to be q-concave at x_0 if there is a neighborhood $U \supset x_0$ and smooth function $\rho : U \to \mathbb{R}$ such that

1) $D \cap U = \{x \in U : \rho(x) < 0\}$;
2) Levi form of ρ at x_0 has at least $n - q + 1$ negative eigenvalues.

Here $n = \dim \Omega$. By the Projection Lemma of Siu, see [Si-T], if x_0 is a q-concave boundary point of D, $q \leq n-1$, one can find a neighborhoods $U \ni x_0$ and $V \ni 0 \in \mathbb{C}^n$ and a proper holomorphic map $\pi : (U, x_0) \to (V, 0)$ such that $\pi(D \cap V)$ will contain a Hartogs figure H, whose associated polydisk P contains the origin. Let d be the branching number of π.

Now suppose that a meromorphic map $f : D \to X$ is given, where X is another complex space. $\pi^{-1} \circ f$ defines a d-valued meromorphic correspondence between V and X.

Definition 3.1.1. A d-valued meromorphic correspondence between complex spaces V and X is an irreducible analytic subset $Z \subset V \times X$ such that restriction $p_1 |_Z$ of the natural projection onto the first factor on Z is proper, surjective and generically d to one.

Thus the extension of f onto the neighborhood of x_0 is equivalent to the extension of Z from H to P. Clear that if f was also a correspondence it will produce no additional complications. Thus we should discuss how far the problem of extending of correspondences go from the extension of mappings.

Let Z be a d-valued meromorphic correspondence between the Hartogs figure H and X. Z defines in a natural way a mapping $f_Z : H \to \mathrm{Sym}^d(X)$ - symmetric power of X of order d. Clearly the extension of Z onto P is equivalent to the extension of f_Z onto P. If X was for example a Kähler manifold. Then by [V] $\mathrm{Sym}^d(X)$ is a Kähler space. So meromorphic correspondences with values in Kähler manifolds are extendable through pseudo-concave boundary points.

For the manifolds from class $\mathcal{G}_1$ this is no longer the case, if even they doesn't contain spherical shells.

3.2. Construction of Example 2: appearance of shells in symmetric products. In this section we shall construct the following

Example 2. *There is a compact complex (elliptic) surface X such that:*

(a) every meromorphic map $f : H^2(r) \to X$ extends meromorphically onto Δ^2, but
(b) there exists a two-valued meromorphic correspondence Z between $\mathbb{C}^2_*$ and X which cannot be extended to origin.

Consider a standard Hopf surface $H = \mathbb{C}^2 \setminus \{0\}/(z \sim 2 \cdot z)$. By $\pi : H \to \mathbb{CP}^1$ denote the standard projection. Let $\phi : C \to \mathbb{CP}^1$ be a nonconstant meromorphic function on the Riemann surface C of positive genus. ϕ will be a d-sheeted ramified covering of $\mathbb{CP}^1$ by C. If we take C to be a torus we can have such ϕ with $d = 2$. Following Kodaira we shall construct an elliptic surface over C in the following way. Put

$$X_1 = \{(z, y) \in C \times H : \phi(z) = \pi(y)\}. \tag{3.2.1}$$

Elliptic structure on X_1 is given by the restriction onto X_1 of the natural projection $p_1 : C \times H \to C$. Note that restriction onto X_1 of the natural projection $p_2 : C \times H \to H$ gives us an d-sheeted covering $p_2 \mid_{X_1}$ of H by X_1 which preserves the elliptic structure. Let $n : X \to X_1$ be a normalization of X_1. Then X is a smooth elliptic surface over C with elliptic fibration $p := p_1 \mid_{X_1} \circ n : X \to C$ and $F := p_2 \mid_{X_1} \circ n : X \to H$ will be a d- sheeted covering.

$Z := F^{-1} \circ \pi : \mathbb{C}^2_* \to X$ is d-valued meromorphic correspondence between $\mathbb{C}^2_*$ and X, which cannot be extended to origin, because the projection $\pi : \mathbb{C}^2_* \to H$ cannot be extended meromorphically to zero.

On the other hand in [Iv-1] it was proved that meromorphic mappings from $H^2(r)$ to an elliptic surface over a Riemann surface of positive genus are extendable onto Δ^2.

One can interpret this example in the way that $\mathrm{Sym}^2(X)$ could have a spherical shell if even X has not.

3.3. Extension theorems for meromorphic correspondences. In view of discussion in 3.1 and 3.2 we can restate our results for meromorphic correspondences.

Definition 3.3.1. By a branched spherical shell of degree d in a complex space X we shall understand an image Σ of $\mathbb{S}^3 \subset \mathbb{C}^2$ under the d-valued meromorphic correspondence between some neighborhood of $\mathbb{S}^3$ and X such that $\Sigma \not\sim 0$ in X.

Corollary 3.3.1. *Let Z be a meromorphic correspondence from the domain D in complex space Ω into the disk-convex complex space $X \in \mathcal{G}_1$ and let x_0 be a concave boundary point of D. Then Z extends onto some neighborhood of x_0 in Ω minus (possibly empty) complex variety C of pure codimension two. If X doesn't contain branched spherical shells then $C = \emptyset$.*

Remark. We would like to point out here that branched shell could be a much violate object that not branched one. For this purpose we consider a smooth complex curve C in the neighborhood of $\bar{\mathbb{B}}^2(2) \setminus \mathbb{B}^2(1/2)$ which doesn't extend to $\mathbb{B}^2(1)$. Let $\pi : W \to \mathbb{B}^2(2)\setminus\bar{\mathbb{B}}^2(1/2)$ be a covering branched along C. The images of $\hat{S} := \pi^{-1}(\mathbb{S}^3)$ in the complex spaces could be a branched shells which doesn't bound an abstract Stein domains.

3.4. Complex Plateau problem for higher-dimensional contours. First let us prove part (a) of *Corollary 3* from *Introduction*. Actually only in the end of the proof in 2.5 we used the fact that the dimension of D is two. The following proposition clearly enables us to finish the proof also of part (a) of *Corollary 3* i.e for $\dim D \geq 3$.

Proposition 3.4.1. *Every holomorphic map f from $H_1^n(r)$ to disk-convex complex space $X \in \mathcal{G}_1$ extends meromorphically onto Δ^n provided $n \geq 2$.*

Remark. In the *Example 2* (see Introduction) the map $F : \mathbb{B}^3 \to X$ was *not holomorphic* ! We shall essentially use the condition of holomorphicity of f in the proof of this *Proposition* .

Proof. It will be convenient for us simultaneously with the proof of the main statement of the *Proposition* to prove also the following weaker statement. Denote by $A^n(a,b) := \Delta^n(b) \setminus \Delta^n(a)$, for $0 \leq a < b$.

Every holomorphic map $f : A^n(\frac{1}{2}, 1) \to X$, where X from Proposition 5.1.1, extends meromorphically onto Δ^n, provided $n \geq 2$ and $f(\partial\Delta^n_{3/4})$ is homologous to zero in X.

We shall prove both statements by induction on n. For $n = 2$ the second statement follows directly from *Theorem 1.* So it is sufficient to prove that for any $n \geq 2$ from the second statement follows the statement of *Proposition* for this n.

So let a holomorphic mapping $f : H_1^n(r) \to X$ is given. For every $z \in \Delta$ restriction f_z of f onto $\Delta^n_z := \{z\} \times \Delta^n$ is holomorphic on $A^n(r,1)$. So, by the assumption f_z meromorphically extends onto Δ^n, because $f(\partial\Delta^n_z) \sim f(\partial\Delta^n_0) \sim 0$ in X! Lemma 2.3.2 immediately gives us (after shrinking Δ^{n+1} and taking different bends of $z_2, ..., z_{n+1}$) the meromorphic extension of f onto $\Delta^{n+1} \setminus S$. Where S is zerodimensional pluripolar compact in Δ^{n+1}.

Because $I(f)$ is an analytic set of positive dimension outside of zerodimensional set, $\overline{I(f)}$ is analytic in $\Delta^{n+1} \setminus H_1^n$, and thus empty. So the fundamental set of f is discrete in $\Delta^{n+1} \setminus S$.

Put $T := f^* w$, where w is pluriclosed metric form on X. T has locally summable coefficients in Δ^{n+1} and its trivial extension $\tilde{T}$ is plurinegative with $dd^c\tilde{T}$ supported on S. Observe that $\tilde{T} = T$ is pluriclosed outside of S.

Lemma 2.6.2 tells us that $dd^c\tilde{T} = 0$ and moreover there is a (1,0)-current γ in any given ball $W \subset \Delta^{n+1}$, $\partial W \cap S = \emptyset$, smooth on $W \setminus S$, such that $\tilde{T} = i(\partial\bar{\gamma} - \bar{\partial}\gamma)$. Remark that the conditions of *Lemma 2.6.2* are satisfied, because S is zerodimensional and $n+1 \geq 3$. All that remained is to repeat the arguments from the proof of *Lemma 2.6.3* to estimate the volume of the graph of f in the neighborhood of S. Namely

$$\mathrm{Vol}(\Gamma_{f|_{W\setminus S}}) = \int_{W\setminus S}(T + dd^c\|z\|^2)^{n+1} = \sum_{j=0}^{n+1} C^j_{n+1} \int_{W\setminus S} T^j \wedge (dd^c\|z'\|^2)^{n+1-j} \leq$$

$$\leq C \cdot \int_{W\setminus S} T^j \wedge (dd^c\|z'\|^2)^{n+1-j} = C \cdot \lim_{\varepsilon \searrow 0} \int_{W\setminus S} \tilde{T}^j_\varepsilon \wedge (dd^c\|z'\|^2)^{n+1-j} \leq$$

$$\leq C \cdot \lim_{\varepsilon \searrow 0} \int_W \tilde{T}^j_\varepsilon \wedge (dd^c\|z'\|^2)^{n+1-j} = C \cdot \lim_{\varepsilon \searrow 0} \int_W (\partial\bar{\gamma}^{1,0}_\varepsilon + \bar{\partial}\gamma^{1,0}_\varepsilon)^j \wedge (dd^c\|z'\|^2)^{n+1-j} =$$

$$= C \cdot \lim_{\varepsilon \searrow 0} \int_W (d(\bar{\gamma}_\varepsilon^{1,0} + \gamma_\varepsilon^{1,0}))^j \wedge (dd^c\|z'\|^2)^{n+1-j} =$$

$$= C \cdot \lim_{\varepsilon \searrow 0} \int_{\partial W} (\bar{\gamma}_\varepsilon^{1,0} + \gamma_\varepsilon^{1,0}) \wedge d(\bar{\gamma}_\varepsilon^{1,0} + \gamma_\varepsilon^{1,0})^{j-1} \wedge (dd^c\|z'\|^2)^{n+1-j} =$$

$$= C \cdot \int_{\partial W} (\bar{\gamma}^{1,0} + \gamma^{1,0}) \wedge d(\bar{\gamma}^{1,0} + \gamma^{1,0})^{j-1} \wedge (dd^c\|z'\|^2)^{n+1-j} < \infty$$

From Bishop theorem we get an extension of the graph of f onto Δ^{n+1}.

q.e.d.

3.5. Coverings of compact complex manifolds. Let X be a compact complex manifold and let D be a domain in complex manifold Ω which covers X. This means that there is some subgroup G of the group $\mathrm{Aut}(D)$ of biholomorphic automorphisms of D, acting properly and discontinuously on D without fixed points, such that $D/G = X$.

Before proving the *Corollary 4* we want to recall that locally pseudoconvex domain in Stein manifold is Stein (Oka), and moreover locally pseudoconvex domain in $\mathbb{CP}^n$ is also Stein, provided it is different from $\mathbb{CP}^n$ itself, see [Ks] and [Hs-2].

Proof of Corollary 4. Let $f : D \to X$ be a covering map. Suppose that one can find a point $p \in \partial D$ such that for any neighborhood V, biholomorphic to a ball, $V \cap D$ is not pseudoconvex. By Docquer-Grauert theorem,[D-G], this means that there exists an embedding $\phi : H_{n-1}^1 \to V$ such that $\phi(\Delta^n) \cap \partial D$ is nonempty, $n = \dim D$. But $f \circ \phi$ extends meromrophically onto $\Delta^n \setminus S$, where S is:

(a) locally finite union of subvarieties of pure codimension two by *Theorem 2* ;
(b) empty, by [Iv-3].

This means that locally $\partial D \subset I(f) \cup S$.

Suppose we can find a point $p \in (I(f) \setminus S) \cap D$. Take an analytic disk $\psi : \Delta \to U$ through p, i.e.$\psi(0) = p$, which is not contained in $I(f)$. Composition $f \circ \psi : \Delta \to X$ is holomorphic. Put $x := (f \circ \psi)(o)$. Choose a neighborhood $G \ni x$ such that *every* branch of f^{-1} is single valued on G. While $f \circ f^{-1} = \mathrm{id}$ on G, the map f must be *holomorphic* in the neighborhood of p. Really, take $p_1 \in \psi(\Delta) \setminus I(f)$ and close enough to p to have $x_1 = f(p_1) \in G$. Then the branch of f^{-1} which sends x_1 to p_1 must send x to p. Contradiction.

Thus $I(f) = \emptyset$ and $D \supset (U \setminus S)$.

q.e.d.

4. Open questions

In this paragraph we list some open problems which seem to us be of significant interest.

Problem 1. Prove the conjecture stated in Itroduction.

Problem 2. Let the complex manifold D is defined as two-sheeted cover of $\Delta^2\setminus\mathbb{R}^2$, i.e. D is "nonschlicht" domain over $\mathbb{C}^2$. Does there exist a compact compex manifold X and a holomorphic (meromorphic) mapping $f : D \to X$ which separates the points?

Note that the results of this paper imply that such X if exists cannot possed a plurinegative metrik form. Thus examples could occur starting from $\dim X \geq 3$.

In the following problems the space X is equipped with some Hermitian metrik. On the subsets of $\mathbb{C}^n$ the metrik is allwayse $dd^c\|z\|^2$.

Problem 3. Consider a class $\mathcal{J}_R$ of meromorphic mappings $f : \Delta^k \to X$, X beeing compact, such that

(a) $\|Df\| \geq R > 0$. Here $\|Df\|$ denotes the norme of the differential of f;

(b) $\mathrm{vol}(f_s(\Delta^k) \leq C_1$ for all $s \in \Delta^k$.

Prove that there is a constant $C_2 = C_2(X, R, C_1)$, not depending on f, such that $\mathrm{vol}(\Gamma_{f_s}) \leq C_2$.

Problem 4. Let $f : \Delta^k_* \to X$ be a meromorphic mapping from a punctured polydisk into a compact complex space X. Suppose that $\mathrm{vol}\, f(\Delta^k_*) < \infty$. Prove that f meromorphically extends to zero.

Problem 5. Let $f : \Delta^{k+1}_* \to X \in \mathcal{G}_k$ be a meromorphic map from punctured $(k+1)$-disk into a compact complex space from class $\mathcal{G}_k$, see Introduction. Prove that $\mathrm{vol}(f(A^k(r,1)) = O(log^{\frac{k+1}{k}}(\frac{1}{r}))$. In particular for equidimensional maps $f : \Delta^n_* \to X^n$ one allwayse should have $\mathrm{vol}(f(A^n(r,1)) = O(log^{\frac{n+1}{n}}(\frac{1}{r}))$.

Problem 6. Fix some $0 < r < 1$ and some constant R. Fix also a compact complex space X. Consider the following class $\mathcal{F}_R$ of meromorphic mappings from $f : \Delta^n \to X$:

(a) $\mathrm{vol}_{2n}(\Gamma_f \cap (A^n(r,1) \times X)) \leq R$*;*

(b) for every k-disk Δ^k_z which has sides parallel to the coordinate planes (i.e. $\Delta^k_z = \{z\} \times \Delta^k$, where $z \in \Delta^{n-k}$) $\mathrm{vol}_{2k}(\Gamma_{f|_{\Delta^k \cap A^n(r,1)}}) \leq R$.

6a. Prove that for any constant l there is a constant A such that for any $f \in \mathcal{F}_R$ satisfying $\mathrm{vol}_{2k}(\Gamma_{f_z}) \leq l$ for all restrictions f_z of f onto the k-disks Δ^k_z one has $\mathrm{vol}_{2n}(\Gamma_f) \leq A$.

6b. Vice versa: for any constant a there is a constant L such that forany $f \in \mathcal{F}_R$ such that $\mathrm{vol}_{2n}(\Gamma_f) \leq a$ one has $\mathrm{vol}_{2k}(\Gamma_{f_s}) \leq L$ for all Δ^k_z.

Problem 7. Let X is a compact complex manifold carrying a plurinegative metrik form, and let $f : \Delta^3 \setminus S \to X$ is a meromrohpic mapping. Suppose that S is a minimal closed subset of Δ^3 such that f extends onto $\Delta^3 \setminus S$. Prove that each connected component of S is a complex curve.

Problem 8. Let K a complet pluripolar compact in pseudoconvex domain in $\mathbb{C}^n, n \geq 2$. Let T a positive current on $D \setminus K$ such that $dd^cT \leq O$ on $D \setminus K$.

(a) Prove that T has locally finite mass in the neighborhood of K.

(b) Suppose that T has locally finite mass in the neighborhood of K. Let $\tilde{T}$ its trivial extension. Prove that $dd^c\tilde{T} \leq O$.

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