CONTINUED FRACTIONS AND UNIQUE ADDITIVE PARTITIONS

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Abstract. A partition of the positive integers into sets A and B *avoids* a set $S \subset \mathbb{N}$ if no two distinct elements in the same part have a sum in S. If the partition is unique, S is *uniquely avoidable.* For any irrational $\alpha > 1$, Chow and Long constructed a partition which avoids the numerators of all convergents to α, and conjectured that the set S_α which this partition avoided was uniquely avoidable. We prove that the set of numerators of convergents is uniquely avoidable if and only if the continued fraction for α has infinitely many partial quotients equal to 1. We also construct the set S_α and show that it is always uniquely avoidable.

1. Introduction

A partition of the positive integers into sets A and B *avoids* a set $S \subset \mathbb{N}$ if no two distinct elements in the same part have a sum in S. We say that S is *avoidable*; if the partition is unique, S is *uniquely avoidable.*

The Fibonacci numbers are uniquely avoidable [1]. Generalized Fibonacci sequences defined by $(s_1, s_2) = 1$, $s_n = s_{n-1} + s_{n-2}$, are also uniquely avoidable provided that $s_1 < s_2$ or $2|s_1 s_2$; Alladi, Erdos and Hoggatt [1] proved this for $s_1 = 1$, and Evans [3] proved the general case. This suggests a connection with continued fractions. Chow and Long [2] studied this connection, and proved that the set of the numerators of continued-fraction convergents to any irrational α with $1 < \alpha < 2$ (easily generalized to any irrational α) is avoidable, although not necessarily uniquely avoidable. Their partition uses the sets

$$\begin{aligned} A_\alpha &= \{n \in \mathbb{N} : \text{the integer multiple of } \alpha \text{ nearest } n \text{ is greater than } n\}, \\ B_\alpha &= \{n \in \mathbb{N} : \text{the integer multiple of } \alpha \text{ nearest } n \text{ is less than } n\}. \end{aligned}$$

Let S_α be the set avoided by the partition $\{A_\alpha, B_\alpha\}$. The main results of Chow and Long are that S_α contains the numerators of all convergents to α, and every other element of S_α is either the numerator of an intermediate fraction or twice the numerator of a convergent.

We give a characterization of a large class of sets which are uniquely avoidable if they are avoidable at all. We use this to show that the set of numerators of convergents to α is uniquely avoidable for any α if and only if infinitely many partial

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quotients of α are 1. We can also use the best approximation property to determine S_α precisely, and show that S_α is uniquely avoidable for any irrational $\alpha > 1$.

2. Results on continued fractions

We will use some elementary results on continued fractions, as given in [4] for example. We use the standard notation for continued fractions, in which $\alpha = [a_0, a_1, \dots]$, $p_{-2} = 0$, $q_{-2} = 1$, $p_{-1} = 1$, $q_{-1} = 0$, and for $n \geq 0$, we have $p_{n+1} = a_{n+1}p_n + p_{n-1}$.

For $1 < \alpha < 2$, let $\alpha' = \alpha/(\alpha-1)$. Then $\alpha = [1, a_1, a_2, \dots]$, and $\alpha' = [a_1+1, a_2, \dots]$. Thus α' and α have the same numerators of convergents; $p_{0,\alpha'} = p_{1,\alpha} = a_1+1$, and the lost numerator $p_{0,\alpha} = 1$ appears as $p_{-1,\alpha'}$. The numerators of intermediate fractions between $p_{0,\alpha} = p_{-1,\alpha'}$ and $p_{1,\alpha} = p_{0,\alpha'}$ are all the integers in this interval. Also, since $1/\alpha + 1/\alpha' = 1$, we have $A_\alpha = B_{\alpha'}$, and thus α and α' give the same partition. Thus generalizing to arbitrary α does not give any extra generality; we may assume $\alpha < 2$ to reduce the number of special cases to be considered.

We will study continued fractions in terms of their approximation properties. For any p, let $E(p) = p - q\alpha$ be the error in approximating p by the closest multiple of α. Then A_α is the set of all n with $E(n) < 0$. Also, if $|E(x) + E(y)| < \alpha/2$, then $E(x+y) = E(x) + E(y)$; otherwise, $E(x+y) = E(x) + E(y) \pm \alpha$.

We can restate several elementary results on continued fractions in terms of E. The best approximation property of convergents is that $|E(x)| \leq |E(y)|$ for all positive integers $y < x$ if and only if x is the numerator of a convergent. The property that alternating convergents approach α from opposite sides is that $E(p_n) < 0$ for n even and $E(p_n) > 0$ for n odd. The property that $|\alpha - p_n/q_n| < 1/(q_n q_{n+1})$ is that $|E(p_n)| < 1/q_{n+1}$.

We also need the following lemma, which appears in similar form in [2].

Lemma 1. *If $p < p_{n+1}$ and $|E(p)| < |E(p_{n-1})|$, then p is either kp_n or the numerator $kp_n + p_{n-1}$ of an intermediate fraction for some $k \in N$.*

Proof. For simplicity of notation, we will assume n is odd, so that $E(p_n) > 0$. If $E(p) > 0$, then let

$$k = \min\left(\left\lfloor \frac{E(p)}{E(p_n)} \right\rfloor, \left\lfloor \frac{p}{p_n} \right\rfloor\right).$$

Then $p' = p - kp_n$ still satisfies $0 < E(p') < |E(p_{n-1})|$, and satisfies either $p' < p_n$ or $0 \leq E(p') < E(p_n)$. But no positive $p' < p_n$ can have $0 < E(p') < |E(p_{n-1})|$, by the best approximation property of p_{n-1}, and no positive $p' < p_{n+1}$ can have $0 < E(p') < E(p_n)$. Thus we have $p' = 0$ and $p = kp_n$.

If $E(p) < 0$, then $E(p - p_{n-1}) = E(p) - E(p_{n-1})$, which is between 0 and $-E(p_{n-1})$, so we can apply the above result to $p - p_{n-1}$. □

We will often need to use the following generalization of this lemma to cases with $p > p_{n-1}$

Lemma 2. *If $|E(p)| < |E(p_{n-1})|$ and $p \le kp_{n+1}$, then $p = ip_{n-1} + jp_n$, where i and j are non-negative integers and $i \le k$.*

Proof. We again assume n is odd, so that $E(p_{n+1}) < 0 < E(p_n)$. The proof is by induction on p. The previous lemma proves the case $p < p_{n+1}$. If $E(p) > 0$, then we let $p = p' + p_n$. Since $E(p') < E(p) < |E(p_{n-1})|$ and $E(p') > -E(p_n) > -|E(p_{n-1})|$, and apply the lemma inductively to p' with the same k. If $E(p) < 0$, then we let $p = p' + p_{n+1} = p' + p_{n-1} + a_{n+1}p_n$. Since $E(p') > E(p) > -|E(p_{n-1})|$ and $E(p') < E(p_{n+1}) < |E(p_{n-1})|$, we can apply the lemma inductively to p' with k reduced by 1. □

3. Characterization of S_α

The characterization of A_α and B_α in terms of E gives a natural characterization of the avoided set S_α.

Theorem 1. *If $E(x) > 0$, then $x \in S_\alpha$ if and only if there is no even $z < 2x$ with $0 < E(z) < E(x)$; likewise, if $E(x) < 0$, then $x \in S_\alpha$ if and only if there is no even $z < 2x$ with $0 > E(z) > E(x)$.*

The following lemma, which we need in the proof of the theorem, is often easier to use in showing that a particular x does occur as a sum.

Lemma 3. *If $E(x) > 0$, then $x \in S_\alpha$ if and only if there is no $y < x$ with $y \ne x/2$ and either $0 < E(y) < E(x)$ or $0 < E(y) + \alpha/2 < E(x)$; likewise, if $E(x) > 0$, then $x \in S_\alpha$ if and only if there is no $y < x$ with $y \ne x/2$ and either $0 > E(y) > E(x)$ or $0 > E(y) - \alpha/2 > E(x)$.*

Proof of Lemma 3 and Theorem 1. If $E(x) > 0$ and $x = y_1 + y_2$ with $y_1, y_2 \in B_\alpha$, and $y_1 \ne y_2$, then $E(y_1)$ and $E(y_2)$ are both positive and thus $E(y_1 + y_2)$ is either negative or equal to $E(y_1) + E(y_2)$. If $E(y_1 + y_2)$ is positive, then $E(y_1)$ and $E(y_2)$ are both less than $E(x)$. Let $E(y_1)$ be the smaller of the two (they are not equal since $y_1 \ne y_2$); then $E(y_1) < E(x)/2 < \alpha/2$, and $0 < E(2y_1) = 2E(y_1) < E(x)$, so we can take $z = 2y_1$ in Theorem 1. Similarly, if $x = y_1 + y_2$ with $y_1, y_2 \in A_\alpha$ and $y_1 \ne y_2$, then $E(y_1)$ and $E(y_2)$ are both negative, and thus $E(y_1 + y_2)$ can only be positive if it is $E(y_1) + E(y_2) + \alpha$; it thus follows that $E(y_1) + \alpha/2$ and $E(y_2) + \alpha/2$, which are both positive, must be less than $E(x)$. Let $E(y_1) + \alpha/2 < E(x)/2$ be the smaller of the two; then $z = 2y_1$ gives $0 < E(z) = 2E(y_1) + \alpha < E(x)$. □

The following two theorems, the main theorems of [2], follow immediately from Lemma 3 and Theorem 1.

Theorem 2. *The numerator p_n of every convergent is in S_α.*

Proof. We apply Theorem 1 to $x = p_n$. The number z would have to satisfy $|E(z)| < |E(p_n)|$, but Lemma 1 shows that any such z must be at least $p_n + p_{n+1}$, which is too large. □

Theorem 3. *If $p \in S_\alpha$, then p is either the numerator p_n of a convergent, twice the numerator of a convergent, or the numerator $p_{n-1}+kp_n$ of an intermediate fraction.*

Proof. If p is not the numerator of a convergent, then $p_n < p < p_{n+1}$ for some n and $|E(p)| > |E(p_n)|$. As in Lemma 1, we assume that n is odd for simplicity of notation. If $0 < E(p_n) < E(p)$, then we can take $y = p_n$ in Lemma 3 unless $p = 2p_n$. If $E(p) < E(p_{n-1}) < 0$, then we can take $y = p_{n-1}$ in Lemma 3 unless $p = 2p_{n-1}$. Otherwise, we have $E(p_n) < E(p) < -E(p_{n-1})$, which is only possible if p is the numerator of an intermediate fraction by Lemma 1. □

The cases which are not resolved by these two theorems can be checked similarly.

Theorem 4. *For p_n a convergent, $2p_n$ is in S_α if and only if p_n is odd, and either (i) p_{n+1} is odd and $a_{n+1} \geq 3$, or (ii) p_{n+1} is even and $a_{n+1} \geq 2$, or (iii) $p_n = 1$.*

Proof. We first assume $|E(p_n)| < \alpha/4$; we will deal with the special case $|E(p_n)| > \alpha/4$ at the end. In the general case, we will again assume that n is odd for simplicity of notation.

We will apply Theorem 1 to $x = 2p_n$. If p_n is even, then we can take $z = p_n$, and thus $2p_n \notin S_\alpha$.

If p_n is odd, we need an even $z < 4p_n$ with $0 < E(z) < 2E(p_n)$. If $E(z) < E(p_n)$, we can apply Lemma 2 to z. If $E(z) > E(p_n)$ with $z > p_n$, then we can apply Lemma 2 to $z - p_n$. If $E(z) > E(p_n)$ with $z < p_n$, then we can take $z' = 2p_n - z$, which has $0 < E(z') < E(p_n)$, and use z' instead of z. Thus, in any of these cases, we must have an even $z = ip_n + jp_{n+1}$ for some $i, j \geq 0$ with $z < 4p_n$ and $0 < E(z) < 2E(p_n)$. If j is even, then since p_n is odd, i must also be even, but this is impossible because $z = 2p_n$ is forbidden and $z = 4p_n$ or $z = 2p_n + 2p_{n+1}$ is too large. If $j = 3$, then we must have $i = 0$ to have $z < 4p_n$, but if $|E(p_{n+1})| < \alpha/6$, we get $E(3p_{n+1}) = 3E(p_{n+1}) < 0$ while we need $E(z) > 0$, and if $\alpha/6 < |E(p_{n+1})| < \alpha/4$, we have $E(3p_{n+1}) = \alpha + 3E(p_{n+1}) > \alpha/4$ while we need $E(z) < E(p_n) < \alpha/4$. If $j = 1$, then we must have $i > 0$ since $E(p_{n+1}) > 0$. The condition $z < 4p_n$ is equivalent to $i < a_{n+1} - 3$, since $z = p_{n+1} + ip_n = p_{n-1} + a_{n+1}p_n + ip_n$, and $p_{n-1} < p_n$ for $p_n \neq 1$. Either choice of z has $E(z)$ in the correct range, since $E(ip_n + p_{n+1}) = iE(p_n) + E(p_{n+1})$, which is positive because $E(p_n) > -E(p_{n+1})$ and less then $2E(p_n) < \alpha/2$ since $E(p_{n+1}) < 0$. Thus we need only check that z is even; it is even if p_{n+1} is odd and $i = 1$, or p_{n+1} is even and $i = 2$.

We now need to check the special cases with $|E(p_n)| > \alpha/4$. Recall that we may assume $\alpha < 2$. Since $|E(p_n)| < 1/q_{n+1}$, there can only be problems if $q_{n+1} \leq 3$, which implies $q_n \leq 2$; thus the only possible p_n/q_n are $1/1$, $2/1$, and $3/2$. Case (iii) covers $p_n = 1$. If $p_n/q_n = 2/1$ is a convergent, then $3/2 < \alpha < 2$; the theorem says that the sum of 4 should occur, and it does occur because 1 and 3 are both in A_α. If $p_n/q_n = 3/2$ is a convergent, then $4/3 < \alpha < 2$, and for any such α, $|E(p_n)| < \alpha/4$. □

Theorem 5. *The numerator $x = p_n + kp_{n+1}$ of an intermediate fraction is in S_α if and only if either (i) p_{n+1} is even, or (ii) $k = 1$ and p_n is odd, or (iii) $k = a_{n+2} - 1$ and p_{n+2} is odd.*

Proof. Again, we will prove the special case $|E(p_n)| > \alpha/3$ at the end, and assume in the general case that n is odd for simplicity of notation.

Since x is the numerator of an intermediate fraction, $0 < -E(p_{n+1}) < E(x) < E(p_n)$. We can apply Theorem 1 to see whether $x \in S_\alpha$; it is not if there is an even $z < 2x$ with $0 < E(z) < E(p_n) + kE(p_{n+1})$. Since $0 < E(z) < E(p_n)$, we can apply Lemma 2 to write $z = ip_n + jp_{n+1}$; since $x < p_{n+2}$, we have $z < 2p_{n+2}$ and thus $i \leq 2$. If $i = 0$ then we must have $j \leq 2k + 1$ to have $z < 2x$, and since $p_n > -a_{n+2}E(p_{n+1})$ and $k < a_{n+2}$, we have $j|E(p_{n+1})| < 2E(p_n)$. If $E(p_n) < \alpha/4$, then $E(z) = E(jp_{n+1}) = jE(p_{n+1}) < 0$, which is impossible; if $\alpha/4 < E(p_n) < \alpha/3$, then we may instead have $E(z) = \alpha + jE(p_{n+1}) > \alpha - 2E(p_n) > E(p_n)$, which is also impossible. If $i = 2$, we have $E(z) > E(z - p_{n+2}) = E(z) - E(p_{n+2}) = E(p_n + (j - a_{n+2})p_{n+1})$, and $E(p_n + (j - a_{n+2})p_{n+1})$ is less than $E(p_n + kp_{n+1})$ only if $j - a_{n+2} > k$, which implies $z > x + p_{n+2}$; this is impossible because x is a numerator of an intermediate fraction between p_{n+1}/q_{n+1} and p_{n+2}/q_{n+2}.

Thus the only possibility is $i = 1$, which gives $z = p_n + jp_{n+1}$. We have $0 < E(z) < E(x)$ for $k < j \leq a_{n+2}$, and $z < 2x$ if $p_n + jp_{n+1} < 2p_n + 2kp_{n+1}$, which requires $j \leq 2k$ since $p_n < p_{n+1}$. If p_{n+1} is even, then p_n is odd, and thus no choice of j gives even z; this is case (i). If p_{n+1} is odd and $k = 1$, the only choice allowed is $j = 2$, which gives odd z if p_n is odd and even z if p_n is even; this is case (ii). If p_{n+1} is odd and $k = a_{n+2} - 1$, the only choice allowed is $j = a_{n+2} = k + 1$, which $z = p_{n+2}$; this is case (iii). If p_{n+1} is odd and we have any other k, we can take either $j = k + 1$ or $j = k + 2$; one of these will give z even.

We now deal with the special cases with $|E(p_n)| > \alpha/3$, which requires $q_{n+1} \leq 2$ and thus $q_n = 1$. If $p_n/q_n = 1/1$, then $n = 1$, and $|E(p_1)| > \alpha/3$ only for $\alpha > 3/2$. We must have $a_2 = 1$, and thus $p_2 = 2$, so case (i) applies to all odd $x < 2a_3 + 1$. We have $2 - 1/(a_3 + 1) < \alpha < 2$. Therefore every odd number up to $2a_3 + 1$ is in A_α and every even number up to $2a_3$ is in B_α, and thus odd numbers less than $2a_3 + 1$ are avoided as required. If $p_n/q_n = 2/1$. then $n = 2$, $a_2 = 1$, and $\alpha > 3/2$, so $|E(2)| < 1/2 < \alpha/3$. □

Theorems 2, 3, 4, and 5 give the complete characterization of the avoided set S_α.

4. Uniquely avoidable sets

The most natural characterization of unique avoidability is the graph-theoretic characterization of [1]. If $S \subset \mathbb{N}$, then the *graph* $G(S)$ of S is the graph with vertex set is $\mathbb{N}$ with an edge between x and y if $x \neq y$ and $x+y \in S$. A partition which avoids S is a 2-coloring of $G(S)$, and thus S is avoidable if and only if $G(S)$ is bipartite, and S is uniquely avoidable if and only if $G(S)$ is bipartite and connected.

Note that, if $a \in S$ and $x < a$, then x and $a - x$ are in the same connected component of $G(S)$; either $x = a - x$, or $x \neq a - x$ and their sum is a. Thus, if a and $a + b \in S$, then x and $x + b$ are in the same connected component for every $x < a$. This allows us to prove the following result.

Theorem 6. *If S contains d, $a + d$, c, and $b + c$ with $(a, b) = 1$, $a \leq c$, and $b \leq d$ (in particular, if S contains a, b, and $a + b$), then all numbers less than $a + b$ are in the same connected component of $G(S)$. If S contains infinitely many such subsets, then $G(S)$ is connected, and therefore S is uniquely avoidable if it is avoidable at all.*

Proof. Assume $a < b$. By the Chinese Remainder Theorem, any integer m with $0 < m \leq ab$ (and thus any m with $0 < m < a + b$) can be uniquely written $m = xa - yb$ with $0 < x \leq b$, $0 \leq y < a$. We prove connectivity for $0 < m < a + b$ by induction on $x + y$. The base case is $m = a$. For any other m, if $m > a$, then m and $m - a$ are in the same component; if $m < a$, then m and $m + b$ are in the same component. □

Our main results on unique avoidability follow.

Theorem 7. *The set of numerators of convergents of α is uniquely avoidable if and only if infinitely many partial quotients a_n are 1.*

Proof. If infinitely many a_n are 1, then we can apply Theorem 6 to p_{n-2}, p_{n-1}, and p_n to show that the graph is connected, and we have already shown that the set of numerators is avoidable. If only finitely many a_n are 1, then there is some N such that for any $n > N$ we have $p_n > 2p_{n-1}$. Thus, regardless of the partition of integers less than p_{n-1}, we can extend the partition up to integers less than p_n by placing integers from p_{n-1} to $\lfloor p_n/2 \rfloor$ arbitrarily, and placing $p_n - x$ in the set which does not contain x for $\lfloor p_n/2 \rfloor < x < p_n$. We can continue inductively for all n. □

Theorem 8. *The set S_α avoided by the partition A_α, B_α is uniquely avoidable for any irrational $\alpha > 1$.*

Proof. If p_{n+1} is even, then $p_n + p_{n+1}$ is either p_{n+2} or the numerator of an intermediate fraction, and if it is the numerator of an intermediate fraction, it is in S_α by case (i) of Theorem 5. If p_{n+1} is odd, then $p_{n+1} - p_n$ is either p_{n-1} or the numerator of an intermediate fraction, and if it is the numerator of an intermediate fraction, it is in S_α by case (i) or (iii) of Theorem 5. In either case, we can apply Theorem 6. □

The unique avoidability of generalized Fibonacci sequences starting with arbitrary s_1, s_2 [3] follows as a special case of these results. If $s_1 < s_2$, we can let s_1 and s_2 be the numerators p_{n-1} and p_n of two consecutive convergents, and let $a_m = 1$ for all $m > n$. If $s_1 > s_2$, we can let $p_n = s_2$, $p_{n+1} = s_1 + s_2$; the set of all the s_i other than s_1 is thus uniquely avoidable, and s_1 is the intermediate fraction $p_{n+1} - p_n$, which is an avoided sum by case (i) of Theorem 5 if $s_2 = p_n$ is even and by case (iii) of

Theorem 5 if $s_1 + s_2 = p_{n+1}$ is odd. Thus the sequence of s_i is uniquely avoidable if $s_1 < s_2$ or either s_1 or s_2 is even, and not avoidable at all otherwise.

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