ON WEIGHTS OF l-ADIC REPRESENTATION

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Abstract. Let J be an abelian variety over a number field k, $[k:\mathbb{Q}]<\infty$. Assume that $\mathrm{Cent}(\mathrm{End}(J\otimes\overline{k}))=\mathbb{Z}$. If the division $\mathbb{Q}$-algebra $\mathrm{End}^0(J\otimes\overline{k})$ splits at a prime number l, then the l-adic representation is defined by the minuscule weights (microweights) of simple classical Lie algebras of types A_m, B_m, C_m or D_m.

Introduction

0.1. Let J be an abelian variety over a number field $k\subset\mathbb{C}$, $[k:\mathbb{Q}]<\infty$. For a given prime number l consider the natural l-adic representation

$$\rho_l:\mathrm{Gal}(\overline{k}/k)\to\mathrm{GL}(H^1_{et}(J\otimes\overline{k},\mathbb{Q}_l)).$$

We recall some conjectures and results concerning ρ_l.

0.2. **Serre - Mumford - Tate conjecture** [5], [9]. *There exists a canonical isomorphism of Lie algebras*

$$\mathrm{Lie}\ \mathrm{Im}(\rho_l)\simeq\mathrm{Lie}[\mathrm{MT}(J\otimes_k\mathbb{C})(\mathbb{Q}_l)],$$

where $\mathrm{MT}(J\otimes_k\mathbb{C})$ *is the Mumford - Tate group associated to the canonical Hodge* $\mathbb{Q}$*-structure on* $H^1(J\otimes_k\mathbb{C},\mathbb{Q})$.

0.3. **P.Deligne theorem** [14, Theorem 0.5.1b]. *Each simple factor g of the reductive Lie algebra* $\mathrm{Lie}\,\mathrm{MT}(J\otimes_k\mathbb{C})\otimes\mathbb{C}$ *is a classical Lie algebra of type* A_m, B_m, C_m *or* D_m, *and the highest weight of any irreducible g-submodule* $E\subset H^1(J\otimes_k\mathbb{C},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$ *is a minuscule weight (microweight).*

We recall the list of minuscule weights:
for type A_m $(m\geq 1)$: $\omega_1,\omega_2,\ldots,\omega_m$;
for type B_m $(m\geq 2)$: ω_m;
for type C_m $(m\geq 3)$: ω_1;
for type D_m $(m\geq 4)$: $\omega_1,\omega_{m-1},\omega_m$;

Supported by the Russian Foundation for Fundamental Research (grant 97-01-00647). Research at MSRI is supported in part by NSF grant DMS-9022140. This paper was written at the period of my stay at the Institut des Hautes Études Scientifiques (Bures-sur-Yvette) and at the Mathematical Sciences Research Institute (Berkeley) in Spring of 1997. I would like to thank the members of both institutions for their warm hospitality and excellent working conditions.

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for type E_6 : ω_1, ω_6;
for type E_7 : ω_7,
where ω_1, ω_2, ..., ω_m are fundamental weights [2, Chapter 8, §7, Section 3].

In virtue of the Deligne theorem the following conjecture may be considered as a weak form of the Serre - Mumford - Tate conjecture:

0.4. **Conjecture on minuscule weights** [14, Conjecture 0.4]. *Let J be an abelian variety over a number field $k \subset \mathbb{C}$. Then each simple factor g of the reductive Lie algebra* $\mathrm{Lie}\ \mathrm{Im}(\rho_l) \otimes \overline{\mathbb{Q}}_l$ *is a classical Lie algebra of type A_m, B_m, C_m or D_m, and the highest weight of any irreducible g-submodule $E \subset H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l) \otimes \overline{\mathbb{Q}}_l$ is a minuscule weight (microweight).*

0.5. If $\mathrm{End}(J \otimes \overline{k}) = \mathbb{Z}$ and $\dim_k J$ is an odd integer, then by the well known J.-P.Serre theorem [3, Section 6.1] there exists a canonical isomorphism of the l-adic Lie algebras

$$\mathrm{Lie}\ \mathrm{Im}(\rho_l) \simeq \mathrm{Lie}\ [\mathrm{MT}(J \otimes_k \mathbb{C})(\mathbb{Q}_l)].$$

In particular, the Serre - Mumford - Tate conjecture and the conjecture on minuscule weights hold for J. The analogous results are obtained for some abelian varieties of another types [10], [11], [12].

In this paper we prove the following main theorem.

0.6. **Theorem** . *Let J be a simple abelian variety over a number field k, $[k : \mathbb{Q}] < \infty$. Assume that* $\mathrm{Cent}(\mathrm{End}(J \otimes \overline{k})) = \mathbb{Z}$. *If the division $\mathbb{Q}$-algebra* $\mathrm{End}^0(J \otimes \overline{k})$ *splits at a prime number l, then each simple factor g of the reductive Lie algebra* $\mathrm{Lie}\ \mathrm{Im}(\rho_l) \otimes \overline{\mathbb{Q}}_l$ *is a simple Lie algebra of type A_m, B_m, C_m or D_m, and the highest weight of any irreducible g-submodule $E \subset H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l) \otimes \overline{\mathbb{Q}}_l$ is a minuscule weight (microweight).*

I am grateful to J.-P.Serre, D.Bertrand and Y.André for the stimulating discussions on this topic.

§1. Some remarks on l-adic representation

1.1. In what lollows $V_l = H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l)$. Let G_{V_l} denote the algebraic envelope of the compact l-adic Lie group $\mathrm{Im}(\rho_l)$. It is well known that G_{V_l} coincides with the closure of $\mathrm{Im}(\rho_l)$ in the Zariski topology of the group $\mathrm{GL}(V_l)$.

Fix a non-Archimedean place v of the field k. Let $k(v)$ is its residue field of characteristic p_v, k_v is the completion of the field k relative to the topology induced by the place v, $\overline{v}$ is any extension of v to the field $\overline{k}$, $D(\overline{v})$ is the decomposition group (an element $\sigma \in \mathrm{Gal}(\overline{k}/k)$ belongs to $D(\overline{v}) \Leftrightarrow \sigma(\overline{v}) = \overline{v} \Leftrightarrow \forall \alpha \in \overline{k}\ \ \overline{v}(\sigma\alpha) = \overline{v}(\alpha)$), $I(\overline{v}) \subset \mathrm{Gal}(\overline{k}/k)$ is the inertia subgroup (an element $\sigma \in \mathrm{Gal}(\overline{k}/k)$ belongs to $I(\overline{v}) \Leftrightarrow \forall \alpha \in \overline{k}$ the relation $\overline{v}(\alpha) \geq 0$ implies $\overline{v}(\sigma\alpha - \alpha) > 0$)).

It is well known that $I(\overline{v})$ is a normal subgroup in $D(\overline{v})$, and $D(\overline{v})/I(\overline{v}) \simeq \mathrm{Gal}(\overline{k(v)}/k(v))$. We may identify $\mathrm{Gal}(\overline{k_v}/k_v)$ with the decomposition group $D(\overline{v})$.

1.2. Assume that v is a non-Archimedean place of k at which J has a good reduction. Let $F_{\overline{v}} \in \mathrm{Gal}(\overline{k}/k)$ be the Frobenius element.

It is well known that the characteristic polynomial of $\rho_l(F_{\overline{v}}^{-1})$ coincides with the characteristic polynomial of the Frobenius endomorphism π_v of the reduction

J_v of J at v. We denote by Δ the set of all eigenvalues of $\rho_l(F_{\overline{v}}^{-1})$ (counting multiplicities). Let Γ_v be a multiplicative subgroup of $\overline{\mathbb{Q}}^{\times}$ generated by Δ.

By the Honda - Tate theory [13] $\mathbb{Q}[\pi_v] = \prod K_i$, K_i are number fields. The multiplicative group $\mathbb{Q}[\pi_v]^{\times}$ determines a $\mathbb{Q}$-torus $T_{\pi_v} = \prod R_{K_i/\mathbb{Q}}(\mathrm{G}_{mK_i})$, where $R_{K_i/\mathbb{Q}}$ are the Weil restrictions of scalar functors. Let H_v be the smallest algebraic subgroup of T_{π_v} defined over $\mathbb{Q}$, such that $\pi_v \in H_v(\mathbb{Q})$. As is well known, H_v is a group of multiplicative type. The connected component of the identity in H_v is called the *Frobenius torus* T_v. Moreover, T_v is a $\mathbb{Q}$-torus in G_{V_l} [3, Section 3b].

1.3. We denote by $\mathbb{C}_{p_v} = (\overline{\mathbb{Q}}_{p_v})^{\wedge}$ the completion of the algebraic closure of the field $\mathbb{Q}_{p_v}$. According to Tate and Raynaud there exists the canonical Hodge - Tate decomposition

$$V_{\mathbb{C}_{p_v}} = V_{p_v} \otimes_{\mathbb{Q}_{p_v}} \mathbb{C}_{p_v} = V_{\mathbb{C}_{p_v}}(0) \oplus V_{\mathbb{C}_{p_v}}(-1),$$

and $\dim V_{\mathbb{C}_{p_v}}(0)$=$\dim V_{\mathbb{C}_{p_v}}(-1)$ ([3, Section 4.3]).

We define a morphism of algebraic $\mathbb{C}_{p_v}$-groups

$$h_{V_{p_v}} : \mathrm{G}_{m\mathbb{C}_{p_v}} \to \mathrm{GL}(V_{\mathbb{C}_{p_v}})$$

by the formula

$$h_{V_{p_v}}(c)(x) = \begin{cases} x & \text{if } \ x \in V_{\mathbb{C}_{p_v}}(0), \\ c^{-1}x & \text{if } \ x \in V_{\mathbb{C}_{p_v}}(-1). \end{cases}$$

The algebraic envelope of $\rho_{p_v}(I(\overline{v}))$ is an algebraic subgroup of $G_{V_{p_v}}$. By the theorem of S.Sen [7], the 1-parameter subgroup $h_{V_{p_v}}$ is contained in the algebraic envelope of $\rho_{p_v}(I(\overline{v}))$. In particular, one has $\mathrm{Im}(h_{V_{p_v}}) \subset G_{V_{p_v}}(\mathbb{C}_{p_v})$.

1.4. **Lemma** . *Assume that* Cent $\mathrm{End}^0(J \otimes \overline{k})$ *is a totally real field, and $J(k)$ contains all points of order p_v^2 in $J(\overline{k})$. Let $M \subset G_{V_{p_v}}$ be the smallest normal algebraic subgroup for which* $\mathrm{Im}(h_{V_{p_v}}) \subset M(\mathbb{C}_{p_v})$. *Then the canonical representation of the semisimple Lie algebra* $[\mathrm{Lie}\ (M)]^{ss} \otimes \overline{\mathbb{Q}}_{p_v}$ *in* $V_{p_v} \otimes \overline{\mathbb{Q}}_{p_v}$ *is defined by the minuscule weights of the classical Lie algebras (of types* A_m, B_m, C_m, D_m).

Proof . By F.A.Bogomolov theorem [1] Lie $\mathrm{Im}(\rho_{p_v})$=Lie$(G_{V_{p_v}})$ and $G_{V_{p_v}}$ containes the group G_m of homotheties. By G.Faltings theorems [4] $G_{V_{p_v}}$ is reductive. In the case under consideration $G_{V_{p_v}}$ is connected [3, Proposition 3.6], and

$$G_{V_{p_v}} = \mathrm{G}_m \cdot S_{V_{p_v}},$$

where $S_{V_{p_v}}$=$[G_{V_{p_v}}, G_{V_{p_v}}]$ is the commutator subgroup of $G_{V_{p_v}}$ [3, Section 1.2.2b]. It is clear that M is a connected reductive algebraic group over $\mathbb{Q}_{p_v}$ with a nontrivial semisimple part $M^{ss} = [M, M]$ (because Im $(h_{V_{p_v}})$ is not contained in $\mathrm{G}_m(\mathbb{C}_{p_v})$); the canonical representation V_{p_v} of M is faithful; if N is a normal connected algebraic subgroup of M, defined over $\mathbb{Q}_{p_v}$, such that Im $(h_{V_{p_v}}) \subset N(\mathbb{C}_{p_v})$, then N is a normal algebraic subgroup of $G_{V_{p_v}}$ (indeed, let $f : S^{sc}_{V_{p_v}} \to S_{V_{p_v}}$ be the universal covering morphism; since M^{ss} is a normal algebraic subgroup of $S_{V_{p_v}}$, the group $(f^{-1}(M^{ss}))^0$ is a normal algebraic subgroup of $S^{sc}_{V_{p_v}}$, and so we get the decomposition

$$S^{sc}_{V_{p_v}} = (f^{-1}(M^{ss}))^0 \times H'',$$

where $(f^{-1}(M^{ss}))^0$ and H'' are simply connected semisimple $\mathbb{Q}_{p_v}$-groups; on the other hand, $(f^{-1}(N^{ss}))^0$ is a normal connected subgroup of $(f^{-1}(M^{ss}))^0$, so we get the decompositions

$$(f^{-1}(M^{ss}))^0 = (f^{-1}(N^{ss}))^0 \times H'$$

and

$$S^{sc}_{V_{p_v}} = (f^{-1}(N^{ss}))^0 \times H' \times H'';$$

consequently N^{ss} is a normal algebraic subgroup of $S_{V_{p_v}}$, and N is a normal algebraic subgroup of $G_{V_{p_v}})$, so $N = M$; the action of $\mathrm{G}_{m\mathbb{C}_{p_v}}$ on $V_{\mathbb{C}_{p_v}} = V_{p_v} \otimes_{\mathbb{Q}_{p_v}} \mathbb{C}_{p_v}$ defined by $h_{V_{p_v}}$ is of weights 0 and -1. In virtue of the well known results of J.-P.Serre [8, §3] the representation V_{p_v} of M is defined by minuscule weights of the classical Lie algebras. Lemma 1.4 is proved.

1.5. **Corollary** . *Assume that* Cent $\mathrm{End}^0(J\otimes\overline{k})$ *is a totally real field. Then for each prime number l the reductive Lie algebra* Lie $\mathrm{Im}(\rho_l)\otimes\overline{\mathbb{Q}}_l$ *has a simple factor g such that the canonical representation of g in $V_l\otimes\overline{\mathbb{Q}}_l$ is defined by the minuscule weights of the classical Lie algebras.*

Indeed, we can choose a non-Archimedean place v such that $p_v = l$. Replacing k by some finite algebraic extension, we may assume that $J(k)$ containes all points of order p_v^2 in $J(\overline{k})$.

§2. On $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subsets of $\Delta\cdot\Delta$

2.1. **Proposition** . *Let J be an abelian variety over a number field k. Suppose that* Cent $\mathrm{End}(J\otimes\overline{k}) = \mathbb{Z}$ *and the division $\mathbb{Q}$-algebra* $\mathrm{End}^0(J\otimes\overline{k})$ *splits at the prime number l.*

After replacing k by some finite extension we may assume that for abelian variety J and for each non-Archimedean place v of k from some set $S^{generic}_{non-Arch}(J,k,l)$ of places of Dirichlet's density 1 the following conditions hold:

1) *for a fixed integer $n\geq 2$ such that $l^n > (2\cdot\dim_k J)^2$, the l^n-torsion points of $J(\overline{k})$ are rational points over k, and k containes all the $(l^n)^{th}$ roots of unity;*
2) $p_v = \mathrm{char}(k(v)) > (2\cdot\dim_k J)^2$;
3) $\mathrm{Norm}_{k/\mathbb{Q}}(v) = p_v$; *$p_v$ is unramified in k;*
4) *J has a good reduction at v;*
5) *Γ_v is torsion-free, G_{V_l} is connected and $\rho_l(F_{\overline{v}}^{-1})\in T_v(\mathbb{Q})$;*
6) *the Frobenius torus T_v is a maximal torus of G_{V_l} and*

$$\mathrm{rank}(\Gamma_v) = \dim(T_v) = \mathrm{rank}(G_{V_l}).$$

Proof . Cent $\mathrm{End}(J\otimes\overline{k}) = \mathbb{Z}$ by the condition of Proposition 2.1. By Albert's classification [6, §21] $\mathrm{End}(J\otimes\overline{k}) = \mathbb{Z}$ or $\mathrm{End}^0(J\otimes\overline{k})$ is a quaternion division algebra over $\mathbb{Q}$ which splits at l. So

$$\mathrm{End}(J\otimes\overline{k})\otimes\mathbb{Q}_l = \begin{cases} \mathbb{Q}_l & \text{if } J \text{ is of type I by Albert's classification,} \\ \mathrm{M}_2(\mathbb{Q}_l) & \text{if } J \text{ is of type II or III by Albert's classification.} \end{cases}$$

After replacing k by some finite extension we may assume that for abelian variety J and for k the condition (2.1.1) holds. Then the Dirichlet density of the set $S^{generic}_{non-Arch}(J,k,l)$ of all non-Archimedean places of k satisfying the conditions (2.1.2) - (2.1.6) is equal to 1 [3, Lemma (2.1), Proposition (3.6), Corollary (3.8)]. On the other hand,

$$G_{V_l} = \mathrm{G}_m \cdot S_{V_l},$$

where S_{V_l}=$[G_{V_l}, G_{V_l}]$ is the commutator subgroup of G_{V_l} [3, Section 1.2.2b]. Moreover, V_l is a direct some of 1 or 2 copies of absolutely irreducible G_{V_l}-module U_l, and $U_l \otimes \mathbb{C}_l$ has the Hodge - Tate decomposition with usual properties [3, Section 4.3]. Proposition 2.1 is proved.

From now on we assume that $v \in S^{generic}_{non-Arch}(J,k,l)$.

2.2. Let $f: S^{sc}_{V_l} \to S_{V_l}$ be the universal covering morphism. An isogeny f extends to an isogeny $f_{\mathrm{G}_m}: \mathrm{G}_m \times S^{sc}_{V_l} \to \mathrm{G}_m \cdot S_{V_l} = G_{V_l}$, which is defined by the formula $f_{\mathrm{G}_m}(a \times s) = a \cdot f(s)$ for $a \in \mathrm{G}_m,\ \ s \in S^{sc}_{V_l}$.

Consider the exact sequence

$$1 \to \mathrm{Ker}(f_{\mathrm{G}_m}) \to \mathrm{G}_m \times S^{sc}_{V_l} \to \mathrm{G}_m \cdot S_{V_l} = G_{V_l} \to 1.$$

It determines the exact sequence of commutative algebraic $\mathbb{Q}_l$-groups

$$1 \to \mathrm{Ker}(f_{\mathrm{G}_m}) \cap [f^{-1}_{\mathrm{G}_m}(T_v)]^0 \to [f^{-1}_{\mathrm{G}_m}(T_v)]^0 \xrightarrow{\varphi} T_v \to 1, \tag{2.2.1}$$

where $[f^{-1}_{\mathrm{G}_m}(T_v)]^0$ is a maximal $\mathbb{Q}_l$-subtorus of $\mathrm{G}_m \times S^{sc}_{V_l}$, which coincides with the connected component of the unity of $f^{-1}_{\mathrm{G}_m}(T_v)$, φ is an isogeny of $\mathbb{Q}_l$-tori, $\mathrm{Ker}(f_{\mathrm{G}_m}) \cap [f^{-1}_{\mathrm{G}_m}(T_v)]^0 = \mathrm{Ker}(\varphi)$ is a finite group, which is isomorphic to some subgroup of $\mathrm{Cent}(S^{sc}_{V_l})$. We rewrite the sequence (2.2.1) as

$$1 \to \mathrm{Ker}(\varphi) \to [f^{-1}_{\mathrm{G}_m}(T_v)]^0 \xrightarrow{\varphi} T_v \to 1. \tag{2.2.2}$$

2.3. Recall that $S_{V_l} \otimes \overline{\mathbb{Q}}_l$ is a $\overline{\mathbb{Q}}_l$-almost simple algebraic group $\Leftrightarrow$ $S_{V_l} \otimes \overline{\mathbb{Q}}_l$ is semisimple and does not have infinite normal algebraic $\overline{\mathbb{Q}}_l$-subgroups different from $S_{V_l} \otimes \overline{\mathbb{Q}}_l$.

If $S_{V_l} \otimes \overline{\mathbb{Q}}_l$ is a $\overline{\mathbb{Q}}_l$-almost simple algebraic group, then by Corollary 1.5 the l-adic representation ρ_l is defined by the minuscule weights of the classical Lie algebras.

2.4. Assume that $S_{V_l} \otimes \overline{\mathbb{Q}}_l$ is not a $\overline{\mathbb{Q}}_l$-almost simple algebraic group. Since $S^{sc}_{V_l} \otimes \overline{\mathbb{Q}}_l$ is a simply connected algebraic $\overline{\mathbb{Q}}_l$-group, it is a product of two proper semisimple $\overline{\mathbb{Q}}_l$-subgroups:

$$S^{sc}_{V_l} \otimes \overline{\mathbb{Q}}_l = S_1 \times S_2.$$

Consider the canonical projections $\mathrm{pr}_i: \mathrm{G}_m \times S_1 \times S_2 \to S_i$ $(i = 1, 2)$ and $\mathrm{pr}_0: \mathrm{G}_m \times S_1 \times S_2 \to \mathrm{G}_m$. Since $[f^{-1}_{\mathrm{G}_m}(T_v \otimes \overline{\mathbb{Q}}_l)]^0$ is a maximal $\overline{\mathbb{Q}}_l$-subtorus of $\mathrm{G}_m \times S_1 \times S_2$, we have:

$$[f^{-1}_{\mathrm{G}_m}(T_v \otimes \overline{\mathbb{Q}}_l)]^0 =$$

$$\mathrm{pr}_0([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0) \times \mathrm{pr}_1([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0) \times \mathrm{pr}_2([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0),$$

and $\mathrm{pr}_i([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0)$ is a maximal $\overline{\mathbb{Q}}_l$-subtorus of S_i $(i = 1, 2)$, $\mathrm{pr}_0([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0) = \mathrm{G}_m$. We get the decomposition

$$[f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0 = \mathrm{G}_m \times \mathrm{pr}_1([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0) \times \mathrm{pr}_2([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0). \quad (2.4.1)$$

We can find an element $y_v \in ([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0)(\overline{\mathbb{Q}}_l)$ such that $\varphi(y_v) = \rho_l(F_{\overline{v}}^{-1})$ (recall that $\rho_l(F_{\overline{v}}^{-1}) \in T_v(\mathbb{Q}) \subset T_v(\overline{\mathbb{Q}}_l)$).

So the decomposition (2.4.1) gives the decomposition $y_v = \lambda \times y_{v1} \times y_{v2}$, where

$$\lambda \in \mathrm{G}_m(\overline{\mathbb{Q}}_l) = \overline{\mathbb{Q}}_l^{\times},$$

$$y_{vi} \in \mathrm{pr}_i([f_{\mathrm{G}_m}^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0)(\overline{\mathbb{Q}}_l) \subset S_i(\overline{\mathbb{Q}}_l).$$

On the other hand,

$$U_l \otimes \overline{\mathbb{Q}}_l = \overline{\mathbb{Q}}_l \otimes W_1 \otimes W_2,$$

where W_1 is an irreducible S_1-module, W_2 is an irreducible S_2-module. Let

$$\rho_0 : \mathrm{G}_m \otimes \overline{\mathbb{Q}}_l \to \mathrm{GL}(\overline{\mathbb{Q}}_l) = \overline{\mathbb{Q}}_l^{\times},$$

$$\rho_i : S_i \to \mathrm{GL}(W_i) \ \ (i = 1, 2)$$

are the corresponding representations. We have a commutative diagram

$$\begin{array}{ccc} (\mathrm{G}_m \otimes \overline{\mathbb{Q}}_l) \times S_1 \times S_2 & \xrightarrow{\rho_0 \otimes \rho_1 \otimes \rho_2} & \mathrm{GL}(\overline{\mathbb{Q}}_l \otimes W_1 \otimes W_2) \\ \downarrow f_{\mathrm{G}_m} & & || \\ G_{V_l} \otimes \overline{\mathbb{Q}}_l & \subset & \mathrm{GL}(\overline{\mathbb{Q}}_l \otimes W_1 \otimes W_2). \end{array} \quad (2.4.2)$$

Since $f_{\mathrm{G}_m}(y_{vi}) \in T_v(\overline{\mathbb{Q}}_l) \subset \mathrm{GL}(U_l)(\overline{\mathbb{Q}}_l)$, we can diagonalize the linear operator $f_{\mathrm{G}_m}(y_{vi})$ (over $\overline{\mathbb{Q}}_l$). Let Δ_i be the set of its eigenvalues. The commutative diagram (2.4.2) shows that the set Δ of all eigenvalues of the linear operator $f_{\mathrm{G}_m}(y_v) = \varphi(y_v) = \rho_l(F_{\overline{v}}^{-1})$ is equal to

$$\Delta = \lambda \cdot \Delta_1 \cdot \Delta_2 \quad (2.4.3)$$

with the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Δ (recall that the characteristic polynomial of $\rho_l(F_{\overline{v}}^{-1})$ belongs to $\mathbb{Z}[t]$).

Let Γ_{Δ_i} be the multiplicative subgroup of $\overline{\mathbb{Q}}^{\times}$ generated by Δ_i, $\Gamma_{\{\lambda\}} = \{\lambda^m \mid m \in \mathbb{Z}\}$. We denote by $\mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0))$ the group of characters of $\overline{\mathbb{Q}}_l$-torus $\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0)$. It is evident that

$$\Gamma_v \subset \Gamma_{\{\lambda\}} \cdot \Gamma_{\Delta_1} \cdot \Gamma_{\Delta_2},$$

so

$$\operatorname{rank}\Gamma_v \leq \sum \operatorname{rank}\ \mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0))(\mathrm{pr}_i(y_v)) \leq$$

$$\sum \operatorname{rank}\ \mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0) = \operatorname{rank}(\mathrm{G}_m \times S_1 \times S_2) = \operatorname{rank}(G_{V_l}). \quad (2.4.4)$$

Condition (2.1.6) implies that

$$\operatorname{rank}\ \mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0))(\mathrm{pr}_i(y_v)) = \operatorname{rank}\ \mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0)). \quad (2.4.5)$$

Since there exists the canonical exact sequence

$$\mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0)) \to \mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_L)]^0))(\mathrm{pr}_i(y_v)) \to 0,$$

and $\mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0))$ is torsion-free, we deduce from (2.4.5) that

$$\Gamma_{\Delta_i} = \mathrm{X}^*(\mathrm{pr}_i([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0))(\mathrm{pr}_i((y_v))$$

is torsion-free. So (2.4.4) implies that

$$\Gamma_{\{\lambda\}} \cdot \Gamma_{\Delta_1} \cdot \Gamma_{\Delta_2} = \Gamma_{\{\lambda\}} \times \Gamma_{\Delta_1} \times \Gamma_{\Delta_2}. \quad (2.4.6)$$

On the other hand, W_i is a symplectic or orthogonal S_i-module, so $W_i^* = W_i$. Hence $\Delta_i^{-1} = \Delta_i$.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts in a natural way on Δ and on $\Delta \cdot \Delta$. For each element $\eta \in \Delta \cdot \Delta$ we define a map $T_\eta : \Delta \to \overline{\mathbb{Q}}^\times$ by the formula $T_\eta(\delta) = \eta\delta^{-1}$. It is easy to see that for each element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have (counting multiplicities)

$$\mathrm{Card}(T_\eta(\Delta) \cap \Delta) = \mathrm{Card}(T_{\sigma(\eta)}(\sigma(\Delta)) \cap \sigma(\Delta)) = \mathrm{Card}(T_{\sigma(\eta)}(\Delta) \cap \Delta),$$

so for any constant c the set

$$\{\eta \in \Delta \cdot \Delta \,|\, \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = c\}$$

is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. It is easy to see that

$$\eta = \lambda^2 \ \Leftrightarrow\ \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = \mathrm{Card}(\Delta).$$

Consequently $\lambda^2 \in \mathbb{Q}$, and λ^2 is of absolute value p_v (by the condition (2.1.3) and the well known A.Weil theorem [6, §21, Theorem 4]), so we get the equality $\lambda^2 = \pm p_v$.

Let ρ be a complex conjugation on $\overline{\mathbb{Q}}$ induced by some embedding $\overline{\mathbb{Q}} \to \mathbb{C}$. According to (2.4.6) each element $\delta \in \Delta$ has the unique decomposition $\delta = \lambda \cdot \delta_1 \cdot \delta_2$, where $\delta_i \in \Delta_i$. We claim that

$$\rho(\lambda\delta_1\delta_2) = \lambda\delta_1^{-1}\delta_2^{-1}. \quad (2.4.7)$$

Indeed, $\rho(\lambda\delta_1\delta_2) = \lambda\delta_1'\delta_2'$ for some $\delta_i' \in \Delta_i$. Then by the A.Weil theorem and by relation (2.1.3) we have $\lambda\delta_1\delta_2\rho(\lambda\delta_1\delta_2) = p_v$, so

$$\lambda^2\delta_1\delta_2\delta_1'\delta_2' = p_v,\ \ \lambda^4(\delta_1\delta_1'\delta_2\delta_2')^2 = p_v^2,\ \ p_v^2(\delta_1\delta_1'\delta_2\delta_2')^2 = p_v^2,\ \ (\delta_1\delta_1'\delta_2\delta_2')^2 = 1.$$

Note that $\lambda^2 = \lambda\delta_1\delta_2 \cdot \lambda\delta_1^{-1}\delta_2^{-1} \in \Gamma_v$, so $\delta_1\delta_1'\delta_2\delta_2' = \lambda\delta_1\delta_2 \cdot \lambda\delta_1'\delta_2'/\lambda^2 \in \Gamma_v$. Since Γ_v is torsion free (by the condition (2.1.5)) the relation $(\delta_1\delta_1'\delta_2\delta_2')^2 = 1$ implies $\delta_1'\delta_2' = (\delta_1\delta_2)^{-1}$. So the claim (2.4.7) is proved.

The A.Weil theorem and (2.4.7) imply that

$$\lambda^2 = \lambda\delta_1\delta_2 \cdot \rho(\lambda\delta_1\delta_2) = p_v. \tag{2.4.8}$$

2.5. **Theorem** . *Assume that the decomposition* $\Delta = \lambda \cdot \Delta_1 \cdot \Delta_2$ *corresponds to the decomposition* $S_{V_l}^{sc} \otimes \overline{\mathbb{Q}}_l = S_1 \times S_2$, *where* S_1 *and* S_2 *are semisimple algebraic* $\overline{\mathbb{Q}}_l$*-groups. Then for each* i $\Delta_i \cdot \Delta_i$ *does not contain a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-invariant subset* $\neq \{1\}$*(without counting multiplicities).*

Proof . Assume that $\Delta_1 \cdot \Delta_1$ containes a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subset $B \neq \{1\}$ (without counting multiplicities). It is evident that

$$\mathrm{Card}(B) < \frac{1}{2} \cdot \mathrm{Card}(\Delta) \cdot \mathrm{Card}(\Delta) \leq \frac{1}{2} \cdot (2 \cdot \dim_k J)^2 < \frac{1}{2} \cdot l^n. \tag{2.5.1}$$

Since $\Delta_2^{-1} = \Delta_2$, we have $1 \in \Delta_2 \cdot \Delta_2$, so

$$\lambda^2 \cdot B \subset \lambda^2 \cdot \Delta_1 \cdot \Delta_1 \subset \lambda^2 \cdot \Delta_1 \cdot \Delta_1 \cdot \Delta_2 \cdot \Delta_2 = \Delta \cdot \Delta.$$

Consequently $\sum_{z\in\lambda^2\cdot B} z \in \mathbb{Z}$.

Assume that p_v does not divide $\sum_{z\in\lambda^2\cdot B} z$. Then for each place w of the field $\overline{\mathbb{Q}}$ lying over p_v we have

$$w\left(\sum_{z\in\lambda^2\cdot B} z\right) = 0.$$

It follows that there exists $z_w \in \lambda^2 \cdot B$ such that $w(z_w) = 0$. Hence for each element $\delta_2 \in \Delta_2$

$$0 = w(z_w) = \frac{1}{2}\{w(z_w\delta_2^2) + w(z_w\delta_2^{-2})\}.$$

Since both summands in the last brackets are nonnegative in virtue of the relations $z_w\delta_2^{\pm 2} \in \Delta \cdot \Delta$, we have the equalities

$$w(z_w\delta_2^2) = w(z_w\delta_2^{-2}) = 0.$$

So $w(\delta_2) = 0$ for *all* $w|p_v$. It follows that δ_2 is a root of 1 [14, Sublemma 3.4.0]. Since $\delta_2 \in \Delta_2$ is an arbitrary element, we conclude that the multiplicative group

Γ_{Δ_2} generated by Δ_2 is a finite subgroup of $\overline{\mathbb{Q}}^\times$. We know that Γ_{Δ_2} is torsion-free. So $\Gamma_{\Delta_2} = \{1\}$, and

$$\text{rank } \Gamma_v \leq 1 + \text{rank } \Gamma_{\Delta_1} = 1 + \text{rank } \mathrm{X}^*(\mathrm{pr}_1([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0))(\mathrm{pr}_1(y_v)) \leq$$

$$1 + \text{rank } \mathrm{X}^*(\mathrm{pr}_1([\varphi^{-1}(T_v \otimes \overline{\mathbb{Q}}_l)]^0)) = 1 + \text{rank}(S_1) < \text{rank } (\mathrm{G}_m \times S_1 \times S_2) = \text{rank}(G_{V_l})$$

contrary to the condition (2.1.6).

Consequently p_v divides $\sum_{z \in \lambda^2 \cdot B} z$. Since $\lambda^2 = p_v$ by (2.4.8) and each $b \in B$ is of absolute value 1 in virtue of the A.Weil theorem [6, §21, Theorem 4], we get the relation

$$\sum_{z \in \lambda^2 \cdot B} z = p_v \cdot b_v,$$

where $b_v \in \mathbb{Z}$ is of absolute value $\leq \text{Card}(B) < \frac{1}{2} \cdot l^n$ in virtue of (2.5.1).

By the condition (2.1.3) p_v is unramified in k, and $\text{Norm}_{k/\mathbb{Q}}(v) = p_v$. Consequently p_v splits completely in k, and we have

$$k \otimes_{\mathbb{Q}} \mathbb{Q}_{p_v} \simeq \mathbb{Q}_{p_v} \times \ldots \times \mathbb{Q}_{p_v}. \tag{2.5.2}$$

By the condition (2.1.1) k containes all the $(l^n)^{th}$ roots of unity. On the other hand it is well known that $(\mathbb{Q}_{p_v}^\times)_{tors} \simeq \mathbb{Z}/(p_v - 1)\mathbb{Z}$. It follows from (2.5.2) that $l^n | (p_v - 1)$ and hence

$$p_v = 1 \ (\bmod\ l^n). \tag{2.5.3}$$

By the condition (2.1.1) all the l^n-torsion points of $J(\overline{k})$ are rational over k. It follows that $\rho_l|_{\mathrm{Gal}(\overline{k}/k)}$ is trivial mod l^n and hence for each $\delta \in \Delta$ we have the relation $\delta = 1\,(\bmod\ l^n)$. The same relation holds for each element of $\Delta \cdot \Delta$.

The known relation $\lambda^2 \cdot B \subset \Delta \cdot \Delta$ implies

$$\sum_{z \in \lambda^2 \cdot B} z = \text{Card}(B) \ (\bmod\ l^n). \tag{2.5.4}$$

In virtue of (2.5.3) we deduce from (2.5.4) that $b_v = \text{Card}(B)\ (\bmod\ l^n)$. It is clear that the inequalities $|b_v| \leq \text{Card}(B) < \frac{1}{2} \cdot l^n$ imply the relation

$$b_v = \text{Card}(B).$$

Consequently all elements of the set $\lambda^2 \cdot B$ are equal to $p_v = \lambda^2$ contradicting to the condition $B \neq \{1\}$. Theorem 2.5 is proved.

2.6. In virtue of the condition (2.1.5) Γ_v is torsion-free. So we may consider the canonical embeddings

$$\Delta \subset \Gamma_v \subset \Gamma_v \otimes \mathbb{Q}.$$

In virtue of Corollary 1.5 there exists a simple simply connected algebraic $\overline{\mathbb{Q}}_l$-group S_2 of classical type such that the highest weight of each irreducible S_2-submodule of $H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l) \otimes \overline{\mathbb{Q}}_l$ is a minuscule weight.

2.7. **Lemma** . *The set Δ_2 contains no three distinct points which are lying on one affine line of the space $\Gamma_{\Delta_2} \otimes \mathbb{Q}$.*

Proof . Assume that S_2 is a simple group of type A_m ($m \geq 1$). Then in N.Bourbaki's notations

$$\omega_i = \varepsilon_1 + ... + \varepsilon_i - \frac{i}{m+1}(\varepsilon_1 + ... + \varepsilon_{m+1}),$$

Weyl's group $W(R)$ is the group of all permutations of $\{\varepsilon_1, ..., \varepsilon_{m+1}\}$ [2, Chapter 6, Section 4.7], dim $E(\omega_r) = \binom{m+1}{r}$ [2, Chapter 8, Table 2], $E(\omega_r)$ is symplectic or orthogonal $\Leftrightarrow r = \frac{m+1}{2}$ [2, Chapter 8, Table 1], and in this case

$$\omega_r = \frac{1}{2}(\varepsilon_1 + ... + \varepsilon_{(m+1)/2} - \varepsilon_{(m+3)/2} - ... - \varepsilon_{m+1}),$$

$$\text{ch } E(\omega_r) = \text{ch } E(\omega_{(m+1)/2}) = \sum_{\substack{a_i \in \{\pm 1\} \\ a_1 + \dots + a_{m+1} = 0}} e^{a_1\varepsilon_1/2 + \dots + a_{m+1}\varepsilon_{m+1}/2} =$$

$$= \sum_{\substack{a_i \in \{\pm 1\} \\ a_1 + \dots + a_m \in \{\pm 1\}}} e^{a_1(\varepsilon_1 - \varepsilon_{m+1})/2 + \dots + a_m(\varepsilon_m - \varepsilon_{m+1})/2}.$$

Since the representation of S_2 in W_2 is symplectic or orthogonal, we have the following relation:

$$\Delta_2 = \{\alpha_1^{a_1} ... \alpha_m^{a_m} \mid a_j \in \{\pm 1\}, a_1 + ... + a_m \in \{\pm 1\}\},$$

where α_1, α_2, ..., α_m are multiplicatively independent algebraic numbers. Hence elements of Δ_2 are vertices of some cube in $\Gamma_{\Delta_2} \otimes \mathbb{Q}$. It is evident that no three such vertices are lying on one affine line of $\Gamma_{\Delta_2} \otimes \mathbb{Q}$.

Assume that S_2 is a simple group of type B_m ($m \geq 2$). Then in N.Bourbaki's notations dim $E(\omega_m) = 2^m$ [2, Chapter 8, Table 2],

$$\omega_m = \frac{1}{2}(\varepsilon_1 + ... + \varepsilon_m),$$

$$\text{ch } E(\omega_m) = \sum_{a_i \in \{\pm 1\}} e^{a_1\varepsilon_1/2 + \dots + a_m\varepsilon_m/2}$$

[2, Chapter 6, Section 4.5], so we may assume that

$$\Delta_2 = \{\alpha_1^{a_1} ... \alpha_m^{a_m} \mid a_j \in \{\pm 1\}\},$$

where α_1, α_2, ..., α_m are multiplicatively independent algebraic numbers. Hence elements of Δ_2 are vertices of some cube in $\Gamma_{\Delta_2} \otimes \mathbb{Q}$, and no three such vertices are lying on one affine line of $\Gamma_{\Delta_2} \otimes \mathbb{Q}$.

Assume that S_2 is a simple group of type C_m $(m \geq 2)$. Then dim $E(\omega_1) = 2m$ [2, Chapter 8, Table 2], $\omega_1 = \varepsilon_1$,

$$\text{ch } E(\omega_1) = \sum_{\substack{a_i \in \{\pm 1\} \\ i \in \{1,\dots,m\}}} e^{a_i \varepsilon_i}$$

[2, Chapter 6, Section 4.6]. So we may assume that

$$\Delta_2 = \{\alpha_j^{a_j} \mid a_j \in \{\pm 1\}, j \in \{1, ..., m\}\},$$

where α_1, α_2, ..., α_m are multiplicatively independent algebraic numbers. Hence elements of Δ_2 are vertices of some cube in $\Gamma_{\Delta_2} \otimes \mathbb{Q}$, and no three such vertices are lying on one affine line of $\Gamma_{\Delta_2} \otimes \mathbb{Q}$.

Assume that S_2 is a simple group of type D_m $(m \geq 3)$. Then dim $E(\omega_1) = 2m$ [2, Chapter 8, Table 2], $\omega_1 = \varepsilon_1$,

$$\text{ch } E(\omega_1) = \sum_{\substack{a_i \in \{\pm 1\} \\ i \in \{1,\dots,m\}}} e^{a_i \varepsilon_i},$$

$$\dim E(\omega_{m-1}) = \dim E(\omega_m) = 2^{m-1},$$

$$\text{ch } E(\omega_{m-1}) = \sum_{\substack{a_i \in \{\pm 1\} \\ \text{Card}(\{i | a_i = -1\}) = 1 (\text{ mod } 2)}} e^{a_1 \varepsilon_1/2 + \ldots + a_m \varepsilon_m/2},$$

$$\text{ch } E(\omega_m) = \sum_{\substack{a_i \in \{\pm 1\} \\ \text{Card}(\{i | a_i = -1\}) = 0 (\text{ mod } 2)}} e^{a_1 \varepsilon_1/2 + \ldots + a_m \varepsilon_m/2}$$

[2, Chapter 6, Section 4.8]. So we may assume that

$$\Delta_2 = \{\alpha_j^{a_j} \mid a_j \in \{\pm 1\}, j \in \{1, ..., m\}\},$$

or

$$\Delta_2 = \{\alpha_1^{a_1} ... \alpha_m^{a_m} \mid a_j \in \{\pm 1\}, \text{Card}(\{j | a_j = -1\}) = 1 (\text{mod } 2)\},$$

or

$$\Delta_2 = \{\alpha_1^{a_1} ... \alpha_m^{a_m} (a_j \in \{\pm 1\}, \text{Card}(\{j \mid a_j = -1\}) = 0 (\text{mod } 2)\},$$

where α_1, α_2, ..., α_m are multiplicatively independent algebraic numbers. Hence in any case elements of Δ_2 are vertices of some cube in $\Gamma_{\Delta_2} \otimes \mathbb{Q}$, and no three such vertices are lying on one affine line of $\Gamma_{\Delta_2} \otimes \mathbb{Q}$. Lemma 2.7 is proved.

2.8. **Lemma** . *If $x, y, z \in \Delta$ are three distinct points lying on some affine line $L \subset \Gamma_v \otimes \mathbb{Q}$, then $x = \lambda \cdot \delta_1' \cdot \delta_2$, $y = \lambda \cdot \delta_1 \cdot \delta_2$, $z = \lambda \cdot \delta_1'' \cdot \delta_2$, where δ_1', δ_1, $\delta_1'' \in \Delta_1$, and $\delta_2 \in \Delta_2$.*

Proof . According to (2.4.6) we may consider the canonical projection

$$\text{pr}_{\Delta_2} : \Gamma_v \otimes \mathbb{Q} = (\Gamma_{\{\lambda\}} \otimes \mathbb{Q}) \times (\Gamma_{\Delta_1} \otimes \mathbb{Q}) \times (\Gamma_{\Delta_2} \otimes \mathbb{Q}) \to \Gamma_{\Delta_2} \otimes \mathbb{Q}.$$

If x, y, z are distinct elements of Δ lying on an affine line L, then $\text{pr}_{\Delta_2}(x)$, $\text{pr}_{\Delta_2}(y)$, $\text{pr}_{\Delta_2}(z)$ are lying on $\text{pr}_{\Delta_2}(L)$. Note that if y is situated between x and z, then $\text{pr}_{\Delta_2}(y)$ is situated between $\text{pr}_{\Delta_2}(x)$ and $\text{pr}_{\Delta_2}(z)$. If $\text{pr}_{\Delta_2}(L)$ is a line, then $\text{pr}_{\Delta_2}(x)$, $\text{pr}_{\Delta_2}(y)$, $\text{pr}_{\Delta_2}(z)$ are *distinct* elements of $\text{pr}_{\Delta_2}(\Delta) = \Delta_2$ lying on an affine line $\text{pr}_{\Delta_2}(L)$ contrary to Lemma 2.7. Consequently $\text{pr}_{\Delta_2}(L)$ is a single point of Δ_2. Lemma 2.8 is proved.

§3. Roots, weights and Frobenius eigenvalues

3.1. Now we start to prove Theorem 0.6.

Let R be the root system of Lie $\mathrm{Im}(\rho_l) \otimes \overline{\mathbb{Q}}_l$ relative to a Cartan subalgebra h. We denote by $(\ ,\ ) : h^* \times h \to \overline{\mathbb{Q}}_l$ the tautological pairing. Let R^V be the dual root system. The sets R and R^V are invariant under the natural action of the Weyl group W. We define the lattice of weights $P(R)$ in h^*:

$$P(R) = \{\lambda \in h^* \mid (\lambda, \alpha^V) \in \mathbb{Z} \ \forall \alpha \in R\}.$$

3.2. Let $\Omega(V_l \otimes \overline{\mathbb{Q}}_l)$ be the collection of weights of $h \subset$ Lie $\mathrm{Im}(\rho_l) \otimes \overline{\mathbb{Q}}_l$. We have

$$\Omega(V_l \otimes \overline{\mathbb{Q}}_l) \subset P(R) \subset h^*.$$

The collection $\Omega(V_l \otimes \overline{\mathbb{Q}}_l)$ is W-invariant and R-saturated [2, Chapter 8, §7, Propositions 3, 4]. This means that, for a weight $\lambda \in \Omega(V_l \otimes \overline{\mathbb{Q}}_l)$ and a root $\alpha \in R$ such that $m = (\lambda, \alpha^V) > 0$, the collection $\Omega(V_l \otimes \overline{\mathbb{Q}}_l)$ containes the arithmetic progression $\lambda,\ \lambda - \alpha, ..., \lambda - m\alpha$ of length m. If $m = (\lambda, \alpha^V) < 0$, then $\Omega(V_l \otimes \overline{\mathbb{Q}}_l)$ contains the arithmetic progression $\lambda,\ \lambda + \alpha, ..., \lambda + (-m)\alpha$ of length $(-m)$.

3.3. Recall that a nonzero weight $\lambda \in P(R)$ is called a minuscule weight (microweight)$\Leftrightarrow(\lambda, \alpha^V) \in \{0, 1, -1\}$ for all $\alpha \in R$ [2, Chapter 8, §7.3].

If the canonical l-adic representation ρ_l is not defined by the minuscule weights, then $\Omega(V_l \otimes \overline{\mathbb{Q}}_l)$ containes the arithmetic progression of type $\lambda,\ \lambda - \alpha,\ \lambda - 2\alpha$ or λ, $\lambda + \alpha,\ \lambda + 2\alpha$. In this case there exist distinct elements $\delta \neq \delta' \neq \delta'' \in \Delta$ such that $\delta^2 = \delta' \cdot \delta''$ (indeed, if $\lambda,\ \lambda - \alpha,\ \lambda - 2\alpha \in \Omega(V_l \otimes \overline{\mathbb{Q}}_l)$, then $2 \cdot (\lambda - \alpha) = \lambda + (\lambda - 2\alpha)$).

3.4. **Remark** . It is evident that δ is the middle point of the segment $[\delta', \delta''] \subset \Gamma_v \otimes \mathbb{Q}$. Hence δ', δ and δ'' are lying on one affine line $L \subset \Gamma_v \otimes \mathbb{Q}$.

3.5. Consider the set

$$\Delta^{line} = \{(\delta', \delta, \delta'') \mid \delta \neq \delta' \neq \delta'' \in \Delta,\ \delta^2 = \delta' \cdot \delta''\}.$$

It is evident that $\Delta^{line} \subset \Delta \times \Delta \times \Delta$ is invariant under the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ because for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have $(\sigma(\delta))^2 = \sigma(\delta^2) = \sigma(\delta') \cdot \sigma(\delta'')$.

The last relation shows that the set

$$B = \{\delta'/\delta,\ \ \delta''/\delta = (\delta'/\delta)^{-1} \mid (\delta', \delta, \delta'') \in \Delta^{line}\} \subset \Gamma_v$$

is invariant under the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

3.6. Let $(\delta', \delta, \delta'') \in \Delta^{line}$. According to Lemma 2.8 we have the following decompositions:

$$\delta' = \lambda \cdot \delta_1' \cdot \delta_2,\ \ \delta = \lambda \cdot \delta_1 \cdot \delta_2,\ \ \delta'' = \lambda \cdot \delta_1'' \cdot \delta_2, \tag{3.6.1}$$

where $\delta_1',\ \delta_1,\ \delta_1'' \in \Delta_1,\ \ \delta_2 \in \Delta_2$.

Let $b = \delta'/\delta \in B$. Then $b = \delta'_1/\delta_1$, and we get from (3.6.1) the following decompositions:

$$\delta' = \lambda \cdot b \cdot \delta_1 \cdot \delta_2, \;\; \delta = \lambda \cdot \delta_1 \cdot \delta_2, \;\; \delta'' = \lambda \cdot b^{-1} \cdot \delta_1 \cdot \delta_2.$$

Since $b \cdot \delta_1 = \delta'_1 \in \Delta_1$ and $\delta_1^{-1} \in \Delta_1$, we have the relation $b \in \Delta_1 \cdot \Delta_1$. So $B \subset \Delta_1 \cdot \Delta_1$. On the other hand, $B \neq \{1\}$, so we have a nontrivial $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subset $B \subset \Delta_1 \cdot \Delta_1$ contrary to Theorem 2.5.

Hence ρ_l is defined by the minuscule weights. A simple factor of type E_6 can't appear because $E(\omega_{1,6})$ is not a symplectic or orthogonal representation [2, Chapter 8, Table 1]. We exclude the case of a simple factor of type E_7 in the next paragraph.

§4. Exclusion of type E_7

4.1. In Bourbaki's notations [2, Chapter 6, §4] a simple Lie algebra of type E_7 has the following system of roots:

$$R = \{\pm\varepsilon_i \pm \varepsilon_j \;\; (1 \le i < j \le 6), \;\; \pm(\varepsilon_7 - \varepsilon_8),$$

$$\pm\frac{1}{2} \cdot (\varepsilon_7 - \varepsilon_8 + \sum_{i=1}^{6} (-1)^{\nu(i)} \varepsilon_i) \;\; (\sum_{i=1}^{6} \nu(i) = 1 \,(\bmod 2))\}.$$

Base of the system of roots is the following set:

$$\{\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7);$$

$$\alpha_2 = \varepsilon_1 + \varepsilon_2; \;\; \alpha_3 = \varepsilon_2 - \varepsilon_1; \;\; \alpha_4 = \varepsilon_3 - \varepsilon_2;$$

$$\alpha_5 = \varepsilon_4 - \varepsilon_3; \;\; \alpha_6 = \varepsilon_5 - \varepsilon_4; \;\; \alpha_7 = \varepsilon_6 - \varepsilon_5\}.$$

For a given $\alpha \in R$ consider the reflection

$$S_\alpha(x) = x - 2 \cdot \frac{(x, \alpha)}{(\alpha, \alpha)} \cdot \alpha.$$

Since $(\alpha, \alpha) = 2$ for all $\alpha \in R$, we get the following formula:

$$S_\alpha(x) = x - (x, \alpha) \cdot \alpha.$$

These reflections S_α $(\alpha \in R)$ generate the Weyl group $W(R)$ of order $2^{10} \cdot 3^4 \cdot 5 \cdot 7$.

4.2. We want to find the set of all weights of irreducible symplectic representation $E(\omega_7)$ of degree $56 = 2^3 \cdot 7$ [2, Chapter 8, Tables 1, 2]. Since

$$\omega_7 = \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7) = \frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)$$

is a munuscule weight (microweight), the set of all weights of $E(\omega_7)$ coincides with $W(R)\cdot\omega_7$ [2, Chapter 8, §7, Proposition 6].

It is easy to find the matrix $A=(a_{ij})=(S_{\alpha_i}(\alpha_j))$:

$$A=\begin{pmatrix} -\alpha_1 & \alpha_2 & \alpha_1+\alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \alpha_1 & -\alpha_2 & \alpha_3 & \alpha_2+\alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \alpha_1+\alpha_3 & \alpha_2 & -\alpha_3 & \alpha_3+\alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \alpha_1 & \alpha_2+\alpha_4 & \alpha_3+\alpha_4 & -\alpha_4 & \alpha_4+\alpha_5 & \alpha_6 & \alpha_7 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4+\alpha_5 & -\alpha_5 & \alpha_5+\alpha_6 & \alpha_7 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5+\alpha_6 & -\alpha_6 & \alpha_6+\alpha_7 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6+\alpha_7 & -\alpha_7 \end{pmatrix}.$$

We consider a positive root

$$\varepsilon_3+\varepsilon_6=\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$$

and the corresponding reflection $S_{\varepsilon_3+\varepsilon_6}$. It is easy to see that

$S_{\varepsilon_3+\varepsilon_6}(\alpha_1)=\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7,$
$S_{\varepsilon_3+\varepsilon_6}(\alpha_2)=\alpha_2,$
$S_{\varepsilon_3+\varepsilon_6}(\alpha_3)=\alpha_3,$
$S_{\varepsilon_3+\varepsilon_6}(\alpha_4)=-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7,$
$S_{\varepsilon_3+\varepsilon_6}(\alpha_5)=\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7,$
$S_{\varepsilon_3+\varepsilon_6}(\alpha_6)=\alpha_6,$
$S_{\varepsilon_3+\varepsilon_6}(\alpha_7)=-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6.$

So we get the following chains:

$2\cdot\omega_7=2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7\xrightarrow{S_{\varepsilon_3+\varepsilon_6}}$
$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_1}}$
$\alpha_2+2\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_3}}$
$\alpha_2+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_5}}$
$\alpha_2+2\alpha_4+\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_4}}$
$\alpha_2+\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_2}}$
$-\alpha_2+\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_6}}$
$-\alpha_2+\alpha_5+\alpha_7;$

$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_4}}$
$2\alpha_1+\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_2}}$
$2\alpha_1+3\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_3}}$
$2\alpha_1+3\alpha_2+4\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_4}}$
$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_5}}$
$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_6}}$
$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+\alpha_7;$

$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_5}}$

$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_6}}$
$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7\xrightarrow{S_{\alpha_7}}$
$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5-\alpha_7;$

$\alpha_2+2\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_4}}$
$\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_2}}$
$3\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$

$\alpha_2+2\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_5}}$
$\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_6}}$
$\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7\xrightarrow{S_{\alpha_3}}$
$\alpha_2+2\alpha_4+\alpha_5+\alpha_7\xrightarrow{S_{\alpha_4}}$
$\alpha_2+\alpha_5+\alpha_7\xrightarrow{S_{\alpha_5}}$
$\alpha_2-\alpha_5+\alpha_7;$

$2\alpha_1+\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\xrightarrow{S_{\alpha_3}}$
$2\alpha_1+\alpha_2+4\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$

$\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7\xrightarrow{S_{\alpha_7}}$
$\alpha_2+2\alpha_3+2\alpha_4+\alpha_5-\alpha_7\xrightarrow{S_{\alpha_3}}$
$\alpha_2+2\alpha_4+\alpha_5-\alpha_7;$

$\alpha_2+\alpha_5+\alpha_7\xrightarrow{S_{\alpha_7}}$
$\alpha_2+\alpha_5-\alpha_7.$

The continuation of this procedure leads to old weights or to opposite weights. Consequently the set $\Omega(E(\omega_7))$ of all weights of $E(\omega_7)$ is equal to $\frac{1}{2}\Omega\cup(-\frac{1}{2}\Omega)$, where Ω is the set of all elements in the chains above. As a result, we obtain the following list of elements of Ω:

$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7;$
$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+\alpha_7;$
$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+2\alpha_6+\alpha_7;$
$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$2\alpha_1+3\alpha_2+4\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$2\alpha_1+3\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$2\alpha_1+\alpha_2+4\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$2\alpha_1+\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+2\alpha_6+\alpha_7;$
$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7;$
$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5-\alpha_7;$
$3\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$\alpha_2+2\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6+\alpha_7;$
$\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+2\alpha_6+\alpha_7;$
$\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7;$

$$\begin{array}{llllll}
\alpha_2 & +\,2\alpha_3 & +\,2\alpha_4 & +\,\alpha_5 & & -\,\alpha_7; \\
\alpha_2 & & +\,2\alpha_4 & +\,3\alpha_5 & +\,2\alpha_6 & +\,\alpha_7; \\
\alpha_2 & & +\,2\alpha_4 & +\,\alpha_5 & +\,2\alpha_6 & +\,\alpha_7; \\
\alpha_2 & & +\,2\alpha_4 & +\,\alpha_5 & & +\,\alpha_7; \\
\alpha_2 & & +\,2\alpha_4 & +\,\alpha_5 & & -\,\alpha_7; \\
\alpha_2 & & & +\,\alpha_5 & +\,2\alpha_6 & +\,\alpha_7; \\
\alpha_2 & & & +\,\alpha_5 & & +\,\alpha_7; \\
\alpha_2 & & & +\,\alpha_5 & & -\,\alpha_7; \\
\alpha_2 & & & -\,\alpha_5 & & +\,\alpha_7; \\
\alpha_2 & & & -\,\alpha_5 & -\,2\alpha_6 & -\,\alpha_7; \\
\alpha_2 & & & -\,\alpha_5 & & -\,\alpha_7.
\end{array}$$

4.3. Let $(x_1, ..., x_7)$ be the coordinates on $\mathbb{Q}\alpha_1 + ... + \mathbb{Q}\alpha_7$ with respect to $\{\alpha_1, ..., \alpha_7\}$. It is evident that the affine hyperplane $\{x \mid x_7 = 1\}$ contains 27 points of $\Omega \cup (-\Omega)$, and the affine hyperplane $\{x \mid x_7 = -1\}$ containes 27 points of $\Omega \cup (-\Omega)$. Consequently there is a partition

$$\Omega \cup (-\Omega) = \{\pm(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)\} \cup$$

$$\{x \in \Omega \cup (-\Omega) \mid x_7 = -1\} \cup \{x \in \Omega \cup (-\Omega) \mid x_7 = 1\}.$$

Since $W(R)$ acts transitively on $\Omega(E(\omega_7)) = \frac{1}{2}(\Omega \cup (-\Omega))$, we see that for each weight $\omega \in \Omega(E(\omega_7))$ there exist parallel affine hyperplanes $L_{-\omega}$, L_ω of dimension 6, which give a partition

$$\Omega(E(\omega_7)) = \{\pm\omega\} \cup \{L_{-\omega} \cap \Omega(E(\omega_7))\} \cup \{L_\omega \cap \Omega(E(\omega_7))\},$$

where

$$\mathrm{Card}(L_{\pm\omega} \cap \Omega(E(\omega_7))) = 27.$$

These hyperplanes separate $-\omega$ and ω like on the following picture (where stars denote elements of $\Omega(E(\omega_7))$):

$$\begin{array}{ccccccccc}
 & & * & | & & & * & | & \\
 & | & *...* & | & & | & *...* & | & \\
* & | & L_{-\omega} & | & & | & L_\omega & | & * \\
-\omega & | & *...* & | & & | & *...* & | & \omega \\
 & | & * & & & | & * & &
\end{array} \tag{4.3.1}$$

4.4. **Definition** . We say that a subset $D \subset \Delta$ has a configuration of type $E_7(\omega_7)$ $\Leftrightarrow$ there exist a 7-dimensional affine subspace $L \subset \Gamma_v \otimes \mathbb{Q}$ and an affine isomorphism $f : L \simeq \mathbb{Q}\alpha_1 + ... + \mathbb{Q}\alpha_7$ such that the following conditions hold:

1) $D = L \cap \Delta$;

2) $f(D) = \Omega(E(\omega_7))$ is the system of all weights of the irreducible representation $E(\omega_7)$ of a simple Lie algebra of type E_7.

4.5. **Remark** . If $S^{sc}_{V_l} \otimes \overline{\mathbb{Q}}_l$ has a simple factor of type E_7, then there exists a subset $D \subset \Delta$ with a configuration of type $E_7(\omega_7)$.

Indeed, if Δ_1' corresponds to a simple factor of type E_7, then due to 2.4 we get the decomposition
$$\Delta = \lambda \cdot \Delta_1' \cdot \Delta_1'' \cdot \Delta_2.$$
It is evident that for each $\delta_1'' \in \Delta_1''$, $\delta_2 \in \Delta_2$ the affine space $L = (\Gamma_{\Delta_1'} \otimes \mathbb{Q}) \cdot \delta_1'' \cdot \delta_2$ contains the subset $D = \Delta_1' \cdot \delta_1'' \cdot \delta_2$ with a configuration of type $E_7(\omega_7)$.

4.6. **Lemma** . *If $D \subset \Delta$ has a configuration of type $E_7(\omega_7)$, then* $\mathrm{pr}_{\Delta_2}(D)$ *is a single point.*

Proof . We denote by Φ the composition of the following maps:
$$\mathbb{Q}\alpha_1 + ... + \mathbb{Q}\alpha_7 \xrightarrow{f^{-1}} L \xrightarrow{\mathrm{pr}_{\Delta_2}} \Gamma_{\Delta_2} \otimes \mathbb{Q}.$$
The map Φ is an affine map (possibly, degenerated). According to the results of Section 2.7
$$\Phi(\Omega(E(\omega_7))) = \mathrm{pr}_{\Delta_2}(f^{-1}(\Omega(E(\omega_7))) = \mathrm{pr}_{\Delta_2}(D)$$
is a subset of the set of all vertices of a cube in $\Gamma_{\Delta_2} \otimes \mathbb{Q}$.

Assume that this subset contains at least two distinct points. Then we get from (4.3.1) the following picture (where affine subspaces $\Phi(L_{\pm\omega})$ are parallel and separate vertices $\Phi(-\omega)$ and $\Phi(\omega)$):
$$\begin{array}{ccccccccc} & & * & | & & * & | & \\ & | & *...* & | & | & *...* & | & \\ * & | & \Phi(L_{-\omega}) & | & | & \Phi(L_\omega) & | & * \\ \Phi(-\omega) & | & *...* & | & | & *...* & | & \Phi(\omega) \\ & | & * & & | & * & & \end{array} \tag{4.6.1}$$
Since all stars are vertices of a cube, we can find an edge of a cube such that the following conditions hold: 1) $\Phi(-\omega)$ is lying on this edge; 2) the image of $\Phi(\omega)$ under the canonical projection of a cube onto this edge is not equal to $\Phi(-\omega)$.

Let Φ' be the composition of Φ and the projection of a cube onto the edge under consideration. We get from (4.6.1) the following picture:
$$\begin{array}{cccc} * & * & * & * \\ \Phi'(-\omega) & \Phi'(L_{-\omega}) & \Phi'(L_\omega) & \Phi'(\omega) \end{array}$$
Consequently we arrive to a contradiction, because there are no “intermediate” vertices between two distinct vertices of a cube lying on one edge. Lemma 4.6 is proved.

4.7. **Remark** . Assume that $D \subset \Delta$ has a configuration of type $E_7(\omega_7)$. Then for each element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the set $\sigma(D) \subset \Delta$ has a configuration of type $E_7(\omega_7)$.

It is evident because $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\Gamma_v \otimes \mathbb{Q}$ as a group of affine transformations.

4.8. Consider the set
$\Delta^{E_7} = \{(\delta', \delta'', ..., \delta^{(56)}) \in \Delta \times ... \times \Delta = \Delta^{56} \mid \delta', \delta'', ...\delta^{(56)}$ are distinct points

of Δ and the set $\{\delta', \delta'', ..., \delta^{(56)}\}$ has a configuration of type $E_7(\omega_7)\}$.

In virtue of Remark 4.7 this set is invariant under the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Consequenly the set

$$B = \{1, \delta''/\delta', ..., \delta^{(56)}/\delta', \delta'/\delta'', ..., \delta^{(56)}/\delta'', ...\} \mid (\delta', ..., \delta^{(56)}) \in \Delta^{E_7}\}$$

is invariant under the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. According to Lemma 4.6 we have the following decompositions:

$$\delta' = \lambda \cdot \delta_1' \cdot \delta_2, \;\; \delta'' = \lambda \cdot \delta_1'' \cdot \delta_2, \;\; ... \;\; \delta^{(56)} = \lambda \cdot \delta_1^{(56)} \cdot \delta_2, \tag{4.8.1}$$

where $\delta_1', \; \delta_1'', \; ..., \delta_1^{(56)} \in \Delta_1, \; \delta_2 \in \Delta_2$.

Hence

$$B = \{1, \delta_1''/\delta_1', ..., \delta_1^{(56)}/\delta_1', \delta_1'/\delta_1'', ..., \delta_1^{(56)}/\delta_1'', ...\} \mid (\delta', ..., \delta^{(56)}) \in \Delta^{E_7}\}$$

is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subset of $\Delta_1 \cdot \Delta_1^{-1} = \Delta_1 \cdot \Delta_1$, and $B \neq \{1\}$ contrary to Theorem 2.5. Thus a simple factor of type E_7 can't appear. Theorem 0.6 is proved.

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