ON RIGIDITY AND THE ALBANESE VARIETY FOR PARALLELIZABLE MANIFOLDS

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Abstract. We study the rigidity questions and the Albanese Variety for Complex Parallelizable Manifolds. Both are related to the study of the cohomology group $H^1(X, \mathcal{O})$. In particular we show that a compact complex parallelizable manifold is rigid iff $b_1(X) = 0$ iff $Alb(X) = \{e\}$ iff $H^1(X, \mathcal{O}) = 0$.

1. Introduction

For a compact Kähler manifold X there exists a compact complex torus $Alb(X)$ and a holomorphic map $\pi : X \to Alb(X)$ such that

1. For every holomorphic map f from X to a compact complex torus T there exists an element $a \in T$ and a holomorphic Lie group homomorphism $F : Alb(X) \to T$ such that $f(x) = a \cdot F \circ \pi(x)$ for all $x \in X$.
2. The holomorphic map $\pi : X \to Alb(X)$ induces isomorphisms on the cohomology groups $H^1(\cdot, \mathbb{R})$, $\Gamma(\cdot, d\mathcal{O})$ and $H^1(\cdot, \mathcal{O})$.

We prove similar statements for parallelizable manifolds. A complex manifold is called parallelizable if its tangent bundle is holomorphically trivial. A compact complex manifold is parallelizable if and only if it is biholomorphic to a quotient G/Γ of a complex Lie group G by a discrete subgroup Γ ([21]). Compact complex parallelizable manifolds are never Kähler unless they are tori ([21]). Nevertheless, concerning the Albanese variety it is possible to obtain results similar to the Kähler case as long as reductive factors of rank 1 are absent from the Lie group G.

More precisely we prove the following.

Theorem 1. *Let G be a connected complex linear-algebraic group and Γ a lattice (i.e. a discrete subgroup such that the quotient manifold $X = G/\Gamma$ has finite volume with respect to Haar measure). Assume that there exists no surjective algebraic group homomorphism from G to $PSL_2(\mathbb{C})$ or $\mathbb{C}^*$.*

Then there exists a compact complex torus A and a surjective holomorphic map $\pi : X \to A$ such that

1. *For every connected complex Lie group T and every holomorphic map $f : X \to T$ there exists an element $a \in T$ and a holomorphic Lie group homomorphism $F : A \to T$ such that $f(x) = a \cdot F(\pi(x))$ for all $x \in X$.*

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2. *The map* $\pi : X \to A$ *induces isomorphisms on the cohomology groups* $H^1(\cdot, \mathbb{C})$ *and* $\Gamma(\cdot, d\mathcal{O})$.
3. *Every meromorphic function on* X *is a pull-back from* A.
4. *If* X *is compact,* π *induces an isomorphism of* $H^1(\cdot, \mathcal{O})$.

Corollary 1. *Let* G *be a connected complex Lie group and* Γ *a lattice. Then there exists an abelian variety* A *and a surjective holomorphic connected* G*-equivariant map* $\pi : G/\Gamma \to A$ *inducing an isomorphism of the respective function fields.*

Furthermore, we determine $H^1(X, \mathcal{O})$ for cocompact lattices in arbitrary complex linear-algebraic groups.

Theorem 2. *Let* G *be a connected complex linear-algebraic group,* $\Gamma \subset G$ *a lattice and* $X = G/\Gamma$. *Let* $G = S \cdot R$ *be a Levi-Malcev-decomposition,* N *the nilradical, and* $A = [S, R] \cdot N'$. *In addition, let* W *denote the maximal subvectorspace of* $\mathcal{L}ie(R'A/A)$ *such that* $Ad(\gamma)|_W$ *is a semisimple linear endomorphism with only real eigenvalues for every* $\gamma \in \Gamma$.

Then $\dim H^1(G/\Gamma, \mathcal{O}) = \dim(G/G') + b_1(\Gamma/(R \cap \Gamma)) + \dim(W)$.

Using this result we may deduce the following vanishing theorem.

Theorem 3. *Let* G *be a connected complex Lie group,* Γ *a discrete cocompact subgroup and* $X = G/\Gamma$.

Then $H^1(X, \mathcal{O}) = \{0\}$ *iff* $b_1(X) = 0$.

Since $H^1(X, \mathcal{T}) \simeq H^1(X, \mathcal{O}^n)$ for an n-dimensional parallelizable manifold, this implies that a compact complex parallelizable manifold admits infinitesimal deformations if and only if $b_1(X) > 0$. In addition, we will prove that for $b_1(X) > 0$ there actually exist small deformations. Thus we obtain the following result characterizing rigidity.

Theorem 4. *Let* G *be a connected complex Lie group,* Γ *a discrete cocompact subgroup and* $X = G/\Gamma$ *the complex quotient manifold. Then the following properties are equivalent.*

1. X *admits no infinitesimal deformations (i.e. there are no non-trivial families over* $Spec\, \mathbb{C}[\epsilon]/(\epsilon^2)$*).*
2. X *admits no small deformations (i.e. there are no non-trivial families over the unit disk).*
3. $b_1(X) = 0$.
4. $H^1(X, \mathcal{O}) = \{0\}$.

This characterization of rigidity of parallelizable manifolds generalizes a number of earlier results. The existence of non-trivial deformations of positive-dimensional tori is classical. Deformations of solv-manifolds have been studied by Nakamura [16]. Raghunathan proved the rigidity of quotients of G/Γ for G semisimple without *rank* 1-factors. Ghys proved theorem 4 for $G = SL_2(\mathbb{C})$.

Theorem 5. *Let* G *and* H *be linear-algebraic Let* G, H *be a complex linear-algebraic groups,* Γ *and* Λ *discrete cocompact subgroups of* G *resp.* H *and* $X = G/\Gamma$ *and* $Y = H/\Lambda$ *the respective quotient manifolds.*

Assume that $\pi_1(X) \simeq \pi_1(Y)$.
Then $\dim H^1(X, \mathcal{O}) = \dim H^1(Y, \mathcal{O})$.

The assumption of G being linear-algebraic is not very restrictive (in fact we do not know any compact complex parallelizable manifold for which it can not be fulfilled), but it is crucial that both X and Y are parallelizable. Nakamura gave an example where X is a compact complex parallelizable manifold and Y a non-parallelizable compact complex manifold diffeomorphic to X (thus in particular $\pi_1(X) \simeq \pi_1(Y)$), but $\dim H^1(X, \mathcal{O}) \neq \dim H^1(Y, \mathcal{O})$ [16].

Parts of the results presented here are already contained in the Habilitationsschrift of the author [22].

2. Preparations

We summarize some basic results which are used in the sequel.

Theorem 6. *(Malcev, [13]) Let G be a real nilpotent Lie group, Γ a discrete subgroup such that G/Γ has finite volume.*

Then G/Γ is compact, Γ' is a lattice in G' and Γ' is of finite index in $G' \cap \Gamma$.

Theorem 7. *(Kodaira, [16]) Let G be a complex nilpotent Lie group and Γ a lattice. Then $H^1(G/\Gamma, \mathcal{O}) \simeq H^1(G/G'\Gamma, \mathcal{O})$.*

Theorem 8. *(Margulis, [14]) Let S be a semisimple complex Lie group without rank 1-factor and Γ a lattice. Then Γ' is of finite index in Γ.*

Theorem 9. *(Mostow, [15]) Let G be a complex connected Lie group, R its radical and N its nilradical and Γ a lattice.*

Then $G\Gamma$ and $N\Gamma$ are closed and $R/(R \cap \Gamma)$ and $N/(N \cap \Gamma)$ are compact.

The following theorem of Akhiezer generalizes earlier results of Raghunathan.

Theorem 10. *(Akhiezer, [2]) Let G be a simply-connected semisimple complex Lie group and Γ a cocompact lattice. Let $X = G/\Gamma$.*

In this case the natural morphism $i : H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O})$ is an isomorphism.

Proposition 1. *Let G be a connected complex Lie group, Γ a lattice and H a closed complex Lie subgroup of G containing Γ.*

Then the connected component H^0 of H is normal in G and therefore G/H is again parallelizable.

Proof. We consider the Tits-fibration, i.e. the natural G-equivariant morphism from G/H to a Grassmann manifold given by $x \mapsto \mathcal{L}ie(G_x)$. From Iwamoto's generalization of the Borel density theorem (see [11]) it follows that this map is constant. This implies that H^0 is normal. □

3. Orbits of the commutator group

Lemma 1. *Let G be a complex Lie group, Γ a discrete subgroup and A a normal simply-connected commutative complex Lie subgroup of G. Assume that $A/(A \cap \Gamma)$ is compact and that the Zariski-closures of $Ad(\Gamma)$ and $Ad(G)$ in $GL(\mathcal{L}ie\, G)$ coincide.*

Then $[\Gamma, A \cap \Gamma]$ is a lattice of $[G, A]$ and a subgroup of finite index in $[G, A] \cap \Gamma$.

Proof. The group A is a complex vector space and conjugation in the group G induces a representation $\rho : G/A \to GL(A)$. This representation is naturally isomorphic to the restriction of the adjoint representation Ad to $\mathcal{L}ie\, A$.

For $g \in G$ define $\zeta_g \in End(A)$ by $\zeta_g = \rho(g) - id_A$. Let $\Lambda = \oplus_{\gamma\in\Gamma}\zeta_\gamma(A \cap \Gamma)$. Clearly, $[\Gamma, A \cap \Gamma]$ is the subgroup generated by Λ. Since $A \cap \Gamma$ is cocompact in A, $< A \cap \Gamma >_{\mathbb{R}} = < A \cap \Gamma >_{\mathbb{C}}$. This implies that $< \Lambda >_{\mathbb{R}}=< \Lambda >_{\mathbb{C}}$ and consequently that $< \Lambda >_{\mathbb{Z}}$ is a lattice in the complex subspace $V =< \Lambda >_{\mathbb{C}}$. We claim that $[A, G'] = V$. Indeed, $Ad(\gamma)(\mathcal{L}ie\, A) \subset \mathcal{L}ie\, V$ for all $\gamma \in \Gamma$ implies that $Ad(g)(\mathcal{L}ie\, A) \subset \mathcal{L}ie\, V$ for all $g \in G$, because we assumed that the Zariski-closures of $Ad(\Gamma)$ and $Ad(G)$ coincide. □

Proposition 2. *Let G be a simply-connected complex Lie group, R its radical and Γ a lattice.*

Then $(G' \cap R)\Gamma$ is closed in G, $R \cap \Gamma'$ is a subgroup of finite index in $G' \cap R \cap \Gamma$ and a cocompact lattice in $G' \cap R$.

Proof. First recall that a normal Lie subgroup in a simply-connected Lie group is also simply-connected.

Let N denote the nilradical and N^k the central series of N. By the theorems of Mostow and Malcev all of the groups N and N^k ($k \in \mathbb{N}$) have compact orbits in G/Γ. Furthermore N^k/N^{k+1} is a simply-connected commutative normal Lie subgroup of G/N^{k+1} for all k.

Thus we may apply the preceding lemma repeatedly and obtain that $G' \cap N \cap \Gamma'$ is of finite index in $G' \cap N \cap \Gamma$ and a lattice in $N \cap G'$.

This is the statement of the proposition, since $G' \cap R \subset N$. □

Corollary 2. *(Barth-Otte* [5]*) Let G be a solvable complex Lie group and Γ a lattice.*

Then $G'\Gamma$ is closed in G.

Theorem 11. *Let G be a simply-connected complex Lie group, R its radical and Γ a lattice. Assume that $\Gamma'/(R \cap \Gamma')$ is a subgroup of finite index in $\Gamma/(R \cap \Gamma)$.*

Then $G'\Gamma$ is closed in G, $G' \cap \Gamma$ is a lattice in G' and Γ' is a subgroup of finite index in $G' \cap \Gamma$.

Corollary 3. *Let G be a simply-connected complex Lie group, R its radical and Γ a lattice. Assume that no simple factor of G/R is isomorphic to $SL_2(\mathbb{C})$.*

Then $G'\Gamma$ is closed in G, $G' \cap \Gamma$ is a lattice in G' and Γ' is a subgroup of finite index in $G' \cap \Gamma$.

Recursive application of the theorem implies the following.

Corollary 4. *Let G and Γ be as in the above theorem. Let G^k denote the k-th term of the derived series of G. Then $G^k\Gamma$ is closed in G, $G^k \cap \Gamma$ is a lattice in G^k and Γ^k is a subgroup of finite index in $G^k \cap \Gamma$.*

Proof. By the above proposition it suffices to consider the case where $R \cap G' = \{e\}$. Then R is central in G. Since G is simply-connected, this implies $G = R \times S$ for a semisimple complex Lie group S. Let $\pi : G \to S$ denote the natural projection. Then $\pi(\Gamma)$ is a lattice in S. Margulis' theorem implies that $\pi(\Gamma')$ is also a lattice in S. It follows that $(\Gamma \cap R) \cdot (\Gamma \cap S) \supset (\Gamma \cap R) \cdot \Gamma'$ is already a lattice in G. In particular, Γ' is a lattice in S and hence of finite index in $S \cap \Gamma = G' \cap \Gamma$. □

Although in general $G' \cap \Gamma$ is not a lattice in G', it is still large in a certain sense.

Proposition 3. *Let G be a complex Lie group, Γ a lattice in G.*
Then every Γ'-invariant plurisubharmonic function on G' is constant.

Proof. Since $(R\cap G')/(R\cap\Gamma')$ is compact, it is clear that Γ'-invariant plurisubharmonic functions are constant along the $R\cap G'$-orbits. Thus we may assume that $R\cap G' = \{e\}$. Then G is a direct product of an abelian complex Lie group R and a semisimple complex Lie group S. Recall that a lattice is Zariski-dense. This implies that Γ' is Zariski-dense in G', because for sufficiently large n the map $\zeta : (G \times G)^n \to G'$ given by

$$\zeta : ((x_1, y_1), \ldots, (x_n, y_n)) \mapsto (x_1 y_1 x_1^{-1} y^{-1}) \cdot \ldots \cdot (x_n y_n x_n^{-1} y_n^{-1}) \tag{1}$$

is a dominant regular morphism and therefore maps $(\Gamma \times \Gamma)^n$ onto a Zariski-dense subset of G'. Now a theorem of Berteloot and Oeljeklaus [4] implies that Γ'-invariant plurisubharmonic functions must be S-invariant. □

In a similar way one can prove the following.

Proposition 4. *Let G be a connected complex Lie group and Γ a lattice.*
Then every plurisubharmonic function on $X = G/\Gamma$ is constant.

4. The Construction of the Albanese torus

Proposition 5. *Let G be a complex Lie group and Γ a closed complex Lie subgroup. Assume that every holomorphic function on G/Γ is constant.*

Let I denote the smallest closed complex Lie subgroup of G containing $G'\Gamma$.

Then G/I is a commutative complex Lie group and the natural projection $\pi : G/\Gamma \to G/I$ has the following properties.

1. *For every connected complex Lie group T and every holomorphic map $f : G/\Gamma \to T$ there exists an element $a \in T$ and a holomorphic Lie group homomorphism $F : G/I \to T$ such that $f(x) = a \cdot F \circ \pi(x)$ for all $x \in X$.*
2. *If $I = G'\Gamma$, then every closed holomorphic 1-form on G/Γ is a pull-back from G/I.*
3. *If Γ is a lattice in G, then G/I is compact.*
4. *If Γ is a lattice in G, then every meromorphic function on G/Γ is a pull-back from G/I.*

Proof. 1. Let Ad denote the adjoint representation of T and Z the connected component of the center of T. Since holomorphic functions on $GL(\mathcal{L}ie\, T)$ separate points, it is clear that $Ad \circ f$ is constant for every holomorphic map $f : G/\Gamma \to T$. Thus there is no loss in generality in assuming $Z = T$.

The cotangent bundle $T^*(Z)$ is spanned by invariant closed holomorphic 1-forms ω_i. Now

$$0 = df^*\omega_i(X,Y) = f^*\omega_i([X,Y]) + X(f^*\omega_i)Y - Y(f^*\omega_i X) = f^*\omega_i([X,Y])$$

for every holomorphic map $f : G/\Gamma \to T$, every closed holomorphic 1-form ω_i and every holomorphic vector fields X, Y (Note that $(f^*\omega_i Y)$ is a holomorphic function on G/Γ and therefore constant.)

It follows that the G'-orbits are contained in the fibers of f.

The map f must be equivariant [23]. Hence f fibers through G/I.

2. The equation $(*)$ implies that $\Gamma(X, d\mathcal{O}) \simeq \mathcal{L}ie(G/G')^*$. Hence the assertion.

3. By a Fubini-type argument it follows that G/I has finite volume. (see [20], Lemma 1.6). Since G/I is an abelian group, this implies compactness.

4. As a homogeneous complex manifold, G/Γ admits a meromorphic reduction $G/\Gamma \to G/J$ with meromorphically separable base G/J ([10]). The base manifold G/J is again parallelizable (prop. 1). Therefore it suffices to show the following:

Lemma 2. *Let H be a connected complex Lie group and Λ a lattice. Assume that $Y = H/\Lambda$ is meromorphically separable.*

Then H is commutative and Y a compact complex torus.

Proof. Recall that the radical R of H has closed orbits in $Y = H/\Lambda$ and that these orbits are moreover compact, because lattices in solvable Lie groups are necessarily cocompact. Thus we obtain a holomorphic fiber bundle $\tau : Y \to Z = H/R\Lambda$ with compact fibers. Using meromorphic separability of Y, for any point $y \in Y$ and any natural number k with $0 \leq k \leq \dim Y$ we can find a closed analytic subset $W \subset Y$ of pure dimension k containing y (simply by taking irreducible components of intersections of zero divisors of meromorphic functions on Y). Since the projection map $\tau : Y \to Z$ is a proper holomorphic map, this implies the existence of analytic hypersurfaces in W unless W is trivial (i.e. a point).

Recall that a theorem of Huckleberry and Margulis [9] states a quotient of a complex semisimple Lie group by a Zariski-dense subgroup never contains analytic hypersurfaces. Hence in our situation W must be a point, i.e. H is solvable. It follows that H/Λ is compact. Finally observe that a meromorphically separable compact homogeneous manifold is automatically projective and in particular Kähler ([8]). But a parallelizable compact complex manifold is Kähler if and only if it is a compact complex torus ([21]). □

□

5. Leray spectral sequence

Let $f : X \to Y$ be a holomorphic map between complex spaces. There is a Leray spectral sequence for the sheaf $\mathcal{O}_X$. The respective lower term sequence yields the following.

$$0 \to H^1(Y, \mathcal{R}^0 f_* \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(Y, \mathcal{R}^1 f_* \mathcal{O}_X) \to H^2(Y, \mathcal{R}^0 f_* \mathcal{O}_X)$$

Assume that f is connected and proper. Then $\mathcal{R}^0 f_* \mathcal{O}_X = \mathcal{O}_Y$. Furthermore, if $\dim H^k(f^{-1}\{p\}, \mathcal{O}) = r$ for a natural number k and all $p \in Y$, then, by Grauert's theorem, $\mathcal{R}^k f_* \mathcal{O}_X$ is a locally free coherent sheaf of rank r. This implies in particular the following observation.

Lemma 3. *Let $f : X \to Y$ be a proper connected holomorphic map and assume that for every fiber F_p the induced cohomology map $i^* : H^1(X, \mathcal{O}) \to H^1(F_p, \mathcal{O})$ vanishes. Assume furthermore that* $\dim H^1(F_p, \mathcal{O})$ *does not depend on $p \in Y$.*

Then $\alpha : H^1(X, \mathcal{O}) \to H^0(\mathcal{R}^1 f_* \mathcal{O})$ is the zero map.

5.1. **Leray spectral sequence for flat bundles.**

Proposition 6. *Let $\pi : E \to X$ be a holomorphic fiber bundle with typical fiber F and structure group S. Assume that F is connected and compact, that S acts on F in such a way that there exists an invariant hermitian metric and that $E \to X$ admits a flat holomorphic connection.*

Then $\alpha : H^1(E, \mathcal{O}) \to H^0(X, \mathcal{R}^1 \pi_* \mathcal{O})$ is surjective.

Proof. Let $\eta \in H^0(X, \mathcal{R}^1 \pi_* \mathcal{O})$. Then there exists an open cover $\mathcal{U} = (U_i)_i$ by contractible open Stein subsets of X such that η is given by $\eta_i \in H^1(\pi^{-1}(U_i), \mathcal{O})$. Using the Dolbeault-isomorphism we may choose corresponding $\bar{\partial}$-closed $(0, 1)$-forms ω_i on $W_i = \pi^{-1}(U_i)$. Of course these forms ω_i are not unique. Now, if there is a *canonical* way to choose ω_i, then the forms ω_i coincide on the intersection of the U_i and yield a globally defined $\bar{\partial}$-closed $(0, 1)$-form on E, implying that η is contained in the image of $H^1(E, \mathcal{O})$.

The assumptions made in the proposition allow us to choose a hermitian metric on each fiber in such a way that for every contractible Stein open subset $U \subset X$ we obtain a trivialization $E|_U \simeq U \times F$ which is compatible with the flat connection and such that the choosen hermitian metric on each fiber is just the pull-back of one fixed S-invariant hermitian metric on F. Then there is a canonical way to choose the forms ω_i. Namely the forms ω_i are to be choosen such that they annihilate horizontal vector fields (with respect to the connection) and are harmonic if restricted to a fiber of π. □

Lemma 4. *Let G be a complex Lie group, Γ a discrete subgroup and A a connected commutative normal complex Lie subgroup such that $A/(A \cap \Gamma)$ is compact. Assume that the short exact sequence of Lie algebras*

$$0 \to \mathcal{L}ie(A) \to \mathcal{L}ie(G) \to \mathcal{L}ie(G/A) \to 0$$

is split.

Then $\pi : E = G/\Gamma \to G/A\Gamma = B$ is a torus bundle with flat holomorphic connection.

Proof. The flat connection is induced by the splitting of the Lie algebra sequence. □

Corollary 5. *Under the assumptions of the lemma the induced map $H^1(E, \mathcal{O}) \to H^0(B, \mathcal{R}^1 \pi_* \mathcal{O})$ is surjective.*

We will also need a description of the structure of $\mathcal{R}^1 f_* \mathcal{O}$ for torus bundles.

Let $f : E \to Y$ be a locally trivial holomorphic fiber bundle with a compact complex torus T is typical fiber. Let $V = \Omega^1(T)$ denote the vector space of holomorphic 1-forms on T. Then there is a exact sequence

$$1 \to \underbrace{Aut^0(T)}_{=T} \to Aut(T) \xrightarrow{\zeta} GL(V)$$

Let $\mathcal{U} = \{U_i\}$ be a trivializing open cover of Y such that E is given by transition functions $\phi_{ij} : U_i \cap U_j \to Aut(T)$.

Claim 1. *Under the above assumptions $\mathcal{R}^1 f_* \mathcal{O}_E$ is a locally free coherent sheaf on Y isomorphic to the sheaf of sections in the flat vector bundle given by the transition functions $\psi_{ij} : U_i \cap U_j \to GL(V)$ defined by $\psi_{ij} = \overline{\zeta \circ \phi_{ij}}$.*

Proof. This is a consequence of the Dolbeault-isomorphism. □

We apply this to parallelizable manifolds.

Proposition 7. *Let G be a complex Lie group, Γ a discrete subgroup and A a normal abelian complex Lie subgroup. Assume that $A/(A \cap \Gamma)$ is compact. Denote the natural projection map $E = G/\Gamma \to G/A\Gamma = B$ by π.*

In this case $\mathcal{R}^1 \pi_* \mathcal{O}$ is a flat vector bundle of rank $\dim A$ over B induced by the representation $\rho : \Gamma \to GL(\mathcal{L}ie\, A^*)$ given by $\gamma \mapsto \overline{Ad^*(\gamma)}$.

Lemma 5. *Let $\pi : X \to Y$ be a finite holomorphic covering.*

Then $\pi^* : H^1(Y, \mathcal{O}) \to H^1(X, \mathcal{O})$ is injective.

Proof. If ω is a $\bar\partial$-closed $(0,1)$-form on Y and $\bar\partial f = \pi^* \omega$ for a function f on X, then $\bar\partial g = \omega$ for $g(y) = \frac{1}{d} \sum_{\pi(x)=y} f(x)$ (where d denotes the degree of π). □

6. Description of $H^1(X, \mathcal{O})$

Theorem 12. *Let G be a connected complex Lie group, $\Gamma \subset G$ a lattice and $X = G/\Gamma$. Let $G = S \cdot R$ be a Levi-Malcev-decomposition, N the nilradical, and $A = [S,R] \cdot N'$. Furthermore let W denote the maximal linear subspace of $\mathcal{L}ie(R'A/A)$ such that $Ad(\gamma)|_W$ is a semisimple linear endomorphism with only real eigenvalues for every $\gamma \in \Gamma$.*

Then $\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim(G/G') + b_1(\Gamma/(R \cap \Gamma)) + \dim(W)$. Equality holds, if G is linear-algebraic.

Remark 1. *We do not know any example of a quotient manifold of a connected complex Lie group by a lattice which is* not *biholomorphic to a quotient of a* linear-algebraic *$\mathbb{C}$-group by a lattice.*

Corollary 6. *Let G be a simply-connected complex Lie group and $\Gamma \subset G$ be a cocompact lattice. Assume that the radical R is nilpotent and that the semisimple group G/R contains no factor S_0 such that $S_0/(S_0 \cap R\Gamma)$ is compact and $S_0 \simeq SL_2(\mathbb{C})$.*

Then $H^1(G/\Gamma, \mathcal{O}) \simeq H^1(G/G'\Gamma, \mathcal{O})$.

Proof. If R is nilpotent, then $R = N$ and therefore $A = R'A$. Since $A \subset H \subset R'A$, it follows that $A = H$. Furthermore the assumption on S implies that $H^1(G/R\Gamma, \mathcal{O}) = 0$. Thus $\dim H^1(G/\Gamma, \mathcal{O}) = \dim G/G'$. The assumption on S also implies that $G'\Gamma$ is closed in G. Thus $H^1(G/G'\Gamma, \mathcal{O}) \simeq H^1(G/\Gamma, \mathcal{O})$. □

We now prove the theorem.

Proof. We study the sequence of fibrations

$$1 \to G/\Gamma \xrightarrow{\pi_1} G/N'\Gamma \xrightarrow{\pi_2} G/A\Gamma \xrightarrow{\pi_3} G/(G' \cap R)\Gamma \xrightarrow{\pi_4} G/R\Gamma \to 1$$

First we must verify the existence of these fibrations, i.e. to show that N', A, $G' \cap R$ and R all have closed orbits. Results of Mostow [15] imply that R and N have closed orbits. By classical results of Malcev [13] this implies closedness of the N'-orbits. From prop. 2 we obtain closedness of the $G' \cap R$-orbits. Finally note that $A = (G^k \cap R)N'$ for k sufficiently large. Hence closedness of A-orbits follows from thm. 11.

We note that all these projections π_i are surjective proper holomorphic maps and for each i we will study the lower term sequence of the Leray spectral sequence for the sheaf $\mathcal{O}$.

Claim 2. *The induced map $\pi_1^* : H^1(G/N'\Gamma, \mathcal{O}) \to H^1(G/\Gamma, \mathcal{O})$ is an isomorphism.*

Proof. By a theorem of Kodaira (see [16]) there is an isomorphism

$$H^1(N/N'(N \cap \Gamma), \mathcal{O}) \simeq H^1(N/(N \cap \Gamma, \mathcal{O}).$$

It follows that the embedding $i : N'/(N' \cap \Gamma) \hookrightarrow N/(N \cap \Gamma)$ induces the zero map between the respective cohomology groups $H^1(\cdot, \mathcal{O})$. Thus the group homomorphism $j^* : H^1(G/\Gamma, \mathcal{O}) \to H^1(N'/(N' \cap \Gamma))$ induced by the inclusion map $j : N'/(N' \cap \Gamma) \to G/\Gamma$ must vanish as well. By homogeneity it follows that for every fiber F of p the induced cohomology map $i^* : H^1(G/\Gamma, \mathcal{O}) \to H^1(F, \mathcal{O})$ is zero. Therefore

$$H^1(G/\Gamma, \mathcal{O}) \to H^0(G/N'\Gamma, \mathcal{R}^1 p_* \mathcal{O})$$

vanishes (lemma 3) and $H^1(G/\Gamma, \mathcal{O}) \simeq H^1(G/N'\Gamma, \mathcal{O})$. □

Claim 3. *The induced map $\pi_2^* : H^1(G/A\Gamma, \mathcal{O}) \to H^1(G/N'\Gamma, \mathcal{O})$ is an isomorphism.*

Proof. Recall that A is defined as $A = [S, R]N'$. Since $[S, R] \subset R \cap G' \subset N$, it is clear that A/N' is abelian. This implies that for $\pi : G/N'\Gamma \to G/A\Gamma$ the higher direct image sheaf $\mathcal{R}^1\pi_*\mathcal{O}$ is the coherent sheaf associated to a flat vector bundle given by a representation ρ which arises in the following way: ρ is the restriction of the complex conjugation of the representation from G on $GL(\mathcal{L}ie(A/N')^*)$ induced by the coadjoint representation. By construction no linear subspace of $\mathcal{L}ie(A/N')$ is invariant under $Ad(S)$. It follows that $\mathcal{R}^1\pi_*\mathcal{O}$ does not admit global sections. Hence the claim. □

The lower term sequence of the Leray spectral sequence for π_3 yields

$$H^1(G/(G' \cap R)\Gamma, \mathcal{O}) \to H^1(G/A\Gamma, \mathcal{O}) \xrightarrow{\alpha} H^0(G/(G' \cap R)\Gamma, \mathcal{R}^1(\pi_3)_*\mathcal{O}) \overset{\phi}{\simeq} W$$

The isomorphism ϕ is a consequence of [24].

Claim 4. *If G is linear-algebraic, then α is surjective.*

Proof. Using cor. 5 it suffices to show that the short exact sequence of complex Lie groups

$$(*) \qquad 1 \to (G' \cap R)/A \to G/A \to G/(G' \cap R) \to 1$$

is split. Let V denote the unipotent radical of G. Then $G' \cap R \subset V \subset N$. Furthermore $A = [S,R]N'$ implies that V/A is commutative. Let L denote a Levi subgroup of G, i.e. a maximal connected reductive subgroup of G. Then L is a reductive group, acting (by conjugation) linearly on the vector group V/A. Hence there is a L-invariant subvector space $W \subset V$ such that $V = W \oplus (G' \cap R)$. Thus $G = (L \times W) \times V$ and we obtain a splitting of $(*)$. $\square$

Next we discuss the lower term sequence for π_4.

$$1 \to H^1(G/R\Gamma, \mathcal{O}) \to H^1(G/(G' \cap R)\Gamma, \mathcal{O}) \xrightarrow{\beta} H^0(G/R\Gamma, \mathcal{R}^1(\pi_4)_*\mathcal{O})$$

First we note that the exact sequence of Lie algebras

$$0 \to \mathcal{L}ie(R/(G' \cap R)) \to \mathcal{L}ie(G/(G' \cap R)) \to \mathcal{L}ie(G/R) \to 0$$

is always split. Hence β is surjective. Furthermore the adjoint action of G on $\mathcal{L}ie(G)$ induces the trivial action on $Lie(R/(G' \cap R))$. Thus $\mathcal{R}^1(\pi_4)_*\mathcal{O}$ is a free $\mathcal{O}$-module sheaf with rank equal to $\dim_{\mathbb{C}}(H^1(R/(G' \cap R)\Gamma, \mathcal{O})$. Note that $G/G' \simeq R/(G' \cap R)$ and that $R/(G' \cap R)$ is a compact complex torus. It follows that $\dim H^0(G/R\Gamma, \mathcal{R}^1(\pi_4)_*\mathcal{O}) = \dim(G/G')$.

Finally we recall that $\dim H^1(G/R\Gamma, \mathcal{O}) = \dim H^1(G/R\Gamma, \mathbb{C}) = b_1(\Gamma/(R \cap \Gamma))$ by a result of Akhiezer [2]. $\square$

We now present a precise vanishing criterion for $H^1(G/\Gamma, \mathcal{O})$.

Proposition 8. *Let G be a complex Lie group and Γ a discrete cocompact subgroup. Then $H^1(G/\Gamma, \mathcal{O}) = 0$ if and only if $b_1(G/\Gamma) = 0$.*

Proof. The exponential sequence yields an embedding $H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathcal{O})$. Hence $H^1(X, \mathcal{O}) = 0$ implies $b_1(X) = 0$ for $X = G/\Gamma$.

Conversely let us assume $b_1(X) = 0$. Then $Hom(\Gamma, \mathbb{Z}) = 0$. Since Γ is a lattice, this implies $Hom(G, \mathbb{C}) = 0$. Hence $G = G'$ and consequently $R = [S,R]$. This implies in particular that $R'A = A$ for $A = [S,R]N'$. Hence $b_1(\Gamma/(R \cap \Gamma)$, $\dim(G/G')$ and $\dim R'A/A$ all equal zero and $H^1(X, \mathcal{O}) = \{0\}$ follows by the above theorem. $\square$

Corollary 7. *Let X be a compact complex parallelizable manifold. Then $H^1(X, \mathcal{T}) = 0$ iff $b_1(X) = 0$.*

7. Topological invariance of $\dim H^1(X, \mathcal{O})$

For compact Kähler manifolds $\dim H^1(X, \mathcal{O})$ equals $b_1(X)$ and therefore depends only on the fundamental group of X. Nakamura has shown (thus answering a question of Iitaka) that for arbitrary compact complex manifolds $\dim H^1(X, \mathcal{O})$ may jump within a smooth family. He gave an example of a parallelizable complex manifold X_0 which admits small deformations X_t for which $\dim H^1(X_t, \mathcal{O}) \neq \dim H^1(X, \mathcal{O})$. In this example the X_t are no longer parallelizable for $t \neq 0$. This is not by coincidence. As an application of the description of $H^1(X, \mathcal{O})$ obtained in the preceding section we deduce the following result concerning the topological invariance of $H^1(X, \mathcal{O})$.

Theorem 13. *Let G, H be a complex linear-algebraic groups, Γ and Λ discrete cocompact subgroups of G resp. H and $X = G/\Gamma$ and $Y = H/\Lambda$ the respective quotient manifolds.*

Assume that $\pi_1(X) \simeq \pi_1(Y)$.

Then $\dim H^1(X, \mathcal{O}) = \dim H^1(Y, \mathcal{O})$.

Proof. Let G_0 denote a complex-linear algebraic group, G ist universal cover, Γ_0 a discrete cocompact subgroup of G_0 and Γ its preimage in G under the natural projection $G \to G_0$.

We have to show that $\dim H^1(G/\Gamma, \mathcal{O})$ is completely determined by Γ.

We observe that simply-connected complex Lie groups are linear and discrete subgroups in linear groups admit maximal normal solvable and maximal normal nilpotent subgroups which we denote by $rad(\cdot)$ resp. $n(\cdot)$. The usual density theorems for lattices in complex Lie groups imply that $rad(\Gamma)$ and $n(\Gamma)$ contain $R \cap \Gamma$ resp. $N \cap \Gamma$ as subgroups of finite index (where R resp. N denotes the radical resp. nilradical of G).

Among all subgroups of finite index of $n(\Gamma)$ we choose one for which the commutator group, which we call $n_1(\Gamma)$ has minimal $\mathbb{Z}$-rank (see e.g. [18], Def.2.9. for the notion of $\mathbb{Z}$-rank for finitely generated nilpotent groups). Then $n_1(\Gamma)$ is commensurable to $N' \cap \Gamma$. Similarly we choose $r_1(\gamma)$. Next recall that for a lattice Λ in a semisimple Lie group S the image $\rho(\Lambda)$ is Zariski-dense in $\rho(S)$. This holds for every *real* representation ρ. Besides $[S, N]N' = [G', N]N'$, since $G' \cap R \subset N$. It follows that $\Gamma_1 := [\Gamma', n(\Gamma)]n_1(\Gamma)$ must be a lattice in $[S, N]N'$.

Now $b_1(G/R\Gamma) = b_1(\Gamma/rad(\Gamma))$,

$$\dim G/G' = \dim R/(R \cap G') = \dim R/[S, N]R' = rank_{\mathbb{Z}}(rad(\Gamma)/\Gamma_1 r_1(\Gamma))$$

and $\dim_{\mathbb{C}}(W)$ equals half the dimension over $\mathbb{Q}$ of the $\mathbb{Q}$-vector space W_0 where W_0 is the maximal $\mathbb{Q}$-linear subspace of $(r_1(\Gamma)\Gamma_1/\Gamma_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the natural linear transformation $\tau(\gamma)$ induced by conjugation is diagonalizable over $\mathbb{R}$ with only real eigenvalues for every $\gamma \in \Gamma$.

Thus $\dim H^1(G/\Gamma, \mathcal{O})$ depends only on Γ. □

We want to emphasize that $\dim H^1(X, \mathcal{O})$ depends on $\pi_1(G/\Gamma)$ in a rather subtle way. As we will see (see section 8) the dimension of $H^1(X, \mathcal{O})$ may even jump within a given commensurability class.

8. Small deformations

So far we proved that there exist infinitesimal deformations iff $b_1(X) > 0$. Now we shall discuss small deformations. By actually constructing a deformation family we will show that every compact parallelizable complex manifold X with $b_1(X) > 0$ admits non-trivial small deformations.

Theorem 14. *Let G be a connected complex Lie group, Γ a discrete cocompact subgroup and $X = G/\Gamma$ the quotient manifold. Assume that $b_1(X) > 0$.*

Then there exist small deformations of X, i.e. there is a proper flat holomorphic family $Y \to \Delta_1$ with $Y_0 \simeq X$ and $Y_t \not\simeq X$ for $t \neq 0$.

Proof. We will reduce the general theorem to two special cases, namely $G \simeq SL_2(\mathbb{C})$ and the case where G is commutative.

Claim 5. *Under the above assumptions there exists a connected complex Lie group H which is either commutative or isomorphic to $SL_2(\mathbb{C})$, a cocompact discrete subgroup $\Lambda \subset H$ and a G-equivariant holomorphic surjection $\pi : X \to Z = H/\Lambda$. Furthermore $b_1(Z) > 0$.*

Proof. Let R denote the radical of G. By a theorem of Mostow there is a fibration $\rho : G/\Gamma \to G/R\Gamma$. If $b_1(G/R\Gamma) > 0$, arithmeticity results for lattices in semisimple Lie groups (see [14]) imply that there exists a fibration $\pi_0 : G/R\Gamma \to SL_2(\mathbb{C})/\Lambda$ such that $\pi = \pi_0 \circ \rho$ is the desired surjection. On the other hand, if $b_1(G/R\Gamma) = 0$, then $G'\Gamma$ is closed in G and Γ' is a subgroup of finite index in $G' \cap \Gamma$. Thus in this case $G/\Gamma \to G/G'\Gamma$ yields a surjection onto a positive-dimensional torus. □

Claim 6. *Let G be either $SL_2(\mathbb{C})$ or $(\mathbb{C}^n, +)$, Γ a discrete cocompact subgroup, $X = G/\Gamma$ the quotient manifold and $\tau : \Gamma \to (\mathbb{Z}, +)$ a surjective group homomorphism.*

Then there exists an open neighbourhood U of e in G such that for every $u \in U$ the Γ-action on G given by

$$\gamma : x \mapsto u^{-\tau(\gamma)} \cdot x \cdot \gamma$$

is free and properly discontinuous. Moreover U contains a subset A of measure zero such that the quotient manifold X_u is not biholomorphic to X for any $u \notin A$.

Proof. For $G = (\mathbb{C}^n, +)$ this is easy to check and for $G = SL_2(\mathbb{C})$ it has been proved by Ghys [7]. □

Thus we deduced that we have a fibration $G/\Gamma \to H/\Lambda$ and non-trivial small deformations of H/Λ. We need to show that these deformations can be lifted to non-trivial deformations of G/Γ.

This is achieved by the following statement.

Claim 7. *Let G, H be connected Lie groups, Γ resp. Λ discrete cocompact subgroups in G resp. H, $\pi : G \to H$ a surjective Lie group homomorphism with $\pi(\Gamma) = \Lambda$, $\tau : \Lambda \to \mathbb{Z}$ a surjective group homomorphism and $u \in G$.*

Assume that the Λ-action on H given by

$$\lambda : h \mapsto (\pi(u))^{-\tau(\lambda)} \cdot h \cdot \lambda$$

is free and properly discontinous.

Then the Γ-action on G given by

$$\gamma : u^{-\tau(\pi(\gamma))} \cdot g \cdot \gamma$$

is also free and properly discontinuous.

Proof. In order to show that the action on G is properly discontinuous, we have to verify that for every compact subset $K \subset G$ the set

$$S = \{\gamma \in \Gamma : u^{-\tau(\pi(\gamma))} \cdot K \cdot \gamma \cap K \neq \emptyset\}$$

is finite. Now $\pi(K)$ is compact and hence

$$S_1 = \{\lambda \in \Lambda : (\pi(u))^{-\tau(\lambda)} \cdot \pi(K) \cdot \lambda \cap \pi(K) \neq \emptyset\}$$

must be finite. Thus $M = \{u^{-\tau(\lambda)} : \lambda \in S_1$ is likewise finite. Now

$$S \subset \{\gamma \in \Gamma : M \cdot K \cdot \gamma \cap K \neq \emptyset\}.$$

Both $M \cdot K$ and K are compact and consequently S is finite. The freeness can be checked in a similar way. □

□

9. Examples

1. Let $\Lambda \subset SL_2(\mathbb{C})$ be a cocompact lattice with a surjective group homomorphism $\rho : \Lambda \to \mathbb{Z}$ (see e.g. [12] for existence of such lattices). Let $G = SL_2(\mathbb{C}) \times \mathbb{C}$ and $\Gamma = \{(s,x) \in G : s \in \Gamma, x - \sqrt{2}\rho(s) \in \mathbb{Z}[i]$. Then Γ is a lattice in G such that $G'\Gamma$ is not closed in G. Furthermore $b_1(G/\Gamma) = rank_{\mathbb{Z}}(\Lambda/\Lambda') + 2 > 0$ and $\dim H^1(G/\Gamma, \mathcal{O}) = rank_{\mathbb{Z}}(\Lambda/\Lambda') + 1 > 0$ although every holomorphic map from G/Γ to a torus is constant.

2. Nakamura ([16]) gave an example of a three-dimensional solvmanifold $X = G/\Gamma$ with $\dim G/G' = 1$ and $\dim H^1(G/\Gamma, \mathcal{O}) = 3$. We will give another such example which in addition demonstrates that $\dim H^1(G/\Gamma, \mathcal{O})$ may jump within a commensurability class of Γ.

Let p be a non-square positive natural number, $L = \mathbb{Q}[\sqrt{p}]$ and $K = L[i]$. By Dirichlet's theorem the group $\mathcal{O}_L^*$ of units of L contains an element of infinite order α. We may assume that $N_{L/Q}(\alpha) = 1$. Fix two embeddings $\sigma_1, \sigma_2 : K \to \mathbb{C}$ such that $\sigma_1 \neq \overline{\sigma_2}$, but $\sigma_1(i) = i$ and $\sigma_2(i) = -i$. Now $\sigma = (\sigma_1, \sigma_2) : K \to \mathbb{C}^2$ embedds $\mathcal{O}_K$ in $\mathbb{C}^2$ as a lattice. Let Λ_i $(i = 1, 2)$ be the subgroups of $\mathcal{O}_K^*$ generated by α resp. α and i. Now σ induces an embedding of the groups Λ_i into $GL(2, \mathbb{C})$. Note that $\det \sigma(\alpha) = N_{L/\mathbb{Q}}(\alpha) = 1$ and $\det \sigma(i) = 1$. Hence both $\sigma_i(\Lambda_i)$ are lattices in a Cartan subgroup T of $SL_2(\mathbb{C})$. Let $\Gamma_i = \Lambda \times \mathcal{O}_K$. Then the groups Γ_i are lattices in a three-dimensional solvable complex Lie group $G = \mathbb{C}^* \times (\mathbb{C}^2, +)$. By construction, the Γ_1-action on $\mathbb{C}^2$ is totally real, while $\sigma(i) \in \Gamma_2$ acts on $\mathbb{C}^2$ without real eigenvalues.

Therefore we obtain a two-to-one covering of three-dimensional solvmanifolds

$$G/\Gamma_1 \to G/\Gamma_2$$

with $\dim H^1(G/\Gamma_1, \mathcal{O}) = 3$ and $\dim H^1(G/\Gamma_2, \mathcal{O}) = 1$.

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