FINITE-VOLUME HYPERBOLIC 4-MANIFOLDS THAT SHARE A FUNDAMENTAL POLYHEDRON

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Abstract. It is known that the volume function for hyperbolic manifolds of dimension ≥ 3 is finite-to-one. We show that the number of nonhomeomorphic hyperbolic 4-manifolds with the same volume can be made arbitrarily large. This is done by constructing a sequence of finite-sided finite-volume polyhedra with side-pairings that yield manifolds. In fact, we show that arbitrarily many nonhomeomorphic hyperbolic 4-manifolds may share a fundamental polyhedron. As a by-product of our examples, we also show in a constructive way that the set of volumes of hyperbolic 4-manifolds contains the set of even integral multiples of $4\pi^2/3$. This is "half" the set of possible values for volumes, which is the integral multiples of $4\pi^2/3$ due to the Gauss-Bonnet formula $\mathrm{Vol}(M) = 4\pi^2/3 \cdot \chi(M)$.

0. Introduction and statement of results

The original aim of research that produced this paper was to construct non-compact hyperbolic 4-manifolds by means of side-pairings of polyhedra. Previous examples of hyperbolic manifolds with dimension higher than three were restricted to constructions via arithmetic groups (see, for example, [A2], [M]), or via "interbreeding" arithmetic groups to get non-arithmetic ones ([G-P]) and there was only one (compact) example using side-pairings, that of Davis in [D]. We were able to produce a number of examples of side-pairings of hyperbolic 4-polyhedra and get, in addition, new information about volumes of hyperbolic 4-manifolds. Further research led to consideration of embedability of these manifolds as complements of surfaces in compact 4-manifolds — we deal with this in [I].

It is known (see [W]) that for every constant $c > 0$ there are only finitely many complete non-homeomorphic hyperbolic n-manifolds with volume $< c$, where $n \geq 4$. For $n = 3$ the set of volumes is a well-ordered (infinite) set, but still only finitely many manifolds may have the same volume. We concern ourselves with whether there is a bound on the number of manifolds that have the same volume.

1991 *Mathematics Subject Classification.* 51M10, 51M25.

This paper represents a part of the author's PhD thesis. The author would like to thank his advisor, Dr. Boris Apanasov, for many helpful conversations and suggestions concerning these results. Research at MSRI is supported in part by NSF grant DMS-9022140.

In dimension 3, this has been answered by Wielenberg (see [Wi1]) for the non-compact case, and by Apanasov and Gutsul ([A-G]) for the compact one. In both papers, for N's that can be made arbitrarily large, polyhedra are constructed in $\mathbb{H}^3$ and different side-pairings are given on them whose quotient spaces are N non-homeomorphic hyperbolic manifolds.

In this paper we prove the analogous result for the non-compact case in dimension 4, namely,

Theorem A. *Given any number N, there exist more than N non-homeomorphic, non-compact, complete hyperbolic 4-manifolds of finite volume that share the same fundamental polyhedron in $\mathbb{H}^4$. In particular, they have the same volume.*

The proof is by constructing polyhedra in $\mathbb{H}^4$ with different side-pairings and utilizing Poincare's polyhedron theorem to see that identifying paired sides yields complete hyperbolic 4-manifolds. The manifolds are then distinguished by how many ends they have. It is known (Theorem 5.39 in [A2] or [A1]) that a complete, hyperbolic, geometrically finite n-manifold has finitely many ends. If the manifold has finite volume, then all of the ends are standard cusp ends, that is, they are of the form $E \times [0, \infty)$, where E is a closed flat manifold. Furthermore, each end of the manifold corresponds to a cycle (equivalence class under identification by side-pairing) of ideal vertices of the polyhedron. We count classes of ideal vertices for each of the side-pairings that we construct and show that we arrive at different numbers for different side-pairings. Therefore, the resulting manifolds are non-homeomorphic, because they have different numbers of ends.

We also give a geometric interpretation of the manifolds we construct. It turns out that each of the manifolds may be obtained by taking two basic manifolds, cutting them along a two-sided totally geodesic embedded 3-manifold and stringing several of these together by gluing them along the isometric cuts. In the process of justifying this interpretation we prove a convenient sufficient condition for when a plane intersecting a fundamental polyhedron for a group G is precisely invariant with respect to some subgroup $J \subset G$ (Theorem 4.4).

After the constructions in this work have been completed the author became aware of two other preprints where non-compact hyperbolic 4-manifolds were obtained by side-pairings of polyhedra in ways different from the one here. Those constructions were then used to prove interesting results. One of the preprints is [N], where B. Nimershiem constructs classes of examples that are used to show that the set of all flat three manifolds that appear as cusps of hyperbolic four-manifolds is dense in the set of all flat three manifolds.

The other, by J. Ratcliffe and S. Tschantz ([R-T]), classifies all non-compact hyperbolic 4-manifolds of minimal volume. In addition to that, it is proved that the set of volumes is the positive integral multiples of $4\pi^2/3$. A by-product of our construction is

Theorem B. *The set of all volumes of hyperbolic 4-manifolds contains the even multiples of $4\pi^2/3$.*

This result is only "half as good" as the quoted one, but an advantage is that we provide an explicit side-pairing to produce manifolds with the desired volumes, whereas Ratcliffe and Tschantz's proof was not constructive and gave only their existence.

1. The polyhedron P and its side-pairings Φ_1 and Φ_2

We use the upper half-space model of hyperbolic space to define a convex four-dimensional hyperbolic polyhedron P as an intersection of some hyperbolic half-spaces.

Recall that the Poincaré upper half-space model of hyperbolic n-space is $\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, t) \in \mathbb{R}^n \mid t > 0\}$ with the metric given by $ds^2 = \frac{dx_1^2 + \cdots + dx_{n-1}^2 + dt^2}{t^2}$. The *boundary at infinity* of a set S is the set of all points in $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$ that are in the (Euclidean) closure of S. In the upper-half-space model hyperbolic hyperplanes are either Euclidean half-spheres or Euclidean half-planes orthogonal to $\partial\mathbb{H}^n$ and they are uniquely determined by their own boundaries at infinity, which are Euclidean $(n-2)$-spheres or $(n-2)$-planes in $\mathbb{R}^{n-1} \cup \{\infty\}$. We will say that the hyperplane is *based*, respectively, on a sphere or a plane. (In our case $n = 4$, so the hyperplanes will be based on 2-spheres and 2-planes in $\mathbb{R}^3$.) The angle between hyperplanes is the same as the angle between their boundaries at infinity.

Every hyperplane in $\mathbb{H}^n$ determines two closed half-spaces: each contains the hyperplane and their interiors are disjoint. The $(n-2)$-sphere or -plane on which the hyperplane is based divides $\mathbb{R}^{n-1}$ into two closed sets, each of which is the boundary of one of the half-spaces that the hyperplane determines.

By a *polyhedron* in $\mathbb{H}^n$ we will mean a connected subset of $\mathbb{H}^n$ with non-empty interior whose boundary is a locally finite collection of hyperplanes. (The polyhedra in our construction are going to be intersections of finitely many half-spaces, so they will also be convex.) A *codimension-one side* S of P is a subset of ∂P such that $S = P \cap X$ and $S = \mathrm{cl}_X(\mathrm{int}_X S)$, where X is a hyperplane that bounds one of the defining half-spaces of P. Then S is an $(n-1)$-dimensional convex polyhedron in X. Proceeding inductively we may define a *codimension-i side* of P to be a codimension-one side of a codimension-$(i-1)$ side of P. (For more details on polyhedra, consult [A2] or [E-P].)

Since every codimension-i side is a polyhedron in dimension $n-i$, we also call it an $(n-i)$-side. Codimension-one sides we will simply call *sides*, codimension-two sides we call *edges*, and we will use the term *vertex* for a 0-side of P. Vertices of P are also called *finite vertices* or *real vertices* as opposed to *vertices at infinity* or *ideal vertices* that are the isolated boundary points of P in $\partial\mathbb{H}^n$. To simplify notation, a hyperbolic hyperplane, the side of P lying on the hyperplane and the boundary at infinity of the hyperplane will be denoted by the same letter. (No confusion should arise here because our P's are convex.)

Consider the planes that bound the rectangular box $R \subset \mathbb{R}^3$, $R = [-2,2] \times [-2,2] \times [-2\sqrt{2}, 2\sqrt{2}]$. Add to them the 12 spheres of radius $\sqrt{2}$ with centers $(\pm1, \pm1, j2\sqrt{2})$ for $j = -1, 0, 1$ and the 18 spheres of the same radius with cen-

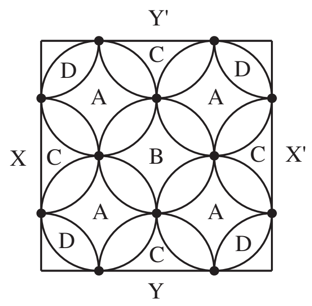

FIGURE 1. Section of P for $t = \sqrt{2}/2$, $z = \sqrt{2}/2$ showing the real vertices

ters $(j, k, \pm\sqrt{2})$ for $j, k = -2, 0, 2$. The upper part of figure 2, going from left to right, depicts intersections of these spheres with planes with constant z-coordinates $-2\sqrt{2}, -\sqrt{2}, 0, \sqrt{2}, 2\sqrt{2}$. Label the spheres by the letters $A_i, A_i', B_i, B_i', C_i, C_i', D_i, D_i'$ in either of the ways suggested by figure 2. Let $X_1, X_1', Y_1, Y_1', Z_1, Z_1'$ be respectively the planes $\{x = -2\}$, $\{x = 2\}$, $\{y = -2\}$, $\{y = 2\}$, $\{z = -2\sqrt{2}\}$, $\{z = 2\sqrt{2}\}$.

Each of the planes that comprise the boundary of R and each of the above spheres determine a hyperplane in $\mathbb{H}^4 = \{(x, y, z, t) \in \mathbb{R}^4 \mid t > 0\}$ that divides $\mathbb{H}^4$ into two half-spaces. For the spheres we choose the half-spaces whose boundary at infinity is unbounded in $\mathbb{R}^3$, for the planes the half-spaces so that the intersection of their boundaries at infinity is the rectangular box R. The polyhedron P is defined as the intersection of those half-spaces. For later convenience, we set $P_- = \{(x, y, z, t) \in P \mid z \leq 0\}$, $P_+ = \{(x, y, z, t) \in P \mid z \geq 0\}$.

The following observations about the spheres and planes that we just defined are easy to check.

(1) Any two spheres that intersect do so at an angle of $\pi/2$.
(2) Whenever the intersection is non-empty, spheres A_i, A_i' intersect planes X_1, X_1', Y_1, Y_1' at angle $\pi/4$. Any other pair of spheres or planes with non-empty intersection intersects at angle $\pi/2$.
(3) R is completely covered by the closed balls bounded by the spheres. This means that P has finite volume and has only finitely many points in its boundary at infinity.
(4) P has 36 vertices at infinity, which correspond to points not covered by the open balls. Their position is illustrated in Figure 9.

It is not obvious right away that P also has finite vertices (that is, 0-sides that are in $\mathbb{H}^4$). This is because many sets of four hyperplanes bounding the polyhedron P meet at one point. For example, sides A_1, B_1, C_1 and A_2' meet at the point $(0, 1, -3\sqrt{2}/2, \sqrt{2}/2) \in \mathbb{H}^4$, and sides A_1, C_1, D_1 and Y_1' meet at the point $(-1, 2, -3\sqrt{2}/2, \sqrt{2}/2)$. Figure 1 depicts the section of P where $t = \sqrt{2}/2$ and

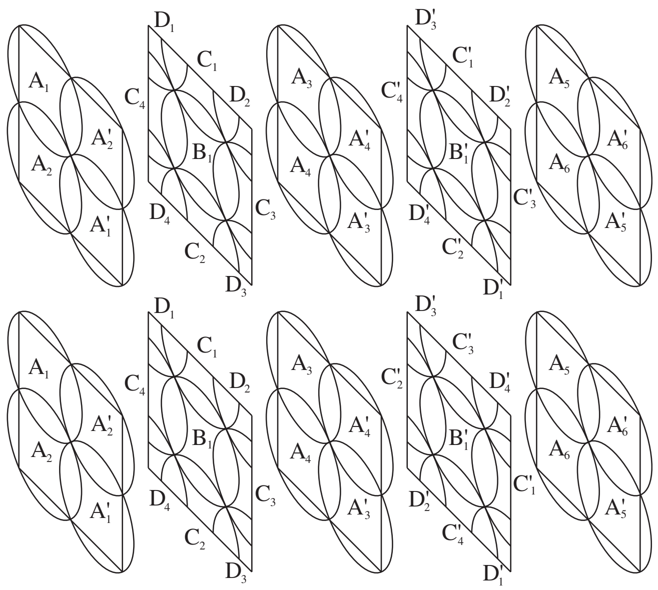

FIGURE 2. Side-pairings Φ_1 (top) and Φ_2 (bottom)

$z = j\sqrt{2}/2$. Here j is any of $-3, -1, 1, 3$ as the section for every j is the same. We can see where four sides of P intersect in a vertex and what letters those sides are labeled by. Figure 1 shows the location of all the vertices in one section — there being $4 \cdot 12 = 48$ in all.

Now we are ready to define two ways to pair sides of P. A *side-pairing* Φ of P is a rule which associates to each side S of P a side S' of P and an isometry s which sends S to S'. This rule is subject to the conditions that $s(\operatorname{int} P) \cap \operatorname{int} P = \emptyset$, that the side associated to S' must be S, that the isometry that takes S' to S must be s^{-1}. For more details, see [E-P] or [R].

First, define the following isometries of $\mathbb{R}^3$:

$$
\begin{aligned}
q_0 &= \text{reflection in plane } \{z = 0\} \\
q_1 &= \text{reflection in plane } \{x - y = 0\} \\
q_2 &= \text{reflection in plane } \{x + y = 0\} \\
s_1 &= \text{rotation by } \pi \text{ about line } \{x + y = 0, z = 0\} \\
s_2 &= \text{rotation by } \pi \text{ about line } \{x - y = 0, z = 0\} \\
t_0 &= \text{translation by } 2\sqrt{2} \text{ in the } z \text{ direction.}
\end{aligned}
$$

Use the same letters to denote the extensions of these maps to $\mathbb{H}^4$. (A Euclidean isometry $f : \mathbb{R}^3 \to \mathbb{R}^3$ extends to a hyperbolic isometry given by $(x, y, z, t) \mapsto (f(x, y, z), t)$.)

Let i_S denote the reflection in the hyperplane S. By s we denote the hyperbolic isometry that pairs the sides S and S' (it sends S to S'). We define Φ_1 to be the side-pairing given in the upper-half-space model by

$$\begin{aligned}
x_1 &= \text{translation by 4 in the } x \text{ direction}\\
y_1 &= \text{translation by 4 in the } y \text{ direction}\\
z_1 &= t_0^2 = \text{translation by } 4\sqrt{2} \text{ in the } z \text{ direction}\\
b_1 &= q_1 \circ t_0 \circ i_{B_1}\\
a_j &= q_l \circ i_{A_j}, \text{ so that } l \equiv j \ (mod\ 2)\\
c_k &= q_0 \circ i_{C_k}\\
d_k &= q_1 \circ t_0 \circ i_{D_k},
\end{aligned}$$

where $j = 1, \ldots, 6$, $k = 1, \ldots, 4$ and $l = 1, 2$. The upper half of Figure 2 shows which sides are paired.

To get another side-pairing, Φ_2, we alter Φ_1 in the way the sides labeled by B's, C's and D's are paired. Refer to the lower half of Figure 2 to see which sides are paired. We define the new pairings b_1, c_k and d_k by

$$\begin{aligned}
b_1 &= q_0 \circ i_{B_1}\\
c_k &= q_1 \circ t_0 \circ i_{C_k}\\
d_k &= s_l \circ q_l \circ i_{D_k}, \text{ so that } l \equiv k \ (mod\ 2),
\end{aligned}$$

where $k = 1, \ldots, 4$ and $l = 1, 2$.

2. Two closed finite-volume hyperbolic 4-manifolds

In this section we prove

Theorem 2.1.

(i) *The side-pairings Φ_1 and Φ_2 generate discrete torsion-free subgroups G_1 and G_2 of* $\operatorname{Isom}\mathbb{H}^4$ *whose fundamental polyhedron is P. Therefore, the quotient of $\mathbb{H}^4$ by the action of either of the groups is a complete hyperbolic 4-manifold.*

(ii) *$\mathbb{H}^4/G_1$ has seven ends while $\mathbb{H}^4/G_2$ has eight. In particular, the two manifolds are not homeomorphic.*

Proof. To prove assertion (i), we use Poincaré's polyhedron theorem. For details, the reader can consult [E-P] and [R] which were our main references, while we will shortly state the version that we are going to use. Other versions of the polyhedron theorem may be found in [A2] and [Ma].

First of all, the maps defined above really do map a side of P isometrically onto a side of P. To verify this for Φ_1, notice that each of its side-pairings

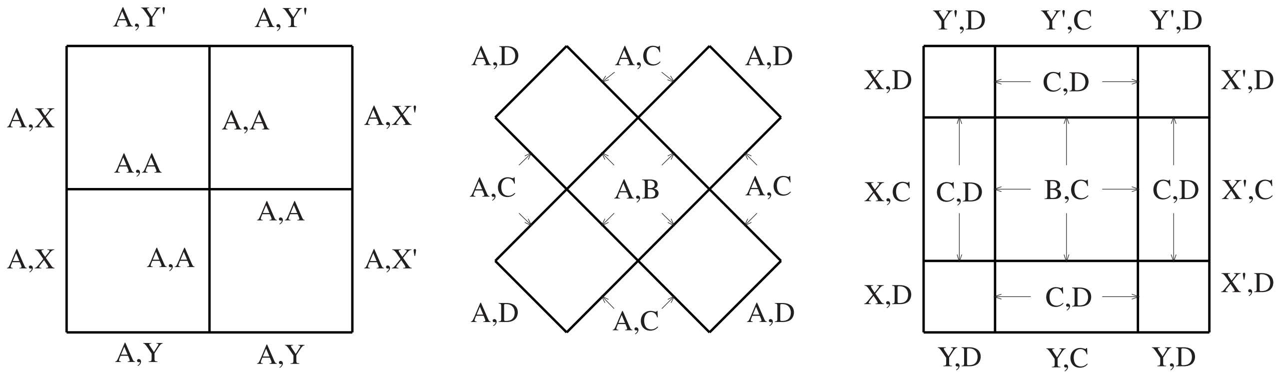

FIGURE 3. Representing edges in P by diagrams

$a_1, \ldots, a_6, c_1, \ldots, c_4$, is of the form $f \circ i_S$ where S is a side and f, *which preserves* P, is an extension to $\mathbb{H}^4$ of a Euclidean transformation on $\mathbb{R}^3$. The i_S keeps S fixed so $f \circ i_S(S) = f(S)$, and f, being an isometry of P, sends its sides to some other sides. We proceed similarly for the the side-pairings $b_1, d_1, \ldots, d_4$: each of them is of the form $f \circ i_S$, but f is now an isometry that takes P_- to P_+, so it sends sides of P_- to sides of P_+. But the sides $B_1, D_1, \ldots, D_4$ are sides of both P_- and P, and, likewise the sides $B'_1, D'_1, \ldots, D'_4$ are sides of both P_+ and P. For the side-pairing x_1 (y_1 and z_1 are done similarly), compose it with a reflection in the side X'_1 to get an isometry of P whose image of X_1 are the same as by x_1. Therefore x_1 carries X_1 to another side of P, namely X'_1. The claim is proved in the same way for the side-pairing Φ_2.

Remark 2.2. Recall that a horosphere in $\mathbb{H}^n$ is either a Euclidean sphere tangent to $\partial\mathbb{H}^n$ or a Euclidean hyperplane $\{t = c\}$ parallel to $\partial\mathbb{H}^n$. The former are said to be *centered* at the point of tangency with $\partial\mathbb{H}^n$, the latter at ∞. Consider a set T of disjoint horospheres, each centered at a vertex at infinity of P. For the vertex ∞ choose, say, the horizontal plane $\{t = 3\}$. For the other vertices choose horospheres of the same radius that is small enough so that the horospheres intersect only those sides of P which contain the center of the horosphere. If the center of a horosphere is on the boundary of a hyperplane S, then i_S preserves the horosphere. This combined with an argument like in the preceding paragraph can be used to show that the side pairings Φ_1 and Φ_2 satisfy the

Consistent horosphere condition: there exists a set T of disjoint horospheres centered at ideal vertices of P so that if g is a side-pairing of a side that contains the center of a horosphere $H \in T$ in its boundary, then $g(H)$ is again a horosphere from T.

Another condition for Poincaré's polyhedron theorem is the "edge cycle condition", called Cyclic in [E-P]. In general, a side-pairing on P induces an equivalence relation on P that is generated by the relation $x \sim s(x)$, where $x \in \partial P \cap S$, S is a side of P and s its side-pairing. The equivalence class $[x]$ of x under this equivalence relation, is called the *cycle* of x. The cycle of an i-side is defined analogously, so that it contains all the i-sides of P that are identified by a string of side-pairings.

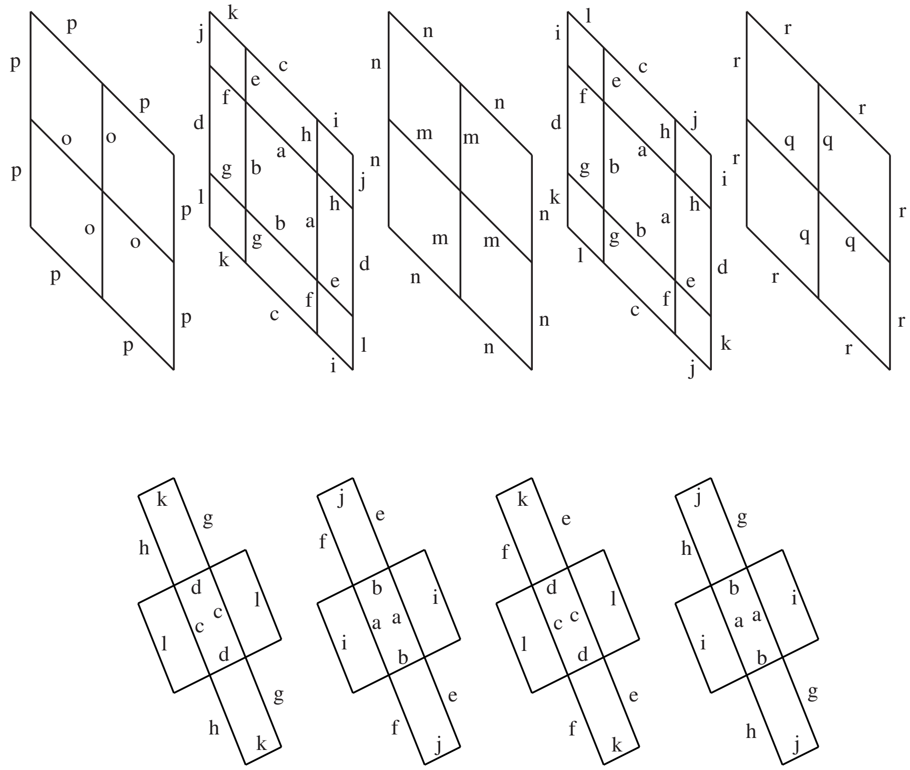

FIGURE 4. Edge chase for Φ_1

Every edge (codimension-2 side) of P is the intersection of two uniquely determined sides of P. The *dihedral angle* at an edge is the angle in the interior of P that the two sides subtend. A cycle of edges can be obtained in the following way. Start with an edge E_1, which is the intersection of sides S_1 and R_1, and let g_1 be the isometry pairing R_1 and some side S_2 of P. We get that $g_1(E_1) = E_2$, where E_2 is an edge determined by S_2 and some other side R_2. Now let g_2 be the isometry pairing R_2 and some side S_3. Continuing in the same way we get a sequence of edges, sides and isometries $\{\sigma_i = (E_i, S_i, R_i, g_i)\}_{i=1,2\ldots}$. This procedure is commonly called "edge-chasing". We require that the above sequence have a period q (called *first cycle length in* [E-P]), that is $\sigma_{q+1} = \sigma_1$ for some q. The cycle of edges will then consist of exactly $E_1, \ldots, E_q$. Due to finite-sidedness of our P, this condition will automatically be satisfied.

It is clear that $g_q \circ \cdots \circ g_1(E_1) = E_1$, but it may happen that the restriction of $g_q \circ \cdots \circ g_1$ on E_1 is not the identity. The second part of the edge cycle condition is that there must be a k so that $(g_q \circ \cdots \circ g_1)^k |_{E_1} = id$. The number kq is called the *second cycle length* in [E-P].

Finally, to fulfill the edge cycle condition we must show that if θ_i, $i = 1, \ldots, q$ is the dihedral angle of edge E_i, then there is a non-zero integer m so that $k(\theta_1 + \cdots + \theta_q) = 2\pi/m$. We may now formulate Poincaré's polyhedron theorem for the case of a hyperbolic polyhedron as follows.

Theorem 2.3. *(Poincaré's polyhedron theorem) Let Φ be a side-pairing on a polyhedron $P \subset \mathbb{H}^n$ that satisfies both the edge cycle condition and the consistent horosphere condition. Then the side-parings of Φ generate a discrete group $G \subset \operatorname{Isom} \mathbb{H}^n$ whose fundamental polyhedron is P.* □

Now we check the edge cycle condition for edges of P and the two side-pairings Φ_1 and Φ_2. With notation as above, we will always have $k = 1$ and $m = 1$. Hence, the second cycle length will always be the same as the first cycle length and they will be 4 and 8 respectively for edges with dihedral angles $\pi/2$ and $\pi/4$. Therefore, the sum of dihedral angles will be exactly 2π.

Firstly, we make sure that all edges are in cycles of said length. For all edges that are intersections of sides based on planes (i.e. "vertical" sides), this check is easy and boils down to checking the conditions of Poincaré's polyhedron theorem for a rectangular Euclidean parallelepiped with parallel sides paired by Euclidean translations. The check for any of the edges of type $A_i \cap Z_1$ is also straightforward. For all the other edges, we use the diagrams in Figures 4 and 5 to simplify and visualize the task of verifying.

Notice that the intersection of two hyperbolic hyperplanes is a codimension-2 hyperbolic subspace whose boundary at infinity is the intersection of the boundaries at infinity of the hyperplanes. Therefore, every edge E of P lies on a codimension-2 subspace determined either by the intersection of a sphere and a plane or by the intersection of two spheres in $\mathbb{R}^3$. (The spheres and planes are the boundaries at infinity of the sides that determine E.) These intersections are circles and they are represented by segments in Figure 3. The letters next to each segment indicate which side-types have generated the edge represented by it. To get the left and right diagrams we take intersections of planes and spheres labeled by the letters in the diagrams and then project them to the plane $\{z = 0\}$. For the middle diagram, we first project to the plane $\{z = 0\}$ the centers of those pairs of spheres whose labels are listed in it. Then we take the perpendicular bisector of the line joining those centers. To account for all the edges, we need several of these diagrams (Figure 4). Edge chasing for Φ_1 and Φ_2 is now performed on Figures 4 and 5 respectively. All edges in one cycle in each horizontal component of the pictures are labeled with the same letter. Some are not labeled because their cycles are similar to other labeled cycles. Also, Figure 5 omits some of the edges because their cycles are the same as for Φ_1.

For example, choose the edge $A_1 \cap A_2$. The edge chase, yielding the cycle labeled o in upper part of Figure 4 is

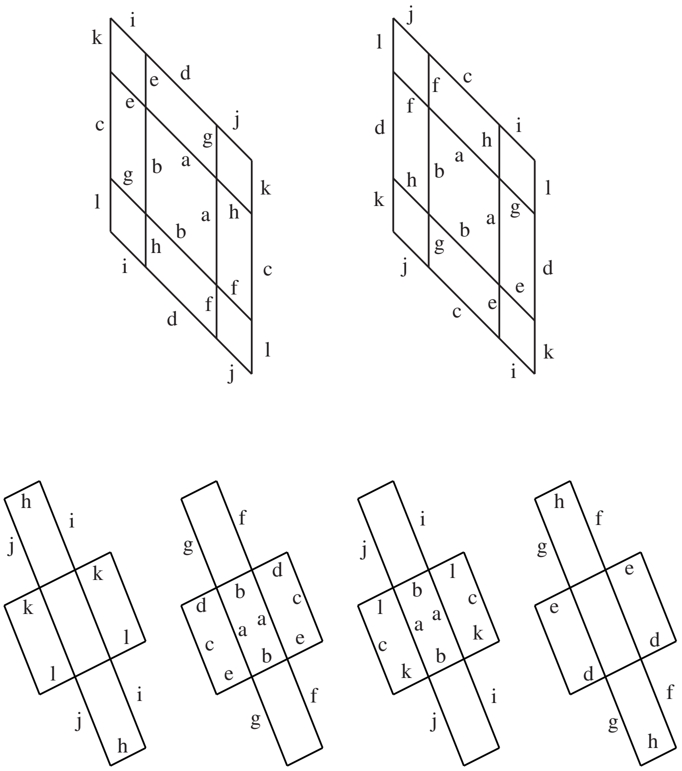

FIGURE 5. Edge chase for Φ_2

$$A_1 \cap A_2 \xrightarrow{a_2} A_2' \cap A_1 \xrightarrow{a_1} A_1' \cap A_2' \xrightarrow{a_2^{-1}} A_2 \cap A_1' \xrightarrow{a_1^{-1}} A_1 \cap A_2$$

As another example, choose the edge $A_1 \cap D_1$. The edge chase, yielding the cycle labeled k in lower part of Figure 4 is

$$A_1 \cap D_1 \xrightarrow{d_1} D_1' \cap A_3' \xrightarrow{a_3^{-1}} A_3 \cap D_3' \xrightarrow{d_3^{-1}} D_3 \cap A_1' \xrightarrow{a_1^{-1}} A_1 \cap D_1.$$

Next, we check that for transformations $g_1, \ldots, g_q$ obtained by edge-chasing we have $g_q \circ \cdots \circ g_1|_{E_1}=1$. As before, $g_i = f_i \circ r_i$, where f_i is the extension to $\mathbb{H}^4$ of a Euclidean transformation on $\mathbb{R}^3$ and r_i is either reflection in a hyperplane containing E_i or the identity. Let $f = f_q \circ \cdots \circ f_1$. It is not difficult to see that f is always orientation preserving. Clearly $g_q \circ \cdots \circ g_1|_{E_1} = f|_{E_1}$, and it will be enough to show that $f = 1$. We will need the following easy lemma.

Lemma 2.4. *Let f be a nontrivial orientation-preserving Euclidean isometry of $\mathbb{R}^3$ that preserves a circle. Then it is a rotation about a line called the* axis *of f. Moreover, we have:*

(i) *If we write f as $Ux + u$, where U is an orthogonal transformation and $u \in \mathbb{R}^3$ then the axis of f is parallel to the axis of U.*

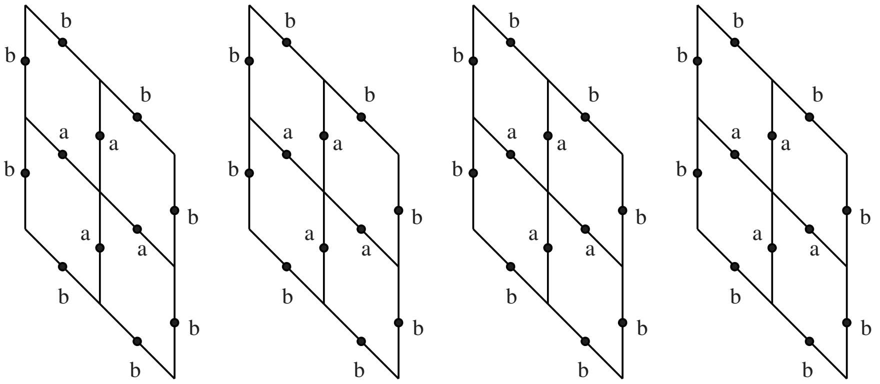

FIGURE 6. Cycles of real vertices

(ii) *The axis of f passes through the center of the circle and is either in the plane of the circle, or perpendicular to it. In the first case, the rotation is by angle π.*
(iii) *If f preserves a line l, then its axis either orthogonally intersects l or it is that line.* $\square$

Let $f = f_q \circ \cdots \circ f_1$ as above and suppose it is nontrivial. In what follows we interchangeably view f as a Euclidean isometry on $\mathbb{R}^3$ or as a hyperbolic isometry of $\mathbb{H}^4$.

We know that f preserves the circle that is the base of the edge $E = E_1$. Also, f preserves the family of planes $V = \{(x, y, z) \in \mathbb{R}^3 \mid x = 4k+2 \text{ or } y = 4k+2,\ k \in \mathbb{Z}\}$ because it is a composite of maps from $\{x_1, y_1, z_1, q_0, q_1, q_2, s_1, s_2\}$. The rotational part U of f is a composite of maps from $\{q_0, q_1, q_2, s_1, s_2\}$, each of which preserves the axes l_1 and l_2 of s_1 and s_2, so U preserves them too. Now, looking at possible positions of the circle we get one of the following cases:

Case 1. When E is one of the sides represented in the lower half of Figure 4 then, by (i) and (ii) of Lemma 2.4, the axis l of U must either have direction vector $(\pm1, \pm1, \sqrt{2})$ or is in the plane perpendicular to that vector. Since U preserves l_1 and l_2, by part (iii) of the lemma, both of l_1 and l_2 must be either perpendicular or identical to l. If $l \perp l_1$ and $l \perp l_2$, then l is the z-axis, which contradicts the possible positions of l. If l is equal to either l_1 or l_2, then one can see that the axis of f is going pass exactly through the segment that represents E in the middle diagram of Figure 3. However, it is clear that no rotation about these segments can preserve the family of planes V so we must have $f = 1$.

Case 2. When E is one of the sides represented in the upper half of Figure 4 the axis l of U lies, by parts (i) and (ii) of Lemma 2.4, in one of the the planes $\{x = 0\}$ or $\{y = 0\}$ or it is the x- or y-axis. Clearly $l \neq l_1$ and $l \neq l_2$, so applying part (iii) of Lemma 2.4 again we get $l \perp l_1$ and $l \perp l_2$, which means l is the z-axis. As long as E is not of the forms $C_i \cap X_1$, $C_i \cap Y_1$, $D_i \cap X_1$ or $D_i \cap Y_1$ the center of the circle on which E is based has odd x and y coordinates. However, a rotation about an axis parallel to z through such points cannot preserve the family of planes V and we again get $f = 1$.

Case 3. In the remaining cases, if E is of form $D_i \cap X_1$ or $D_i \cap Y_1$, regard f as an isometry of $\mathbb{H}^4$ and examine its action on vertices of P that are on E. Looking at Figure 1, it is clear that f, being by the above a rotation about an axis parallel to z, must send vertices that are on E to points whose either x or y coordinate falls out of $[-2, 2]$, a contradiction with the fact that f preserves E. Finally, if E is of form $C_i \cap X_1$ or $C_i \cap Y_1$ we just compute f: it is always $q_0^2 = 1$ for the side-pairing Φ_1 and it is always $q_1^2 = 1$ for the side-pairing Φ_2.

Thus, we have shown that $f = 1$ in all possible cases and the edge cycle condition has been verified.

Since the consistent horosphere condition is fulfilled by Remark 2.2, we may apply theorem 2.3 to get that the groups G_1 and G_2 generated by the side-pairings Φ_1 and Φ_2 are discrete, and that P is the fundamental polyhedron for both of them.

What we do not yet know is whether G_1 and G_2 are torsion-free, that is, whether $\mathbb{H}^4/G_i$, $i = 1, 2$ are hyperbolic manifolds and not just orbifolds.

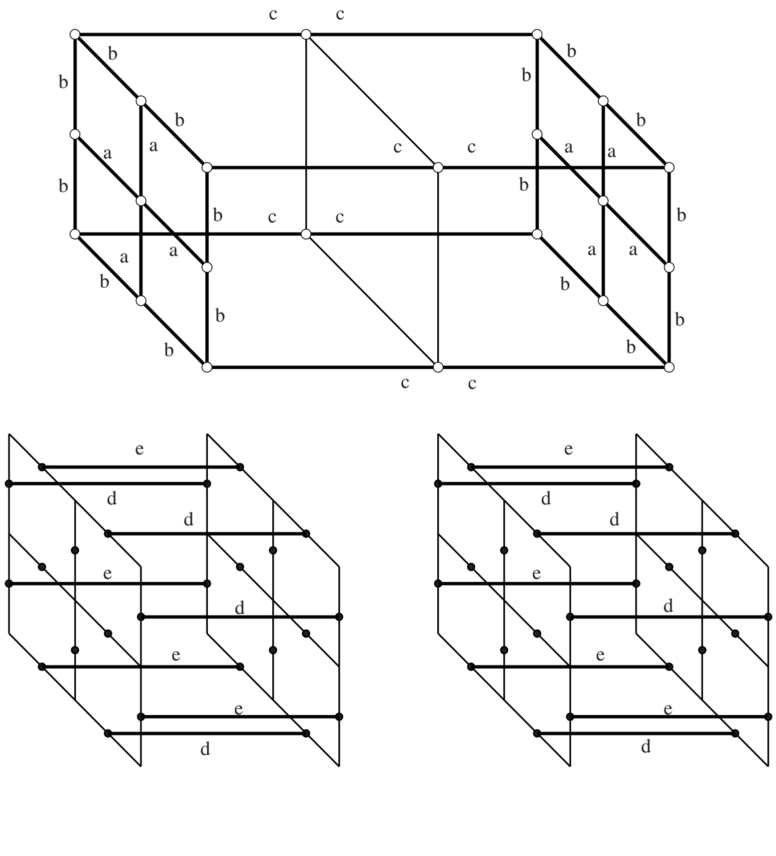

FIGURE 7. Cycles of some 1-sides for Φ_1

Recall (see [R]) that the *normalized solid angle at point* x of a polyhedron P is defined as $\omega(x) = (\operatorname{Vol} B(x, r) \cap P)/\operatorname{Vol} B(x, r)$. Here $B(x, r)$ is a hyperbolic ball about x or radius r, Vol is hyperbolic volume and r is taken small enough so that $B(x,r)$ intersects only those sides of P on which x lies. Let $[x] = \{x_1, \dots, x_n\}$ be the cycle of x for some side-pairing of P. We define the *normalized solid angle sum of* $[x]$ as $\omega[x] = \sum_{y\in[x]} \omega(y)$. We are going to use Theorem 11.1.1 from [R] which says that

Theorem 2.5. *If $\omega[x] = 1$ for every $x \in P$ then the group G generated by the side-pairings of P is torsion-free.* $\square$

Knowing that P is a fundamental polyhedron for a discrete group G implies $\omega[x] \le 1$ for every point of P. Really, for every $x_i \in [x]$, choose an isometry $g_i \in G$ taking x_i to x. (In general, there may be many ways to make the choices.) We now have a injective map from $\{x_1, \dots, x_n\}$ to $\{$translates of P under G containing $x\}$ given by $x_i \mapsto g_i(P)$. Since $\{g_1(P) \cap B(x,r), \dots, g_n(P) \cap B(x,r)\}$ fill out maybe only a portion of $B(x,r)$, we get $\omega[x] \le 1$. (Note that the strict inequality will occur if and only if x is the fixed point of an element in G.)

Therefore, it is enough to see that $\omega[x] \ge 1$. For an $x \in P$ that is in the interior of 3 or 4-sides of P, it is clear that $\omega[x] = 1$. For an x in the interior of 2-sides, this is the edge-cycle condition. This leaves 0- and 1-sides to be checked.

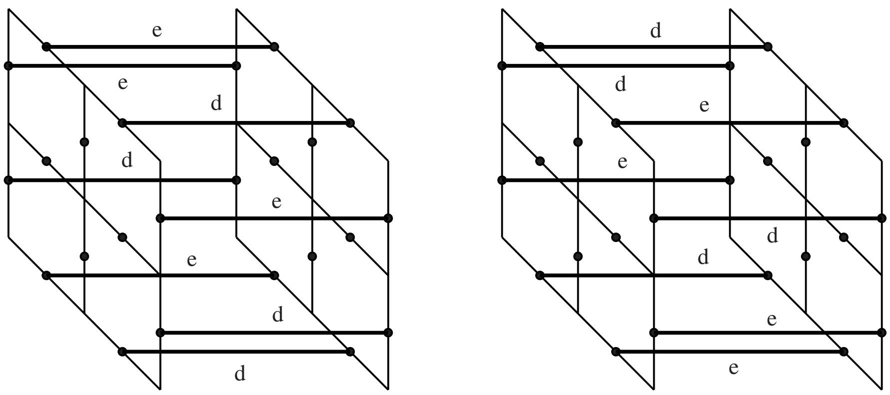

FIGURE 8. Cycles of some 1-sides for Φ_2

Figure 6 shows the sections of P for $t = \sqrt{2}/2$ and $z = -3\sqrt{2}/2, -\sqrt{2}/2, \sqrt{2}/2, 3\sqrt{2}/2$. These sections contain all the vertices (0-sides) of P. All vertices in the same cycle (there are only two cycles) are labeled by the same letter — the cycles are the same for both Φ_1 and Φ_2. Vertices in the cycle labeled a occur as the intersection of four hyperplanes, each pair of which meets at angle $\pi/2$. Normalize P so that a vertex x from cycle a is the origin in the ball model $\mathbb{B}^4$ of hyperbolic space. We see that the normalized solid angle at x is the same as the normalized solid angle at $0 \in \mathbb{B}^4$ subtended by the four coordinate planes, and that is $1/16$. Vertices in the cycle labeled b are always intersections of four hyperplanes where one pair of them intersects at angle $\pi/4$ and all other pairs intersect at angle $\pi/2$. Normalizing as before, we see that the normalized solid angle at x is the same as the one at 0 subtended by the hyperplanes $\{x_2 = 0\}$, $\{x_3 = 0\}$, $\{x_4 = 0\}$, $\{x_1 - x_2 = 0\}$, and that is $1/32$. Since cycles a and b contain 16 and 32 points respectively, we are done.

Now consider 1-sides. There are three cases depending on whether a 1-side F connects an ideal and a real vertex, two ideal vertices, or two real vertices.

For the first case, let F be a 1-side of P that is a geodesic half-line between one real and one ideal vertex of P. If for some $x \in \operatorname{int} F$ we have $\omega[x] < 1$, then x is a fixed point of some nontrivial $g \in G$. The isometry g must preserve F — otherwise, we'd have $g(F) \cap F = \{x\}$ and this contradicts the fact that translates of P meet only along i-sides. However, this implies that the real vertex on F is fixed under g, a possibility we just proved cannot happen. So, we are left with checking 1-sides that have as endpoints either both real or both ideal vertices of P.

Every 1-side is the intersection of three different sides of P. Hence, to find all 1-sides with both endpoints real or ideal, we have to find pairs of real or ideal vertices lying on the same three sides. Figure 7 schematically depicts those 1-sides of P. A boldface line segment joining the real or ideal vertices indicates the existence of a 1-side joining them. The three sides on which the 1-sides lie are easily deduced from their position in the picture. (For example, the 1-sides labeled c are intersections of sides labeled by D's, X's and Y's.) As before, the letters on the 1-sides indicate to which cycle of 1-sides they belong.

It takes a bit of checking to see that we have found all the needed 1-sides. For example, to see that no 1-side joins an ideal vertex in the plane $\{z = 0\}$ to an ideal vertex in the plane $\{z = \sqrt{2}\}$ we note that every vertex in the plane $\{z = 0\}$ lies on only one of the B, C or D-sides, and some X, Y, Z or A-sides, while every vertex in the plane $\{z = \sqrt{2}\}$ lies on only one of the A-sides and some number of B, C or D-sides. Therefore, any pair of vertices from those two hyperplanes cannot belong to the same three sides.

The 1-sides in the cycle labeled a are intersections of three sides meeting pairwise at angles $\pi/2$ (two sides labeled by A and one by Z) . Taking an x from a side in the cycle, and normalizing in $\mathbb{B}^4$ so that $x = 0$ and the three sides are the first three coordinate hyperplanes, we see that $\omega(x) = 1/8$. Since there are 8 1-sides in the cycle, we get $\omega[x] \geq 1$.

Other cycles are checked in the same way. For an x on a 1-side in the cycles b, c, d, e we get normalized solid angles of respectively $1/16, 1/8, 1/8, 1/8$ with $16, 8, 8, 8$ 1-sides in the cycle, so $\omega[x] \geq 1$. Thus, we have proved that side-pairings Φ_1 and Φ_2 give rise to hyperbolic 4-manifolds.

Assertion (ii) of Theorem 2.1 is verified by counting cycles of ideal vertices for Φ_1 and Φ_2. The cycles are shown in Figure 9. $\square$

3. Proof of Theorem A

The crucial observation that the proof uses is that the side-pairings Φ_1 and Φ_2 yield manifolds with respectively 7 and 8 ends. This is immediate from the fact

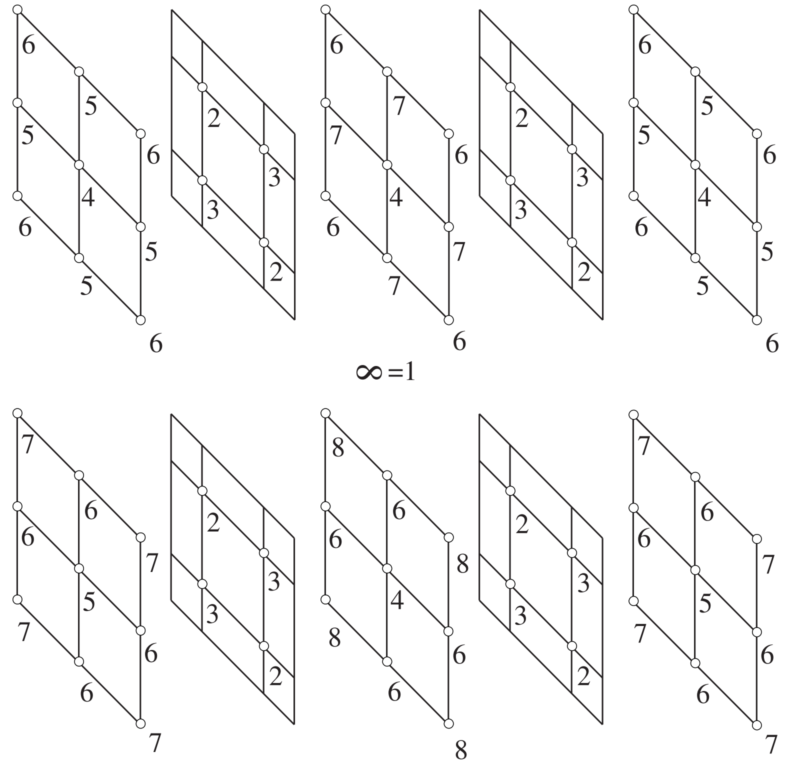

FIGURE 9. Cycles of ideal vertices for Φ_1 (top) and Φ_2 (bottom). In both cases, the vertex ∞ is labeled by 1

that there are 7 and 8 classes of vertices at infinity for the side-pairings Φ_1 and Φ_2, as shown in Figure 9.

Choose a positive integer n, and let $Q_1, \dots, Q_n$ be n copies of the polyhedron P. Call each of them a *block*. To each block, assign either of the side-pairings Φ_1 or Φ_2 and call it a block *of type* Φ_1 or Φ_2. Now form a new polyhedron Q by attaching side Z_1 of Q_i to side Z_1' of Q_{i-1}, $i = 2, \dots, n$, i.e. by stringing the blocks together in a linear fashion in the direction of the z-axis. Q has on it a side-pairing induced by the side-pairings on each block. Clearly, the sides that were attached have vanished, so they don't fall under this rule: the remaining sides Z_1 of Q_1 and Z_1' of Q_n are paired by the translation t^{2n}. A moment's reflection convinces us that the side-pairing on Q generates a torsion-free group G — the proof is basically an n-fold repetition of the proof of the same result for the side-pairings Φ_1 and Φ_2. Except for i-sides, $i = 0, 1, 2$, that are contained in the sides Z_1' of Q_1 and Z_1 of Q_n, all the cycles are just inherited cycles from pairings on each block. The special cases are easily dealt with — they follow patterns established for Φ_1 and Φ_2.

How many cycles of vertices at infinity does Q have? Let there be k blocks of type Φ_1 and $n - k$ blocks of type Φ_2 among $Q_1, \dots, Q_n$. Starting with $Q_1, \dots, Q_l$, adding a block of type Φ_1 onto Q_l will add three new cycles of vertices at infinity:

there are seven cycles on Φ_1 but four fall into cycles already existing on $Q_1, \dots, Q_l$. Similarly, adding a block of type Φ_2 adds four new cycles of vertices at infinity. It is now easy to see that Q will have $4+3k+4(n-k) = 4+4n-k$ cycles of vertices at infinity. We have complete freedom of choice for k, so by varying k from 0 to n we can get manifolds with anywhere from $4+3n$ to $4+4n$ ends. Therefore, we have obtained at least $n+1$ nonhomeomorphic manifolds with the same fundamental polyhedron Q, which is what we set out to prove. (We likely get many more, since we completely ignored the various orderings of blocks of the two types that are possible when constructing Q.) $\square$

4. A geometric interpretation of the construction

In this section we analyze the construction of the manifolds in the previous section from a gluing-of-manifolds perspective.

Let $M = \mathbb{H}^n/G$, where G is a discrete torsion-free subgroup of $\operatorname{Isom}\mathbb{H}^n$. A *totally geodesic hypersurface* is a subset $N \subset M$ so that for every $x, y \in N$ every geodesic connecting x and y is also contained in N. We are interested in embedded totally geodesic hypersurfaces which are the ones for which $p^{-1}(N)$ is a disjoint union of hyperplanes in $\mathbb{H}^n$, where $p : \mathbb{H}^n \to M$ is the standard projection. Let H be one of those hyperplanes, and $J \subset G$ its stabilizer in G, that is the subgroup $J = G_H = \{j \in G \mid j(H) = H\}$. Then H is *precisely invariant under* J, i.e. $g(H) = H$ when $g \in J$ and $g(H) \cap H = \emptyset$ when $g \in G \setminus J$. In particular, we want to look at some hypersurfaces with $\operatorname{Vol} N < \infty$, so they will correspond to subgroups $G_H \subset G$ that act on a hyperplane H as a hyperbolic lattice.

Conversely, we may start with a subgroup $J \subset G$ and a hyperplane H precisely invariant under J. Then H/J is an embedded totally geodesic hypersurface in M.

Let $M_i = \mathbb{H}^4/G_i$, where G_i is generated by the side-pairing Φ_i, $i = 1, 2$, and let H be the supporting hyperplane of side Z_1 of the polyhedron P. Suppose we know that H is precisely invariant under the subgroup $J \subset G_1$, $J = < a_1, a_2, x_1, x_2 >$. Then H/J is a two-sided totally geodesic hypersurface in M_1 and we may cut M_1 along this hypersurface to get a connected manifold M_1' that has two boundary components which are isometric 3-dimensional hyperbolic manifolds given by H/J. We may do the same with M_2 to get M_2' which is also connected (the subgroup J in question doesn't change).

Now let $M = \mathbb{H}^4/G$, where G is a subgroup generated by any of the side-pairings of the polyhedron Q defined in the previous section. We identify the block Q_1 with P and its side-pairing with Φ_1 or Φ_2. Let H_i be the hyperplane supporting the side Z_1 of the block Q_i, $i = 1, \dots, n$. So, $H_i = z_1^{i-1}(H)$ and it is kept invariant by the subgroup $J_i \subset G$, $J_i = z_1^{i-1} J z_1^{-(i-1)}$ (see Figure 10). Suppose we know that H_i is precisely invariant under the subgroup $J_i \subset G$. Then we may cut M along the hypersurfaces H_i/J_i. The hyperplanes H_i are exactly the ones that separate the polyhedron Q into blocks $Q_1, \dots, Q_n$ (see Figure 10). Under identification of paired sides, we see that each block yields a submanifold of type M_1' or M_2'. Therefore, after cutting along H_i/J_i, $i = 1, \dots, n$, we will get n pieces, each isometric to either M_1' or M_2'. Thus, we will have shown

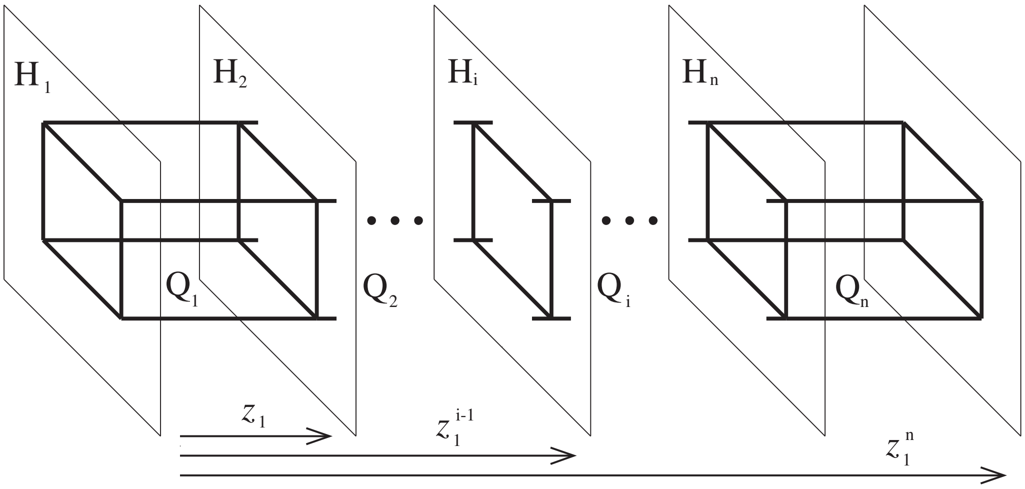

FIGURE 10. The planes H_i in the polyhedron Q

Proposition 4.1. *Any of the manifolds constructed in section 3 are obtained by gluing n copies of either M_1' or M_2' so that each copy of M_1' or M_2' is glued to another copy along one of their two totally geodesic boundaries i.e. they are strung together in a circular fashion. Here M_1' and M_2' are manifolds that we obtain by cutting hyperbolic 4-manifolds M_1 and M_2 along a totally geodesic hypersurface.* □

Remark 4.2. Let $G_{0i},\ i=1,2$ be the subgroup generated by the same generators as G_i, but with z_1 omitted. Then, by using Maskit's combination theorems (see theorems 6.19 and 6.24 in [A2]) and fairly standard arguments (see, for example, [A3]) it is possible to show that

$$G = \left(G_{0i_1} \underset{J_2}{*} (z_1 G_{0i_2} z_1^{-1}) \underset{J_3}{*} (z_1^2 G_{0i_3} z_1^{-2}) \underset{J_4}{*} \cdots \underset{J_n}{*} (z_1^{n-1} G_{0i_n} z_1^{-(n-1)}) \right) \underset{z_1^n}{*} .$$

Here $\underset{J_i}{*}$ denotes the free product with amalgamation, while the last $\underset{z_1^n}{*}$ is the HNN-extension of the free amalgamated product in the parentheses by z_1^n, where z_1^n conjugates subgroups J_1 and $z_1^n J_1 z_1^{-n}$. The index i_k is 1 or 2 depending on whether the block Q_k is of type Φ_1 or Φ_2.

Remark 4.3. The hyperbolic 3-manifold H/J has been described by Wielenberg, see Example 3 in [Wi2]. It is the complement of a certain four-component link in S^3. Thus, we are gluing 4-manifolds along boundaries that are link complements in S^3.

The only thing left to do is to verify that the hyperplanes $H, H_1, \ldots, H_n$ are precisely invariant, respectively, under the subgroups $J, J_1, \ldots, J_n \subset G$. To this end we use the technical theorem stated below.

Theorem 4.4. *Let P be a fundamental polyhedron for a discrete group $G \subset \mathrm{Isom}(\mathbb{H}^n)$ that is generated by some side-pairing of P. Let H be a hyperplane in $\mathbb{H}^n$ so that $\mathrm{int}_H(H \cap P) \neq \emptyset$ and let J be a subgroup of G that keeps H invariant. Assume the following three conditions hold:*

1. *$H \cap P$ is a fundamental polyhedron for the action of J in H.*
2. *If H contains a side S of P and s is the side-pairing corresponding to S then $s(H) \neq H$.*
3. *Suppose H contains an edge E of P. Let $\{\sigma_i = (E_i, S_i, R_i, g_i)\}_{i=1,2\dots}$ be the sequence obtained by the edge-chase corresponding to E as in the edge cycle condition. (That is, $E_1 = E$, S_i and R_i are the sides that determine E_i, $E_i = S_i \cap R_i$, $g_i(E_i) = E_{i+1}$ and g_i is the side-pairing that pairs R_i and S_{i+1}.) Let α be the angle between H and S_1 and let θ_i be the dihedral angle of P at the edge E_i. If another edge E_{l+1} in the cycle of E is contained in H and β is the angle between H and S_l, then $\theta_1 + \cdots + \theta_l + \beta - \alpha = k\pi$ must be satisfied for some integer k.*

Then H is precisely invariant under J.

Proof. Let $f \in G$ and suppose that $K = f(H) \cap H \neq \emptyset$. We want to show that $f \in J$. There are three cases depending on how K intersects the elements of the tiling $\{g(P),\ g \in G\}$. We will repeatedly use the fact that $H \subset \cup_{j \in J}\, j(P)$ and $f(H) \subset \cup_{j \in J}\, fj(P)$ which follows from the assumption that $H \cap P$ is a fundamental polyhedron for J in H.

Case 1. There exists an $x \in K$ so that $x \in \mathrm{int}\, g(P)$ for some $g \in G$. Since $x \in H$, there must be a translate of P under J that contains x. The only possible candidate is g, so we conclude $g \in J$. Likewise, since $x \in f(H)$, there is a $j \in J$ so that $x \in fj(P)$. But this can only happen if $fj = g$, so $f = gj^{-1} \in J$.

Case 2. There exists an $x \in K$ that is contained in the interior of a side R of some translate of P. Then R is common to exactly two translates of P. If R is not contained in H then H cuts into the interior of both of those translates. Again, the parts of H that are in the interiors of these translates must be covered by translates of P under J so the translates abutting R are of form $j(P)$ and $j'(P)$ for some $j, j' \in J$. Furthermore, there must be a $j'' \in J$ so that $x \in fj''(P)$. Since x is in only two translates of P, this means that either $fj'' = j$ or $fj'' = j'$. In both cases we get $f \in J$.

If, on the other hand, H does contain R, then at least one of the two translates of P abutting R is of form $j(P)$, $j \in J$. Assuming $f(H) \neq H$ gives us that $f(H)$ intersects the interior of $j(P)$. The portion of $f(H)$ in $\mathrm{int}\, j(P)$ must be in some $fj'(P)$ for some $j' \in J$ so we get $fj' = j$, which forces $f \in J$, contradicting $f(H) \neq H$. Therefore $f(H) = H$. Now the two translates of P that abut R are of form $j(P)$ and $js^{-1}(P)$, where s is the side-pairing of the side S for which $j(S) = R$. One of those translates is also of form $fj'(P)$ for some $j' \in J$. If $fj' = j$ then $f \in J$. The other case, $fj' = js^{-1}$ implies $s = j'^{-1}f^{-1}j$, so s preserves H. This, however, contradicts assumption (2), because H contains S, since $R \subset H$, $S = j^{-1}(R)$ and $j^{-1}(H) = H$.

Case 3. If neither case 1 nor 2 occurs, we get that K is contained in translates of edges of P, which are $n-2$-dimensional. Since $\dim K \geq n-2$ we get that K must be $(n-2)$-dimensional, which implies $f(H) \neq H$. Furthermore, there exists an $x \in K$ and an edge E' of some $g(P)$ so that x is in the interior of E'. As before, one of the translates of P that contains x must be of the form $j(P)$. Move everything by j^{-1} so that x is now on an edge E of P and $E \subset j^{-1}f(H) \cap H$. The translates of P that abut E are P, $g_1^{-1}(P)$, $g_1^{-1}g_2^{-1}(P), \ldots$, so as before, there must be a $j' \in J$ and an integer l so that $j^{-1}fj' = g_1^{-1} \circ \cdots \circ g_l^{-1}$. But then $j^{-1}f(H) = j^{-1}fj'(H) = g_1^{-1} \circ \cdots \circ g_l^{-1}(H)$, so $E \subset g_1^{-1} \circ \cdots \circ g_l^{-1}(H) \cap H$. From $E \subset g_1^{-1} \circ \cdots \circ g_l^{-1}(H)$ we get that $E_{l+1} = g_l \circ \cdots \circ g_1(E) \subset H$, so E_{l+1} is in the cycle of E and is contained in H. Let $K^{\perp}$ be the 2-dimensional orthogonal complement of K through x. The intersections of translates of P that abut E with $K^{\perp}$ are angles with rays emanating from a single vertex x. Intersections of H and $g_1^{-1} \circ \cdots \circ g_l^{-1}(H)$ with $K^{\perp}$ are two lines and the angle between them is $\theta_1 + \cdots + \theta_l + \beta - \alpha$. Condition 2) now says that this angle is $k\pi$, so the lines are identical and so are the hyperplanes that they represent. From here it follows that $f(H) = H$, a contradiction with $f(H) \neq H$. Therefore, case 3 never occurs and $f \in J$ by cases 1 and 2. $\square$

Remark 4.5. The proof of Theorem 4.4 did not use any hyperbolic space-specific properties, only the fact that P was a fundamental polyhedron. Therefore it also applies in the other two constant curvature settings, that is, for fundamental polyhedra of discrete isometry groups of the n-sphere and Euclidean n-space.

Remark 4.6. Notice that the group G in the theorem did not have to be torsion-free. However, if H contains an edge of P, condition (3) allows the number k that was defined in the edge cycle condition to only be 1 or 2.

Examples. We give several applications of the theorem that include the claims of precise invariantness needed for Proposition 4.2. All except example 4.8 have as P the polyhedron defined in section 1.

Example 4.7. Let $G = G_1$ or G_2, H=the hyperplane based on the plane $\{z = 0\}$, $J = < x_1, y_1, a_3, a_4 >$. Clearly H is invariant under J. By applying (now in dimension 3) Poincaré's polyhedron theorem to $H \cap P$ and restrictions of x_1, y_1, a_3, a_4 to H we may easily see that $H \cap P$ is a fundamental polyhedron for J in H. (Here conditions (2) and (3) from the theorem do not apply.) Therefore, H/J is a totally geodesic hypersurface embedded in M_1 or M_2.

Example 4.8. Let $G = G_1$ or G_2, $H = Z_1$ and $J = < x_1, y_1, a_1, a_2 >$. As in Example 4.7 we check that $H \cap P$ is a fundamental polyhedron for J in H. Here we also need to verify condition (3) of Theorem 4.4. (Condition (2) clearly holds.) Taking, for example, $E = Z_1 \cap A_1$ whose cycle is $\{Z_1 \cap A_1,\ A_1' \cap Z_1,\ Z_1' \cap A_5,\ A_5' \cap Z_1',\}$ we see that $\alpha = 0$, $l = 1$ and $\beta = \pi/2$, so condition (3) is satisfied. Using Theorem 4.4 gives us that H/J is a totally geodesic hypersurface embedded in M_1 or M_2.

Example 4.9. It is now easy to see that the hyperplanes $H_1, \ldots, H_n$ (in above notation) are precisely invariant under the subgroups $J_1, \dots, J_n \subset G$. The proof for H_1 and J_1 corresponds to the one in Example 4.8, while the other cases correspond to Example 4.7. This completes the proof of Proposition 4.2.

Example 4.10. Let $G = G_1$ or G_2 and let H_1, H_2 be the hyperplanes based respectively on the planes $\{x - y = 0\}$ and $\{x + y = 0\}$. We may use Theorem 4.4 to verify that H_1 is precisely invariant under $< a_2, a_4, a_6, b_1, d_2, d_4, y_1 x_1 >$ and that H_2 is precisely invariant under $< a_1, a_3, a_5, b_1, d_1, d_3, y_1^{-1} x_1 >$. Again, condition (1) of Theorem 4.4 is verified by the Poincaré polyhedron theorem in dimension 3. Note that condition (3) of that same theorem applies.

5. Proof of Theorem B

We will show that the polyhedron P has volume $2 \cdot 4\pi^2/3$. Then every manifold obtained from any of the side-pairings of Q described in section 3 will have volume $2n \cdot 4\pi^2/3$. The Gauss-Bonnet formula (see [G], page 84, and [H]) applied to a non-compact hyperbolic 4-manifold M gives $\operatorname{Vol} M = \chi(M) \cdot 4\pi^2/3$. Here $\chi(M)$ is the Euler characteristic of the compact part of M, i.e. the manifold with boundary obtained by retracting every end $E \times [0, \infty)$ of M to $E \times \{0\}$. It will be enough to show that the manifold obtained from either of the side-pairings Φ_1 or Φ_2 has Euler characteristic 2.

Recall that for a finite CW-complex X, $\chi(X)$ may be computed either as an alternating sum of the numbers of i-cells in X, or as the alternating sum of the ranks of the i-th homology groups of X.

Let P be finite-sided n-dimensional hyperbolic polyhedron with a side-pairing defined on it. Now let X be the CW-complex obtained from P in the obvious way, with 0-cells the real and ideal vertices of P, 1-cells the 1-sides of P together with their points at infinity, and so on. Then X inherits identifications by side-pairings of P, which give rise to a quotient space Y, also a CW-complex (even if P yields a manifold by identification, Y will not be one). We then have

Lemma 5.1. *If M is obtained from a side-pairing of P and Y is as above, then*

$$\chi(M) = \chi(Y) - \textit{number of ends of } M.$$

In other words, we may compute $\chi(M)$ directly from the polyhedron by taking the alternating sum of numbers of cycles of i-sides and ignoring the cycles of ideal vertices.

Proof. Using M to also denote the compact part of the hyperbolic manifold, we see that $Y = M \cup V$, where V is a disjoint union of cones over Euclidean manifolds that are the boundary of M, so that $M \cap V$ is a disjoint union of Euclidean manifolds. Consider the absolute Mayer-Vietoris sequence for M and V (see [Do], Proposition 8.15). Enumerate the terms so that the k-th homologies of $M \cup V$, the sum of M and V, and $M \cap V$ correspond to indices $3k$, $3k + 1$ and $3k + 2$ respectively ($k \geq 0$). Let c_j and z_j denote respectively the rank of the j-th term and the rank of the kernel of the homomorphism joining the j-th and the $(j - 1)$-st terms of the sequence. By exactness of that sequence we have $c_j = z_{j-1} + z_j$. Use this equality to see that $\sum(-1)^k c_{3k} - \sum(-1)^k c_{3k+1} + \sum(-1)^k c_{3k+2} = 0$, which is exactly $\chi(M \cup V) - (\chi(M) + \chi(V)) + \chi(M \cap V) = 0$. Since V is contractible, $\chi(V)$ =number of components of V=number of of boundary components of M. The fact that $M \cap V$ is a disjoint union of Euclidean manifolds implies $\chi(M \cap V) = 0$ which yields the desired formula. $\square$

Now to finish the proof of Theorem B, we just have to count cycles of i-sides for Φ_1 or Φ_2. The polyhedron P from section 1 has one 4-side, 36 3-sides, 168 2-sides, 216 1-sides and 48 real 0-sides. Each 3-side is paired to exactly one other one, which yields 18 cycles of 3-sides. Among 2-sides, there are 24 with dihedral angle $\pi/4$, giving 3 cycles, and 144 with dihedral angle $\pi/2$, yielding 36 cycles. Among 1-sides, there are 80 with normalized solid angle 1/16 giving 5 cycles and 136 with normalized solid angle 1/8, yielding 17 cycles. From Figure 6 we know there are 2 cycles of 0-vertices. Thus, $\chi(M) = 2 - (17 + 5) + (36 + 3) - 18 + 1 = 2$, which completes the proof. $\square$

Remark 5.2. Note that the same reasoning as in the above paragraph may be used to see that, when the side-pairing of an n-polyhedron P yields a manifold M, $\chi(M)$ depends only on the alternating sum of normalized solid angles of P over all the i-sides of P. In particular, it doesn't depend on the side-pairing of P. Indeed, the sum of normalized solid angles for each cycle of an i-side is exactly 1, which is contributed to the count of cycles of i-sides.

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