COMPLEXES OF NOT i-CONNECTED GRAPHS

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Abstract. Complexes of (not) connected graphs, hypergraphs and their homology appear in the construction of knot invariants given by V. Vassiliev [V1, V2, V3]. In this paper we study the complexes of not i-connected k-hypergraphs on n vertices. We show that the complex of not 2-connected graphs has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $2n-5$. This answers one of the questions raised by Vassiliev [V3] in connection with knot invariants. For this case the S_n-action on the homology of the complex is also determined. For complexes of not 2-connected k-hypergraphs we provide a formula for the generating function of the Euler characteristic, and we introduce certain lattices of graphs that encode their topology. We also present partial results for some other cases. In particular, we show that the complex of not $(n-2)$-connected graphs is Alexander dual to the complex of partial matchings of the complete graph. For not $(n-3)$-connected graphs we provide a formula for the generating function of the Euler characteristic.

1. Introduction

In this paper we study the homotopy type and homology of simplicial complexes whose simplices are the edge sets of not i-connected graphs and hypergraphs on n vertices. The case $i = 1$ is already well understood (see Proposition 2.1), and here we begin the examination of the topological structure of such complexes for $i \geq 2$.

Although our point of view is mainly combinatorial, our original motivation for studying these complexes comes from the theory of Vassiliev invariants in knot theory. By determining the homotopy type of the complex of not 2-connected graphs on n vertices we answer a question posed by V. Vassiliev in [V3], where he presents a new approach to Vassiliev knot invariants using a filtration of the simplicial resolution of the space of not-knots as in [V2]. More precisely, he studies the space Σ of maps $f : S^1 \to \mathbb{R}^3$ such that $f(S^1)$ has multiple points or cusps. The simplicial resolution $\widetilde{\Sigma}$ of Σ is obtained roughly speaking as follows: singular knots are resolved by blowing up each r-fold self-intersection to an $(\binom{r}{2} - 1)$-simplex, and similarly for the set of cusps. A suitable filtration (see [V3]) of $\widetilde{\Sigma}$, combinatorially defined in terms of these

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simplices, gives rise to a spectral sequence that contains the homology of the complex of not 2-connected graphs on n vertices as a basic ingredient.

Our work continues the already fruitful interaction between the theory of Vassiliev invariants and questions in topological and homological combinatorics of graph complexes (see [V1]). The study of complexes of not i-connected graphs has intriguing combinatorial and algebraic aspects as well. For example, such aspects become apparent when considering the complex of not $(n-2)$-connected graphs on n vertices. In Section 7 this complex is shown to be Alexander dual to the complex of partial matchings of the complete graph on n vertices. These matching complexes, along with complexes of partial matchings of bipartite graphs, have previously been studied for other reasons, see [BLVZ]. In each case for which we calculate the Betti numbers, we detect nontrivial homology. For $(n-3)$-connected graphs (see Section 8) and for most complexes of not 2-connected hypergraphs (see Section 6) we have been unable to compute the Betti numbers explicitly, but we do determine the generating function of their reduced Euler characteristics. The homology is seen to be nontrivial in almost all of these cases.

Surprisingly, these non-vanishing phenomena are suggested by a result motivated by a conjecture in complexity theory. The conjecture states that complexes of graphs on n vertices having some non-trivial monotone graph property – like being not i-connected – are evasive (see for example [KSS]). Kahn, Saks & Sturtevant [KSS] showed that non-evasive complexes are contractible. In many naturally arising cases, including those examined here, the converse is true and evasive complexes in fact have non-vanishing reduced Euler characteristics.

In Section 4 we study the action of the symmetric group on the complex of not 2-connected graphs induced by its natural action on the vertices. This action induces a representation of S_n on the homology groups of the complex, which we determine. Using the representation, we deduce upper bounds on the number of Vassiliev invariants of a given bi-order. This representation coincides with a recently well studied representation which appears in the work of Robinson & Whitehouse [RW, Wh], Kontsevich [K], Getzler & Kapranov [GK], Mathieu [Ma], Hanlon & Stanley [H2, HS] and Sundaram [Su].

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2. Preliminaries

We now introduce the basic concepts used in this paper. By a graph $G = (V(G), E(G))$ we mean a loopless graph without multiple edges on the vertex set $V(G)$ and with edge set $E(G) \subseteq \binom{V(G)}{2}$. Our standard vertex set will be the set $[n] := \{1, 2, \ldots, n\}$. A graph G is called *connected* if for any two distinct vertices $v, v' \in V(G)$ there is a path from v to v' in G, that is, a sequence of edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\ldots$, $\{v_{l-1}, v_l\}$ $\in E(G)$ such that $v = v_1$ and $v' = v_l$. Such a path will sometimes be denoted by $v_1, v_2, \ldots, v_l$. The *size* of a graph G is $|V(G)|$.

A graph G is called *i-connected*, for a number i such that $0 < i < |V(G)|$, if for any j vertices $v_1, \ldots, v_j \in V(G)$, $j < i$, the graph G' that is obtained from G by deleting the vertices $v_1, \ldots, v_j$ and their adjacent edges is connected. Equivalently, G is i-connected if and only if for every pair v, v' of not adjacent vertices there are at least i paths from v to v' that are pairwise vertex disjoint except at their endpoints.

A graph with at least $i+1$ vertices which is not i-connected is also called *(i-1)-separable*, and a 1-separable (that is, not 2-connected) graph will often be called just separable. Of course, if $G = (V(G), E(G))$ is a graph that is not i-connected for some $i \geq 1$ then for any subset $E' \subseteq E(G)$ the graph $G' = (V(G), E')$ on the same vertex set is not i-connected either. Hence if we fix an n-element vertex set V and identify a graph with the set of its edges, then we may regard the set of not i-connected graphs on V as a simplicial complex.

Definition: Δ_n^i is the complex of not i-connected graphs on n vertices.

For a graph G and a vertex v we denote by $G - v$ the graph that is obtained from G by deleting the vertex v from its set of vertices and deleting all edges emerging from v from the set of edges. If v and w are two distinct vertices of G then we denote by vw the two-element set $\{v, w\}$, by $G \setminus vw$ the graph $(V(G), E(G) \setminus \{vw\})$, and by $G + vw$ the graph $(V(G), E(G) \cup \{vw\})$. Note that (by definition) if $xy \in E(G)$ then $G + xy = G$ and if $xy \notin E(G)$ then $G \setminus xy = G$. A subset $V' \subseteq V(G)$ of the vertex-set of a graph $G = (V(G), E(G))$ is called a *cutset* if the graph obtained from G by deleting the vertices in V' and all adjacent edges is not connected. In particular, a graph is i-separable if and only if there is a cutset of cardinality i. A cutset of cardinality 1 is also called a *cutpoint*.

More generally, one may consider complexes of not i-connected k-uniform hypergraphs. Recall that a *k-uniform hypergraph* on a vertex set V is a subset E of the set of k-element subsets $\binom{V}{k}$ of V. We will call the k-uniform hypergraphs *k-graphs* for short. Note that a 2-graph is just a graph. A k-graph is called *i-connected* if its underlying 2-graph is i-connected. The underlying 2-graph of a k-graph E is the graph on V whose edge set contains a k-clique on $\{v_1, \ldots, v_k\}$ for each hyperedge $\{v_1, \ldots, v_k\} \in E$.

Definition: $\Delta_{n,k}^i$ is the complex of all not i-connected k-graphs on n vertices.

Cutsets and cutpoints are defined analogously for k-graphs as they were for graphs.

For the notation related to simplicial complexes and partially ordered sets – posets for short – used in this paper, we refer the reader to Section 10.

Let us now review some known results. For $i = 1$ we have that Δ_n^1 and $\Delta_{n,k}^1$ are the complexes of disconnected graphs, resp., disconnected k-graphs. The topology of $\Delta_{n,k}^1$ is well understood up to homotopy type.

Proposition 2.1. *Let $n \geq 2$. Then*

(i) *The complex Δ_n^1 is homotopy equivalent to a wedge of $(n-1)!$ spheres of dimension $n-3$. In particular, $\widetilde{H}_i(\Delta_n^1) = 0$ for $i \neq n-3$ and $\widetilde{H}_{n-3}(\Delta_n^1) \cong \mathbb{Z}^{(n-1)!}$.*

(ii) *The complex $\Delta_{n,k}^1$ is homotopy equivalent to a wedge of spheres of dimensions $n-(k-2)\cdot t-3$, $1 \leq t \leq \lfloor \frac{n}{k} \rfloor$. In particular, the homology of $\Delta_{n,k}^i$ is free and concentrated in dimensions $n-(k-2)\cdot t-3$, $1 \leq t \leq \lfloor \frac{n}{k} \rfloor$.*

Part (i) follows from well-known properties of partition lattices (see [B, BWa, St2]) together with the crosscut theorem (see [B]). An alternative proof is provided in [V1]. Part (ii) was established by Björner and Welker in [BWe]. See Theorem 4.5 and Section 7.8 of [BWe] for exact numerical information on the homology of $\Delta_{n,k}^1$.

The character of the symmetric group for the representation on $\widetilde{H}_{n-3}(\Delta_n^1)$ was determined by Stanley in [St2] in terms of the character of S_n on the homology of the partition lattice. These two characters are equal by an equivariant version of the crosscut theorem. The character of the symmetric group on the homology of $\Delta_{n,k}^1$ was given by Sundaram & Wachs [SW].

Unless otherwise explicitly stated, all homology groups in this paper have integer coefficients.

3. Homology and homotopy type of Δ_n^2

The main theorem of this section gives a complete description of the homotopy type of Δ_n^2.

Theorem 3.1. *Let $n \geq 3$. Then Δ_n^2 has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $2n-5$.*

Remark: This result was circulated for several months as a conjecture. During that time, the Euler characteristic of Δ_n^2 was calculated by Rodica Simion [Si]. The theorem was proved independently and simultaneously, almost to the day, by V. Turchin in Moscow, in a homology version [V3] that is equivalent to our result by some general arguments from homotopy theory.

For any natural number k, let B_k be the Boolean algebra on k elements (i.e., the lattice of subsets of a k-element set) and let Π_k be the lattice of partitions of a k-set into subsets, ordered by refinement. It is well-known that $\Delta(\overline{B_k})$ — being the barycentric subdivision of a simplex boundary — is homeomorphic to a $(k-2)$-sphere, and that $\Delta(\overline{\Pi_k}) \simeq \Delta_k^1$ has the homotopy type of a wedge of $(k-1)!$ spheres of dimension $k-3$ (see Proposition 2.1 (i) and its references). These facts imply the following.

Lemma 3.2. *$\Delta(\overline{B_k \times \Pi_k})$ has the homotopy type of a wedge of $(k-1)!$ spheres of dimension $2k-3$.*

Proof: Let $\emptyset$ and $[k]$ be the least element and top element of B_k, and let $1|\cdots|k$ and $|1\cdots k|$ be the least and top elements of Π_k. Apply the Homotopy Complementation Formula 10.3 (ii) to $p = (\emptyset, |1\cdots k|)$. The set of complements of p in $B_k \times \Pi_k$

consists of the single element $q = ([k], 1|\cdots|k)$. Obviously, $\Delta((\hat{0}, q)) \cong \Delta(\overline{B_k})$ and $\Delta((q, \hat{1})) \cong \Delta(\overline{\Pi_k})$. Then by Formula 10.3 (i) we have

$$\Delta(\overline{B_k \times \Pi_k}) \simeq \Sigma\big(\Delta(\overline{B_k}) * \Delta(\overline{\Pi_k})\big).$$

Since the join of a wedge of n spheres of dimension i with a wedge of m spheres of dimension j is homotopy equivalent to a wedge of nm spheres of dimension $i+j+1$ (see for example [BWe, Lemma 2.5 (ii)]) the assertion follows. Recall that suspension can be regarded as a join with a 0-sphere and that the join operation is associative. □

Thus, in order to prove Theorem 3.1 it suffices to demonstrate that Δ_n^2 is homotopy equivalent to $\Delta(\overline{B_{n-1} \times \Pi_{n-1}})$. In order to state more precisely what we will prove, we make the following definitions.

Definition: For $x \in [n]$ and any graph G on $[n]$, $N_G(x)$ is the *neighborhood* of x in G, i.e. $N_G(x) = \{y \in [n] : xy \in E(G)\}$, and $\pi(x, G)$ is the partition of the set $[n] \setminus \{x\}$ determined by the connected components of $G - x$.

Definition: $\phi : \overline{\mathcal{L}at(\Delta_n^2)} \to \overline{B_{n-1} \times \Pi_{n-1}}$ is the map of posets given by $G \mapsto (N_G(1), \pi(1, G))$, and $\phi^* : \Delta(\overline{\mathcal{L}at(\Delta_n^2)}) \to \Delta(\overline{B_{n-1} \times \Pi_{n-1}})$ is the simplicial map induced by ϕ.

Note that if G is a graph on $[n]$ such that $N_G(1) = \{2, \ldots, n\}$ and $G-1$ is connected, then G is 2-connected. On the other hand, if $N_G(1) = \emptyset$ and $\pi(1, G) = 2|3|\ldots|n$ then G is the empty graph. Thus ϕ is well-defined. It is clear that ϕ is order preserving, so ϕ^* is well-defined. We can now state the key technical result, from which (in view of Lemma 3.2) Theorem 3.1 follows.

Lemma 3.3. *The simplicial map ϕ^* is a homotopy equivalence.*

To prove Lemma 3.3 we use Quillen's Fiber Lemma (see Proposition 10.1). In our situation this says that if for each $(S, \pi) \in \overline{B_{n-1} \times \Pi_{n-1}}$ the order complex of the poset $\phi_{\leq}^{-1}(S, \pi) = \left\{G \in \overline{\mathcal{L}at(\Delta_n^2)} : \phi(G) \leq (S, \pi)\right\}$ is contractible, then ϕ^* is a homotopy equivalence. If $\pi \neq |2\cdots n|$ then $\phi_{\leq}^{-1}((S, \pi))$ has a top element, namely the graph G such that $1t$ is an edge of G for $t \in S$ and G induces the complete graph on each block of π. So assume that $\pi = |2\cdots n|$. If $|S| \leq 1$ then there is also a top element in $\phi_{\leq}^{-1}((S, \pi))$, namely the graph G which induces a clique on $\{2, \ldots, n\}$ and has $N_G(1) = S$. If $S = \{2, \ldots, n\}$ then (S, π) does not lie in the proper part of $B_{n-1} \times \Pi_{n-1}$. In summary, it remains to consider the fibers $\phi_{\leq}^{-1}(S, \pi)$ for pairs (S, π) such that $\pi = |2\cdots n|$ and $S \subseteq \{2, \ldots, n\}$ with $2 \leq |S| \leq n-2$. To handle these remaining cases, we make the following definitions.

Definition:

(1) For $2 \leq k \leq n-1$, $\Delta(k) = \{G \in \Delta_n^2 : N_G(1) \subseteq \{2, \ldots, k\}\}$.
(2) For $3 \leq k \leq n-1$, $\Delta(k-1, k) = \{G \in \Delta(k-1) : G + 1k \in \Delta(k)\}$.

Note that if $(S, \pi) = (\{2, \ldots, k\}, |2\cdots n|)$ then $\Delta(k) = \phi_{\leq}^{-1}(S, \pi)$. Also, $\Delta(k-1, k)$ consists of those graphs in $\Delta(k-1)$ which do not become 2-connected when the edge $1k$ is added.

By the above discussion and the fact that the natural action of S_n on $\mathcal{L}at(\Delta_n^2)$ is order preserving, Lemma 3.3 follows immediately from the next lemma.

Lemma 3.4. *For $2 \leq k \leq n-1$, $\Delta(k)$ is contractible.*

The proof of Lemma 3.4 proceeds by induction on k, the case $k = 2$ having been handled above. The inductive proof is therefore achieved by the combination of the following two lemmas.

Lemma 3.5. *Let $3 \leq k \leq n-1$. If $\Delta(k-1)$ and $\Delta(k-1,k)$ are contractible, then so is $\Delta(k)$.*

Proof: Let $\star(1k)$ be the subcomplex of $\Delta(k)$ consisting of graphs that either contain the edge $1k$ or else can be extended within $\Delta(k)$ to contain $1k$. Then $\star(1k)$ is a cone with base $\Delta(k-1,k)$ and apex $1k$, and we have

$$\Delta(k) = \Delta(k-1) \cup \star(1k),$$
$$\Delta(k-1,k) = \Delta(k-1) \cap \star(1k).$$

Thus, $\Delta(k)$ is a union of two contractible complexes with contractible intersection, and hence $\Delta(k)$ is itself contractible (see e.g. [B, Lemma 10.3]). □

Lemma 3.6. *For $3 \leq k \leq n-1$, $\Delta(k-1,k)$ is contractible.*

To prove Lemma 3.6 we will use a special case of Forman's discrete Morse theory (see [F], and for this case also [Ch]). The following works for regular cell complexes, but we will only need the simplicial case.

Definition: Let Σ be a simplicial complex.

(1) $D(\Sigma)$ is the digraph whose vertex set is Σ and whose edges are the edges in the Hasse diagram of $\mathcal{L}at(\Sigma) \setminus \{\hat{1}\}$, all directed downward.
(2) For any set X of edges in $D(\Sigma)$, $D_X(\Sigma)$ is the digraph obtained from $D(\Sigma)$ by reversing the direction of the edges in X, so these edges are directed upward while the remaining edges are directed downward.

Before we can formulate the following lemma we have to recall some basic facts about collapsibility (see for example [B]). Given a simplicial complex Σ, a face $\sigma \in \Sigma$ is called *free* if σ is not maximal and is contained in a unique maximal face of Σ. If σ is free in Σ then passing from Σ to the complex $\Sigma \setminus \{\tau : \tau \supseteq \sigma\}$ is called an *elementary collapse* of Σ. If we can obtain a single vertex by applying a sequence of elementary collapses to a complex Σ, then Σ is called *collapsible*. Since it is easily seen that an elementary collapse of Σ is a strong deformation retraction it follows that collapsible complexes are contractible.

Proposition 3.7. *Let Σ be a simplicial complex. If $D(\Sigma)$ contains a perfect matching M such that $D_M(\Sigma)$ is acyclic, then Σ is collapsible.*

Proof: This is a special case of Corollary 3.5 of [F], and this case is easily proved by induction on $|\Sigma|$. If $\Sigma = \{\emptyset, \{x\}\}$ then the claim is clearly true. If $|\Sigma| > 2$, let x be a source in $D_M(\Sigma)$, which must exist since $D_M(\Sigma)$ contains no directed cycle. It is easy to see that x must be a free face of Σ which is properly contained in a unique face $y \in \Sigma$. Now Σ is collapsible to the complex obtained by removing x and y, and we can apply the inductive hypothesis. □

We call a perfect matching of the type described in Proposition 3.7 an *acyclic perfect matching* on $D(\Sigma)$. Our goal is to produce an acyclic perfect matching on $D(\Delta(k-1,k))$. The following easy result will be useful.

Lemma 3.8. *Let Σ be a simplicial complex, let M be a matching on $D(\Sigma)$ and let $F_0 \to F_1 \to \ldots \to F_r \to F_0$ be a directed cycle in $D_M(\Sigma)$. Then there is some dimension d such that $dim(F_i) \in \{d, d+1\}$ for all $i \in [r]$.*

Proof: If the F_i have more than two distinct dimensions then some F_i must be incident to two upward directed edges. This contradicts the fact that M is a matching, and the result follows immediately. □

Before proceeding with the proof of Lemma 3.6 we make some technical definitions.

Definition: Consider separable graphs on the vertex set $[n]$.

(1) We denote the set of cutpoints of such a graph G by $\mathrm{Cut}(G)$.
(2) For fixed $k \in \{3, \ldots, n-1\}$, let
    (a) $I(k) := \{G \in \Delta(k-1,k) \mid N_G(1) = \emptyset\}$.
    (b) $J(k) := \{G \in \Delta(k-1,k) \mid N_G(1) \neq \emptyset$ and $\mathrm{Cut}(G + 1k) \neq \{1\}\}$.
    (c) $F(k) := \{G \in \Delta(k-1,k) \mid \mathrm{Cut}(G + 1k) = \{1\}\}$.

Note that $\Delta(k-1,k)$ is the disjoint union of $I(k)$, $J(k)$ and $F(k)$, and that both $I(k)$ and $I(k) \cup J(k)$ are subcomplexes of $\Delta(k-1,k)$.

The following lemma implies Lemma 3.6, and therefore completes the proof of Theorem 3.1.

Lemma 3.9. *For any $k \in \{3, \ldots, n-1\}$, $D(\Delta(k-1,k))$ admits an acyclic perfect matching.*

Proof: This proof will be carried out in three steps. We will construct an acyclic perfect matching first for $D(I(k))$, then for $D(I(k) \cup J(k))$, and finally for $D(\Delta(k-1,k))$.

Step 1: $D(I(k))$ admits an acyclic perfect matching.

Note that $I(k)$ contains a unique maximal face, namely the complete graph on $\{2, \ldots, n\}$. Thus $I(k)$ is a simplex and it is easy to see that the matching $M = \{G + 23 \to G \setminus 23 | G \in I(k)\}$ is an acyclic perfect matching on $D(I(k))$.

Step 2: $D(I(k) \cup J(k))$ admits an acyclic perfect matching.

It suffices to show that there exists a matching M^* consisting of edges between elements of $J(k)$ which covers all the elements of $J(k)$, and such that $D_{M^*}(I(k) \cup J(k))$ is acyclic. If M^* is such a matching, let M° be an acyclic perfect matching on $D(I(k))$ and set $M = M^* \cup M^\circ$. Then M is a perfect matching on $D(I(k) \cup J(k))$ which contains no edges between $I(k)$ and $J(k)$, so that any directed cycle in $D_M(I(k))$ cannot cover points from both $I(k)$ and $J(k)$. It follows immediately that M is acyclic.

Now let $G \in J(k)$ and let $c \in \mathrm{Cut}(G + 1k)$, $c \neq 1$. Let $x = \min\{N_G(1)\}$. If $xk \in E(G)$ then clearly $G \setminus xk \in J(k)$. If $c \notin \{x, k\}$ then since $1k$ and $1x$ are edges of $G + 1k$, x and k lie in the same connected component of $(G + 1k) - c$. If $c \in \{x, k\}$ then clearly c is a cutpoint of $G + xk + 1k$. In any case, $c \in \mathrm{Cut}(G + xk + 1k)$ and $G + xk \in J(k)$.

Let M^* consist of all edges $G + xk \to G \setminus xk$, where x is determined as above. Clearly M^* is a matching which covers all points in $J(k)$. Assume for contradiction that $A_1 \to B_1 \to A_2 \to B_2 \to \ldots \to B_r \to A_1$ is a directed cycle in $D_{M^*}(I(k) \cup J(k))$. Clearly all the A_i and all the B_i are in $J(k)$, and by Lemma 3.8 we may assume that for each i there are edges α_i and β_i such that $B_i = A_i + \alpha_i$ and $A_{i+1} = B_i \setminus \beta_i$. Thus $A = A + \alpha_1 \setminus \beta_1 + \ldots + \alpha_r \setminus \beta_r$ and $\{\alpha_i\} = \{\beta_i\}$. By the definition of M^*, no $\alpha_i = x_i k$ contains 1, so no β_i contains 1. It follows that $N_{A_1}(1) = N_{B_1}(1) = N_{A_2}(1) = \ldots = N_{B_r}(1)$. By the choice of the x_i's this forces $\alpha_1 = \alpha_2 = \ldots = \alpha_r$, which is clearly impossible.

Step 3: $D(\Delta(k-1, k))$ admits an acyclic perfect matching.

As in Step 2, it suffices to produce a matching M^* on edges connecting elements of $F(k)$ which covers all points in $F(k)$ and such that $D_{M^*}(\Delta(k-1, k))$ is acyclic.

Let $G \in F(k)$. Then $(G+1k)-1$ splits into connected components $C_1, \ldots, C_s$ such that for each $i \in [s]$ the subgraph of $G + 1k$ induced on $V(C_i) \cup \{1\}$ is 2-connected. We may assume that $n \in V(C_1)$. Note that since $k < n$, $1n \notin G + 1k$. Define $S(G)$ to be the set of all $x \in V(C_1) \cap N_{G+1k}(1)$ such that there is a path $P = 1, x, \ldots, n$ in $G + 1k$ with $P \cap N_{G+1k}(1) = \{x\}$.

We claim that $|S(G)| > 1$. Indeed, let $1, x, \ldots, n$ be a shortest path from 1 to n in $G + 1k$. Clearly $x \in S(G)$. Since the subgraph of $G + 1k$ induced on $V(C_1) \cup \{1\}$ is 2-connected and $x \neq n$, there exists a path from 1 to n in this graph which does not contain x. Let $1, y, \ldots, n$ be a shortest such path. Then $y \in S(G)$.

Let x, y be the two smallest elements of $S(G)$. If $xy \notin G$ then clearly $G+xy \in F(k)$ and $S(G+xy) = S(G)$. Now assume $xy \in G$ and let H be the subgraph of $G \setminus xy + 1k$ induced on $V(C_1) \cup \{1\}$. If d is a cutpoint of H then x and y are in different components of $H - d$ (otherwise d is a cutpoint of the subgraph of $G + 1k$ induced on $V(C_1) \cup \{1\}$). However, there is a cycle $1, x, \ldots, y, 1$ in H. Thus there is no such cutpoint d and H is 2-connected. It follows that $G \setminus xy \in F(k)$ and $S(G \setminus xy) = S(G)$.

Now, let M^* consist of the edges $G + xy \to G \setminus xy$ where x, y are determined as above. Then M^* is a matching which consists of edges connecting points in $F(k)$ and covers all points in $F(k)$. It remains to show that $D_{M^*}(\Delta(k-1, k))$ is acyclic.

Assume for contradiction that $A_1 \to B_1 \to A_2 \to \ldots \to B_r \to A_1$ is a directed cycle in $D_{M^*}(\Delta(k-1, k))$. As in Step 2, we may assume that there are edges α_i and β_i such that $B_i = A_i + \alpha_i$, $A_{i+1} = B_i \setminus \beta_i$ and $\{\alpha_i\} = \{\beta_i\}$.

By the definition of M^*, each α_i connects two elements of $N_{B_i+1k}(1) = N_{A_i+1k}(1)$, so no β_i contains 1. Thus $N_{A_i}(1) = N_{B_j}(1)$ for all i, j, and each β_i connects two elements of $N_{B_i+1k}(1) = N_{A_{i+1}+1k}(1)$. Write $\alpha_1 = xy$. Then $\beta_1 \neq xy$, and in $A_2 + 1k$, x and y are still the two smallest neighbors of 1 which are contained in paths from 1 to n which intersect $N_{A_2+1k}(1)$ exactly once. Thus $\alpha_2 = xy = \alpha_1$, giving the desired contradiction. □

4. The character for the action of S_n on $\tilde{H}_{2n-5}(\Delta_n^2)$

In view of Theorem 3.1 it is natural to investigate the representation of the symmetric group S_n on the only non-zero homology group of Δ_n^2, induced by the obvious action. In this section we consider homology with complex coefficients, hence all representations are over $\mathbb{C}$. In many of the computations below, we actually determine character values for the representation of S_n on the only non-zero homology group of $\Delta(\overline{\mathcal{L}at(\Delta_n^2)})$, which is easily seen to be the same as the representation described above.

Definition:

(i) We denote by ω_n^2 the character of S_n given by $g \mapsto Trace(g, \tilde{H}_{2n-5}(\Delta_n^2))$.

(ii) Let C_n be a cyclic subgroup of S_n generated by a full n-cycle. We denote by ℓie_n the character of S_n induced from the character on C_n which takes the value $e^{\frac{2\pi i}{n}}$ on a fixed generator. It is well known (see e.g. [Re, Chapter 8]) that ℓie_n is the character of S_n on the multigraded piece of the free Lie algebra generated by n variables.

For the rest of this section we let S_{n-1} be the stabilizer of the point 1 in the natural action of S_n on the set $[n]$.

Theorem 4.1. *The character ω_n^2 is given by*

$$\omega_n^2 = \ell ie_{n-1} \uparrow_{S_{n-1}}^{S_n} - \ell ie_n.$$

The proof will follow a sequence of lemmas establishing the main steps.

Lemma 4.2. *If $g \in S_{n-1}$ then $\omega_n^2(g) = \ell ie_{n-1}(g)$.*

Proof: It is easily seen that the map $\phi : \overline{\mathcal{L}at(\Delta_n^2)} \to \overline{B_{n-1} \times \Pi_{n-1}}$, defined in the previous section, commutes with the actions of S_{n-1} on the two posets. Thus the induced map on homology is S_{n-1}-equivariant and is an S_{n-1}-module isomorphism by Lemma 3.3. Thus, the characters of S_{n-1} on the homology of Δ_n^2 and on the homology of $\Delta(\overline{B_{n-1} \times \Pi_{n-1}})$ coincide. By an equivariant version of Proposition 10.3 (see [We]), $\Delta(\overline{B_{n-1} \times \Pi_{n-1}})$ has the S_{n-1}-homotopy type of $\Sigma(\Delta(\overline{B_{n-1}}) * \Delta(\overline{\Pi_{n-1}}))$, where the group S_{n-1} acts diagonally on $\Delta(\overline{B_{n-1}}) * \Delta(\overline{\Pi_{n-1}})$. Thus the character of S_{n-1} on the homology of $\Delta(\overline{B_{n-1} \times \Pi_{n-1}})$ is given by the product of the characters of S_{n-1} on $\widetilde{H}_*(\Delta(\overline{B_{n-1}}))$ and $\widetilde{H}_*(\Delta(\overline{\Pi_{n-1}}))$. The character of S_{n-1} on $\widetilde{H}_*(\Delta(\overline{B_{n-1}}))$ is rather easily seen to be the sign-character of S_{n-1} (see [St2]). The character of S_{n-1} on $\widetilde{H}_*(\Delta(\overline{\Pi_{n-1}}))$ was determined in [St2] as $sign_{n-1} \cdot \ell ie_{n-1}$. This implies the assertion. □

Since every element of S_n which has a fixed point is conjugate to an element of S_{n-1}, it remains to determine $\omega_n^2(g)$ for all fixed-point-free $g \in S_n$.

Definition: Let $g \in S_n$. We denote by L^g the poset of faces of Δ_n^2 which are fixed by g, and by g^* the element of S_{n+1} which fixes $n+1$ and acts as g does on $[n]$.

Write $\hat{0}$ for the empty graph in L^g, which is the unique minimum element of L^g, and for any poset P let μ_P be the Möbius function on P.

Lemma 4.3. *For $g \in S_n$, $\omega_n^2(g) = \sum_{G \in L^g} \mu_{L^g}(\hat{0}, G)$.*

Proof: It is well-known (see e.g. [B, (13.5)]) that if a group acts on a bounded poset P then for any group element g we have

$$\mu_{P^g}(\hat{0}, \hat{1}) = \sum_i (-1)^i Tr(g, \tilde{H}_i(\Delta(\overline{P}))).$$

In the case under consideration, the only nonzero reduced homology group is the one in dimension $2n-5$, so the lemma follows immediately from the definition of the Möbius function. □

The next two lemmas will be used to determine $\omega_n^2(g)$ when g is fixed-point-free.

Lemma 4.4. *Let G be a graph whose automorphism group acts transitively on $V(G)$. If G is connected then G is 2-connected.*

Proof: Let v be a leaf of some spanning tree in the connected graph G. Then v is not a cutpoint. Since $\mathrm{Aut}(G)$ is transitive on vertices there cannot be any other cutpoints. Hence G is 2-connected. □

Lemma 4.5. *Let $g \in S_n$ be fixed-point-free. Write g as a product of disjoint cycles, $g = g_1 \dots g_r$. Let $V_i = supp(g_i)$. Let $G \in L^g$ be connected and let $x \in \mathrm{Cut}(G)$ with $x \in V_j$. Then there exists some connected component C of $G - x$ such that $V_j \setminus \{x\} \subseteq C$ and $C \cap V_i \neq \emptyset$ for all $i \in [r]$.*

Proof: Let G_j be the graph on V_j such that an edge yz is in $E(G_j)$ if $yz \in E(G)$ or if there is a path P from y to z in $E(G)$ such that $P \cap V_j = \{y, z\}$. Since G is connected, so is G_j. Also, the group generated by g_j is a group of automorphisms of G_j which acts transitively on V_j. By Lemma 4.4, G_j is 2-connected. It follows that all elements of $V_j \setminus \{x\}$ are in the same connected component of $G - x$. Now for $i \neq j$, let P be a path of shortest length connecting some $y \in V_i$ with some $z \in V_j$. If $z = x$ replace P with $g(P)$. Now P contains no vertices from $V_i \cup V_j$ other than y and $z \neq x$. Thus P is a path in $G - x$ and y lies in the component of $G - x$ containing $V_j \setminus \{x\}$. □

We can now determine the values of ω_n^2 on fixed-point-free elements of S_n.

Lemma 4.6. *Let $g \in S_n$ be fixed-point-free. Then $\omega_n^2(g) = -\omega_{n+1}^2(g^*)$.*

Proof: As usual we write $\hat{0}$ for the empty graph. By Lemma 4.3 we have

$$\omega_{n+1}^2(g^*) = \sum_{G \in L^{g^*}} \mu_{L^{g^*}}(\hat{0}, G).$$

Let M^g be the poset of all graphs on $[n]$ which are fixed by g. Note that if $G \in L^{g^*}$ then $G-(n+1) \in M^g$. For $F \in M^g$ let $D(F)$ be the set of all $G \in L^{g^*}$ such that $G-(n+1)=F$. We have

$$\omega_{n+1}^2(g^*) = \sum_{F \in M^g} \sum_{G \in D(F)} \mu_{L^{g^*}}(\hat{0}, G).$$

Any $G \in L^{g^*}$ is a union of $\langle g^* \rangle$-orbits on $\binom{[n+1]}{2}$. Let $o(G)$ be the number of such orbits. It is easy to see that $\mu_{L^{g^*}}(\hat{0}, G) = (-1)^{o(G)}$. Let $p(G)$ be the number of such orbits containing edges covering the point $n+1$. Applying the previous argument to M^g, we get for any $F \in M^g$

$$\sum_{G \in D(F)} \mu_{L^{g^*}}(\hat{0}, G) = \mu_{M^g}(\hat{0}, F) \sum_{G \in D(F)} (-1)^{p(G)}.$$

We will examine this sum for each $F \in M^g$, looking separately at the cases where F is disconnected, connected but not 2-connected, and 2-connected. Write g as a product of disjoint cycles, $g = g_1 \dots g_r$ and let $V_i = supp(g_i)$. Note that if $v \in V_i$ and $G \in L^{g^*}$ with $\{v, n+1\} \in G$, then the $\langle g^* \rangle$-orbit containing $\{v, n+1\}$ consists of the edges $\{w, n+1\}$ for all $w \in V_i$, and is contained in $E(G)$. Also, $p(G)$ is simply the number of such orbits. Let $O(g)$ be the set of all such orbits, and for $S \subseteq O(g)$ let $G(S)$ be the graph induced on the edges which are contained in elements of S. For $F \in M^g$ define

$$\Sigma(F) := \left\{ S \subseteq O(g) : F \cup G(S) \in L^{g^*} \right\}.$$

Note that $\Sigma(F)$ is a simplicial complex on $O(g)$. Let $P(F) = \mathcal{L}at(\Sigma(F)) \setminus \{\hat{1}\}$. By the above arguments we have

$$\sum_{G \in D(F)} \mu_{L^{g^*}}(\hat{0}, G) = \mu_{M^g}(\hat{0}, F) \sum_{S \in P(F)} \mu_{P(F)}(\hat{0}, S).$$

We now examine the three cases.

Case 1: F is not connected.

Then $P(F)$ is the Boolean algebra on $O(g)$, since $n+1$ is a cutpoint of $F \cup G(S)$ for all $S \subseteq O(g)$. It follows immediately that

$$\sum_{G \in D(F)} \mu_{L^{g^*}}(\hat{0}, G) = 0.$$

Case 2: F is connected but not 2-connected.

We will use the block decomposition described in Proposition 5.1 of the following section. Given a connected but not 2-connected graph $F \in M^g$, let $T(F)$ be the bipartite graph whose vertices are the vertices of F and the blocks of F, with $\{v, W_i\}$ an edge if and only if $v \in W_i$. It is easy to see that $T(F)$ is a tree and that $\langle g \rangle$ is a group of automorphisms of $T(F)$ which preserves each part of the given bipartition. It follows that g fixes a vertex of $T(F)$ (see [L]). Since g fixes no vertex of F, g must fix some block B of F. This means that there is some nonempty $J \subset [r]$ such that

$B = \cup_{j \in J} V_j$. Let S be the set of all orbits in $O(g)$ which contain edges that include vertices in B. We will show that every maximal element of $P(F)$ contains S, from which it follows immediately that

$$\sum_{G \in D(F)} \mu_{L^{g^*}}(\hat{0}, G) = 0.$$

Let $G \in D(F)$ and let $c \in \mathrm{Cut}(G)$. Since F is connected, $c \neq n+1$. Also, if $N_G(n+1) \neq \emptyset$ then since g is fixed-point-free c must be a cutpoint of F. If $N_G(n+1) = \emptyset$ then every $x \in [n]$ cuts G, so in any case we may assume $c \in \mathrm{Cut}(F)$. Let $c \in V_i$. By Lemma 4.5, there is some connected component C of $F - c$ which contains $V_i \setminus \{c\}$ and at least one element of each V_j. Since B is 2-connected and $B \cap C \neq \emptyset$, we must have $B \subseteq C$.

We will now show that c must be a cutpoint of $G \cup S$. If $N_G(n+1) = \emptyset$ then adding S to G simply moves the previously isolated point $n+1$ into the connected component of $G-c$ which contains C. However, there is a component of $F-c$ besides C, which remains separated from C in $G-c$. Now, assume that $N_G(n+1) \neq \emptyset$. Then there exists some set I such that $N_G(n+1) = \cup_{i \in I} V_i$. The component of $G-c$ containing C contains elements of each V_i, and it follows that $n+1$ must also be in this component. Thus, adding S to G does not reduce the number of components of $G-c$.

Case 3: F is 2-connected.

In this case the only $G \in D(F)$ is that for which $N_G(n+1) = \emptyset$. Indeed, since each V_i has at least two elements we cannot have $|N_G(n+1)| = 1$, and the claim follows. Thus

$$\sum_{G \in D(F)} \mu_{L^{g^*}}(\hat{0}, G) = \mu_{M^g}(\hat{0}, F).$$

Let K^g be the set of 2-connected graphs in M^g. Combining the information from the three cases we have shown that

$$\omega^2_{n+1}(g^*) = \sum_{F \in K^g} \mu_{M^g}(\hat{0}, F).$$

By definition of the Möbius function and the fact that M^g has a maximum element, we have

$$\sum_{F \in K^g} \mu_{M^g}(\hat{0}, F) = -\sum_{F \in L^g} \mu_{L^g}(\hat{0}, F) = -\omega^2_n(g),$$

and the proof is complete. □

Proof of Theorem 4.1: Set $\rho_n = \ell ie_{n-1} \uparrow^{S_n}_{S_{n-1}} - \ell ie_n$. We must show that $\rho_n(g) = \omega^2_n(g)$ for all $g \in S_n$.

By the definition of induced characters, if $g \in S_n$ is not the product of disjoint cycles of the same length then $\ell ie_n(g) = 0$. We will assume from now on that any $g \in S_n$ which fixes a point is contained in S_{n-1} (so by our convention it fixes the point 1).

By the definition of induced characters and Theorem 3.1, we have

$$\rho_n(\mathrm{id}) = (n-2)!\ [S_n : S_{n-1}] - (n-1)! = (n-2)! = \omega_n^2(\mathrm{id}).$$

If $g \neq \mathrm{id}$ and g has at least two fixed points, then $\ell ie_{n-1}(g) = \ell ie_n(g) = 0$, so $\rho_n(g) = \omega_n^2(g)$ by Lemma 4.2.

If $g \neq \mathrm{id}$ has exactly one fixed point, then $\ell ie_n(g) = 0$. For $h \in S_n$ we have $g^h := h^{-1}gh \in S_{n-1}$ if and only if $h \in S_{n-1}$. By the definition of induced characters and Lemma 4.2,

$$\rho_n(g) = \ell ie_{n-1} \uparrow_{S_{n-1}}^{S_n} (g) = \frac{1}{(n-1)!} \sum_{h \in S_{n-1}} \ell ie_{n-1}(g^h) = \ell ie_{n-1}(g) = \omega_n(g).$$

If $g \in S_n$ has no fixed points then $\ell ie_{n-1} \uparrow_{S_{n-1}}^{S_n} (g) = 0$ and $\rho_n(g) = -\ell ie_n(g)$. As before, let g^* be the element of S_{n+1} which fixes $n+1$ and acts as g does on $[n]$. We have shown above that $\omega_{n+1}^2(g^*) = \ell ie_n(g)$. Hence, by Lemma 4.6, $\rho_n(G) = \omega_n^2(g)$. □

According to Vassiliev, the number of linearly independent knot invariants of bi-order $(n, n-1)$, modulo lower bi-order invariants, is bounded from above by the multiplicity of the trivial representation in the restriction of ω_n^2 to the cyclic group C_n generated by $(12\cdots n)$. See [V3] for all details. As a corollary of Theorem 4.1 we obtain a formula for this multiplicity. We write $\langle \xi, 1\rangle$ for the multiplicity of the trivial character in any character ξ of C_n.

Corollary 4.7.

$$\langle \omega_n^2 \downarrow_{C_n}^{S_n}, 1\rangle = (n-2)! - \frac{1}{n} \sum_{d|n} \mu(d)\phi(d)(\frac{n}{d} - 1)!\, d^{\frac{n}{d}-1}$$

Proof: As a consequence of a result by Hanlon [H1] (see also [St2]), it is straightforward to show that

$$\langle \ell ie_n \downarrow_{C_n}^{S_n}, 1\rangle = \frac{1}{n} \sum_{d|n} \mu(d)\phi(d)(\frac{n}{d} - 1)!\, d^{\frac{n}{d}-1},$$

where μ is the usual number-theoretic Möbius function and ϕ is Euler's function. On the other hand,

$$\langle \ell ie_{n-1} \uparrow_{S_{n-1}}^{S_n} \downarrow_{C_n}^{S_n}, 1\rangle = \frac{1}{n} \sum_{g \in C_n} \ell ie_{n-1} \uparrow_{S_{n-1}}^{S_n} (g) = \frac{1}{n}\, \ell ie_{n-1} \uparrow_{S_{n-1}}^{S_n} (\mathrm{id}) = (n-2)!.$$

Now the assertion follows immediately from Theorem 4.1. □

The values of $w_n = \langle \omega_n^2 \downarrow_{C_n}^{S_n}, 1\rangle$ for small n are given in the table below.

n	3	4	5	6	7	8	9	10	11
w_n	1	1	2	6	18	96	564	4,072	32,990

Table 1: Multiplicity w_n of the trivial character in $\omega_n^2 \downarrow_{C_n}^{S_n}$

The character ω_n^2 and the tensor product of ω_n^2 with the sign character have recently appeared in various different settings, see Section 9.4.

5. The lattice of block-closed graphs

In this section we will obtain information on the topology of $\Delta^2_{n,k}$ by producing a lattice $\Sigma_{n,k}$ such that $\Delta(\overline{\Sigma_{n,k}})$ is homotopy equivalent to $\Delta^2_{n,k}$ and examining the structure of $\Sigma_{n,k}$. For lattice and poset terminology not explained in Section 10 we refer to [St3].

We begin by recalling some elements of the well known structure theory of separable graphs, which appears e.g. in [L].

Definition: Let G be any graph. A *block* of G is a subset W of $V(G)$ such that the subgraph of G induced on W is 2-connected or W is a singleton or a pair of points connected by an edge, and the subgraph of G induced on any proper superset of W is separable. We will say that G is *block-closed* if the subgraph induced on each block is a clique.

Given a graph G, say that $e \equiv e'$ for two of its edges e and e' if they both lie in some circuit of G. This is easily seen to be an equivalence relation on $E(G)$. If W is the set of nodes underlying an equivalence class then W is a block, and all non-singleton blocks correspond to equivalence classes of edges in this way. From this it is easy to derive the following basic facts about the "block decomposition" of G, see [L] for more details.

Proposition 5.1. *Let G be a graph. Then there exists a unique decomposition of $V(G)$ into blocks $W_1, \dots, W_r$, and if $i \neq j$ we have $|W_i \cap W_j| \leq 1$. Moreover, if B_G is the graph with vertex set $\{w_1, \dots, w_r\}$ such that $\{w_i, w_j\} \in E(B_G)$ if and only if $|W_i \cap W_j| = 1$, then B_G is a forest (that is, B_G contains no cycles).*

Note that if K is a k-graph with underlying graph G, then every block of G has size at least k or is a single vertex.

Definition: Let K be a k-graph with underlying graph G, and let $W_1, \dots, W_r$ be the blocks of G. We define K^* to be the k-graph which induces the complete k-graph on each W_i and contains no other hyperedges. We also define $\Sigma_{n,k}$ to be the poset of all graphs on vertex set $[n]$ in which every block is either an isolated vertex or a clique of size at least k, ordered by inclusion.

The first part of the following lemma is immediate from the definition, and the second follows via a standard argument for closure operators on lattices.

Lemma 5.2. (i) *The map $K \mapsto K^*$ defines a closure operator on $\mathcal{L}at(\Delta^2_{n,k})$ whose image is isomorphic to $\Sigma_{n,k}$.*
(ii) *$\Sigma_{n,k}$ is a lattice.*

The meet operation in the lattice $\Sigma_{n,k}$ is intersection of edge-sets followed by deletion of the edges in all blocks of size smaller than k. Note that the elements of $\Sigma_{n,2}$ are the block-closed graphs, and that we have a tower of embeddings as subposets (not sublattices):

$$\Sigma_{n,k} \subseteq \cdots \subseteq \Sigma_{n,3} \subseteq \Sigma_{n,2}.$$

Hence, in view of the following result the topology of all the complexes $\Delta^2_{n,k}$ is encoded into the lattice $\Sigma_{n,2}$ of block-closed graphs.

Theorem 5.3. *The complexes $\Delta^2_{n,k}$ and $\Delta(\overline{\Sigma_{n,k}})$ are homotopy equivalent.*

Proof: K^* is the complete graph (the top element of $\Sigma_{n,k}$) if and only if K is 2-connected. Hence, the map $K \mapsto K^*$ restricts to a closure operator on $\overline{\mathcal{L}at(\Delta^2_{n,k})}$ whose image is isomorphic to $\overline{\Sigma}_{n,k}$. The theorem then follows from Corollary 10.2. □

We will now investigate the structure of $\Sigma_{n,k}$. The next two lemmas follow immediately from the definition of $\Sigma_{n,k}$. We write $\hat{0}$ for the empty graph, which is the minimum element of $\Sigma_{n,k}$, and $\hat{1}$ for the complete graph, which is its maximum.

Lemma 5.4. *Let $G, H \in \Sigma_{n,k}$. Then G covers H if and only if one of the following conditions holds:*

(i) *$E(G) \setminus E(H)$ is a clique on k vertices belonging to k pairwise different components of H.*
(ii) *$E(G) \setminus E(H)$ is a complete bipartite graph on parts A and B, and there is a vertex v such that $A \cup \{v\}$ and $B \cup \{v\}$ are blocks in H.*
(iii) *Only if $k > 2$: $E(G) \setminus E(H)$ is a star (that is, a connected graph with at most one vertex of degree more than one), and the vertices of degree one in this star form a block in H belonging to a component of H distinct from that of the center of the star.*

The three types of coverings can informally be described as follows:

(i) select a vertex from each of k pairwise disjoint components of H and then create a k-clique on these vertices;
(ii) complete the union of two overlapping blocks of H to a clique;
(iii) for $k > 2$: select a block and a vertex from different components of H and complete their union to a clique.

The lattices $\Sigma_{n,k}$ are neither upper nor lower semimodular. However, they exhibit a recursive structure on lower intervals, and certain upper intervals are upper semimodular, as the following lemma shows.

Lemma 5.5. *Let $G \in \Sigma_{n,k}$.*

(i) *If G has r non-singleton blocks of sizes $m_1, \dots, m_r$ then the interval $[\hat{0}, G]$ is isomorphic to the direct product $\Sigma_{m_1,k} \times \cdots \times \Sigma_{m_r,k}$*
(ii) *If G is connected then the interval $[G, \hat{1}]$ is isomorphic to a direct product of partition lattices. More precisely, suppose that G has s cutpoints and that the i-th cutpoint lies in $t_i \geq 2$ blocks. Then, $[G, \hat{1}] \cong \Pi_{t_1} \times \cdots \times \Pi_{t_s}$.*

The following description of the coatoms of $\Sigma_{n,k}$, that is, the elements which are covered by $\hat{1}$, follows immediately from the two preceding lemmas.

Lemma 5.6. *Let M be a coatom of $\Sigma_{n,k}$. Then one of the following conditions holds:*

(i) *M is connected and has two blocks of size l, m with $k \leq l \leq m \leq n-k+1$ and $l + m = n + 1$. In this case, the interval $\left[\hat{0}, M\right]$ is isomorphic to $\Sigma_{l,k} \times \Sigma_{m,k}$.*
(ii) *M consists of an $(n-1)$-clique and an isolated vertex. In this case, $k > 2$ and the interval $\left[\hat{0}, M\right]$ is isomorphic to $\Sigma_{n-1,k}$.*

For any graph G let $c(G)$ be the number of connected components, and $b(G)$ the number of blocks of size ≥ 2.

Theorem 5.7. (i) *The lattice $\Sigma_{n,2}$ is graded with rank function*

$$\rho(G) = 2n - 2c(G) - b(G).$$

In particular, its length is $\rho(\hat{1}) = 2n - 3$.

(ii) *The lattice $\Sigma_{n,3}$ is graded with rank function*

$$\rho(G) = n - c(G) - b(G).$$

In particular, its length is $\rho(\hat{1}) = n - 2$.

(iii) *If $k > 3$ and $n < 2k - 1$, then $\Sigma_{n,k}$ is isomorphic to the lower-truncated Boolean algebra $\{A \subseteq [n] : |A| \geq k\} \cup \{\emptyset\}$. In particular, $\Sigma_{n,k}$ is graded of length $n - k + 1$.*

(iv) *If $k > 3$ and $n \geq 2k - 1$, then ℓ is the length of a maximal chain of $\Sigma_{n,k}$ if and only if*

$$\ell = (n-2) - t(k-3), \text{ for some } 1 \leq t \leq \lfloor \frac{n-1}{k-1} \rfloor.$$

In particular, $\Sigma_{n,k}$ is of length $n - k + 1$ and is not graded.

(v) *If $k > 3$ then $G \in \overline{\Sigma_{n,k}}$ is contained in a chain of length $n - k + 1$ if and only if G consists of a clique of size $l \geq k$ and $n - l$ isolated vertices.*

Proof: For claims (i) and (ii) it suffices to check that the given rank functions increase by 1 for each type of covering given in Lemma 5.4 and take value zero at the empty graph. Claim (iii) is clear from the definition.

Claims (iv) and (v) are implied by the following description of the maximal chains in $\Sigma_{n,k}$. We will here view $\Sigma_{n,k}$ as a subposet of $\Sigma_{n,3}$, and we let ρ denote the restriction of the rank function of claim (ii) from $\Sigma_{n,3}$ to $\Sigma_{n,k}$.

A maximal chain from $\hat{0}$ to $\hat{1}$ in $\Sigma_{n,k}$ is a sequence of covering steps. By Lemma 5.4 there are three possibilities for each step. The rank function ρ will increase by 1 for coverings of types (ii) or (iii), and by $k - 2$ for coverings of type (i). Hence, the length of a maximal chain must be $n - 2 - t(k-3)$, where t is the number of covering steps of type (i). Note that $t \geq 1$ since the first covering in the chain must be of type (i), and that $t \leq \lfloor \frac{n-1}{k-1} \rfloor$ since each step of type (i) reduces the number of connected components by $k - 1$ and the total reduction of components along the whole chain is $n - 1$.

Now, suppose that $1 \leq t \leq \lfloor \frac{n-1}{k-1} \rfloor$. A maximal chain of length $n - 2 - t(k-3)$ is constructed as follows. First perform a sequence of t covering steps of type (i) producing the graph with k-cliques on the sets $\{1, \ldots, k\}$, $\{k, \ldots, 2k-1\}$, ... $\{(t-1)k - (t-2), \ldots, tk - (t-1)\}$. Then continue from there via a sequence of $t - 1$ covering steps of type (ii) leading to the graph with a $(tk - (t-1))$-clique on the set $[tk - (t-1)]$. Finally, $n - (tk - (t-1))$ covering steps of type (iii) will lead to the complete graph. The total number of steps taken, i.e. the length of the constructed chain, is $t + (t-1) + n - (tk - (t-1)) = n - 2 - t(k-3)$. □

The above result yields some nontrivial information about the topology of $\Delta^2_{n,k}$. For instance, part (i) shows that the order complex of $\overline{\Sigma_{n,2}}$ is pure of dimension $2n - 5$.

With Theorem 5.3 this implies that the homology of $\Delta^2_{n,2}$ vanishes in dimensions greater than $2n-5$ and is free in dimension $2n-5$. Of course, in this case we already have more precise knowledge from Theorem 3.1. By similar reasoning we can conclude the following new information about the $k=3$ case from part (ii) of Theorem 5.7.

Theorem 5.8. *$\widetilde{H}_i(\Delta^2_{n,3}) = 0$ for all $i > n-4$, and $\widetilde{H}_{n-4}(\Delta^2_{n,3})$ is free.*

In the remaining cases the following can be deduced.

Theorem 5.9. *Assume that $k > 3$.*
(i) *$\widetilde{H}_i(\Delta^2_{n,k}) = 0$ if $i > n-k-1$ or $n-k-1 > i > n-2k+2$.*
(ii) *$\widetilde{H}_{n-k-1}(\Delta^2_{n,k})$ is free of dimension $\binom{n-1}{k-1}$.*
(iii) *If $n < 2k-1$ then $\Delta^2_{n,k}$ has the homotopy type of a wedge of $\binom{n-1}{k-1}$ spheres of dimension $n-k-1$.*
(iv) *If $n = 2k-1$ then $\Delta^2_{n,k}$ has the homotopy type of a wedge of spheres. This wedge consists of $\binom{n-1}{k-1}$ $(n-k-1)$-spheres and $\frac{1}{2}n\binom{n-1}{k-1}$ 1-spheres.*

Proof: We use Theorem 5.3 without reference throughout the proof. Claim (i) follows immediately from Theorem 5.7(iii),(iv),(v). By Theorem 5.7(v), the subposet of $\Sigma_{n,k}$ generated by chains of length $n-k+1$ is isomorphic to the poset obtained by removing all sets of sizes $1, 2, \ldots, k-1$ from the Boolean algebra B_n. Claims (ii) and (iii) now follow immediately from the rank selection results in [B, St2], along with Theorem 5.7(iii),(iv).

If $n = 2k-1$, let W be the set of vertices in $\Delta(\overline{\Sigma_{n,k}})$ corresponding to graphs which consist of two k-cliques intersecting in a single vertex, and let Δ_0 be the complex obtained by removing all simplices containing an element of W from $\Delta(\overline{\Sigma_{n,k}})$. Then Δ_0 is the order complex of the subposet of $\overline{\Sigma_{n,k}}$ generated by chains of length $n-k+1$, and is therefore homotopy equivalent to a wedge of $\binom{n-1}{k-1}$ $(n-k-1)$-spheres, as above. If $G \in \Sigma_{n,k}$ corresponds to an element $w \in W$, then by Lemma 5.4, $(\overline{\Sigma_{n,k}})_{<G}$ consists of two graphs which contain a k-clique and $n-k$ isolated vertices. It follows that $link_{\Delta(\overline{\Sigma_{n,k}})}(w)$ consists of two vertices in Δ_0. There is a homotopy equivalence between Δ_0 and a wedge of $\binom{n-1}{k-1}$ $(n-k-1)$-spheres which maps $\cup_{w\in W} link_{\Delta(\overline{\Sigma_{n,k}})}(w)$ to the wedge point. It is easy to see that $|W| = \frac{1}{2}n\binom{n-1}{k-1}$, and claim (iv) follows. $\square$

The homology of $\Delta^2_{n,3}$ has been computed for $4 \le n \le 7$. It is concentrated in dimension $n-4$, see Table 2.

$n \backslash i$	0	1	2	3
2	0	0	0	0
3	0	0	0	0
4	$\mathbb{Z}^3$	0	0	0
5	0	$\mathbb{Z}^{21}$	0	0
6	0	0	$\mathbb{Z}^{180}$	0
7	0	0	0	$\mathbb{Z}^{2010}$

Table 2: Homology groups $\widetilde{H}_i(\Delta^2_{n,3})$

We believe that the concentration of homology in dimension $n-4$ is true in general, see the discussion in Section 9.2. One approach to proving this could be via the following lemma. Recall that a graph is called a *forest* if it is free of circuits. This is equivalent to saying that every block in its block decomposition has at most two vertices.

Lemma 5.10. *Suppose that the order complex of the open interval $(G, \hat{1})$ in $\Sigma_{n,2}$ is topologically $(n-5)$-connected for every forest G. Then $\Delta^2_{n,3}$ is homotopy equivalent to a wedge of $(n-4)$-spheres.*

Proof: By Theorem 5.3 we may replace $\Delta^2_{n,3}$ by $\Delta(\overline{\Sigma_{n,3}})$, which by Theorem 5.7(ii) is $(n-4)$-dimensional. Hence by known reductions (see [B, (9.19)]) it suffices to prove that $\Delta(\overline{\Sigma_{n,3}})$ is $(n-5)$-connected. By Theorem 3.1 we know that $\overline{\Sigma_{n,2}}$ is $(n-5)$-connected, and we will show how to transfer this connectivity to the subposet $\overline{\Sigma_{n,3}}$ under the given hypothesis.

Let P_n be the subposet of $\overline{\Sigma_{n,2}}$ consisting of all elements which contain at least one block of size greater than two. The elements in $\overline{\Sigma_{n,2}} \setminus P_n$ are the forests G, so a version of Quillen's fiber lemma (see [B, Lemma 11.12]) together with our hypothesis about the intervals $(G, \hat{1})$ shows that P_n is $(n-5)$-connected.

Now, note that $\overline{\Sigma_{n,3}} \subseteq P_n$. Let $\rho : P_n \to \overline{\Sigma_{n,3}}$ be the map which sends $H \in P_n$ to the subgraph obtained by removing from H all edges which are not contained in a block of size at least three. Then ρ is a lower closure operator on P_n (that is, a closure operator on P_n with the opposite order) whose image is $\overline{\Sigma_{n,3}}$. Hence, by Corollary 10.2 $\overline{\Sigma_{n,3}}$ is $(n-5)$-connected also. $\square$

We end this section with an easy result which shows that the homology of $\Delta^2_{n,k}$ vanishes in all sufficiently low dimensions. For this the posets $\Sigma_{n,k}$ are not used.

Lemma 5.11. *Let E be a k-graph on n vertices. If E is 2-connected, then E contains at least $\lceil \frac{n}{k-1} \rceil$ hyperedges.*

Proof: If E is 2-connected then for each k-edge $X = \{v_1, \dots, v_k\} \in E$ there exist at least two v_i which are contained in some k-edge of E other than X. It follows easily that

$$n \leq |E|(k-1).$$

$\square$

Corollary 5.12. *The complex $\Delta^2_{n,k}$ is topologically $(\lceil \frac{n}{k-1} \rceil - 3)$-connected, implying that $\widetilde{H}_i(\Delta^2_{n,k}) = 0$ for $i = 0, 1, \dots, \lceil \frac{n}{k-1} \rceil - 3$.*

Proof: Let $m = \lceil \frac{n}{k-1} \rceil - 2$. By Lemma 5.11 $\Delta^2_{n,k}$ contains the full m-skeleton on its vertex set. The corollary follows immediately. $\square$

6. The Euler characteristic of the complex $\Delta^2_{n,k}$

In Section 5 we were able to determine the homotopy type of $\Delta^2_{n,k}$ for $k > 3$ when $n \leq 2k-1$, but not for $k = 3$, nor for $k > 3$ and $n > 2k-1$. Indeed, other than the connectivity result given in Corollary 5.12, in the case $k = 3$ our only information on the topology of $\Delta^2_{n,k}$ is given by Theorem 5.8 (unless n is very small), and in the case $k > 3$ and $n > 2k-1$ we were only able to determine the homology group $\widetilde{H}_{n-k-1}(\Delta^2_{n,k})$.

In this section, we investigate the reduced Euler characteristic of $\Delta^2_{n,k}$. We will determine a formula for the exponential generating function

$$M_k(x) := \sum_{n=1}^{\infty} \tilde{\chi}(\Delta^2_{n,k}) \frac{x^n}{n!},$$

for all $k \geq 2$. That formula is stated in the following theorem.

Theorem 6.1. *For $k \geq 2$, we have*

$$M'_k\left(x\frac{p_{k-1}(x)}{p_k(x)}\right) = \ln\left(\frac{p_{k-1}(x)}{p_k(x)}\right),$$

where $p_k(x) := 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!}$.

Theorem 6.1 gives another proof that $\tilde{\chi}(\Delta^2_{n,2}) = -(n-2)!$. It also implies

$$M'_3(x) = \ln\left(\frac{-x(x-2)}{(x-1)+\sqrt{2-(x-1)^2}}\right),$$

which gives the sequence $0, 0, -1, 3, -21, 180, -2010, 27090, -430290, \ldots$ for $\tilde{\chi}(\Delta^2_{n,3})$, cf. Table 2. To obtain these corollaries set $y := x\frac{p_{k-1}(x)}{p_k(x)}$ and solve for x to get $x = \frac{1}{1-y}$ and $x = \frac{y-1+\sqrt{2-(y-1)^2}}{2-y}$, when $k = 2$ and 3 respectively.

To prove Theorem 6.1 we will use the posets $\Sigma_{n,k}$ defined in Section 5. Note that $\tilde{\chi}(\Delta^2_{n,k}) = \mu_{\Sigma_{n,k}}(\hat{0}, \hat{1})$. We will write $\mu_k(G)$ for $\mu_{\Sigma_{n,k}}(\hat{0}, G)$ and $\mu_k(n)$ for $\mu_{\Sigma_{n,k}}(\hat{0}, \hat{1})$.

Let $\Pi_{n,k}$ be the k-equal lattice, which is the lattice of partitions of $[n]$ into subsets such that each subset has size one or at least k. Let $\tau_k(n) := \mu_{\Pi_{n,k}}(\hat{0}, \hat{1})$ for $n \geq k$ and $\tau_k(2) = \cdots = \tau_k(k-1) = 0$, but $\tau_k(1) = 1$. The exponential generating function, $T_k(x) := \sum_{n=1}^{\infty} \tau_k(n)\frac{x^n}{n!}$, for the Möbius function of $\Pi_{n,k}$ is known to be

$$T_k(x) = \ln(p_k(x)),$$

where $p_k(x)$ is as above. It was first calculated in [BL].

Let $\mathcal{C}$ be the set of connected graphs in $\Sigma_{n,k}$. Now let

$$\sigma : \Sigma_{n,k} \setminus \mathcal{C} \longrightarrow \Pi_{n,k}$$

be the function which maps a disconnected graph in $\Sigma_{n,k}$ to the partition determined by its connected components. It is easily seen that for each $x \in \Pi_{n,k}$, $\sigma_{\leq}^{-1}(x)$ has a unique maximum element and therefore has a contractible order complex. Thus by Proposition 10.1 and the definition of the Möbius function, we have

$$\tau_k(n) = -\sum_{G\in\Sigma_{n,k}\setminus\mathcal{C}} \mu_k(G) = \sum_{G\in\mathcal{C}} \mu_k(G).$$

Thus it suffices to concentrate on the connected but not 2-connected graphs. First we need a simple lemma.

Lemma 6.2. *If $G \in \mathcal{C}$ has blocks $W_1, \dots, W_r$ with $|W_i| = w_i$, then*

$$\mu_k(G) = \prod_{i=1}^{r} \mu_k(w_i).$$

Proof: By Lemma 5.5 the interval $\left[\hat{0}, G\right]$ is isomorphic to the product poset $\Sigma_{w_1,k} \times \dots \times \Sigma_{w_r,k}$. The lemma now follows from the well known multiplicativity of the Möbius function. □

Now define $\alpha_k(n) := \sum_{G\in\mathcal{C}_1} \mu_k(G)$, where $\mathcal{C}_1 := \{G \in \mathcal{C} : n \text{ is not a cutpoint of G}\}$. Also, set $\alpha_k(1) = \dots = \alpha_k(k-1) = 0$ and $A_k(x) := \sum_{n=1}^{\infty} \alpha_k(n)\frac{x^n}{n!}$.

Lemma 6.3. *We have*

$$A'_k(x) = \ln\left(\frac{p_{k-1}(x)}{p_k(x)}\right).$$

Proof: If n is a cutpoint of $G \in \mathcal{C}$, let $P_1, \dots, P_t$ be the connected components of $G - n$. Then for each $i \in [t]$, n is not a cutpoint of the connected subgraph of G induced on $P_i \cup \{n\}$. Using Lemma 6.2, we get the recursive formula

$$\tau_k(n) = \alpha_k(n) + \sum_{t=2}^{\left\lfloor \frac{n-1}{k-1} \right\rfloor} \sum_{P_1|\dots|P_t\in\Pi_{n-1}} \alpha_k(|P_1|+1)\cdots\alpha_k(|P_t|+1),$$

where each summand in the double sum on the right counts the Möbius functions of all elements $G \in \mathcal{C}$ such that $G - n$ has connected components $P_1, \dots, P_t$. By the definition of $\alpha_k(n)$ we can rewrite this formula as

$$\tau_k(n) = \sum_{P_1|\dots|P_t\in\Pi_{n-1}} \alpha_k(|P_1|+1)\cdots\alpha_k(|P_t|+1).$$

The exponential formula (Proposition 10.5) and easy power series manipulations then give

$$T'_k(x) = e^{A'_k(x)}.$$

□

Proof of Theorem 6.1: We will establish a recurrence relation for α_k involving μ_k. Let $G \in \mathcal{C}_1$ and let W be the block of G containing n. Then W is the unique maximal clique in G which contains n. Let $S = W \setminus \{n\}$. Let B_G be the graph on the blocks of G defined in Proposition 5.1. Let $T_1, \dots, T_t$ be the connected components of $B_G - W$ and for each T_i let V_i be the set of vertices of G which are contained in a block that is contained in T_i. For each T_i there is a unique $j_i \in S$ such that the subgraph of G induced on $V_i \cup \{j_i\}$ is a connected union of blocks of G. Conversely, any $G \in \mathcal{C}_1$ can be obtained by choosing S, T_i and j_i as above and then choosing graphs H_i on $T_i \cup \{j_i\}$ such that each H_i is either a clique of size at least k or isomorphic to a connected element of $\overline{\Sigma}_{|T_i|+1,k}$. These choices can all be made independently, so we get the recurrence relation

$$\alpha_k(n) = \sum_{S \uplus T = [n-1]} \mu_k(|S|+1) \sum_{T_1 \uplus \ldots \uplus T_t = T} (\alpha_k(|T_1|+1)|S|) \cdots (\alpha_k(|T_t|+1)|S|).$$

We cannot apply the exponential formula directly at this point due to the factors $|S|$ which appear on the right hand side of the above equation. However, we get

$$A'_k(x) = \sum_{n=1}^{\infty} \frac{\alpha_k(n) x^{n-1}}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} \frac{\mu_k(i+1)x^i}{(n-1)!} \sum_{T_1|\ldots|T_t \in \Pi_{n-i-1}} (\alpha_k(|T_1|+1)i) \cdots (\alpha_k(|T_t|+1)i) x^{n-i-1}$$

$$= \sum_{i=1}^{\infty} \frac{\mu_k(i+1)x^i}{i!} \sum_{n=i+1}^{\infty} \sum_{T_1|\ldots|T_t \in \Pi_{n-i-1}} (\alpha_k(|T_1|+1)i) \cdots (\alpha_k(|T_t|+1)i) \frac{x^{n-i-1}}{(n-i-1)!}$$

Applying the exponential formula, for each i we get

$$\sum_{n=i+1}^{\infty} \sum_{T_1|\ldots|T_t \in \Pi_{n-i-1}} (\alpha_k(|T_1|+1)i) \cdots (\alpha_k(|T_t|+1)i) \frac{x^{n-i-1}}{(n-i-1)!} = e^{iA'_k(x)}.$$

Thus

$$A'_k(x) = \sum_{i=1}^{\infty} \frac{\mu_k(i+1)x^i}{i!} e^{iA'_k(x)} = M'_k\left(xe^{A'_k(x)}\right).$$

The theorem now follows from Lemma 6.3. □

7. $(n-2)$-connected graphs and matching complexes

Before we proceed to consider $(n-2)$-connected graphs, let us state some simple but useful facts about the general situation. What do maximal $(i-1)$-separable graphs on the n element set $[n]$ look like? Is is clear that each such graph is described by an $(i-1)$-set A and a partition $B \uplus C$ of $[n] \setminus A$ into two non-empty blocks B, C. The corresponding maximal $(i-1)$-separable graph is the complete graph on $[n]$ with all edges connecting B and C removed.

Now let G be an $(n-2)$-connected graph on n vertices, so $G \notin \Delta_n^{n-2}$. Then by the above description of maximal $(n-3)$-separable graphs the induced subgraph on any three vertices must contain at least two edges. Thus the complementary graph (i.e., the graph containing precisely the edges that are not in G) is a matching. The graphs on n vertices that are matchings form a simplicial complex, that we denote by M_n. We conclude the following.

Proposition 7.1. *The matching complex M_n is Alexander dual (in the sense of Proposition 10.4) to the complex Δ_n^{n-2}. In particular, there is an isomorphism*

$$\widetilde{H}_i(M_n) \cong \widetilde{H}^{\binom{n}{2}-i-3}(\Delta_n^{n-2}).$$

The matching complexes M_n have attracted attention for various reasons. In [BLVZ, Theorem 4.1] the matching complex M_n is shown to be topologically $(\lfloor \frac{n+1}{3} \rfloor - 2)$-connected, which implies that $\widetilde{H}_i(M_n) = 0$, for $i = 0, \ldots, \lfloor \frac{n+1}{3} \rfloor - 2$. We thus get the following corollary.

Corollary 7.2. *The cohomology of Δ_n^{n-2} vanishes in dimensions $i \geq \binom{n}{2} - \lfloor \frac{n+1}{3} \rfloor - 1$.*

The following table shows what we know about the homology groups $\widetilde{H}_i(M_n)$, based on the results of [BLVZ] for $n \leq 6$ and $n = 8$, and our own computations.

$n \backslash i$	0	1	2	3	4	5
2	0	0	0	0	0	0
3	$\mathbb{Z}^2$	0	0	0	0	0
4	$\mathbb{Z}^2$	0	0	0	0	0
5	0	$\mathbb{Z}^6$	0	0	0	0
6	0	$\mathbb{Z}^{16}$	0	0	0	0
7	0	torsion[1]	$\mathbb{Z}^{20}$	0	0	0
8	0	0	$\mathbb{Z}^{132}$	0	0	0
9	0	0	$\mathbb{Z}^{42} \oplus$ torsion[2]	$\mathbb{Z}^{70}$	0	0
10	0	0	torsion[3]	$\mathbb{Z}^{1216}$	0	0
11	0	0	0	$\mathbb{Z}^{1188} \oplus$ torsion[4]	$\mathbb{Z}^{252}$	0
12	0	0	0	torsion[5]	$\mathbb{Z}^{12440}$	0

Table 3: Homology groups $\widetilde{H}_i(M_n)$ of matching complexes

We see that the complexes Δ_n^{n-2} can have torsion, and that this phenomenon begins with Δ_7^5.

8. The Euler characteristic of the complex Δ_n^{n-3}

Consider the complex $(\Delta_n^{n-3})^*$ which is the Alexander dual of Δ_n^{n-3}, and the exponential generating function of its reduced Euler characteristic

$$F_n^{n-3}(x) := \sum_{n \geq 0} \widetilde{\chi}((\Delta_n^{n-3})^*) \frac{x^n}{n!}.$$

[1] There is $\mathbb{Z}_3$-torsion of rank 1. No $\mathbb{Z}_p$-torsion for $p = 2$, $5 \leq p \leq 17$.
[2] There is $\mathbb{Z}_3$-torsion of rank 8. No $\mathbb{Z}_p$-torsion for $p = 2$, $5 \leq p \leq 17$.
[3] There is $\mathbb{Z}_3$-torsion of rank 1. No $\mathbb{Z}_p$-torsion for $p = 2, 5, 7$.
[4] There is $\mathbb{Z}_3$-torsion of rank 35. No $\mathbb{Z}_p$-torsion for $p = 2, 5, 7$.
[5] There is $\mathbb{Z}_3$-torsion of rank 56. No $\mathbb{Z}_p$-torsion for $p = 2, 5, 7$.

The values of $\widetilde{\chi}((\Delta_n^{n-3})^*)$ in the degenerate cases $n \leq 3$ will appear from an explicitly calculated expansion below. We will express the reduced Euler characteristic of Δ_n^{n-3} in terms of an expression for this series.

Theorem 8.1. *We have that:*

$$F_n^{n-3}(x) = x - \frac{\exp(\frac{x}{2(1+x)}) + x - \frac{1}{4}x^2 - \frac{1}{8}x^4}{\sqrt{1+x}}.$$

The exponential generating function of the reduced Euler characteristic of Δ_n^{n-3} is then the sum of the real and imaginary parts of $-F_n^{n-3}(ix)$.

Proof: We will argue as we did for Δ_n^{n-2} in Section 7. If a graph G is $(n-3)$-connected then the induced subgraph on any 4 of its vertices contains either a vertex of degree 3 or a path of length 3. Thus in the complementary graph the induced subgraph on any 4 vertices is either contained in a 3-cycle or in a path of length 3. In particular, there are no 4-cycles and no vertices of degree 3 in the complementary graph. Thus the connected components of the complementary graph are paths of any length and cycles of length different from 4. Moreover, any graph in which every connected component is a cycle not of length 4 or a path is the complement of an $(n-3)$-connected graph, so $(\Delta_n^{n-3})^*$ consists of all such graphs. There are exactly $n!/2$ different paths of length n and $(n-1)!$ different n-cycles on an n-element vertex set. Now a direct application of the Exponential Formula 10.5 gives the result for the generating function of $(\Delta_n^{n-3})^*$. The remaining assertion follows from the fact that when passing from Δ_n^i to its Alexander dual the Euler characteristic changes by a factor of $-(-1)^{n(n-1)/2}$. □

The complex $(\Delta_n^{n-3})^*$ has maximal simplexes of dimensions $n-1$ and $n-2$ only. It is easily collapsible to a pure complex of dimension $n-2$. A Maple computation (see below) shows that neither the Euler characteristic of $(\Delta_n^{n-3})^*$ nor the Euler characteristic of Δ_n^{n-3} alternate in sign, so the pure complexes are certainly not all Cohen-Macaulay. The calculation shows that

$$F_n^{n-3}(x) = -1 - x + \frac{1}{4}x^4 + \frac{1}{20}x^5 + \frac{1}{20}x^6 - \frac{1}{27}x^7 - \frac{1}{224}x^8 - \frac{1}{480}x^9 + O(x^{10}).$$

We have also studied the slightly larger complex of graphs which are the disjoint union of cycles and paths of any lengths (i.e., graphs with maximum vertex degree at most 2). This is also a reasonable generalization of the matching complex, which is the complex of all graphs with maximum vertex degree at most 1. The Euler characteristic of the corresponding Alexander dual has almost the same generating function as for $(\Delta_n^{n-3})^*$. That generating function is

$$x - \frac{\exp(\frac{x}{2(1+x)}) + x - \frac{1}{4}x^2}{\sqrt{1+x}} =$$

$$-1 - x + \frac{1}{8}x^4 - \frac{3}{40}x^5 + \frac{1}{20}x^6 - \frac{1}{28}x^7 - \frac{17}{896}x^8 - \frac{7}{1920}x^9 - \frac{23}{2400}x^{10} + O(x^{11}).$$

The maximal simplices in the cycles-and-paths complex have dimension $n-1$ or $n-2$, and the complex can be collapsed to a pure $(n-2)$-dimensional complex. The generating function for the Euler characteristic shows that these collapsed complexes are not all Cohen-Macaulay.

9. Final Remarks

9.1. **Homology and Topology of Δ_n^i.** The results and computations presented in this paper suggest that there is probably no uniform statement that covers the topology of all complexes Δ_n^i. However, for $i \leq 2$ the homotopy type calculations for Δ_n^i give very nice answers. This is consistent with the graph theoretical study of not i-connected graphs, where there is a good structure theory only when $i \leq 3$ (see for example Chapter 6 of Lovász' book [L], or the survey article by Oxley [O] and the references therein).

As mentioned, there is a structure theory for 3-connected graphs. The 3-connected graphs on n vertices for which neither the deletion nor the contraction of an edge leads to a 3-connected graph were classified by Tutte (see Theorem 2.3 in [O]) as "wheels" and "whirls," both having $2n-2$ edges. Note that this does not provide a characterization of the deletion-minimally 3-connected graphs (the minimal non-faces of Δ_n^3), however it does show that no graph with less than $2n-2$ edges can be 3-connected. Hence, Δ_n^3 has a complete $2n-4$-skeleton, which shows that $\widetilde{H}_i(\Delta_n^3) = 0$ for $i < 2n-4$. This fact together with the following table leads to an interesting conjecture.

$n \backslash i$	0	1	2	3	4	5	6	7	8	9	10
3	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	$\mathbb{Z}^1$	0	0	0	0	0	0
5	0	0	0	0	0	0	$\mathbb{Z}^6$	0	0	0	0
6	0	0	0	0	0	0	0	0	$\mathbb{Z}^{36}$	0	0
7	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}^{240}$

Table 4: Homology groups $\widetilde{H}_i(\Delta_n^3)$

Conjecture 9.1: Δ_n^3 has the homotopy type of a wedge of $\frac{(n-3)(n-2)!}{2}$ spheres of dimension $2n-4$.

For general i the situation (concerning Δ_n^i) seems to be far more complicated. However, we would like to remark that for $i = 2, 3$ the known Betti numbers are (up to sign) the Lah-numbers $L_{n-2,1}$ and $L_{n-2,2}$ (see [Co, p.165-166]). This coincidence unfortunately fails for $i = 4$, which is easily seen by comparing $L_{n-2,3}$ with $\widetilde{\chi}(\Delta_n^4)$ for $n = 6$. For $i > 3$ no good structure theory for i-connected graphs is known, and the results of Section 7 on $(n-2)$-connected graphs indicate that the topology of Δ_n^i will not behave nicely for all i. Nevertheless, the Alexander Duality with the matching complexes M_n encourages a closer look at the complexes Δ_n^{n-2}. Surprisingly, the prime 3 seems to play a special role in the topology of these complexes. We have not detected p-torsion in the homology of Δ_n^{n-2} for any prime $p \neq 3$.

Question 9.2: Does the homology of M_n have p-torsion for any prime $p \neq 3$?

Even more surprising is the fact that the same prime 3 seems to play an analogous role for the matching complexes $M_{n,n}$ on complete bipartite graphs $K_{n,n}$, also called chessboard complexes (see [BLVZ]).

$n\backslash i$	0	1	2	3	4	5	6
1	0	0	0	0	0	0	0
2	$\mathbb{Z}$	0	0	0	0	0	0
3	0	$\mathbb{Z}^4$	0	0	0	0	0
4	0	0	$\mathbb{Z}^{15}$	0	0	0	0
5	0	0	$\mathbb{Z}_3$	$\mathbb{Z}^{56}$	0	0	0
6	0	0	0	$\mathbb{Z}^{25} \oplus$ torsion[1]	$\mathbb{Z}^{210}$	0	0
7	0	0	0	0	$\mathbb{Z}^{588} \oplus$ torsion[2]	$\mathbb{Z}^{792}$	0
8	0	0	0	0	torsion[3]	?	?

Table 5: Homology groups $\widetilde{H}_i(M_{n,n})$ for the bipartite matching complexes

It is easy to see that $M_{n,n}$ collapses to an $(n-2)$-dimensional complex. Hence homology is free in dimension $n-2$ and vanishes in higher dimensions. Looking at the table and the footnotes the following question naturally occurs.

Question 9.3: Is the homology of $M_{n,n}$ free except for 3-torsion ?

9.2. **Homology and Topology of $\Delta^2_{n,k}$.**

For k-graph complexes the problem of determining the topology of the complex of separable k-graphs (the $i = 2$ case) seems to be the most important. The complexes $\Delta^2_{n,k}$ play the same role in the study of spaces of "knots" for which k-fold self-intersections are forbidden as the complexes Δ^2_n play for ordinary knots.

Question 9.4: What is the homology and homotopy type of $\Delta^2_{n,k}$?

The evidence from Section 5 leads us to anticipate the following answer for $k = 3$.

Conjecture 9.5: $\Delta^2_{n,3}$ is homotopy equivalent to a wedge of $(n-4)$-spheres.

A natural approach to this question is through further combinatorial study of the lattices $\Sigma_{n,k}$ defined in Section 5.

Conjecture 9.6: The lattice $\Sigma_{n,k}$ is shellable.

This is open in all cases, except for the somewhat degenerate cases $n < 2k-1$ when $\Sigma_{n,k}$ is a truncated Boolean algebra. If Conjecture 9.6 were verified for $k = 2$ it would reprove Theorem 3.1, and it would via Lemma 5.10 imply the truth of Conjecture 9.5. If Conjecture 9.6 were verified for $k = 3$ it would also imply the truth of Conjecture 9.5. If Conjecture 9.6 were verified for $k > 3$ it would via Theorem 5.7(iv) and the results of [BW] imply the truth of the following.

[1] There is $\mathbb{Z}_3$-torsion of rank 10. No $\mathbb{Z}_p$-torsion for $p = 2, 5, 7$

[2] There is $\mathbb{Z}_3$-torsion of rank 66. No $\mathbb{Z}_p$-torsion for $p = 2, 5, 7$

[3] There is $\mathbb{Z}_3$-torsion of rank 1. This group is finite according to [FH].

Conjecture 9.7: If $k > 3$ and $n \geq 2k-1$, then $\Delta^2_{n,k}$ is homotopy equivalent to a wedge of spheres. Furthermore, the dimensions of the spheres are precisely $n-4-t(k-3)$ for $1 \leq t \leq \lfloor \frac{n-1}{k-1} \rfloor$.

9.3. **Generating series of Euler characteristics.** Let $F_i(x) = \sum_{n\geq 0} \widetilde{\chi}(\Delta^i_n)\frac{x^n}{n!}$ and $G_i(x) = \sum_{n\geq 0} \widetilde{\chi}((\Delta^{n-i}_n)^*)\frac{x^n}{n!}$. By the results presented in this paper we get the following table:

i	$F_i(x)$	$G_i(x)$
1	$ln(1+x)$	-1
2	$(1-x)\log(1-x)+1+x$	$-\exp(x-\frac{x^2}{2})$
3		$x - \frac{\exp(\frac{x}{2(1+x)})+x-\frac{1}{4}x^2-\frac{1}{8}x^4}{\sqrt{1+x}}$

Table 6: Generating functions of the Euler characteristics of Δ^i_n and $(\Delta^{n-i}_n)^*$

We cannot formulate a conjecture about the entries in this table for $i > 3$. Nevertheless, even though the actual homology computation may be too difficult, the generating series may be computable for a few more cases. Assuming a positive answer to Conjecture 9.1, we get

$$F_3(x) = (x - \frac{3}{2})\log(1-x) + 1 - \frac{3}{2}x + \frac{1}{4}x^2.$$

9.4. **The representation of the symmetric group.** All complexes $\Delta^i_{n,k}$ are invariant under the action of the symmetric group S_n. This action determines a linear representation of S_n on each homology group $\widetilde{H}_j(\Delta^i_{n,k})$. For fixed n, k and i, the alternating sum of the characters of the given representations is a virtual character of S_n that we denote by $\omega^i_{n,k}$. For $k = 2$ and $i = 1, 2$ this is an actual character (up to sign) and satisfies

$$(-1)^{n+1}\omega^2_{n+1,k} = (sign_n\omega^1_{n,k}) \uparrow^{S_{n+1}}_{S_n} + sign_{n+1}\omega^1_{n+1,k}.$$

From looking at the dimensions of the homology modules it is clear that the analogous formula for $k \geq 3$ does not hold. We have also seen that homology of $\Delta^i_{n,k}$ is not torsion-free in general. On the other hand, [RR] has demonstrated that for the closely related matching complexes it is possible to determine the representations on the rational homology. Thus it is reasonable to ask:

Question 9.8: What is the character of S_n on each non-vanishing rational homology group of $\Delta^i_{n,k}$?

The character $\omega^2_n = \omega^2_{n,2}$ determined in Section 4 has recently appeared in several different areas of mathematics. First in the work of C. A. Robinson & S. Whitehouse [RW] and S. Whitehouse [Wh] on gamma-homology of algebras and later in work of E. Getzler & M. Kapranov [GK] on operads, O. Mathieu [Ma] on hyperplane arrangements and symplectic geometry, in the work of M. Kontsevich [K] on Lie algebras and symplectic geometry, and in the work of P. Hanlon [H2], P. Hanlon & R.P. Stanley [HS] and S. Sundaram [Su] in a combinatorial and representation-theoretic context. It seems mysterious that the same character pops up in so many seemingly unrelated places.

Question 9.9: What are the deeper connections between the various contexts where the character ω_n^2 appears ?

The analogous question for the character $\omega_{n,2}^1 = sign_n \cdot \ell ie_n$ has been studied quite extensively (see for example [Ba, BaBe, Re]), and for that case much detailed information is known. An important aspect of the work in [Ba] and [BaBe] is the construction of explicit bases for the modules under consideration. Thus a first step towards an answer to Question 9.9 could be a solution of the following problem.

Problem 9.10: Describe a combinatorial basis for the homology of Δ_n^2.

A positive answer to the shellability conjecture 9.6 for $k = 2$ could via the induced shelling basis (see [BW]) lead to progress on Problem 9.10.

10. Notation and Tools

In this short section we will summarize the main tools that we use in the study of the complexes $\Delta_{n,k}^i$. We refer the reader to the survey paper [B] for more details and references.

Let P be a finite partially ordered set – *poset* for short. If P has a unique minimum element $\hat{0}$ and a unique maximum element $\hat{1}$, we denote by $\overline{P}$ the *proper part* of P, that is the poset obtained by removing from P the elements $\hat{0}$ and $\hat{1}$. By $\Delta(P)$ we denote the simplicial complex of all chains in P. The complex $\Delta(P)$ is called the *order complex* of P.

By convention we include the empty set $\emptyset$ in every simplicial complex. For any simplicial complex Δ, the *face lattice* $\mathcal{L}at(\Delta)$ is the poset of faces of Δ, ordered by inclusion and enlarged by an additional greatest element $\hat{1}$. Then the order complex $\Delta(\overline{\mathcal{L}at(\Delta)})$ of the proper part of $\mathcal{L}at(\Delta)$ is homeomorphic to Δ. Indeed, $\Delta(\overline{\mathcal{L}at(\Delta)})$ is the barycentric subdivision of Δ.

For a poset P and $p \in P$ we denote by $P_{\leq p}$ the sub-poset $\{p' \mid p' \in P;\ p' \leq p\ \}$. The posets $P_{\geq p}$, $P_{<p}$ and $P_{>p}$ are analogously defined. For $p \leq p'$ in P we denote by $[p,p']$ the *closed interval* $P_{\geq p} \cap P_{\leq p'}$ in P, and by (p,p') the *open interval* $P_{>p} \cap P_{<p'}$.

For a poset P we denote by μ_P the $\mathbb{Z}$-valued *Möbius function* (see [St3]), defined recursively on the intervals of P by $\mu_P(x,x) = 1$ and $\mu_P(x,y) = -\sum_{x \leq z < y} \mu_P(x,z)$ if $x < y$.

By a map $f : P \to Q$ of posets we always mean a poset homomorphism (i.e., $x \leq y$ implies $f(x) \leq f(y)$). For an element $q \in Q$ we denote by $f_{\leq}^{-1}(q)$ the preimage of $Q_{\leq q}$ under f. The poset $f_{\geq}^{-1}(q)$ is analogously defined.

Proposition 10.1 (Quillen Fiber Lemma [Q])**.** *Let $f : P \to Q$ be a map of posets. If $\Delta(f_{\leq}^{-1}(q))$ is contractible for all $q \in Q$ then $\Delta(P)$ and $\Delta(Q)$ are homotopy equivalent.*

A map $f : P \to P$ from a poset to itself is called a *closure operator* if $f(x) \geq x$ and $f(f(x)) = f(x)$ for all $x \in P$. The Quillen Fiber Lemma immediately implies the fact that closure operators preserve the homotopy type.

Corollary 10.2 (Closure Lemma). *Let $f : P \to P$ be a closure operator on the partially ordered set P. Then $\Delta(P)$ and $\Delta(f(P))$ are homotopy equivalent.*

If the poset P is a *lattice* (i.e., suprema, denoted by "$\vee$", and infima, denoted by "$\wedge$", exist) then there is another tool for computing the homotopy type. Note that if P is a finite lattice then there is a least element $\hat{0}$ and a largest element $\hat{1}$ in P. For an arbitrary element $p \in P$ we say that $a \in P$ is a *complement* of p if $p \wedge a = \hat{0}$ and $p \vee a = \hat{1}$.

Proposition 10.3 (Homotopy Complementation Formula [BWa]).

(i) *Let P be a poset and $A \subseteq P$ an antichain. Assume $\Delta(P \setminus A)$ is contractible. Then $\Delta(P)$ is homotopy equivalent to*

$$\bigvee_{x \in A} \Sigma\Big(\Delta(P_{<x}) * \Delta(P_{>x})\Big).$$

(ii) *Let P be the proper part of a lattice and let Co be the set of complements of some element $p \neq \hat{0}, \hat{1}$. Then $\Delta(P \setminus Co)$ is contractible.*

In the formulation of the proposition $\bigvee$ denotes the wedge product, Σ denotes the suspension and $*$ denotes the join of topological spaces.

Our next tool is the combinatorial version of a standard duality theorem from algebraic topology.

Proposition 10.4 (Combinatorial Alexander Duality). *Let Δ be a finite simplicial complex on vertex set V and define*

$$\Delta^* = \{B \subseteq V \mid V \setminus B \notin \Delta\}.$$

Then

$$\widetilde{H}_i(\Delta) \cong \widetilde{H}^{|V|-i-3}(\Delta^*).$$

This is derived as follows. The usual Alexander duality theorem (see e.g. Munkres [Mu]) says that

$$\widetilde{H}_i(A) \cong \widetilde{H}^{n-i-1}(S^n \setminus A)$$

for any compact subset A of the n-sphere S^n. In our situation, let $P = 2^V \setminus \{\emptyset, V\}$. This truncated Boolean algebra is the proper part of the face lattice of the boundary complex of a simplex, so $\Delta(P) \cong S^{|V|-2}$. Now let A be the realization of $\Delta(\overline{\mathcal{L}at(\Delta)})$ as a subspace of $\Delta(P)$. It is easy to see that $\Delta(P \setminus \overline{\mathcal{L}at(\Delta)})$ is a strong deformation retract of $S^{|V|-2} \setminus A$, and since $P \setminus \overline{\mathcal{L}at(\Delta)} \cong \overline{\mathcal{L}at(\Delta^*)}$ the result follows.

Finally, we recall a result from enumerative combinatorics. For a number sequence $(a_n)_{n\geq 0}$ the formal power series $\sum_{n\geq 0} a_n \frac{x^n}{n!}$ is called its *exponential generating function.*

Proposition 10.5 (Exponential formula)**.** *Suppose that two functions $a, b : \mathbb{N} \longrightarrow \mathbb{Z}$ are given such that*

$$b(n) = \sum_{S_1|\cdots|S_t \in \Pi_n} a(|S_1|) \cdots a(|S_t|), \quad n \geq 1,$$

where the sum ranges over all set partitions of $[n]$ and $a(0) = 0$, $b(0) = 1$. Then the exponential generating functions $A(x) := \sum_{n=0}^{\infty} \frac{a(n)x^n}{n!}$ and $B(x) := \sum_{n=0}^{\infty} \frac{b(n)x^n}{n!}$ satisfy

$$B(x) = e^{A(x)}.$$

For the proof see [St1, St4].

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