2-STACK SORTABLE PERMUTATIONS WITH A GIVEN NUMBER OF RUNS

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ABSTRACT. Using earlier results we prove a formula for the number $W_{(n,k)}$ of 2-stack sortable permutations of length n with k runs, or in other words, $k-1$ descents. This formula will yield the suprising fact that there are as many 2-stack sortable permutations with $k-1$ descents as with $k-1$ ascents. We also prove that $W_{(n,k)}$ is unimodal in k, for any fixed n.

1. INTRODUCTION

1.1. **Our main results.** In this paper we are going to show that the number of 2-stack sortable permutations of length n with $k-1$ ascents is equal to the number $T_{(n,k)}$ of $\beta(1,0)$-trees on $n+1$ nodes with k leaves. (See Section 2 for the definition of $\beta(1,0)$-trees). This, and results from [2] and [5] will enable us to easily show that

$$W_{(n,k)} = \frac{(n+k-1)!(2n-k)!}{k!(n+1-k)!(2k-1)!(2n-2k+1)!}, \tag{1}$$

which formula is symmetric in k and $n+1-k$.

1.2. **Background and Definitions.** In what follows, permutations of length n will be called n-permutations. We say that i is a *descent* of the permutation $p = (p_1, p_2, \cdots, p_n)$ if $p_i > p_{i+1}$. Similarly, j is an ascent of p if $p_j < p_{j+1}$. If p has d descents, then p decomposes into $d+1$ increasing sequences of consecutive entries, and we then say that p has $d+1$ *runs*. We will use the concept of descents and runs interchangeably, according to the current context.

The stack-sorting operation Π can be defined on the set of all n-permutations as follows. Let $p = p_L n p_R$ be an n-permutation, with p_L and p_R respectively denoting its subword before and after the maximal entry. Let $\Pi(p) = \Pi(p_L)\Pi(p_R)n$, where p_L and p_R are defined recursively by this same rule. For a nonrecursive, algorithmic definition, or the origin of the notion see [6], [8].

A permutation p is called t-stack sortable if $\Pi^t(p)$ is the identity permutation.

The set of 1-stack sortable n-permutations is easy to characterize by the following notion of pattern avoidance. Let $q = (q_1, q_2, \cdots, q_k)$ be a k-permutation and let $p = (p_1, p_2, \ldots, p_n)$ be an n-permutation. We say that p contains a q-subsequence (or q-pattern) if there exists $1 \le i_{q_1} < i_{q_2} < \cdots < i_{q_k} \le n$ such that $p_{i_1} < p_{i_2} < \cdots < p_{i_k}$. We say that p avoids q if p contains no q-subsequence.

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For example, p avoids 231 if it cannot be written as $\cdots, a, \cdots, b, \cdots c, \cdots$ so that $c < a < b$. It is easy to show [6] [8] [9] that a permutation is 1-stack sortable if and only if it avoids the pattern 231. In paticular, the number of 1-stack sortable permutations is therefore C_n, the n-th Catalan number.

The set of 2-stack sortable permutations is much more complex. For example, it is not true that a subword of a 2-stack sortable permutation is always 2-stack sortable; for 35241 is 2-stack sortable, while 3241 is not. Similarly, it is far more difficult to find a formula for the number W_n of these permutations than for that of 1-stack sortable ones. See [10] [4] or [3] for a proof of the formula $W_n = 2(3n)!/((n+1)!(2n+1)!)$ and see [8] for a characterization of 2-stack sortable permutations.

In this paper we are going to construct a bijection between the set of 2-stack sortable n-permutations with k runs and that of $\beta(1,0)$-trees on $n+1$ vertices having k leaves. This bijection will be based on a bijection in [4], which mapped these permutations into a class of labeled binary trees. (Our trees are not binary). The set of these trees has recently been shown to be equinumerous [2] to a certain class of planar maps, and the number of those planar maps is known [5]. Therefore, we are going to obtain a *formula* for the number $W_{(n,k)}$ of 2-stack sortable permutations with k runs, which is a bit surprising, given that even the formula for W_n was fairly difficult to prove.

What is even more surprising is that this formula reveals that *there are as many 2-stack sortable permutations with k descents as with k ascents.* We do not see any obvious reason why this should be true. A direct combinatorial proof of this fact would be highly desirable, but it may not be easy as we do not even have such a proof for the same fact in the much simpler case of 132-avoiding permutations. We also prove that $W_{(n,k)}$ is unimodal in k for any fixed n.

We point out that this is the second case recently [1] when a class of the description trees introduced in [2] to enumerate planar maps has been used to enumerate permutations.

2. The correspondence between trees and permutations

Let p be an n-permutation. A *right-to-left maximum* of p is an entry which is larger than all entries it precedes. If the right-to-left maxima of p are $a_1 > a_2 > \cdots > a_t$, then clearly $a_1 = n$ and a_t is the last entry of p. In the next three paragraphs we closely follow the characterization of 2-stack sortable permutation given in [4].

Write p in the form $s_1 a_1 s_2 a_2 \cdots s_t a_t$. Here the s_i are strings of entries located between two right-to-left maxima. We say that p is of type 1 if the entry $a_t - 1$ is part of the string s_t. Otherwise we say that p is of type 2. The 1-permutation 1, denoted ϵ, is of type 2. Then the following holds.

Lemma 1. [4] *There is a bijection F from the set of 2-stack sortable n-permutations of type 1 to that of 2-stack sortable $(n-1)$-permutations with a right-to-left maximum marked. Moreover, F preserves the number of descents and does not decrease the number of right-to-left maxima.*

Proof:[4] Let p be of type 1, with the above $s_1 a_1 s_2 a_2 \cdots s_t a_t$ decomposition. Define F as follows. Delete a_t from the end of p and decrement all entries larger than a_t

by 1. Then $a_t - 1 \in s_t$ becomes a right-to-left maximum in the new permutation, in fact, it becomes the tth right-to-left maximum. Mark this entry to get $F(p)$. This map is reversible by simply adding an entry one larger than the marked vertex to the end of $F(p)$ and increment all the larger entries by one.

So F is a bijection. Moreover, F does not create or destroy any existing descents as it deletes the last entry of p, and that entry was larger than its predecessor. The number of right-to-left maxima is not decreased by F, because $a_t - 1$ becomes a right-to-left maximum instead of a_t.

Finally, one sees easily [4] that $F(p)$ is 2-stack sortable if and only if p is, and the proof is complete. □

We need one more lemma from [4]. We omit its proof as we will not need any of its rather complicated machinery. Let $rl(p)$ be the number of right-to-left maxima of p and let $desc(p)$ be the number of descents in p.

Lemma 2. *Let $n \geq 2$. Then there is a bijection G from the set of 2-stack sortable n-permutations of type 2 onto the set of ordered pairs (p_1, p_2) where*

1. *p_1 is any 2-stack sortable n_1-permutation, p_2 is either a 2-stack sortable n_2-permutation of type 1, or ϵ,*
2. *$n_1 + n_2 = n$,*
3. *$rl((p_1) + rl(p_2) = rl(p)$,*
4. *$desc(p_1) + desc(p_2) + 1 = desc(p)$.*

We are going to iterate the decomposition of the above Lemma. In other words, we take a 2-stack sortable n-permutation p of type 2, decompose it into (p_1, p_2) by the bijection G of [4], then, if p_1 was of type 2 (and not ϵ), then we apply G to p_1 to get (p_{11}, p_{12}). Again, if p_{11} is of type 2 (and not ϵ), the we apply G to p_{11}, and so on. We stop when both elements of the current decomposition are either of type 1, or ϵ. As each step of this algorithm uses a bijection, p can be recovered from its final decomposition. This proves the following Corollary.

Corollary 1. *There is a bijection H from the set of 2-stack sortable n-permutations onto the set of string $(q_1, q_2, \cdots, q_r)$, where $r \geq 1$ and*

- *for all i, q_i is either a 2-stack sortable permutation of type 1, or ϵ and*
- *$\sum_{i=1}^{r} |q_i| = n$,*
- *$\sum_{i=1}^{r} rl(q_i) = rl(p)$,*
- *the total number of runs in the q_i equals the number of runs in p.*

The last part follows from the last part of Lemma 2 and the fact that the number of runs in any permutation is one more than that of descents.

Cori, Jacquard and Schaeffer [2] give the following definition in their study of planar maps.

Definition 1. *A rooted plane tree with nonnegative integer labels $l(v)$ on each of its vertices v is called a $\beta(1,0)$-tree if it satisfies the following conditions:*

- *if v is a leaf, then $l(v) = 1$,*
- *if v is the root and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) = \sum_{i=1}^{k} l(v_k)$,*

- *if v is a nonroot internal node and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) \leq \sum_{i=1}^{k} l(v_k)$.*

Example 1. Two $\beta(1,0)$-trees are shown in Figures 1a and 1b.

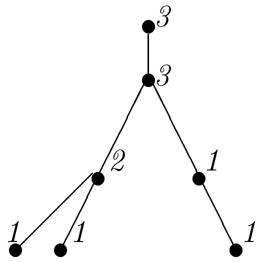

Figure 1a

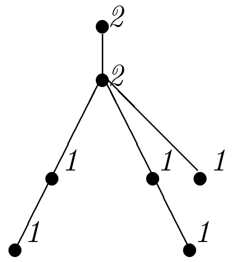

Figure 1b

Now we are in position to state and prove the main result of this paper.

Theorem 1. *For any positive integer n and for any positive integer $k \leq n$, there is a bijection B from the set of 2-stack sortable n-permutations with k runs onto that of $\beta(1,0)$-trees on $n+1$ nodes having k leaves. Therefore, $W_{(n,k)} = T_{(n,k)}$. Moreover, for any 2-stack sortable n-permutation p, the label of the root of $B(p)$ equals $rl(p)$.*

Proof: We are going to prove the statements of the Theorem by induction on n and k. For $k = 1$ and any n, the statements are clearly true: there is one n-permutation with one run, namely $12\cdots n$, and there is one $\beta(1,0)$-tree having only one leaf, namely the one consisting of a single path. The root label of this tree is 1, and the number of right-to-left maxima of this permutation is 1, so the initial case is proved.

Now let $k > 1$ and suppose we already know the statements for all permutations shorter than n. Let p be any 2-stack sortable n-permutation with k runs.

- First suppose p is of type 1. Consider $F(p)$, where F is the bijection defined in the proof of Lemma 1. Forget for now which vertex of $F(p)$ is marked, then $F(p)$ is a 2-stack sortable $(n-1)$-permutation with k runs. So, by our induction hypothesis, B associates a $\beta(1,0)$-tree T' on n nodes to it, which has k leaves and whose root label is equal to $rl(F(p)) \geq rl(p)$. Now add one node above the root x of T' and "recall" which right-to-left maximum of $F(p)$ was marked. If it was the tth right-to-left maximum, that is, if p had exactly t right-to-left maxima, then let the label of this new root x (and necessarily, that of the old root, which is the only child of x) be t. Then we certainly get a $\beta(1,0)$-tree T as $t \leq rl(F(p))$. We set $B(p) = T$. By the argument of Lemma 1 we see that p can be recovered from T (as T', and thus $F(p)$ can), and that T has k leaves, and its root label is $rl(p)$ as it should be.
- If $p = \epsilon$, then $B(p)$ is the only $\beta(1,0)$-tree on two nodes. We point out that here, as well as in the previous case, the root of $B(p)$ has only one child.
- Now suppose p is of type 2. Take $H(p) = (q_1, q_2, \cdots, q_r)$ as defined in Corollary 1 and consider $B(q_i)$ for all i as defined by the previous two cases. $B(q_i)$ has $q_i + 1$ nodes, so by contracting the roots of all $B(q_i)$ to one node (keeping the $B(q_i)$ ordered from left to right), we get a tree T with $1 + \sum_{i=1}^{r} |q_i| = n + 1$

nodes. We set $B(p) = T$. The components q_i can be recovered from T by simply cutting off its root and adding a new root to the top of each component obtained. (Recall that in all the $B(q_i)$, the root has only one child). The number of leaves of T equals the sum of the leaves of the $B(q_i)$, so by the previous two cases and by Corollary 1, it equals the number of runs of p. The root label of T is by definition the sum of the labels of its children, which is, again by the previous two cases and by Corollary 1, equal to $rl(p)$.

This completes the proof of the Theorem. □

We remark that our algorithm certainly sets up a bijection between the set of *all* 2-stack sortable n-permutations and that of *all* $\beta(1,0)$-trees on $n+1$ vertices.

3. New enumerative results

Exercise 2.9.8.b in [5] shows that the number of nonseparable rooted planar maps with $f+1$ faces and $p+1$ vertices is equal to

$$\frac{(2f+p-2)!(2p+f-2)!}{f!p!(2f-1)!(2p-1)!}. \tag{2}$$

It is shown in [2] that the number of these maps equals the number of $\beta(1,0)$-trees on $n+1$ nodes with $k=f+1$ leaves and $n+1-k=p+1$ internal nodes. (Here the root is counted as an internal node). By Theorem 1 this implies the result we announced in the introduction.

Theorem 2. *Let $W_{(n,k)}$ be the number of 2-stack sortable n-permutations with k runs. Then for all $1 \leq k \leq n$ we have*

$$W_{(n,k)} = \frac{(n+k-1)!(2n-k)!}{k!(n+1-k)!(2k-1)!(2n-2k+1)!}.$$

In particular, equation (2) shows that the roles of $f+1=k$ and $p+1=n+1-k$ are symmetric. In other words, $W_{(n,k)} = W_{(n,n+1-k)}$. A permutation with k runs has $k-1$ descents, so have proved the following surprising Corollary.

Corollary 2. *The number of 2-stack sortable n-permutations with k descents equals that of those with k ascents.*

A routine computation yields that $W_{(n,k)}/W_{(n,k-1)} > 1$ if and only if $k \leq (n+1)/2$, and we already know from the previous Corollary that if $n=2k$, then $W_{(n,k)} = W_{(n,k+1)}$. So we have proved the following Corollary.

Corollary 3. *For fixed n, the sequence $W_{(n,k)}$ is unimodal in k, and its peak is at $k = [(n+1)/2]$.*

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