PIERI-TYPE FORMULAS FOR MAXIMAL ISOTROPIC GRASSMANNIANS VIA TRIPLE INTERSECTIONS

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Abstract. We give an elementary proof of the Pieri-type formula in the cohomology of a Grassmannian of maximal isotropic subspaces of an odd orthogonal or symplectic vector space. This proof proceeds by explicitly computing a triple intersection of Schubert varieties. The decisive step is an exact description of the intersection of two Schubert varieties, from which the multiplicities (which are powers of 2) in the Pieri-type formula are immediately obvious.

Introduction

The goal of this paper is to give an elementary geometric proof of Pieri-type formulas in the cohomology of Grassmannians of maximal isotropic subspaces of odd orthogonal or symplectic vector spaces. For this, we explicitly compute a triple intersection of Schubert varieties, where one is a special Schubert variety. Previously, Sertöz [16] had studied such triple intersections in orthogonal Grassmannians, but was unable to determine the intersection multiplicities and obtain a formula.

These multiplicities are either 0 or powers of 2. Our proof explains them as the intersection multiplicity of a linear subspace (defining the special Schubert variety) with a collection of quadrics and linear subspaces (determined by the other two Schubert varieties). This is similar to triple intersection proofs of the classical Pieri formula (*cf.* [7][5, p. 203][4, §9.4]) where the multiplicities (0 or 1) count the number of points in the intersection of linear subspaces. A proof of the Pieri-type formula for classical flag varieties [17] was based upon those ideas. Similarly, the ideas here provide a basis for a proof of Pieri-type formulas in the cohomology of symplectic flag varieties [1].

These Pieri-type formulas are due to Hiller and Boe [6], whose proof used the Chevalley formula [2]. Another proof, using the Leibnitz formula for symplectic and orthogonal divided differences, was given by Pragacz and Ratajski [14]. These formulas also arise in the theory of projective representations of symmetric groups [15, 9] as product formulas for Schur P- and Q-functions, and were first proven in this context by Morris [11]. The connection of Schur P- and Q-functions to geometry was established by Pragacz [12] (see also [13]).

In Section 1, we give the basic definitions and state the Pieri-type formulas in both the orthogonal and symplectic cases, and conclude with an outline of the proof in the

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orthogonal case. Since there is little difference between the proofs in each case, we only do the orthogonal case in full. In Section 2, we describe the intersection of two Schubert varieties, which we use in Section 3 to complete the proof.

1. The Grassmannian of maximal isotropic subspaces

Let V be a $(2n+1)$-dimensional complex vector space equipped with a non-degenerate symmetric bilinear form β and W a $2n$-dimensional complex vector space equipped with a non-degenerate alternating bilinear form, also denoted β. A subspace H of V or of W is *isotropic* if the restriction of β to H is identically zero. Isotropic subspaces have dimension at most n. The *Grassmannian of maximal isotropic subspaces* B_n or $B(V)$ (respectively C_n or $C(W)$) is the collection of all isotropic n-dimensional subspaces of V (respectively of W). The group $So_{2n+1}\mathbb{C} = \mathrm{Aut}(V,\beta)$ acts transitively on B_n with the stabilizer P_0 of a point a maximal parabolic subgroup associated to the short root, hence $B_n = So_{2n+1}\mathbb{C}/P_0$. Similarly, $C_n = Sp_{2n}\mathbb{C}/P_0$, where P_0 is a maximal parabolic associated to the long root.

Both B_n and C_n are smooth complex manifolds of dimension $\binom{n+1}{2}$. While they are not isomorphic if $n > 1$, they have identical Schubert decompositions and Bruhat orders. Another, deeper, connection is discussed in Remark 2.8. We describe the Schubert decomposition. For an integer j, let $\overline{\jmath}$ denote $-j$. Choose bases $\{e_{\overline{n}}, \ldots, e_n\}$ of V and $\{f_{\overline{n}}, \ldots, f_n\}$ of W for which:

$$\beta(e_i, e_j) \;=\; \begin{cases} 1 & \text{if } i = \overline{\jmath} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta(f_i, f_j) \;=\; \begin{cases} j/|j| & \text{if } i = \overline{\jmath} \\ 0 & \text{otherwise} \end{cases}.$$

For example, $\beta(e_1, e_0) = \beta(f_{\overline{2}}, f_1) = 0$ and $\beta(e_0, e_0) = \beta(f_{\overline{1}}, f_1) = -\beta(f_1, f_{\overline{1}}) = 1$. Schubert varieties are determined by sequences

$$\lambda :\; n \geq \lambda_1 > \lambda_2 > \cdots > \lambda_n \geq \overline{n}$$

whose set of absolute values $\{|\lambda_1|, \ldots, |\lambda_n|\}$ equals $\{1, 2, \ldots, n\}$. Let $\mathbb{SY}_n$ denote this set of sequences. The Schubert variety X_λ of B_n is

$$\{H \in B_n \mid \dim H \cap \langle e_{\lambda_j}, \ldots, e_n\rangle \geq j \text{ for } 1 \leq j \leq n\}$$

and the Schubert variety Y_λ of C_n

$$\{H \in C_n \mid \dim H \cap \langle f_{\lambda_j}, \ldots, f_n\rangle \geq j \text{ for } 1 \leq j \leq n\}.$$

Both X_λ and Y_λ have codimension $|\lambda| := \lambda_1 + \cdots + \lambda_k$, where $\lambda_k > 0 > \lambda_{k+1}$. Given $\lambda, \mu \in \mathbb{SY}_n$, we see that

$$X_\mu \supset X_\lambda \iff Y_\mu \supset Y_\lambda \iff \mu_j \leq \lambda_j \text{ for } 1 \leq j \leq n.$$

Define the *Bruhat order* by $\mu \leq \lambda$ if $\mu_j \leq \lambda_j$ for $1 \leq j \leq n$. Note that $\mu \leq \lambda$ if and only if $\mu_j \leq \lambda_j$ for those j with $0 < \mu_j$.

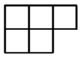
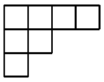
Example 1.1. Suppose $n = 4$. Then $X_{3\,2\,\overline{1}\,\overline{4}}$ consists of those $H \in B_4$ such that

$$\dim H \cap \langle e_3, e_4\rangle \geq 1,\; \dim H \cap \langle e_2, e_3, e_4\rangle \geq 2,\; \text{and} \dim H \cap \langle e_{\overline{1}}, \ldots, e_4\rangle \geq 3.$$

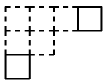
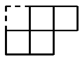
Define $P_\lambda := [X_\lambda]$, the cohomology class Poincaré dual to the fundamental cycle of X_λ in the homology of B_n. Likewise set $Q_\lambda := [Y_\lambda]$. Since Schubert varieties are closures of cells from a decomposition into (real) even-dimensional cells, these *Schubert classes* $\{P_\lambda\}$, $\{Q_\lambda\}$ form bases for integral cohomology:

$$H^*B_n \;=\; \bigoplus_\lambda P_\lambda \cdot \mathbb{Z} \qquad \text{and} \qquad H^*C_n \;=\; \bigoplus_\lambda Q_\lambda \cdot \mathbb{Z}.$$

Each $\lambda \in \mathbb{SY}_n$ determines and is determined by its diagram, also denoted λ. The diagram of λ is a left-justified array of $|\lambda|$ boxes with λ_j boxes in the jth row, for $\lambda_j > 0$. Thus

$3\,2\,\overline{1}\,\overline{4} \longleftrightarrow$  and $4\,2\,1\,\overline{3} \longleftrightarrow$ .

The Bruhat order corresponds to inclusion of diagrams. Given $\mu \le \lambda$, let λ/μ be their set-theoretic difference. For instance,

$4\,2\,1\,\overline{3}/3\,2\,\overline{1}\,\overline{4} \longleftrightarrow$  and $3\,2\,\overline{1}\,\overline{4}/1\,\overline{2}\,\overline{3}\,\overline{4} \longleftrightarrow$ .

Two boxes are connected if they share a vertex or an edge; this defines *components* of λ/μ. We say λ/μ is a *skew row* if $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \cdots \ge \mu_n$ equivalently, if λ/μ has at most one box in each column. Thus $4\,2\,1\,\overline{3}/3\,2\,\overline{1}\,\overline{4}$ is a skew row, but $3\,2\,\overline{1}\,\overline{4}/1\,\overline{2}\,\overline{3}\,\overline{4}$ is not.

The *special Schubert class* $p_m \in H^*B_n$ ($q_m \in H^*C_n$) is the class whose diagram consists of a single row of length m. Hence, $p_2 = P_{2\,\overline{1}\,\overline{3}\,\overline{4}}$. A *special Schubert variety* X_K (Y_K) is the collection of all maximal isotropic subspaces which meet a fixed isotropic subspace K nontrivially. If $\dim K = n+1-m$, then $[X_K] = p_m$ and $[Y_K] = q_m$.

Theorem 1.2 (Pieri-type Formula)**.** *For any $\mu \in \mathbb{SY}_n$ and $1 \le m \le n$,*

1. $P_\mu \cdot p_m \;=\; \displaystyle\sum_{\lambda/\mu \ \textit{skew row}} 2^{\delta(\lambda/\mu)-1}\, P_\lambda$ *and*
2. $Q_\mu \cdot q_m \;=\; \displaystyle\sum_{\lambda/\mu \ \textit{skew row}} 2^{\varepsilon(\lambda/\mu)}\, Q_\lambda,$

where $\delta(\lambda/\mu)$ counts the components of the diagram λ/μ and $\varepsilon(\lambda/\mu)$ counts the components of λ/μ which do not contain a box in the first column.

Example 1.3. For example,

$$\begin{aligned} P_{3\,2\,\overline{1}\,\overline{4}} \cdot p_2 &= 2 \cdot P_{4\,2\,1\,\overline{3}} \;+\; P_{4\,3\,\overline{2}\,\overline{1}} \qquad \text{and} \\ Q_{3\,2\,\overline{1}\,\overline{4}} \cdot q_2 &= 2 \cdot Q_{4\,2\,1\,\overline{3}} \;+\; 2 \cdot Q_{4\,3\,\overline{2}\,\overline{1}}. \end{aligned}$$

For $\lambda, \mu, \nu \in \mathbb{SY}_n$, there exist integral constants $g^\lambda_{\mu,\nu}$ and $h^\lambda_{\mu,\nu}$ defined by the identities

$$P_\mu \cdot P_\nu \;=\; \sum_\lambda g^\lambda_{\mu,\nu}\, P_\lambda \qquad \text{and} \qquad Q_\mu \cdot Q_\nu \;=\; \sum_\lambda h^\lambda_{\mu,\nu}\, Q_\lambda.$$

These constants were first given a combinatorial formula by Stembridge [18].

Define λ^c by $\lambda^c_j := \overline{\lambda_{n+1-j}}$. Let [pt] be the class dual to a point. The Schubert basis is self-dual with respect to the intersection pairing: If $|\lambda| = |\mu|$, then

$$P_\mu \cdot P_{\lambda^c} \;=\; Q_\mu \cdot Q_{\lambda^c} \;=\; \left\{ \begin{array}{ll} [\mathrm{pt}] & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{array} \right. . \tag{1}$$

Define the Schubert variety X'_{λ^c} to be

$$\{H \in B_n \mid \dim H \cap \langle e_{\overline{n}}, \ldots, e_{\lambda_j}\rangle \geq n+1-j \text{ for } 1 \leq j \leq n\}.$$

This is a translate of X_{λ^c} by an element of $So_{2n+1}\mathbb{C}$. In a similar fashion, define Y'_{λ^c}, a translate of Y_{λ^c} by an element of $Sp_{2n}\mathbb{C}$. For any λ, μ, $X_\mu \bigcap X'_{\lambda^c}$ is generically transverse [10]. This is because if X_μ and X'_{λ^c} are *any* Schubert varieties in general position, then there is a basis for V such that these varieties and the quadratic form β are as given.

We see that to establish the Pieri-type formula, it suffices to compute the degree of

$$X_\mu \bigcap X'_{\lambda^c} \bigcap X_K \quad \text{and} \quad Y_\mu \bigcap Y'_{\lambda^c} \bigcap Y_K$$

where K is a general isotropic $(n+1-m)$-plane and $|\lambda| = |\mu| + m$.

We only do the (more difficult) orthogonal case of Theorem 1.2 in full, and indicate the differences for the symplectic case. We first determine when $X_\mu \bigcap X'_{\lambda^c} \neq \emptyset$. Let $\mu, \lambda \in \mathbb{SY}_n$. Then, by the definition of Schubert varieties, $H \in X_\mu \bigcap X'_{\lambda^c}$ implies $\dim H \bigcap \langle e_{\mu_j}, \ldots, e_{\lambda_j}\rangle \geq 1$, for every $1 \leq j \leq n$. Hence $\mu \leq \lambda$ is necessary for $X_\mu \bigcap X'_{\lambda^c}$ to be nonempty. In fact,

$$X_\mu \bigcap X'_{\lambda^c} \;=\; \left\{ \begin{array}{ll} \langle e_{\lambda_1}, \ldots, e_{\lambda_n}\rangle & \text{if } \lambda = \mu \\ \emptyset & \text{otherwise,} \end{array} \right.$$

which establishes (1).

Suppose $\mu \leq \lambda$ in $\mathbb{SY}_n$. For each component d of λ/μ, let col(d) index the columns of d together with the column just to the left of d, which is 0 if d meets the first column, in that it has a box in the first column. For each component d of λ/μ, define a quadratic form β_d:

$$\beta_d \;:=\; \sum_{j \in \mathrm{col}(d)} x_j x_{\overline{j}},$$

where $x_{\overline{n}}, \ldots, x_n$ are coordinates for V dual to the basis $e_{\overline{n}}, \ldots, e_n$. For each *fixed point* of λ/μ (j such that $\lambda_j = \mu_j$), define the linear form $\alpha_j := x_{\overline{\lambda_j}}$. If there is no component meeting the first column, then we say that 0 is a fixed point of λ/μ and define $\alpha_0 := x_0$. Let $Z_{\lambda/\mu}$ be the common zero locus of these forms α_j and β_d.

Lemma 1.4. *Let $\mu \leq \lambda$ and $H \in X_\mu \bigcap X'_{\lambda^c}$. Then $H \subset Z_{\lambda/\mu}$.*

Let $\mathcal{Q}$ be the isotropic points in V, the zero locus of β. For each $0 \leq i \leq n$, there is a unique form among the α_j, β_d in which one (or both) of the coordinates $x_i, x_{\overline{\imath}}$ appears. Thus β is in the ideal generated by these forms α_j, β_d and we see that they are dependent on $\mathcal{Q}$. However, if $\delta = \delta(\lambda/\mu)$ counts the components of λ/μ and φ

the number of fixed points, then the collection of φ forms α_j and $\delta-1$ of the forms β_d *are* independent on $\mathcal{Q}$. Moreover, Lemma 2.2 shows that

$$n+1 \;=\; \varphi+\delta+\#\text{columns of } \lambda/\mu.$$

Thus, if $m=|\lambda|-|\mu|$, then $\varphi+\delta-1\leq n-m$, with equality only when λ/μ is a skew row. Since $\mathcal{Q}$ has dimension $2n$, it follows that a general isotropic $(n+1-m)$-plane K meets $Z_{\lambda/\mu}$ only if λ/μ is a skew row. We deduce

Theorem 1.5. *Let $\mu,\lambda\in\mathbb{SY}_n$ and suppose K is a general isotropic $(n+1-m)$-plane with $|\mu|+m=|\lambda|$. Then*

$$X_\mu\bigcap X'_{\lambda^c}\bigcap X_K \;\neq\; \emptyset$$

only if $\mu\leq\lambda$ and λ/μ is a skew row.

Under the hypotheses of Theorem 1.5, the forms α_j and β_d define $2^{\delta(\lambda/\mu)-1}$ isotropic lines in K. Theorem 3.1 asserts that a general isotropic line in $Z_{\lambda/\mu}$ determines a unique $H\in X_\mu\bigcap X'_{\lambda^c}$, which completes the proof of Theorem 1.2.

Example 1.6. We show that if $K\subset\mathcal{Q}$ is a general 3-plane, then

$$\#\, X_{3\,2\,\overline{1}\,\overline{4}}\bigcap X'_{(4\,2\,1\,\overline{3})^c}\bigcap X_K \;=\; 2.$$

Note that 2 is the coefficient of $P_{4\,2\,1\,\overline{3}}$ in the product $P_{3\,2\,\overline{1}\,\overline{4}}\cdot p_2$ of Example 1.3.

First, the local coordinates for $X_{3\,2\,\overline{1}\,\overline{4}}\bigcap X'_{(4\,2\,1\,\overline{3})^c}$ described in Lemma 3.3 show that, for any $x,z\in\mathbb{C}$, the row span H of the matrix with rows g_i and columns e_j

$$\begin{array}{c|cccc|c|cccc}
 & e_{\overline{4}} & e_{\overline{3}} & e_{\overline{2}} & e_{\overline{1}} & e_0 & e_1 & e_2 & e_3 & e_4\\ \hline
g_1 & 0&0&0&0&0&0&0&-x&1\\
g_2 & 0&0&0&0&0&0&1&0&0\\
g_3 & 0&0&0&1&2z&-2z^2&0&0&0\\
g_4 & x&1&0&0&0&0&0&0&0
\end{array}$$

is in $X_{3\,2\,\overline{1}\,\overline{4}}\bigcap X'_{(4\,2\,1\,\overline{3})^c}$. Suppose K is the row span of the matrix with rows v_i

$$\begin{array}{c|cccc|c|cccc}
 & e_{\overline{4}} & e_{\overline{3}} & e_{\overline{2}} & e_{\overline{1}} & e_0 & e_1 & e_2 & e_3 & e_4\\ \hline
v_1 & 0&1&0&1&0&0&1&0&1\\
v_2 & 1&1&0&1&2&-2&1&1&-1\\
v_3 & 0&0&1&0&0&-1&0&0&0
\end{array}$$

Then K is an isotropic 3-plane, and the forms

$$\begin{aligned}
\beta_0 &= 2x_{\overline{1}}x_1+x_0^2\\
\beta_d &= x_{\overline{4}}x_4+x_{\overline{3}}x_3\\
\alpha_2 &= x_{\overline{2}}
\end{aligned}$$

define the 2 isotropic lines $\langle v_1\rangle$ and $\langle v_2\rangle$ in K. Lastly, for $i=1,2$, there is a unique $H_i\in X_{3\,2\,\overline{1}\,\overline{4}}\bigcap X'_{(4\,2\,1\,\overline{3})^c}$ with $v_i\in H_i$. In these coordinates,

$$H_1\;:\;x=z=0\quad\text{and}\quad H_2\;:\;x=z=1.$$

In the symplectic case, isotropic K are not contained in a quadric $\mathcal{Q}$, the form $\alpha_0 = x_0$ does not arise, only components which do not meet the first column give quadratic forms β_d, and the analysis of Lemma 3.2 (2) in Section 3 is (slightly) different.

2. The intersection of two Schubert varieties

We study the intersection $X_\mu \bigcap X'_{\lambda^c}$ of two Schubert varieties. Our main result, Theorem 2.4, expresses $X_\mu \bigcap X'_{\lambda^c}$ as a product whose factors correspond to components of λ/μ, and each factor is itself an intersection of two Schubert varieties. These factors are described in Lemmas 2.5 and 2.6, and in Corollary 2.7. These are crucial to the proof of the Pieri-type formula that we complete in Section 3. Also needed is Lemma 2.1, which identifies a particular subspace of $H \cap \langle e_1, \ldots, e_n\rangle$ for $H \in X_\mu \bigcap X'_{\lambda^c}$.

For Lemma 2.1, we work in the (classical) Grassmannian $G_k(V^+)$ of k-planes in $V^+ := \langle e_1, \ldots, e_n\rangle$. For basic definitions and results see any of [8, 5, 4]. Schubert subvarieties $\Omega_\sigma, \Omega'_{\sigma^c}$ of $G_k(V^+)$ are indexed by partitions $\sigma \in \mathbb{Y}_k$, where $n-k \geq \sigma_1 \geq \cdots \geq \sigma_k \geq 0$. For $\sigma \in \mathbb{Y}_k$ define $\sigma^c \in \mathbb{Y}_k$ by $\sigma^c_j = n-k-\sigma_{k+1-j}$. For $\sigma, \tau \in \mathbb{Y}_k$, define

$$
\begin{aligned}
\Omega_\tau &:= \{H \in G_k(V^+) \mid \dim H \cap \langle e_{k+1-j+\tau_j}, \ldots, e_n\rangle \geq j,\ 1 \leq j \leq k\}\\
\Omega'_{\sigma^c} &:= \{H \in G_k(V^+) \mid \dim H \cap \langle e_1, \ldots, e_{j+\sigma_{k+1-j}}\rangle \geq j,\ 1 \leq j \leq k\}.
\end{aligned}
$$

Let $\lambda, \mu \in \mathbb{SY}_n$ with $\mu \leq \lambda$, and suppose $\mu_k > 0 > \mu_{k+1}$. Define partitions σ and τ in $\mathbb{Y}_k$ (which depend upon λ and μ) by

$$
\begin{aligned}
\tau &:= \mu_1 - k \geq \cdots \geq \mu_k - 1 \geq 0\\
\sigma &:= \lambda_1 - k \geq \cdots \geq \lambda_k - 1 > 0.
\end{aligned}
$$

Lemma 2.1. *Let $\mu \leq \lambda \in \mathbb{SY}_n$, and define $\sigma, \tau \in \mathbb{Y}_k$, and k as above. If $H \in X_\mu \bigcap X'_{\lambda^c}$, then $H \cap V^+$ contains a k-plane $L \in \Omega_\tau \bigcap \Omega'_{\sigma^c}$.*

Proof. Suppose first that $H \in X_\mu$ with $\dim H \cap \langle e_{\mu_j}, \ldots, e_n\rangle = j$ for $j = k$ and $k+1$. Since $\mu_k > 0 > \mu_{k+1}$, we see that $L := H \cap V^+$ has dimension k. If $H \in X'_{\lambda^c}$ in addition, it is an exercise in the definitions to verify that $L \in \Omega_\tau \bigcap \Omega'_{\sigma^c}$. The lemma follows as such H are dense in X_μ. ∎

The first step towards Theorem 2.4 is the following combinatorial lemma.

Lemma 2.2. *Let φ count the fixed points and δ the components of λ/μ. Then*

$$
n+1 \;=\; \varphi + \delta + \#\text{columns of } \lambda/\mu,
$$

and $\mu_j > \lambda_{j+1}$ precisely when $|\mu_j|$ is an empty column of λ/μ.

Proof. Suppose k is a column not meeting λ/μ. Thus, there is no i for which $\mu_i < k \leq \lambda_i$. Let j be the index such that $|\mu_j| = k$. If $\mu_j = k$, then we must also have $\lambda_{j+1} < k$, as $\mu_{j+1} < k$. Either $\mu_j = \lambda_j$ is a fixed point of λ/μ or else $\mu_j < \lambda_j$, so that k is the column immediately to the left of a component d which does not meet the first column.

If $\mu_j = -k$, then $\lambda_j = -k$, for otherwise $k = \lambda_i$ for some i, and for this i we must necessarily have $\mu_i < k$, contradicting k being an empty column. This proves the lemma, as 0 is either a fixed point of λ/μ or else λ/μ has a component meeting the first column, but not both. ▛

Let d_0 be the component of λ/μ meeting the first column (if any). Define mutually orthogonal subspaces V_φ, V_0, and V_d, for d a component of λ/μ not meeting the first column ($0 \notin \mathrm{col}(d)$) as follows:

$$\begin{aligned}
V_\varphi &:= \langle e_{\mu_j}, e_{\overline{\mu_j}} \mid \mu_j = \lambda_j \rangle,\\
V_0 &:= \langle e_0, e_k, e_{\overline{k}} \mid k \in \mathrm{col}(d_0) \rangle,\\
V_d^- &:= \langle e_k \mid k \in \mathrm{col}(d) \rangle,\\
V_d^+ &:= \langle e_{\overline{k}} \mid k \in \mathrm{col}(d) \rangle,
\end{aligned}$$

and set $V_d := V_d^- \oplus V_d^+$. Then

$$V = V_\varphi \oplus V_0 \oplus \bigoplus_{0 \notin \mathrm{col}(d)} V_d.$$

For each fixed point $\mu_j = \lambda_j$ of λ/μ, define the linear form $\alpha_j := x_{\overline{\mu_j}}$. For each component d of λ/μ, let the quadratic form β_d be the restriction of the form β to V_d. Composing with the projection of V to V_d gives a quadratic form (also written β_d) on V. If there is no component meeting the first column, define $\alpha_0 := x_0$ and call 0 a fixed point of λ/μ. If $0 \notin \mathrm{col}(d)$, then the form β_d identifies V_d^+ and V_d^- as dual vector spaces.

Lemma 2.3. *Let $H \in X_\mu \bigcap X'_{\lambda^c}$. Then*

1. $H \bigcap V_\varphi = \langle e_{\mu_j} \mid \mu_j = \lambda_j \rangle$.
2. $\dim H \bigcap V_0 = \#col(d_0)$.
3. *For all components d of λ/μ which do not meet the first column,*

$$\begin{aligned}
\dim H \bigcap V_d^+ &= \#\mathit{rows\ of\ } d,\\
\dim H \bigcap V_d^- &= \#col(d) - \#\mathit{rows\ of\ } d,
\end{aligned}$$

and $\left(H \bigcap V_d^-\right)^\perp = H \bigcap V_d^+$.

Proof of Lemma 2.3. Let $H \in X_\mu \bigcap X'_{\lambda^c}$. Suppose $\mu_j > \lambda_{j+1}$ so that $|\mu_j|$ is an empty column of λ/μ. Then the definitions of Schubert varieties imply

$$H = H \cap \langle e_{\overline{n}}, \ldots, e_{\lambda_{j+1}} \rangle \oplus H \cap \langle e_{\mu_j}, \ldots, e_n \rangle.$$

Suppose d is a component not meeting the first column. If the rows of d are $j, \ldots, k$, then

$$\begin{aligned}
H \cap V_d^+ &= H \cap \langle e_{\mu_k}, \ldots, e_{\lambda_j} \rangle\\
&= H \cap \langle e_{\overline{n}}, \ldots, e_{\lambda_j} \rangle \cap \langle e_{\mu_k}, \ldots, e_n \rangle,
\end{aligned}$$

and so has dimension at least $k - j + 1$.

Similarly, if $l, \ldots, m$ are the indices i with $\overline{\lambda_j} \leq \mu_i, \lambda_i \leq \overline{\mu_k}$, then $H \bigcap V_d^-$ has dimension at least $l - m + 1$. Hence $\dim V_d/2 = \#\mathrm{col}(d) = k + m - l - j + 2$, as $\lambda_j, \ldots, \lambda_k, \overline{\lambda_l}, \ldots, \overline{\lambda_m}$ are the columns of d.

Since H is isotropic, $\dim H_d^+ + \dim H_d^- \leq \#\mathrm{col}(d)$, which proves the first part of (3). Moreover, $H \bigcap V_d^+ = \left(H \bigcap V_d^-\right)^\perp$: Since H is isotropic, we have $\subset$, and equality follows by dimension counting.

Similar arguments prove the other statements. ▰

For $H \in X_\mu \bigcap X'_{\lambda^c}$, define $H_\varphi := H \bigcap V_\varphi$, $H_0 := H \bigcap V_0$, $H_d^+ := H \bigcap V_d^+$, and $H_d^- := H \bigcap V_d^-$. Then $H_\varphi \subset V_\varphi$ is the zero locus of the linear forms α_j, H_0 is isotropic in V_0, and, for each component d of λ/μ not meeting the first column, $H_d := H_d^+ \oplus H_d^-$ is isotropic in V_d, which proves Lemma 1.4. Moreover, H is the orthogonal direct sum of H_φ, H_0, and the H_d.

Theorem 2.4. *The map*

$$\{H_0 \mid H \in X_\mu \bigcap X'_{\lambda^c}\} \times \prod_{0 \notin col(d)} \{H_d \mid H \in X_\mu \bigcap X'_{\lambda^c}\} \longrightarrow X_\mu \bigcap X'_{\lambda^c}$$

defined by

$$(H_0, \ldots, H_d, \ldots) \longmapsto \langle H_\varphi, H_0, \ldots, H_d, \ldots \rangle$$

is an isomorphism.

Proof. By the previous discussion, it is an injection. For surjectivity, note that both sides have the same dimension. Indeed, $\dim X_\mu \bigcap X'_{\lambda^c} = |\lambda| - |\mu|$, the number of boxes in λ/μ. Lemmas 2.5 and 2.6 show that the factors of the domain each have dimension equal to the number of boxes in the corresponding components. ▰

Suppose there is a component, d_0, meeting the first column. Let l be the largest column in d_0, and define $\lambda(0), \mu(0) \in \mathbb{SY}_l$ as follows: Let j be the first row of d_0 so that $l = \lambda_j$. Then, since d_0 is a component, for each $j \leq i < j + l - 1$, we have $\lambda_{i+1} \geq \mu_i$ and $l = \overline{\mu_{j+l-1}}$. Set

$$\begin{aligned} \mu(0) &:= \mu_j > \cdots > \mu_{j+l-1} \\ \lambda(0) &:= \lambda_j > \cdots > \lambda_{j+l-1} \end{aligned}$$

Define $\lambda(0)^c$ by $\lambda(0)^c_p := \overline{\lambda(0)_{l+1-p}} = \overline{\lambda_{j+l-p}}$. The following lemma is a straightforward consequence of these definitions.

Lemma 2.5. *With the above definitions,*

$$\{H_0 \mid H \in X_\mu \bigcap X'_{\lambda^c}\} = X_{\mu(0)} \bigcap X'_{\lambda(0)^c},$$

as subvarieties of $B_k \simeq B(V_0)$, and $\lambda(0)/\mu(0)$ has a unique component meeting the first column and no fixed points.

We similarly identify $\{H_d \mid H \in X_\mu \bigcap X'_{\lambda^c}\}$ as an intersection $X_{\mu(d)} \cap X'_{\lambda(d)^c}$ of Schubert varieties in $B_{\#\mathrm{col}(d)} \simeq B(\langle e_0, V_d \rangle)$. Let $j, \ldots, k$ be the rows of d and $l, \ldots, m$

be the indices i with $\overline{\lambda_j} \leq \mu_i, \lambda_i \leq \overline{\mu_k}$, as in the proof of Lemma 2.3. Let $p = \#\mathrm{col}(d)$ and define $\lambda(d), \mu(d) \in \mathbb{SY}_p$ as follows. Set $a = \mu_k$, and define

$$\begin{array}{rcl} \mu(d) & := & \mu_j - a + 1 > \cdots > \quad 1 \quad > \mu_l + a - 1 > \cdots > \mu_m + a - 1 \\ \lambda(d) & := & \lambda_j - a + 1 > \cdots > \lambda_k - a + 1 > \lambda_l + a - 1 > \cdots > \lambda_m + a - 1 \end{array}$$

As with Lemma 2.5, the following lemma is straightforward.

Lemma 2.6. *With these definitions,*

$$\{H_d \mid H \in X_\mu \bigcap X'_{\lambda^c}\} \;\simeq\; X_{\mu(d)} \bigcap X'_{\lambda(d)^c}$$

as subvarieties of $B_p \simeq B(\langle e_0, V_d\rangle)$ and $\lambda(d)/\mu(d)$ has a unique component not meeting the first column and no fixed points.

Suppose now that $\mu, \lambda \in \mathbb{SY}_n$ where λ/μ has a unique component d not meeting the first column and no fixed points. Suppose λ has k rows. A consequence of Lemma 2.3 is that the map $H_d^+ \mapsto \langle H_d^+, (H_d^+)^\perp\rangle$ gives an isomorphism

$$\{H_d^+ \mid H \in X_\mu \bigcap X'_{\lambda^c}\} \;\xrightarrow{\sim}\; X_\mu \bigcap X'_{\lambda^c}. \tag{2}$$

The following corollary of Lemma 2.1 identifies the domain.

Corollary 2.7. *With μ, λ as above and σ, τ, and k as defined in the paragraph preceding Lemma 2.1, we have:*

$$\{H_d^+ \mid H \in X_\mu \bigcap X'_{\lambda^c}\} \;=\; \Omega_\tau \bigcap \Omega_{\sigma^c},$$

as subvarieties of $G_k(V^+)$.

Remark 2.8. The symplectic analogs of Lemma 2.6 and Corollary 2.7, which are identical save for the necessary replacement of Y for X and C_p for B_p, show an unexpected connection between the geometry of the symplectic and orthogonal Grassmannians. Namely, suppose λ/μ has no component meeting the diagonal. Then the projection map $V \twoheadrightarrow W$ defined by

$$e_i \;\longmapsto\; \begin{cases} 0 & \text{if } i = 0 \\ f_i & \text{otherwise} \end{cases}$$

and its right inverse $W \hookrightarrow V$ defined by $f_j \mapsto e_j$ induce isomorphisms

$$X_\mu \bigcap X'_{\lambda^c} \;\stackrel{\sim}{\longleftrightarrow}\; Y_\mu \bigcap Y'_{\lambda^c}.$$

3. Pieri-type intersections of Schubert varieties

Fix λ/μ to be a skew row with $|\lambda| - |\mu| = m$. Let $Z_{\lambda/\mu} \subset \mathcal{Q}$ be the zero locus of the forms α_j and β_d of §2. If λ/μ has δ components, then as a subvariety of $\mathcal{Q}$, $Z_{\lambda/\mu}$ is the generically transverse intersection of the zero loci of the forms α_j and any $\delta - 1$ of the forms β_d. It follows that a general $(n+1-m)$-plane $K \subset \mathcal{Q}$ meets $Z_{\lambda/\mu}$ in $2^{\delta-1}$ lines. Thus if $\langle v\rangle \subset Z_{\lambda/\mu}$ is a general line, then

$$\# X_\mu \bigcap X'_{\lambda^c} \bigcap X_K \;=\; 2^\delta \cdot \# X_\mu \bigcap X'_{\lambda^c} \bigcap X_{\langle v\rangle}.$$

Theorem 1.2 is a consequence of this observation and the following:

Theorem 3.1. *Let λ/μ be a skew row, $Z_{\lambda/\mu}$ be as above, and $\langle v\rangle$ a general line in $Z_{\lambda/\mu}$. Then $X_\mu \bigcap X'_{\lambda^c} \bigcap X_{\langle v\rangle}$ is a singleton.*

Proof. Let $\mathcal{Q}_0$ be the cone of isotropic points in V_0 and $\mathcal{Q}_d$ the cone of isotropic points in V_d for $d \neq d_0$. Since

$$Z_{\lambda/\mu} \;=\; H_\varphi \oplus \mathcal{Q}_0 \oplus \bigoplus_{0\notin \mathrm{col}(d)} \mathcal{Q}_d,$$

we see that a general non-zero vector v in $Z_{\lambda/\mu}$ has the form

$$v \;=\; \sum_{\mu_j=\lambda_j} a_j e_{\mu_j} \;+\; v_0 \;+\; \sum_{0\notin\mathrm{col}(d)} v_d,$$

where $a_j \in \mathbb{C}^\times$ and $v_0 \in \mathcal{Q}_0$, $v_d \in \mathcal{Q}_d$ are general vectors.

Thus, if $H \in X_\mu \bigcap X'_{\lambda^c} \bigcap X_{\langle v\rangle}$, we see that $v_0 \in H_0$ and $v_d \in H_d$. By Theorem 2.4, H is determined by H_0 and the H_d, thus it suffices to prove that H_0 and the H_d are uniquely determined. The identifications of Lemmas 2.5 and 2.6 show that this is just the case of the theorem when λ/μ has a single component, which is Lemma 3.2 below. ∎

Lemma 3.2. *Suppose $\lambda, \mu \in \mathbb{SY}_n$ where λ/μ is a skew row with a unique component and no fixed points.*

1. *If λ/μ does not meet the first column and $v \in \mathcal{Q}_d$ is a general vector, then $X_\mu \bigcap X'_{\lambda^c} \bigcap X_{\langle v\rangle}$ is a singleton.*
2. *If λ/μ meets the first column and $v \in \mathcal{Q}$ is general, then $X_\mu \bigcap X'_{\lambda^c} \bigcap X_{\langle v\rangle}$ is a singleton.*

Proof of (1). Let $v \in \mathcal{Q}_d$ be a general vector. Since $\mathcal{Q}_d \subset V^+ \oplus V^-$, $v = v^+ \oplus v^-$ with $v^+ \in V^+$ and $v^- \in V^-$. Suppose $\mu_k > 0 > \mu_{k+1}$. Consider the set

$$\{H^+ \in G_k(V^+) \mid v \in H^+ \oplus \left(H^+\right)^\perp\} \;=\; \{H^+ \mid v^+ \in H^+ \subset (v^-)^\perp\}.$$

This is a Schubert variety $\Omega''_{h(n-k,k)}$ of G_kV^+, where $h(n-k,k)$ is the partition of hook shape with a single row of length $n-k$ and a single column of length k.

Under the isomorphisms of (2) and Lemma 2.6, and with the identification of Corollary 2.7, we see that

$$X_\mu \bigcap X'_{\lambda^c} \bigcap X_{\langle v\rangle} \;\simeq\; \Omega_\tau \bigcap \Omega'_{\sigma^c} \bigcap \Omega''_{h(n-k,k)},$$

where σ, τ are as defined in the paragraph preceding Corollary 2.7. For $\rho \in \mathbb{Y}_k$, let $S_\rho := [\Omega_\rho]$ be the cohomology class Poincaré dual to the fundamental cycle of Ω_ρ in $H^*G_kV^+$. The multiplicity we wish to compute is

$$\deg(S_\tau \cdot S_{\sigma^c} \cdot S_{h(n-k,k)}). \tag{3}$$

By a double application of the classical Pieri's formula (as $S_{h(n-k,k)} = S_{n-k} \cdot S_{1^{k-1}}$), we see that (3) is either 1 or 0, depending upon whether or not σ/τ has exactly one box in each diagonal. But this is the case, as the transformation $\mu, \lambda \longmapsto \tau, \sigma$ takes columns to diagonals. ∎

Our proof of Lemma 3.2 (2) uses a system of local coordinates for $X_\mu \bigcap X'_{\lambda^c}$. Let λ/μ be as in Lemma 3.2 (2), and suppose $\lambda_{k+1} = 1$. For $y_2, \ldots, y_n, x_0, \ldots, x_{n-1} \in \mathbb{C}$, define vectors $g_j \in V$ as follows:

$$g_j \ := \ \begin{cases} e_{\lambda_j} + \displaystyle\sum_{i=\mu_j}^{\lambda_j - 1} x_i\, e_i & j \leq k \\ -2x_0^2 e_1 + 2x_0 e_0 + e_{\overline{1}} + \displaystyle\sum_{i=\mu_{k+1}}^{\overline{2}} y_i\, e_i & j = k+1 \\ e_{\lambda_j} + \displaystyle\sum_{i=\mu_j}^{\lambda_j - 1} y_i\, e_i & j > k+1 \end{cases} . \tag{4}$$

Lemma 3.3. *Let $\lambda, \mu \in \mathbb{SY}_n$ where λ/μ is a skew row meeting the first column with no fixed points, and define $\tau, \sigma \in \mathbb{Y}_k$, and k as for Lemma 2.1. Then*

1. *For any $x_1, \ldots, x_{n-1} \in \mathbb{C}$, we have $\langle g_1, \ldots, g_k\rangle \ \in \ \Omega_\tau \bigcap \Omega'_{\sigma^c}$.*
2. *For and $x_0, \ldots, x_{n-1} \in \mathbb{C}$ with $x_{\overline{\mu_{k+1}}}, \ldots, x_{\overline{\mu_{n-1}}} \neq 0$, the condition that $H := \langle g_1, \ldots, g_n\rangle$ is isotropic determines a unique $H \in X_\mu \bigcap X'_{\lambda^c}$.*

Moreover, these coordinates parameterize dense subsets of the intersections.

Proof. The first statement is immediate from the definitions.

For the second, note that each $g_j \in \mathcal{Q}$. The conditions that $\langle g_1, \ldots, g_n\rangle$ is isotropic are

$$\beta(g_i, g_j) \ = \ 0 \quad \text{for} \quad i \leq k < j.$$

Only $n-1$ of these are not identically zero. Indeed, for $i \leq k < j$,

$$\beta(g_i, g_j) \ \not\equiv \ 0 \quad \Longleftrightarrow \quad \begin{cases} \text{either} & \overline{\lambda_j} < \mu_i < \overline{\mu_j}, \\ \text{or} & \mu_i < \overline{\mu_j} < \lambda_i. \end{cases}$$

Moreover, if we order the variables $y_2 < \cdots < y_n < x_0 < \cdots < x_{n-1}$, then, in the lexicographic term order, the leading term of $\beta(g_i, g_j)$ for $i \leq k < j$ is

$$\begin{array}{rl} y_{\lambda_i} & \text{if } \overline{\lambda_j} < \mu_i < \overline{\mu_j}, \\ y_{\overline{\mu_j}} x_{\overline{\mu_j}} & \text{if } \mu_i < \overline{\mu_j} < \lambda_i, \quad \text{or} \\ y_n = y_{\overline{\mu_n}} & \text{if } i = 1, \ j = n. \end{array}$$

Since $\{2, \ldots, n\} = \{\lambda_2, \ldots, \lambda_{k-1}, \overline{\mu_k}, \ldots, \overline{\mu_n}\}$, each y_l appears as the leading term of a unique $\beta(g_i, g_j)$ with $i < k \leq j$, thus these $n-1$ non-trivial equations $\beta(g_i, g_j) = 0$ determine $y_2, \ldots, y_n$ uniquely.

These coordinates parameterize an n-dimensional subset of $X_\mu \bigcap X'_{\lambda^c}$. Since $\dim(X_\mu \bigcap X'_{\lambda^c}) = n$ and $X_\mu \bigcap X'_{\lambda^c}$ is irreducible [3], this subset is dense, which completes the proof. ∎

Example 3.4. Let $\lambda = 6\,5\,3\,1\,\overline{2}\,\overline{4}$ and $\mu = 5\,3\,1\,\overline{2}\,\overline{4}\,\overline{6}$ so $k = 3$. We display the components of the vectors g_i in a matrix

	$e_{\overline{6}}$	$e_{\overline{5}}$	$e_{\overline{4}}$	$e_{\overline{3}}$	$e_{\overline{2}}$	$e_{\overline{1}}$	e_0	e_1	e_2	e_3	e_4	e_5	e_6
g_1	0	0	0	0	0	0	0	0	0	0	0	x_5	1
g_2	0	0	0	0	0	0	0	0	0	x_3	x_4	1	0
g_3	0	0	0	0	0	0	0	x_1	x_2	1	0	0	0
g_4	0	0	0	0	y_2	1	$2x_0$	$-2x_0^2$	0	0	0	0	0
g_5	0	0	y_4	y_3	1	0	0	0	0	0	0	0	0
g_6	y_6	y_5	1	0	0	0	0	0	0	0	0	0	0

Then there are 5 non-zero equations $\beta(g_i, g_j) = 0$ with $i \leq 3 < j$:

$$\begin{array}{rcccl} 0 &=& \beta(g_3, g_4) &=& y_2x_2 + x_1 \\ 0 &=& \beta(g_3, g_5) &=& y_3 + x_2 \\ 0 &=& \beta(g_2, g_5) &=& y_4x_4 + y_3x_3 \\ 0 &=& \beta(g_2, g_6) &=& y_5 + x_4 \\ 0 &=& \beta(g_1, g_6) &=& y_6 + x_5y_5 \end{array}$$

Solving, we obtain:

$$y_2 = -x_1/x_2,\ y_3 = -x_2,\ y_4 = -y_3x_3/x_4,\ y_5 = -x_4,\ \text{and } y_6 = -x_5y_5.$$

Proof of Lemma 3.2 (2). Suppose $\lambda, \mu \in \mathbb{SY}_n$ where λ/μ is a skew row with a single component meeting the first column and no fixed points. Let $v \in \mathcal{Q}$ be a general vector and consider the condition that $v \in H$ for $H \in X_\mu \bigcap X'_{\lambda^c}$. Let $\sigma, \tau \in \mathbb{Y}_k$ be defined as in the paragraph preceding Lemma 2.1. We first show that there is a unique $L \in \Omega_\tau \bigcap \Omega_{\sigma^c}$ with $L \subset H$, and then argue that H is unique.

The conditions on μ and λ imply that $\mu_n = \overline{n}$ and $\mu_j = \lambda_{j+1}$ for $j < n$. We further suppose that $\lambda_{k+1} = 1$, so that the last row of λ/μ has length 1. This is no restriction, as the isomorphism of V defined by $e_j \mapsto e_{\overline{j}}$ sends $X_\mu \bigcap X'_{\lambda^c}$ to $X_{\lambda^c} \bigcap X'_{(\mu^c)^c}$ and one of λ/μ or μ^c/λ^c has last row of length 1.

Let $v \in \mathcal{Q}$ be general. If necessary, scale v so that its $e_{\overline{1}}$-component is 1. Let $2z$ be its e_0-component, then necessarily its e_1-component is $-2z^2$. Let $v^- \in V^-$ be the projection of v to V^-. Similarly define $v^+ \in V^+$. Set $v' := v^+ + 2z^2e_1$, so that $\beta(v^-, v') = 0$ and

$$v \;=\; v^- + 2z(e_0 - ze_1) + v'.$$

Let $H \in X_\mu \bigcap X'_{\lambda^c}$, and suppose that $v \in H$. In the notation of Lemma 2.1, let $L \in \Omega_\tau \bigcap \Omega_{\sigma^c}$ be a k-plane in $H \bigcap V^+$. If H is general, in that

$$\dim H \bigcap \langle e_{\overline{n}}, \ldots, e_{\lambda_{k+2}} \rangle \;=\; \dim H \bigcap \langle e_{\overline{n}}, \ldots, e_0 \rangle \;=\; n - k - 1,$$

then $\langle L, e_1 \rangle$ is the projection of H to V^+. As $v \in H$, we have $v^+ \in \langle L, e_1 \rangle$. Since $L \subset v^\perp \bigcap V^+ = (v^-)^\perp$, we see that $v' \in L$, and hence

$$v' \;\in\; L \;\subset\; (v^-)^\perp.$$

As in the proof of part (1), there is a (necessarily unique) such $L \in \Omega_\tau \bigcap \Omega_{\sigma^c}$ if and only if σ/τ has a unique box in each diagonal. But this is the case, as the transformation $\mu, \lambda \longrightarrow \tau, \sigma$ takes columns (greater than 1) to diagonals.

To complete the proof, we use the local coordinates for $X_\mu \bigcap X'_{\lambda^c}$ and $\Omega_\tau \bigcap \Omega_{\sigma^c}$ of Lemma 3.3. Since v is general, we may assume that that the k-plane $L \in \Omega_\tau \bigcap \Omega_{\sigma^c}$ determined by $v' \in L \subset (v^-)^\perp$ has non-vanishing coordinates $x_{\overline{\mu_{k+1}}}, \ldots, x_{\overline{\mu_{n-1}}}$, so that there is an $H \in X_\mu \bigcap X'_{\lambda^c}$ in this system of coordinates with $L = H \cap V^+$.

Such an H is determined up to a choice of coordinate x_0. The requirement that $v \in H$ forces the projection $\langle e_{\overline{1}} + 2x_0 e_0 \rangle$ of H to $\langle e_{\overline{1}}, e_0 \rangle$ to contain $e_{\overline{1}} + 2ze_0$, the projection of v to $\langle e_{\overline{1}}, e_0 \rangle$. Hence $x_0 = z$, and it follows that there is at most one $H \in X_\mu \bigcap X'_{\lambda^c}$ with $v \in H$. Let $g_1, \ldots, g_n$ be the vectors (4) determined by the coordinates $x_1, \ldots, x_{n-1}$ for L with $x_0 = z$. We claim $v \in H := \langle g_1, \ldots, g_n \rangle$.

Indeed, since $v' \in L$ and $v^- \in L^\perp = \langle g_{k+1} - 2z(e_0 - ze_1), g_{k+2}, \ldots, g_n \rangle$, we see there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with

$$v^- + v' \;=\; \alpha_1 g_1 + \cdots + \alpha_{k+1}(g_{k+1} - 2z(e_0 - ze_1)) + \cdots + \alpha_n g_n.$$

We must have $\alpha_{k+1} = 1$, since the $e_{\overline{1}}$-component of both v and g_{k+1} is 1. It follows that

$$v \;=\; \sum_{i=1}^{n} \alpha_i g_i \quad \in \; H. \qquad \blacksquare$$

Remarks.

1. Our desire to give elementary proofs led us to restrict ourselves to the complex numbers. With the appropriate modifications, these same arguments give the same results for Chow groups of these same varieties over any field of characteristic $\neq 2$. For example, the appropriate intersection-theoretic constructions and the properness of a general translate provide a substitute for our use of transversality. Then one could argue for the multiplicity of $2^{\delta-1}$ as follows:

   If $\lambda, \mu \in \mathbb{SY}_n$ and K is a linear subspace in $\mathcal{Q}$, then the scheme-theoretic intersection $X_\mu \bigcap X'_{\lambda^c} \bigcap X_K$ is $pr_* \pi^*(K)$, where

$$\begin{tikzcd} & & \Xi = \{(p, H) \mid p \in H \in X_\mu \bigcap X'_{\lambda^c}\} \arrow[dl, "\pi"'] \arrow[dr, "pr"] & \\ K \arrow[r, hook] & \mathcal{Q} & & C_n \end{tikzcd}$$

   Then intersection theory on the quadric $\mathcal{Q}$ (a homogeneous space) and Kleiman's Theorem that the intersection with a general translate is proper [10] gives a factor of $2^{\delta-1}$ from the intersection multiplicity of K and the subvariety $Z_{\lambda/\mu}$ of $\mathcal{Q}$ consisting of the image of π. The arguments of Section 3 show that π has degree 1 onto its image.
2. Conversely, similar to the proof of Lemma 3.3, we could give local coordinates for any intersection $X_\mu \bigcap X'_{\lambda^c}$. Such a description would enable us to establish

transversality directly, and to dispense with the intersection theory of the classical Grassmannian. This would work over any field whose characteristic is not 2, but would complicate the arguments we gave.

3. We have not investigated to what extent these methods would work in characteristic 2.

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