TOPOLOGY OF TWO-CONNECTED GRAPHS AND HOMOLOGY OF SPACES OF KNOTS

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ABSTRACT. We propose a new method of computing cohomology groups of spaces of knots in $\mathbb{R}^n$, $n \geq 3$, based on the topology of configuration spaces and two-connected graphs, and calculate all such classes of order ≤ 3. As a byproduct we define the higher indices, which invariants of knots in $\mathbb{R}^3$ define at arbitrary singular knots. More generally, for any finite-order cohomology class of the space of knots we define its principal symbol, which lies in a cohomology group of a certain finite-dimensional configuration space and characterizes our class modulo the classes of smaller filtration.

1. INTRODUCTION

The knots, i.e. smooth embeddings $S^1 \to \mathbb{R}^n$, $n \geq 3$, form an open dense subset in the space $\mathcal{K} \equiv C^\infty(S^1, \mathbb{R}^n)$. Its complement Σ is the *discriminant set*, consisting of maps, having selfintersections or singularities. Any cohomology class $\gamma \in H^i(\mathcal{K} \setminus \Sigma)$ of the space of knots can be described as the linking number with an appropriate chain of codimension $i+1$ in $\mathcal{K}$ lying in Σ.

In [V2] a method of constructing some of such (co)homology classes (in particular, knot invariants) was proposed. For $n = 3$ the 0-dimensional classes, arising from this construction, are exactly the "finite-type knot invariants", and for $n > 3$ this method provides a complete calculation of all cohomology groups of knot spaces in $\mathbb{R}^n$.

However, the precise calculations by this method are very complicated. The strongest results obtained by now are as follows.

D. Bar-Natan calculated the $\mathbb{C}$-valued knot invariants of orders ≤ 9 (and used for this several weeks of computer work and the Kontsevich's realization theorem), see [BN].

T. Stanford wrote a program, realizing the algorithm of [V2] for computing $\mathbb{Z}$-invariants, and found all such invariants of orders ≤ 7.

D. Teiblum and V. Turchin (also using a computer) found the first non-trivial 1-dimensional cohomology class of the space of knots in $\mathbb{R}^3$ (which is of order 3) and

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proved that there are no other positive-dimensional cohomology classes of order 3. For a description of this class, see § 4.4.2.

By the natural periodicity of cohomology groups of spaces of knots in $\mathbb{R}^n$ with different n (similar to the fact that all algebras $H^*(\Omega S^m)$ with m of the same parity are isomorphic up to a scaling of dimensions) these results can de extended to cohomology of spaces of knots in $\mathbb{R}^n$ with arbitrary odd n: to any knot invariant of order i in $\mathbb{R}^3$ there corresponds a $(n-3)i$-dimensional cohomology class for any odd n, and the Teiblum–Turchin class is the origin of a series of $((n-3)i+1)$-dimensional cohomology classes in spaces of knots in $\mathbb{R}^n$, n odd.

Below I describe another, in a sense opposite method of calculating these cohomology groups, based on a different filtration of the discriminant variety. On the level of knot invariants in $\mathbb{R}^3$, this method is more or less equivalent to the Bar-Natan's calculus of Chinese Character Diagrams and gives a partial explanation of its geometrical sense. Using this method, it is possible to repeat "by hands" the calculation of Teiblum–Turchin (which was nontrivial even for computer) and to obtain a similar result for the case of even n.

The idea of reducing the cohomology of spaces of nonsingular geometrical objects (such as e.g. polynomials of a fixed degree in $\mathbb{C}^1$ or $\mathbb{R}^1$ without multiple roots) to the (Alexander dual) homology of the complementary discriminant set was proposed by V. I. Arnold in [A1], see also [A2]. This idea proved to be very fruitful, because the discriminant is a naturally stratified set (whose open strata consist of "equisingular" objects). Many homology classes of the discriminant can be calculated with the help of its filtration, defined by this stratification.

The "reversed" filtrations of (some natural resolutions of) the discriminant sets, introduced in [V1], simplify very much these calculations and allow to solve similar problems in the multidimensional case, see e.g. [V5], [V6]. Moreover, the spectral sequences, defined by such filtrations, are functorial with respect to the inclusion of (finite-dimensional) functional spaces, and thus give rise to stable sequences, calculating cohomology of complements of discriminants in infinite-dimensional spaces like the space of knots or of C^∞-functions without complicated singularities.

This reversion is a continuous generalization of the combinatorial formula of inclusions and exclusions: if we wish to calculate the cardinality of a finite union of finite sets, then, instead of counting the points in "open strata" (i.e., for any k, in all k-fold intersections of these sets, from which all the $(k+1)$-fold intersections are removed) we can to count separately all points in all sets, then to distract from the obtained sum the correction term, corresponding to all 2-fold intersections (not taking into account the fact that some of these double intersections points are also triple points), then to add the correction terms, corresponding to all triple points, etc. All results of [V1]–[V7], concerning the cohomology of complements of discriminants, were obtained by this method.

However, in the calculation of the first term E_1 of the spectral sequence from [V2], converging to the cohomology of knot spaces (in particular to knot invariants), we used an auxiliary spectral sequence, based on the "natural" sub-stratification of the resolved discriminant. The main portion of hard calculations in the theory of finite-order knot invariants is the calculation in this auxiliary sequence (especially in its terms responsible for the 0-dimensional cohomology). In this paper we reverse also this filtration. From the point of view of the new method the previous calculations are just the computation of cellular homology of certain configuration spaces, which can be studied by more "theoretical" methods.

As a byproduct, we obtain the notion of the *index*, which any knot invariant assigns to any (finitely-degenerate) singular knot in $\mathbb{R}^3$. In the standard theory of finite-order invariants this index was considered for the simplest points of Σ, i.e. for immersions having only transverse double points or (in a more implicit way) at most one triple point.

In the general situation this function is not numerical: it takes values in certain homology groups related with singular knots. Say, for an immersion having one generic k-fold self-intersection point this group is the $(2k-4)$-th homology group of the complex of *two-connected graphs*[1] with k nodes, in particular, according to [BBLSW] and [T], is $(k-2)!$-dimensional. In fact, the information provided by the index of such a singular knot is equivalent to the "totality of all extensions of our invariant to all Chinese Character Diagrams of order $k-1$ with exactly k legs", see [BN2].

More generally, for any finite-order cohomology class of the space of knots we define its *principal symbol*, which lies in a cohomology group of a certain finite-dimensional configuration space and characterizes our class modulo the classes of smaller filtration.

Our first calculations lead to some essential problems in the homological combinatorics and representation theory, see [BBLSW], [T], and the first answers indicate the existence of a rich algebraic structure behind it, which probably will allow one to guess many invariants and cohomology classes of arbitrary orders.

Notation. For any topological space X, $\bar{H}_*(X)$ denotes the *Borel–Moore homology group* of X, i.e. the homology group of its one-point compactification reduced modulo the added point.

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[1]Unlike the topological terminology, in the combinatorics a graph with k nodes is called two-connected if it is connected (i.e. joins any two of these k nodes), and, moreover, removing from it any node together with all incident edges, we obtain also a connected graph.

2. Complexes of connected and two-connected graphs

We consider only graphs without loops and multiple edges, but maybe with isolated nodes.

A graph with k nodes is *connected,* if any two of its nodes are joined by a chain of its edges. A graph is *l-connected* if it is connected, and removing from it any $l-1$ nodes together with all incident edges, we obtain again a connected graph (with $k-l+1$ nodes).

The set of all graphs with given k nodes generates an (acyclic) simplicial complex. Namely, consider the simplex $\Delta(k)$, whose vertices are in the one-to-one correspondence with all $\binom{k}{2}$ edges of the complete graph with these k nodes. The faces of this simplex are subgraphs of the complete graph: indeed, any face is characterized by the set of its vertices, i.e. by a collection of edges of the complete graph. All faces of the simplex $\Delta(k)$ form an acyclic simplicial complex, which also will be denoted by $\Delta(k)$. A generator of this complex is a graph with ordered edges, while permuting the edges we send such a generator to $\pm$ itself depending on the parity of the permutation. We will always choose the generator, corresponding to the lexicographic order of edges induced by some fixed order of initial k nodes. The boundary of a graph is a formal sum of all graphs obtained from it by removing one of its edges, taken with coefficient 1 or -1.

Faces, corresponding to non-connected graphs, form a subcomplex of the complex $\Delta(k)$. The corresponding quotient complex is denoted by $\Delta^1(k)$ and is called the *complex of connected graphs.* In a similar way, the complex $\Delta^l(k)$ of *l-connected graphs* is the quotient complex of $\Delta(k)$, generated by all faces, corresponding to l-connected graphs.

Theorem 1. a) For any k, the group $H_i(\Delta^1(k))$ is trivial for all $i \neq k-2$, and $H_{k-2}(\Delta^1(k)) \simeq \mathbb{Z}^{(k-1)!}$.

b) A basis in the group $H_{k-2}(\Delta^1(k))$ consists of all linear (homeomorphic to a segment) graphs, one of whose endpoints is fixed.

Statement a) of this theorem is a corollary of a theorem of Folkman [Fo], see e.g. [B]. A proof of b) and another proof of a), based on the Goresky–MacPherson formula for the homology of plane arrangements, is given in [V3].

Theorem 2. For any k, the group $H_i(\Delta^2(k))$ is trivial if $i \neq 2k-4$ and is isomorphic to $\mathbb{Z}^{(k-2)!}$ if $i = 2k-4$.

This theorem was proved independently and almost simultaneously by Eric Babson, Anders Björner, Svante Linusson, John Shareshian, and Volkmar Welker, on one hand, and by Victor Turchin on the other, see [BBLSW], [T].

Example 1. Suppose that $k=3$ and the original nodes are numbered by 1, 2 and 3. The simplex $\Delta(3)$ is a triangle, whose vertices are called $(1,2)$, $(1,3)$ and

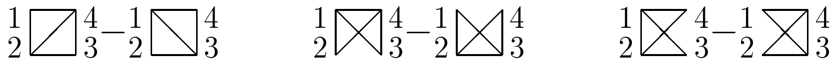


FIGURE 1. Basic chains for 2-connected graphs with 4 vertices

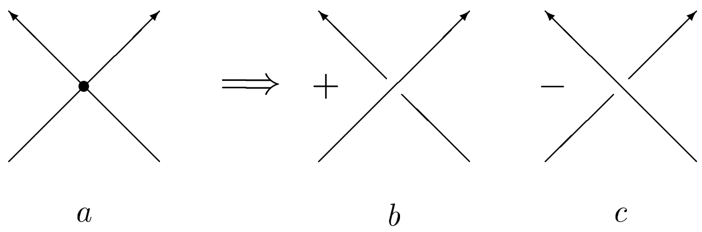


FIGURE 2. Local perturbations of a self-intersection point

$(2,3)$. Among its 7 faces only four correspond to connected graphs, namely, all faces of dimension 1 or 2. In particular, the homology group $H_i(\Delta^1(3))$ is trivial if $i \neq 1$ and is isomorphic to $\mathbb{Z}^2$ if $i = 1$.

The unique face of $\Delta(3)$, corresponding to a 2-connected graph, is the triangle itself. Thus $H_i(\Delta^2(3))$ is trivial if $i \neq 2$ and is isomorphic to $\mathbb{Z}$ if $i = 2$.

EXAMPLE 2. The simplex $\Delta(4)$ has $\binom{4}{2} = 6$ vertices. The corresponding complex of 2-connected graphs consists of the simplex itself, all 6 its faces of dimension 4, and 3 faces of dimension 3, corresponding to all cycles of length 4. It is easy to calculate that $H_i(\Delta^2(4)) = 0$ for $i \neq 4$ and $H_4(\Delta^2(4)) \simeq \mathbb{Z}^2$. Namely, this homology group is generated by three 4-chains, any of which is the difference of two graphs with 5 edges, obtained from the complete graph by removing edges, connecting complementary pairs of points, see Fig. 1.

Such basic chains are numbered by unordered partitions of four points into two pairs and satisfy one relation: the sum of all three chains is equal to the boundary of the complete graph.

3. SIMPLEST EXAMPLES OF INDICES OF SINGULAR KNOTS

For any immersion $\phi : S^1 \to \mathbb{R}^3$ with exactly i transverse crossings we can consider all 2^i possible small resolutions of this singular immersion, replacing any of its self-intersection points (see Fig. 2a) by either undercrossing or overcrossing. Any such local resolution can be invariantly called positive (see Fig. 2b)) or negative (Fig. 2c)), see [V2]. The sign of the entire resolution is defined as $(-1)^{\#-}$, where $\#-$ is the number of *negative* local resolutions in it.

Given a knot invariant, the index (or "i-th jump") of our singular immersion is defined as the sum of values of this invariant at all positive resolutions minus the

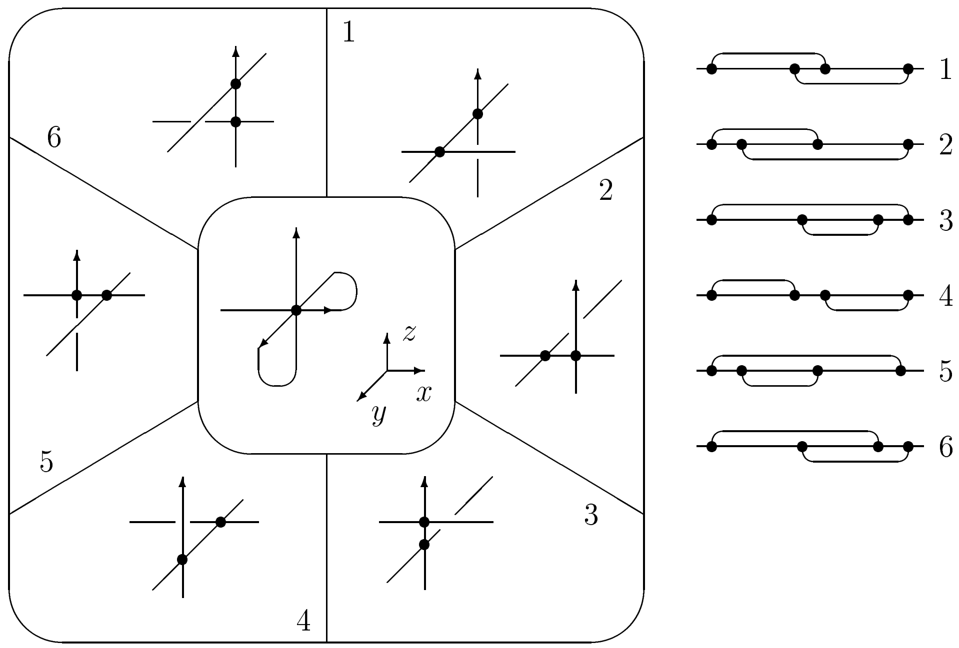


FIGURE 3. Splittings of a triple point

similar sum over negative resolutions. Invariants of order $\leq j$ can be defined as those taking zero index at all immersions with $> j$ transverse crossings.

Further, let $\phi : S^1 \to \mathbb{R}^3$ be an immersion with l transverse double self-intersections and one generic triple point. There are 6 different perturbations of this triple point, splitting it into a pair of double points, see Fig. 3 (= Fig. 15 in [V2]). Thus we get 6 immersions with $l + 2$ double crossings each, and, given a knot invariant, 6 their indices $I(1), \ldots, I(6)$ of order $l + 2$. These indices are dependent: they satisfy the *four-term relations*

$$I(1) - I(4) = I(2) - I(5) = I(6) - I(3). \tag{1}$$

The common value of these three differences is a characteristic assigned by our invariant to the initial singular immersion with a triple point and also is called its index, see [V2].

An equivalent definition of the order of invariants is as follows: an invariant is of order j if all such indices assigned by it to all immersions with $\geq j - 1$ double and one triple generic self-intersection points vanish.

In § 5.3 we define similar indices for maps $S^1 \to \mathbb{R}^3$ having an arbitrary finite number of multiple self-intersections or singular points.

4. The discriminant set and its resolution

In this section we recall basic facts from [V2] concerning the topological structure of the discriminant set in the space $\mathcal{K}$ of all smooth maps $S^1 \to \mathbb{R}^n$, $n \geq 3$.

For simplicity we mention this space as a space of very large but finite dimension ω. A partial justification of this assumption uses finite-dimensional approximations of $\mathcal{K}$, see [V2]. Below we indicate by the single quotes ‘, ’ the non-rigorous assertions using this assumption and needing a reference to [V2] for such an justification.

4.1. **Simplicial resolution of the discriminant.** The *resolution* σ of the discriminant set Σ is constructed as follows. Denote by Ψ the space of all unordered pairs (x, y) of points of S^1 (allowing $x = y$); it is easy to see that Ψ is diffeomorphic to the closed Möbius band. Consider a generic embedding I of Ψ in a space $\mathbb{R}^N$ of $(very)^2$ large dimension $N \gg \omega^2$. For any discriminant map $\phi : S^1 \to \mathbb{R}^n$ consider all points $(x, y) \subset \Psi$ such that either $x \neq y$ and $\phi(x) = \phi(y)$, or $x = y$ and $\phi'(x) = 0$. Denote by $\Delta(\phi)$ the convex hull in $\mathbb{R}^N$ of images of all such points under the embedding I. If I is generic, then this convex hull is a simplex, whose vertices coincide with all these images. The space σ is defined as the union of all simplices of the form $\phi \times \Delta(\phi) \subset \mathcal{K} \times \mathbb{R}^N$. The restriction on σ of the obvious projection $\mathcal{K} \times \mathbb{R}^N \to \mathcal{K}$ is proper and induces a ‘homotopy equivalence’ $\pi : \bar{\sigma} \to \bar{\Sigma}$ of one-point compactifications of spaces σ and Σ. By the ‘Alexander duality’, the homology groups $\bar{H}_*(\sigma) \equiv \bar{H}_*(\Sigma)$ of these compactifications ‘coincide’ (up to a change of dimensions) with the cohomology groups of the space of knots:

$$H^i(\mathcal{K} \setminus \Sigma) \simeq \bar{H}_{\omega-i-1}(\Sigma) \equiv \bar{H}_{\omega-i-1}(\sigma). \tag{2}$$

The calculation of these groups is based on the natural stratification of Σ and σ. Let A be a non-ordered finite collection of natural numbers, $A = (a_1, a_2, \dots, a_{\#A})$, any of which is not less than 2, and b a non-negative integer. Set $|A| = a_1 + \dots + a_{\#A}$. An (A, b)-*configuration* is a collection of $|A|$ distinct points in S^1 separated into groups of cardinalities $a_1, \dots, a_{\#A}$, plus a collection of b distinct points in S^1 (some of which can coincide with the above $|A|$ points). For brevity, $(A, 0)$-configurations are called simply A-configurations. A map $\phi : S^1 \to \mathbb{R}^n$ *respects* an (A, b)-configuration if it glues together all points inside any of its groups of cardinalities $a_1, \dots, a_{\#A}$, and its derivative ϕ' is equal to 0 at all the b points of this configuration. For any (A, b)-configuration the set of all maps, respecting it, is an affine subspace in $\mathcal{K}$ of codimension $n(|A| - \#A + b)$; the number $|A| - \#A + b$ is called the *complexity* of the configuration. Two (A, b)-configurations are *equivalent* if they can be transformed into one another by an orientation-preserving homeomorphism $S^1 \to S^1$.

For any (A, b)-configuration J, $A = (a_1, \dots, a_{\#A})$, the corresponding simplex $\Delta(J) \subset \mathbb{R}^N$ is defined as the simplex $\Delta(\phi)$ for an arbitrary *generic* map ϕ respecting J; it has exactly $\binom{a_1}{2} + \dots + \binom{a_{\#A}}{2} + b$ vertices.

For any class $\mathbf{J}$ of equivalent (A, b)-configurations, the corresponding $\mathbf{J}$-*block* in σ is defined as the union of all pairs $\{\phi, x\}$, where ϕ is a map $S^1 \to \mathbb{R}^n$, respecting some (A, b)-configuration J of this equivalence class, and x is a point of the simplex $\Delta(J)$.

4.2. **The main filtration in the resolved discriminant.** The *main filtration* in σ is defined as follows: its term σ_i is the union of all $\mathbf{J}$-blocks over all equivalence classes $\mathbf{J}$ of (A, b)-configurations of complexities $\leq i$.

This filtration is the unique useful filtration in entire σ and will not be revised. Indeed, its term $\sigma_i \setminus \sigma_{i-1}$ is the space of an affine bundle of dimension $\omega - ni$ over a finite-dimensional base, which can easily be described: the one-point compactification of this base is a finite cell complex, see [V2]. The spectral sequence, 'calculating' the groups (2) and induced by this filtration, satisfies the condition

$$E^1_{p,q} = 0 \text{ for } p(n-2) + q > \omega - 1. \tag{3}$$

(By the definition of the spectral sequence, $E^1_{p,q} = \bar{H}_{p+q}(\sigma_i \setminus \sigma_{i-1})$.)

The Alexander dual *cohomological* spectral sequence $E_r^{p,q}$ is obtained from this one by the formal change of indices,

$$E_r^{p,q} \equiv E^r_{-p,\omega-1-q}; \tag{4}$$

it 'converges' to some subgroups of groups $H^{p+q}(\mathcal{K} \setminus \Sigma)$, and the support of its term E_1 belongs to the wedge

$$p \leq 0, p(n-2) + q \geq 0. \tag{5}$$

4.2.1. *Kontsevich's realization theorem.* M. Kontsevich [K2] proved that this spectral sequence (over $\mathbb{C}$) degenerates at the first term: $E_1^{p,q} \equiv E_\infty^{p,q}$. At least if $n = 3$, then for the groups $E^{p,q}$ with $p + q = 0$, which provide knot invariants, this follows from his integral realization of finite-order invariants, described in [K1], [BN2]. Moreover, for arbitrary n and $p < 0$ exactly the same construction proves the degeneration of the "leading term" of the column $E^{p,*}$:

$$E_1^{p,p(2-n)}(\mathbb{C}) \simeq E_\infty^{p,p(2-n)}(\mathbb{C}). \tag{6}$$

In several other cases (in particular if $-p$ is sufficiently small with respect to n) this follows also from dimensional reasons. However, generally for greater values of q the proof is more complicated.

4.2.2. *The main filtration and connected graphs.* This filtration induces an infinite filtration in (some subgroup of) the cohomology group $H^*(\mathcal{K} \setminus \Sigma) \simeq \bar{H}_{\omega-*-1}(\sigma)$: an element of this group has order i if it can be defined as the linking number with a cycle lying in the term σ_i. In particular, for $* = 0$ we get a filtration in the space of knot invariants; it is easy to see that this filtration coincides with the elementary characterization of finite-order invariants given in § 3.

Let J be an (A, b)-configuration of complexity i. By construction, the corresponding **J**-block is a fiber bundle, whose base is the space of (A, b)-configurations equivalent to J, and the fiber over such a configuration J' is the direct product of an affine space of dimension $\omega - ni$ (consisting of all maps respecting this configuration) and the simplex $\Delta(J')$. Consider this **J**-block as the space of the fiber bundle, whose base is the $(\omega - ni)$-dimensional affine bundle over the previous base, and the fibers are simplices $\Delta(J')$. Some points of these simplices belong not only to the i-th term σ_i of our filtration, but even to the $(i-1)$-th term. These points form a simplicial subcomplex of the simplex $\Delta(J')$, let us describe it. Any face of $\Delta(J')$ can be depicted by $\#A$ graphs with $a_1, \dots, a_{\#A}$ vertices and b signs $\pm$. Indeed, to any vertex of $\Delta(J')$ there corresponds either an edge, connecting two points inside some of our $\#A$ groups of points, or one of last b points of the (A, b)-configuration J'.

PROPOSITION 1 (see [V2]). *A face of $\Delta(J')$ belongs to σ_{i-1} if and only if either at least one of corresponding $\#A$ graphs is non-connected or at least one of b points does not participate in its picture (= participates with sign $-$).* □

Denote by $\Delta^1(J')$ the quotient complex of $\Delta(J')$ by the union of all faces belonging to σ_{i-1}. Theorem 1 implies immediately the following statement.

PROPOSITION 2. *The quotient complex $\Delta^1(J')$ is acyclic in all dimensions other than $|A| - \#A + b - 1 \equiv i - 1$, while*

$$\bar{H}_{|A|-\#A+b-1}(\Delta^1(J')) \simeq \otimes_{j=1}^{\#A} \mathbb{Z}^{(a_j-1)!}. \tag{7}$$

4.3. **Non-compact knots.** Simultaneously with the usual knots, i.e., embeddings $S^1 \to \mathbb{R}^n$, we will consider *non-compact knots*, i.e. embeddings $\mathbb{R}^1 \to \mathbb{R}^n$ coinciding with the standard linear embedding outside some compact subset in $\mathbb{R}^1$. The space of all smooth maps with this behavior at infinity will be denoted by K, and the discriminant $\Sigma \subset K$ again is defined as the set of all such maps, having singularities and self-intersections. In this case the configuration space Ψ, participating in the construction of the resolution σ, is not the Möbius band, but the closed half-plane $\mathbb{R}^2/\{(t, t') \equiv (t', t)\}$, which we usually will realize as the half-plane $\{(t, t')|t \le t'\}$.

There is an obvious one-to one correspondence between isotopy classes of standard and non-compact knots in $\mathbb{R}^3$, and spaces of invariants, provided by the above spectral sequences in both theories, naturally coincide. On the other hand, the CW-structure on resolved discriminants in spaces of non-compact knots is easier; this is the reason why in [V2] only the non-compact knots were considered. In the next section 5 we develop some new techniques for calculating cohomology groups of spaces of compact or non-compact knots in $\mathbb{R}^n$, and in section 6 (respectively, 7) apply it to the calculation of cohomology classes of order ≤ 3 of the space of non-compact knots in $\mathbb{R}^n$ (respectively, to the classes of order ≤ 2 of the space of compact knots).

4.4. **Ancient auxiliary filtration in the term $\sigma_i \setminus \sigma_{i-1}$ of the main filtration.** By definition, the term $E^1_{i,q}$ of the main spectral sequence is isomorphic to $\bar{H}_{i+q}(\sigma_i \setminus \sigma_{i-1})$. To calculate these groups, the *auxiliary filtration* in the space $\sigma_i \setminus \sigma_{i-1}$ was defined in [V2]. Namely, its term G_α is the union of all **J**-blocks, such that **J** is an equivalence class of (A, b)-configurations of complexity i, consisting of $\leq \alpha$ geometrically distinct points. In particular, $G_{2i} = \sigma_i \setminus \sigma_{i-1}$ and $G_{i-1} = \emptyset$. By Proposition 2, $\bar{H}_j(G_\alpha) = 0$ if $j \geq \omega - i(n-1) + \alpha$.

4.4.1. *Example: knot invariants in* $\mathbb{R}^3$. If $n = 3$, then $\bar{H}_j(G_{2i-2}) = 0$ for $j \geq \omega - 2$, in particular, calculating the group $\bar{H}_{\omega-1}(\sigma_i \setminus \sigma_{i-1})$, we can ignore all **J**-blocks with auxiliary filtration $\leq 2i - 2$:

$$\bar{H}_{\omega-1}(\sigma_i \setminus \sigma_{i-1}) \simeq \bar{H}_{\omega-1}((\sigma_i \setminus \sigma_{i-1}) \setminus G_{2i-2}). \tag{8}$$

The remaining part $(\sigma_i \setminus \sigma_{i-1}) \setminus G_{2i-2}$ consists of **J**-blocks satisfying the following condition.

Definition. An (A, b)-configuration J (and the corresponding **J**-block) of complexity i is *simple*, if one of three is satisfied:[2]

I. $A = (2^{\times i})$, $b = 0$;

II. $A = (2^{\times (i-1)})$, $b = 1$ and the last point does not coincide with any of $2i - 2$ points, defining the A-part of the configuration;

III. $A = (3, 2^{\times (i-2)})$, $b = 0$.

The complex $\Delta^1(J)$, corresponding to a configuration J of type I or II, consists of unique cell of dimension $i-1$, i.e. of the simplex $\Delta(J)$ itself, while for $A = (3, 2, \dots, 2)$ it consists of one i-dimensional simplex and certain three of its faces of dimension $i - 1$.

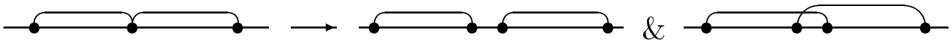
A chain complex, calculating groups (8) (and, moreover, all the groups $\bar{H}_*(\sigma_i \setminus \sigma_{i-1})$) was written out in [V2]; its part calculating the top-dimensional group (8) can be described in the following terms.

An (A, b)-configuration of type I (or the equivalence class of such configurations) can be depicted by a *chord diagram*, i.e. a collection of i chords spanning some i pairs of distinct points of S^1 or $\mathbb{R}^1$. (A, b)-configurations of type III are depicted in [V2] by similar diagrams with $i + 1$ chords, three of which form a triangle and remaining $i - 2$ have no common endpoints. In [BN2] this configuration is depicted by the same collection of $i - 2$ chords and an **Y**-wise star connecting the points of the triple with a point not on the line (or circle). Configurations of types I and III are called $[i]$- and $\langle i \rangle$-configurations, respectively, see [V2].

[2] We denote by $2^{\times m}$ the expression $2, \dots, 2$ with 2 repeated m times

An element of the group (8) is a linear combination of several **J**-blocks of type I and interior parts (swept out by open i-dimensional simplices) of blocks of type III. This linear combination should satisfy the homology condition close to all strata of codimension 1 in σ_i : any such stratum should participate in the algebraic boundary of this combination with coefficient 0. Close to such a stratum, corresponding to an (A, b)-configuration J of type II, there is unique stratum of maximal dimension: this is the stratum of type I, whose (A, b)-configuration can be obtained from that of J by replacing its unique singular point by a small chord, connecting two points close to it. The corresponding homological condition is as follows: any stratum of maximal dimension, whose chord diagram can be obtained in this way (i.e. has a chord, whose endpoints are not separated by endpoints of other chords) should participate in the linear combination with zero coefficient.

Further, any configuration of type III defines three strata of codimension 1 in σ_i (some of which can coincide) swept out by three $(i-1)$-dimensional faces of corresponding simplices $\Delta(J')$. Such a stratum can be described by an $\langle i \rangle$-configuration, in which one chord, forming the triangle, is erased. This stratum is incident to three strata of maximal dimension: one stratum swept out by open i-dimensional simplices from the same **J**-block of type III, defined by the same $\langle i \rangle$-configuration, and two strata of type I corresponding to chord diagrams, obtained from our configuration by diversing the endpoints of our non-complete triangle:



The homological condition is as follows: the difference of coefficients, with which two latter strata participate in the linear combination, should be equal to the coefficient of the stratum swept out by triangles. In particular, all three such differences, corresponding to erasing any of three edges of the chord triangle, should be equal to one another. These *three* equalities (among which only two are independent) are called the *4-term relations,* cf. § 3.

The common value of these three differences is a characteristic of our $(3, 2, \ldots, 2)$-configuration (and of the element α of the homology group (8)): it is equal to the coefficient, with which the main stratum of the **J**-block of type III participates in the cycle α. It is natural to call it the *index* of this cycle at this configuration.

In a similar way, any element α of the group (8) assigns an index to any i-chord diagram: this is the coefficient, with which the corresponding **J**-block of type I participates in the cycle realizing α.

The group (8) is thus canonically isomorphic to the group of all $\mathbb{Z}$-valued functions on the set of i-chord diagrams, which a) take zero value on all diagrams having chords not crossed by other chords of the diagram, and b) satisfy 4-term relations defined by all possible $\langle i \rangle$-configurations.

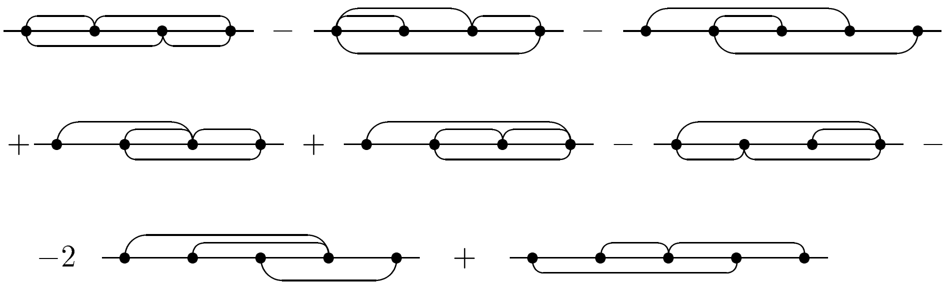


FIGURE 4. The Teiblum–Turchin cocycle

When i grows, this system of equations grows exponentially, and, which is even worse, the answers do not satisfy any transparent rule, see [BN].

4.4.2. *The Teiblum–Turchin cocycle.* In this subsection we consider the resolution of the discriminant in the space of non-compact knots.

Let $\mathbf{J}$ be an arbitrary equivalence class of (A, b)-configurations of complexity i in $\mathbb{R}^1$. Denote by $\tilde{B}(\mathbf{J})$ the intersection of the corresponding $\mathbf{J}$-block with $\sigma_i \setminus \sigma_{i-1}$. This set $\tilde{B}(\mathbf{J})$ consists of several cells, which are in one-to-one correspondence with all possible collections of $\#A$ connected graphs with $a_1, \dots, a_{\#A}$ vertices respectively; for some examples see Fig. 4. Such cells, corresponding to all equivalence classes $\mathbf{J}$ of (A, b)-configurations of complexity i, define a cell decomposition of the quotient space σ_i/σ_{i-1}.

THEOREM (D. Teiblum, V. Turchin). *The linear combination of suitably oriented (see* [V2]*) cells of $\sigma_3 \setminus \sigma_2$ shown in Fig. 4 defines a non-trivial $(\omega - 2)$-dimensional Borel–Moore homology class of $\sigma_3 \setminus \sigma_2$ (and hence also an $(\omega - 2 - 3(n-3))$-dimensional homology class of the term $\sigma_3 \setminus \sigma_2$ of the similar resolution of the space of singular non-compact knots in $\mathbb{R}^n$ for any odd $n \geq 3$).*

On the other hand, the group $\bar{H}_{\omega-3-3(n-3)}(\sigma_2)$ is trivial (see [V2]), hence this class can be extended to a well–defined class in $\bar{H}_{\omega-2-3(n-3)}(\sigma)$ and in $H^{1+3(n-3)}(K \setminus \Sigma)$. If $n > 3$, then by dimensional reasons this class is nontrivial, however its nontriviality for $n = 3$ and any explicit realization are unknown yet.

5. NEW AUXILIARY FILTRATION

For any (A, b)-configuration J denote by $\rho(J)$ the number of *geometrically distinct* points in it. In particular, for any A-configuration J, $\rho(J) = |A|$. The greatest possible value of $\rho(J)$ over all A- or (A, b)-configurations of complexity i is equal to $2i$.

DEFINITION. The *stickiness* of an A-configuration J of complexity i is the number $2i - \rho(J)$. The *reversed auxiliary filtration* $\Phi_0 \subset \cdots \subset \Phi_{i-1}$ in the term $\sigma_i \setminus \sigma_{i-1}$ of the main filtration is defined as follows: its term Φ_α is the closure of the union of $\mathbf{J}$-blocks over all equivalence classes $\mathbf{J}$ of A-configurations of stickiness $\leq \alpha$.

If J is an (A,b)-configuration with $b>0$, then the corresponding $\mathbf{J}$-block belongs to the closure of an $\tilde{\mathbf{J}}$-block, where $\tilde{J}$ is an $\tilde{A}$-configuration, $\tilde{A} = A \cup 2^{\times b}$. Thus $\Phi_{i-1} = \sigma_i \setminus \sigma_{i-1}$.

So, to any finite-order cohomology class v of the space of knots in $\mathbb{R}^n$ (in particular, to any invariant of knots in $\mathbb{R}^3$) there correspond two numbers: the first, $i(v)$, is its order, and the second, $r(v)$, is its reversed filtration, i.e. the minimal reversed filtration in $\sigma_{i(v)} \setminus \sigma_{i(v)-1}$ of realizing it cycles in $\sigma_{i(v)}$.

5.1. **First examples.**

NOTATION. For any topological space X, the *k-th configuration space* $B(X,k)$ is the space of all subsets of cardinality k in X. $\pm\mathbb{Z}$ is the local system of groups on the configuration space $B(X,k)$, locally isomorphic to $\mathbb{Z}$ and such that elements of $\pi_1(B(X,k))$, defining odd permutations of k points, act on its fibers as multiplication by -1. The Borel–Moore homology group $\bar{H}_*(B(\Psi,k),\pm\mathbb{Z})$ is the homology group of locally finite chains with coefficients in this local system, cf. [V6].

In the following two examples we consider the discriminant in the space of non-compact knots.

EXAMPLE 3. If $i=1$, then the entire space $\sigma_1 \equiv \Phi_0$ is the space of an $(\omega-n)$-dimensional affine fiber bundle over the manifold Ψ, diffeomorphic to the closed half-plane. In particular, all its Borel–Moore homology groups are trivial and there are no non-trivial cohomology classes of order 1 for an arbitrary n.

EXAMPLE 4. Our filtration of the set $\sigma_2 \setminus \sigma_1$ consists of two terms $\Phi_0 \subset \Phi_1$. Φ_0 is the space of a fiber bundle, whose base is the configuration space $B(\Psi,2)$, and the fiber is the product of an open interval and an affine space of codimension $2n$ in the space K. The generator of the group $\pi_1(B(\Psi,2)) \simeq \mathbb{Z}$ changes the orientation of the first factor of the fiber and multiplies the orientation of the second by $(-1)^n$. In particular, the Borel–Moore homology group of the term Φ_0 is isomorphic (up to the shift of dimensions) to the group $\bar{H}_*(B(\Psi,2))$ if n is odd and to $\bar{H}_*(B(\Psi,2),\pm\mathbb{Z})$ if n is even. It is easy to calculate (see also Lemma 1 below) that both these groups are trivial in all dimensions.

The term $\Phi_1 \setminus \Phi_0$ is the space of a fiber bundle, whose base is the configuration space $B(\mathbb{R}^1,3)$ of triples of points in the line, and the fiber is the product of an affine subspace of codimension $2n$ in K and the interior part of a triangle. The Borel–Moore homology of this term is obviously isomorphic to $\mathbb{Z}$ in the dimension $\omega-2n+5$ and is trivial in all other dimensions. Thus the group of cohomology classes of order 2 is isomorphic to $\mathbb{Z}$ in dimension $2n-6$ and is trivial in all other dimensions.

A comparison. Calculating the homology of $\sigma_2 \setminus \sigma_1$ with the help of the ancient auxiliary filtration, we assign the edges of these triangles to the same term as their interior parts; as a consequence, they separate the space $B(\Psi, 2)$ into several cells, corresponding to combinatorial types of chord diagrams. In the new approach all these cells are joined together and form unique manifold $B(\Psi, 2)$ with a simple topology.

In a similar way, for any i the space $\sigma_i \setminus \sigma_{i-1}$ consists of i terms $\Phi_0 \subset \cdots \subset \Phi_{i-1}$ of the reversed filtration, and the ultimate term $\Phi_{i-1} \setminus \Phi_{i-2}$ is the space of a fiber bundle, whose base is the configuration space $B(\mathbb{R}^1, i+1)$, and the fiber is the direct product of an affine subspace of codimension ni in the space K and an $\binom{i+1}{2}$-vertex simplex, from which all faces belonging to Φ_{i-2} are removed. These faces are exactly those corresponding to not 2-connected graphs, see Theorem 3 below.

Similarly, to any immersion $\mathbb{R}^1 \to \mathbb{R}^n$, having $j < \infty$ self-intersection points of finite multiplicities $a_1, \ldots, a_j$, we associate the homology group of the corresponding component of the term $\Phi_\alpha \setminus \Phi_{\alpha-1}$ of our filtration in $\sigma_i \setminus \sigma_{i-1}$, where $i = (\sum_{m=1}^j a_m - j)$, $\alpha = \sum_{m=1}^j a_m - 2j$. This group is isomorphic (up to a shift of dimensions) to the tensor product of homology groups of complexes of two-connected graphs with $a_1, \ldots, a_j$ nodes.

5.2. **Structure of the reversed filtration.** The set $\Phi_\alpha \setminus \Phi_{\alpha-1}$ of the new filtration in $\sigma_i \setminus \sigma_{i-1}$ consists of several components, numbered by unordered decompositions of the number $2i - \alpha$ into $i - \alpha$ subsets of cardinalities ≥ 2. To any such decomposition

$$A = (a_1, a_2, \ldots, a_{\#A}), \quad \#A = i - \alpha, \quad \sum a_l = 2i - \alpha, \quad a_l \geq 2, \tag{9}$$

there corresponds the closure of all **J**-blocks defined by all equivalence classes **J** of A-configurations. Let us describe this closure explicitly.

Definition. An *A-collection* is a collection of $|A|$ points in $\mathbb{R}^1$ separated into $\#A$ groups of cardinalities $a_1, \ldots, a_{\#A}$, such that the points inside any group of cardinality > 2 are pairwise distinct.

A map $\phi : \mathbb{R}^1 \to \mathbb{R}^n$ *respects* an A-collection, if $\phi(x) = \phi(y)$ for any two different points x, y of the same group, and $\phi'(x) = 0$ for any point x such that (x, x) is a group of cardinality $a_i = 2$ of this collection.

The set of all maps $\phi : \mathbb{R}^1 \to \mathbb{R}^n$, respecting an A-collection, is an affine subspace in K of codimension $\leq n(|A| - \#A)$. Indeed, any group of cardinality a_j gives us $n(a_j - 1)$ independent restrictions on ϕ (although the union of all $n \sum (a_j - 1) \equiv n(|A| - \#A)$ such conditions can be dependent).

An A-collection is an *A-set* if these conditions are independent, i.e. the codimension of the space of maps $\phi : \mathbb{R}^1 \to \mathbb{R}^n$, respecting this collection, is equal to $n(|A| - \#A)$.

For example, $(2,2,2)$-sets are any subsets of cardinality 3 in Ψ not of the form $((x < y), (x < z), (y < z))$.

More generally, given an A-collection, consider the graph with $|A|$ vertices, corresponding to its points, and edges of following two types: all points inside any group are connected by black edges, and geometrically coincident points of different groups are connected by white edges. (If there is a degenerate group of the form (x, x) and the point x belongs also to other group(s), then it is depicted by two points connected by a black edge, and only one of these points is connected by white edges with corresponding points of other groups.)

PROPOSITION 3. *Our A-collection is an A-set if and only if any cycle of this graph, having no repeating nodes, contains no white edges. In particular, two different groups cannot have two common points.*

Proof. Suppose that we have such a cycle containing white edges. Removing an arbitrary its vertex, incident to a white edge of the cycle, from the corresponding group of the A-collection (say, from the group of cardinality a_l) we obtain an $\tilde{A}$-collection, $\tilde{A} = (a_1, \dots, a_l - 1, \dots, a_{\#A})$, defining the same subspace in K. Thus the codimension of this subspace is no greater than $n(|A| - \#A - 1)$. Conversely, if we have no such cycles, then there exists a node of our graph, not incident to white edges. Removing the corresponding point from the A-collection (and cancelling the group containing this point if it is a group of cardinality 2) we obtain an $\tilde{A}$-collection with $|\tilde{A}| - \#\tilde{A} = |A| - \#A - 1$. On the other hand, erasing this point we loss exactly n linear conditions on the corresponding subspace in K, which are independent on the others. Thus the "if" part of Proposition 3 follows by induction. $\square$

An A-set and A'-set are *related* if the sets of maps $\mathbb{R}^1 \to \mathbb{R}^n$, respecting them, coincide. For example, the (3)-set $((x < y < z))$ and (2, 2)-set $((x, y), (x, z))$ are related. There is a partial order on any class of related sets: the set $\bar{\Upsilon}$ is a *completion* of the related set Υ if $\bar{\Upsilon}$ is obtained from Υ by replacing some two intersecting groups by their union or by a sequence of such operations.

The space of all A-sets with given A is denoted by $C(A)$.

An open dense subset in $C(A)$ consists of A-configurations, see § 4.1 above. E.g., such a subset in $C(2, 2, 2)$ consists of all points $((x, x'), (y, y'), (z, z')) \in B(\Psi, 3)$ such that none two of six numbers $x, x', \dots, z'$ coincide.

5.2.1. *The complex $\Lambda(\Upsilon)$.* Given an A-set Υ in $\mathbb{R}^1$, consider the simplex $\Delta(\Upsilon)$ with $\sum_{j=1}^{\#A} \binom{a_j}{2}$ vertices, corresponding to all pairs $(x, y) \in \Psi$ such that x and y belong to one group of Υ. A face of this simplex can be encoded by a collection of $\#A$ graphs, whose edges span the pairs corresponding to vertices of this face.

Such a face is called 2-*connected*, if all these graphs are 2-connected.

We will identify the nodes of these graphs with corresponding points in $\mathbb{R}^1$.

Denote by $\Lambda(\Upsilon)$ the union of interior points of all 2-connected faces of our simplex.

Theorem 2 implies immediately the following statement.

COROLLARY 1. *1. The group $\bar{H}_l(\Lambda(\Upsilon))$ is trivial for all $l \neq 2|A| - 3\#A - 1$ and is isomorphic to $\otimes_{j=1}^{\#A} \mathbb{Z}^{(a_j-2)!}$ for $l = 2|A| - 3\#A - 1$; this isomorphism is defined canonically up to permutations of factors $\mathbb{Z}^{(a_j-2)!}$, corresponding to coinciding numbers a_j.*

2. For any decomposition of the set Υ into two non-intersecting subsets Υ', Υ'', any of which is the union of several groups of Υ, we have

$$\bar{H}_{*-1}(\Lambda(\Upsilon)) \simeq \bar{H}_*(\Lambda(\Upsilon')) \otimes \bar{H}_*(\Lambda(\Upsilon'')), \tag{10}$$

where the lower index $*-1$ denotes the shift of grading. □

(In other words, if we introduce the notation

$$\mathcal{H}(\Upsilon) \equiv \bar{H}_{2|A|-3\#A-1}(\Lambda(\Upsilon))$$

for any A and any A-set Υ, then $\mathcal{H}(\Upsilon) \simeq \mathcal{H}(\Upsilon') \otimes \mathcal{H}(\Upsilon'')$.)

Namely, if we have two cycles $\gamma' \subset \Lambda(\Upsilon') \subset \Delta(\Upsilon')$, $\gamma'' \subset \Lambda(\Upsilon'') \subset \Delta(\Upsilon'')$, then the corresponding *join* cycle $\gamma' * \gamma'' \subset \Delta(\Upsilon) \equiv \Delta(\Upsilon') * \Delta(U'')$ is realized as the union of all open intervals, connecting the points of γ' and γ''. Similarly, if γ^j, $j = 1, \dots, \#A$, are cycles defining some elements of groups $\bar{H}_*(\Lambda(\Upsilon_j)) \simeq \mathbb{Z}^{(a_j-2)!}$, then their join $\gamma^1 * \dots * \gamma^{\#A}$ is swept out by open $(\#A-1)$-dimensional simplices, whose vertices are the points of these cycles γ^j. The corresponding homology map $\otimes \bar{H}_*(\Lambda(\Upsilon_j)) \to \bar{H}_{*+\#A-1}(\Lambda(\Upsilon))$ is an isomorphism and depends on the choice of the orientation of these simplices (= the ordering of their vertices or, equivalently, of the groups $\Upsilon_j \subset \Upsilon$).

Let A be a multiindex (9) of complexity i, and Υ an A-set in $\mathbb{R}^1$. The simplex $\Delta(\Upsilon)$ can be realized as the simplex in $\mathbb{R}^N$, spanned by all points $I(x,y)$, $(x,y) \in \Psi$, such that x and y belong to the same group of Υ. For any map $\phi \in K$, respecting Υ, the simplex $\phi \times \Delta(\Upsilon) \subset K \times \mathbb{R}^N$ belongs to the term σ_i of the main filtration, and its intersection with the space $\sigma_i \setminus \sigma_{i-1}$ belongs to the term Φ_α of the reversed filtration of this space, where $\alpha = 2i - \rho(\Upsilon)$.

PROPOSITION 4. *If $\bar{\Upsilon}$ is a completion of Υ and $\bar{\Upsilon} \neq \Upsilon$, then the simplex $\Delta(\Upsilon)$ is a not two-connected face of $\Delta(\bar{\Upsilon})$.*

Proof. Consider the two-color graph of the A-set Υ mentioned in Proposition 3. The graph, representing the face $\Delta(\Upsilon) \subset \Delta(\bar{\Upsilon})$, is obtained from it by contracting all white edges. Since $\Upsilon \neq \bar{\Upsilon}$, there is at least one white edge connecting two points of the same group of $\bar{\Upsilon}$. Removing from $\Delta(\Upsilon)$ the vertex, obtained from this edge, we get a graph, which by Proposition 3 is not connected. □

THEOREM 3. *For any multiindex A of the form (9) and any A-set Υ, a point of the simplex $\phi \times \Delta(\Upsilon)$ belongs to $\Phi_\alpha \setminus \Phi_{\alpha-1}$ if and only if it is an interior point of a 2-connected face.*

Proof. First we prove the "only if" part. Consider a face of the simplex $\Delta(\Upsilon)$ such that one of corresponding graphs $g_1, \dots, g_{\#A}$, say g_m, is connected but not two-connected. We need to prove that this face belongs to $\Phi_{\alpha-1}$. Since $\Phi_{\alpha-1}$ is closed in Φ_α, it is sufficient to prove this property for all A-sets Υ from an arbitrary dense subset in $C(A)$. In particular we can assume that Υ is an A-configuration (see § 4.1), i.e. all its $|A|$ points are pairwise distinct.

Let $y \in \mathbb{R}^1$ be a node of our graph g_m such that removing it we split g_m into two non-empty graphs g'_m, g''_m with a'_m and a''_m nodes respectively, $a'_m + a''_m = a_m - 1$. Let A' be the multiindex $(a_1, \dots, a_{m-1}, a'_m + 1, a''_m + 1, a_{m+1}, \dots, a_{\#A})$ and Υ' an A'-set, related to Υ, whose groups of cardinalities $a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_{\#A}$ coincide with these for Υ, and the last two coincide with the sets of vertices of g'_m, g''_m, augmented by y. Then our face of the simplex $\phi \times \Delta(\Upsilon) \simeq \Delta(\Upsilon)$ lies also in the simplex $\phi \times \Delta(\Upsilon')$. This simplex belongs to the closure of the set swept out by similar simplices $\phi' \times \Delta(\Upsilon'(\tau))$, $\tau \in (0, \varepsilon]$, where $\Upsilon'(\tau)$ are A'-sets with $\rho(\Upsilon'(\tau)) > \rho(\Upsilon') = \rho(\Upsilon)$, namely, some their $|A'| - 2$ points coincide with these for Υ', and only two points y of the m-th and $(m+1)$-th groups are replaced by $y - \varepsilon$ and $y + \varepsilon$ respectively. Thus our face belongs to a lower term of the reversed filtration.

Conversely, for any multiindex A of the form (9), the closure of the union of all **J**-blocks over all A-configurations J consists of simplices of the form $\phi \times \Delta(\Upsilon)$, where Υ is an A-set, and ϕ a map respecting it. By Proposition 4, if Υ is such an A-set and $\bar{\Upsilon}$ a completion of Υ, $\bar{\Upsilon} \neq \Upsilon$, then such a simplex is a not two-connected face of $\Delta(\bar{\Upsilon})$. □

COROLLARY 2. *The component of $\Phi_\alpha \setminus \Phi_{\alpha-1}$, corresponding to the multiindex A of the form (9), is the space of a fiber bundle over the space of all A-sets, whose fiber over the A-set Υ is the direct product of the complex $\Lambda(\Upsilon)$ and the affine subspace of codimension ni in K, consisting of all maps ϕ, respecting Υ.* □

Denote this component by $S(A)$.

COROLLARY 3. *For any A, the group $\bar{H}_*(S(A))$ is trivial in all dimensions greater than $\omega - 1 - (n-3)i$.*

This follows immediately from Theorems 2 and 3 and implies formulas (3) and (5).

5.3. **Index of a knot invariant at a complicated singular knot.** Let A be a multiindex $(a_1, \dots, a_{\#A})$ of complexity i, J an A-configuration, consisting of groups $J_1, \dots, J_{\#A}$, and ϕ an immersion $\mathbb{R}^1 \to \mathbb{R}^3$, respecting J but not respecting more complicated configurations. Define the group $\Xi_*(J)$ as the tensor product

$$\otimes_{m=1}^{\#A} \bar{H}_*(\Lambda(J_m))$$

(where $\Lambda(J_m)$ is the complex of two-connected graphs, whose nodes are identified with points of the group J_m).

By Theorem 2 this graded group has unique nontrivial term in dimension $2|A|-4\#A$ and is isomorphic to $\otimes_{m=1}^{\#A}\mathbb{Z}^{(a_j-2)!}$ in this dimension. In particular, there is an isomorphism

$$\Xi_*(J) \equiv \bar{H}_{*+\#A-1}(\Lambda(J)). \tag{11}$$

By the *join* construction (see § 5.2.1) this isomorphism is defined almost canonically, only up to a sign, which can be specified by any ordering of our groups J_m and is different for orders of different parity.

Any knot invariant V of order i defines an element of this group $\Xi_*(J)$, the *index* of the singular knot ϕ; this index generalizes similar indices of not very complicated singular knots considered in §§ 3 and 4.4.1. Indeed, consider any small open contractible neighborhood U of our point ϕ in the space of discriminant maps equisingular to ϕ (i.e. respecting equivalent configurations). The complete pre-image of U in the space σ_i is homeomorphic to the direct product of U and the simplex $\Delta(\phi)$. Remove from this simplex all faces, lying in lower terms of the main and reversed auxiliary filtrations. Remaining domain will be homeomorphic to $U\times\Lambda(J)$. The intersection of our invariant (i.e. of the cycle $\gamma\in\bar{H}_{\omega-1}(\sigma_i)$ realizing it) with this domain defines (via the Künneth isomorphism) a class in $\bar{H}_{2|A|-3\#A-1}(\Lambda(J))$ and hence, by the isomorphism (11), also in the group $\Xi_*(J)$; this class is the desired index $\langle V|\phi\rangle$.

To make this definition correct, we need to specify the orientation of U, participating in the construction of the Künneth isomorphism. This orientation consists of a) an orientation of the space $\mathbf{J}$ of configurations equivalent to J, and b) an orientation of the space $\chi(J)$ of maps $\mathbb{R}^1\to\mathbb{R}^3$ respecting J. To define the first orientation, we order all groups J_m of J in an arbitrary way and order points of any group by their increase in $\mathbb{R}^1$. Thus we get an order of all points of J (first there go points of the first group in increasing order, then of the second, etc.), and thus also the orientation of the configuration space $\mathbf{J}$.

Further, we will suppose that an orientation of the functional space K is fixed, then orientations of the subspace $\chi(J)$ can be identified with its coorientations, i.e. orientations of the normal bundle. Let X,Y,Z be fixed coordinates in $\mathbb{R}^3$, so that the map ϕ consists of three real functions $X(t),Y(t),Z(t)$. Then a coorientation of $\chi(J)$ can be specified by the differential form $[d(X($the second point $t_{1,2}$ of the first group $J_1\subset J)-X($the first point $t_{1,1}$ of this group$))\wedge d(Y(t_{1,2})-Y(t_{1,1}))\wedge d(Z(t_{1,2})-Z(t_{1,1}))]\wedge\ldots\wedge[d(X(t_{1,a_1})-X(t_{1,a_1-1}))\wedge d(Y(t_{1,a_1})-Y(t_{1,a_1-1}))\wedge d(Z(t_{1,a_1})-Z(t_{1,a_1-1}))]\wedge$ (the same for the second group J_2) $\wedge\ldots\wedge$ (the same for $J_{\#A}$).

Lemma 1. *1. The orientation of U, defined by this pair of orientations of $\mathbf{J}$ and $\chi(J)$, depends on the choice of the order of groups $J_m\subset J$, namely, an odd permutation of these groups multiplies it by -1.*

2. This orientation will be preserved if we change the increasing order of points in an arbitrary group J_m by means of any cyclic permutation.

The proof is elementary. $\square$

The first statement of lemma implies that our index $\langle V|\phi\rangle$ does not depend on the choice of the order of groups $J_m \subset J$ (formally participating in its construction). Indeed, changing this order by an odd permutation, we multiply by -1 both the Künneth isomorphism and the isomorphism (11).

The second statement allows us to define a similar index also in the case of compact knots $S^1 \to \mathbb{R}^3$.

Moreover, suppose that our discriminant point ϕ is generic in its stratum of complexity i, i.e. it does not belong to the closure of the set of maps, respecting configurations of complexities $> i$. (Typical examples of non-generic points are immersions, having self-tangency points or triple points, at which three local branches are coplanar.) Then *any* knot invariant (of arbitrary order) defines in the same way its generalized index, taking its value in the same homology group. Indeed, in this case the complete intersection of our neighborhood U in entire $\sigma\setminus\sigma_{i-1}$ (and not in $\sigma_i\setminus\sigma_{i-1}$) has the same structure of the direct product $U \times \Lambda(J)$.

For any class $\mathbf{J}$ of equivalent A-configurations, all groups $\Xi_*(J)$, $J \in \mathbf{J}$, are canonically isomorphic to one another, thus we can define the group $\Xi_*(\mathbf{J})$ as any of them.

Lemma 2. *Let V be an invariant of order i of non-compact knots in $\mathbb{R}^3$, and $\mathbf{J}$ a class of equivalent A-configurations in $\mathbb{R}^1$ of complexity $|A| - \#A = i$. Then all indices $\langle V|\phi\rangle \in \Xi_*(\mathbf{J})$ for all generic immersions ϕ, respecting A-configurations $J \in \mathbf{J}$, coincide.* $\square$

This (obvious) lemma allows us to define the index $\langle V|\mathbf{J}\rangle \in \Xi_*(\mathbf{J})$ as the common value of these indices $\langle V|\phi\rangle$.

These indices satisfy the natural homological condition (the analog of the STU-relation of [BN2]). Let J, J' be two A-configurations from neighboring equivalence classes, i.e. J' can be obtained from J by the permutation of exactly two neighboring points $t_1 < t_2$ of different groups $J_m, J_{m+1} \subset J$. Then groups $\Xi_*(\mathbf{J}), \Xi_*(\mathbf{J}')$ are obviously identified. Let $A!$ be the multiindex, obtained from A by replacing two numbers a_m, a_{m+1} by one number $a_m + a_{m+1} - 1$, and $J!$ the $A!$-configuration obtained from J by contracting the segment $[t_1, t_2]$ into a point τ, which will thus belong to the group of cardinality $a_m + a_{m+1} - 1$. The index $\langle V|J!\rangle$ is an element of the group $\Xi_*(J!)$, i.e. a linear combination of collections of $\#A!$ two-connected graphs (with ordered edges). Its boundary $\partial\langle V|J!\rangle$ is a linear combination of similar collections of graphs, exactly one of which is not two-connected. Denote by $\partial_\tau\langle V|J!\rangle$ the part of this linear combination, spanned by all such collections, that removing from any of them the point τ we split the corresponding graph Γ_m with $a_m + a_{m+1} - 1$ vertices into two disjoint graphs, whose nodes are the points of $J_m \setminus t_1$ and $J_{m+1} \setminus t_2$. Any such graph Γ_m is the union of two graphs with a_m and a_{m+1} nodes and common node τ, thus the space of all such linear combinations also can be identified with any of groups $\Xi_*(\mathbf{J}), \Xi_*(\mathbf{J}')$. The promised homological condition consists in the fact that the chain in σ_i, Alexander dual to V, actually is a cycle, and in particular its algebraic boundary close to the common boundary of $\mathbf{J}$-, $\mathbf{J}'$- and $\mathbf{J}!$-blocks is equal to zero. This condition is as follows:

$$\langle V|\mathbf{J}\rangle - \langle V|\mathbf{J}'\rangle = \pm\partial_\tau \langle V|\mathbf{J}!\rangle, \tag{12}$$

where the coefficient $\pm$ depends on the choice of local orientations of all participating strata.

The system of equations (12), corresponding to all points τ of the configuration $\mathbf{J}!$, is strong enough to determine the index $\langle V|\mathbf{J}!\rangle$ if we know all indices $\langle V|\mathbf{J}\rangle$ for all configurations $\mathbf{J}$ of smaller stickiness. Indeed, any knot invariant V of order i is determined (up to terms of lower orders) by its indices at chord diagrams, i.e., the configurations of stickiness 0.

This equality allows us also to give the following characterization of the reversed filtration. Let $\mathbf{J}$, $\mathbf{J}'$ be two classes of equivalent A-configurations with the same A. An arbitrary correspondence between their groups $J_m, J'_{m'}$ of equal cardinalities allows us to identify the groups $\Xi_*(\mathbf{J})$ and $\Xi_*(\mathbf{J}')$. This identification is not unique if some of numbers a_m of the multiindex A coincide. Moreover, in the last case we can define a group of automorphisms, acting on $\Xi_*(\mathbf{J})$ and generated by all possible permutations of groups of the same cardinalities a_m, preserving the order of points inside any group, cf. Corollary 1 above.

Proposition 5. *For any knot invariant V of order i, the following conditions are equivalent:*

1) V has reversed filtration $\leq \alpha$;

2) for any multiindex A with $|A| - \#A = i$, $i - \#A = \alpha$, all the indices $\langle V|\mathbf{J}\rangle \in \Xi_*(\mathbf{J})$, defined by V in all $\mathbf{J}$-blocks over all A-configurations $\mathbf{J}$, are invariant under all the identifications and automorphisms described in the previous paragraph.

Proof. (1) $\Rightarrow$ (2). If $\bar{V}$ is the class in $\bar{H}_{\omega-1}(\sigma_i)$, representing V, then for any A as above its restriction on $\Phi_\alpha \setminus \Phi_{\alpha-1}$ defines a class in $\bar{H}_{\omega-1}(S(A))$. The indices $\langle V|\mathbf{J}\rangle$ for all classes $\mathbf{J}$ of equivalent A-configurations are defined by restrictions of this class to all corresponding $\mathbf{J}$-blocks. In particular, these indices for neighboring classes of A-configurations (i.e. for those obtained one from another by a change of orders of only two points from different groups) should coincide: otherwise our cycle in $S(A)$ would have nontrivial boundary at the border between these two blocks.

The implication (2) $\Rightarrow$ (1) means that there are no non-trivial elements of the group $\bar{H}_{\omega-1}(\sigma_i \setminus \sigma_{i-1})$, represented by cycles with support in $(\sigma_i \setminus \sigma_{i-1}) \setminus \Phi_\alpha$; this follows from the fact that any invariant V of order i is determined up to invariants of lower orders by its indices at all i-chord diagrams (i.e., at all $\mathbf{J}$-blocks with $A = (2, \dots, 2)$). $\square$

5.4. **The symbol of a cohomology class of a finite order.** Let again n be an arbitrary natural number greater than 2, and K the space of non-compact knots $\mathbb{R}^1 \to \mathbb{R}^n$. Let $V \in H^a(K \setminus \Sigma)$ be any cohomology class of order i, i.e., a class, Alexander dual to an $(\omega - a - 1)$-dimensional cycle $\bar{V} \in \sigma_i$. Let α be the reversed filtration of this class, i.e. we can choose this cycle $\bar{V}$ in such a way that it lies in $\Phi_\alpha \cup \sigma_{i-1}$. Then for an arbitrary multiindex A with $|A| - \#A = i$ and $\#A = i - \alpha$, the *symbol* $s(A, V)$ is defined as follows.

Let us denote by $\Xi(A)$ the local system of groups on the configuration space $C(A)$, whose fiber over an A-set Υ is identified with $\Xi_*(\Upsilon)$. This local system is not constant if some of numbers a_m coincide: a path in $C(A)$, permuting some groups of the same cardinality, acts nontrivially on the fiber.

Further, let $\tilde{\Xi}(A)$ be a similar local system with a slightly more complicated monodromy action. Namely, any path in $C(A)$, permuting exactly two groups of points, acts exactly as in the system $\Xi(A)$ (i.e., permutes the corresponding factors $\bar{H}_*(\Lambda(J_m))$) if the cardinality of any of these groups is odd, and additionally multiplies such an operator by -1 if these cardinalities are even.

The symbol $s(A, V)$, which we are going to define, is an element of the group $H^a(C(A), \Xi(A))$ if n is odd and of $H^a(C(A), \tilde{\Xi}(A))$ if n is even.

The construction repeats that for the index of a knot invariant given in the previous subsection. Namely, we consider the class of the realizing V dual cycle $\bar{V}$ in the group $\bar{H}_{\omega-a-1}(\Phi_\alpha \setminus \Phi_{\alpha-1})$, then, using Corollary 2 of Theorem 3, Thom isomorphism for the fiber bundle mentioned in this Corollary, and Poincaré duality, we reduce this group to the cohomology group of $C(A)$ with coefficients in a local system associated to the homology bundle of the fiber bundle $\Phi_\alpha \setminus \Phi_{\alpha-1} \to C(A)$.

For instance, if $n = 3$ and $a = 0$, i.e. V is just a knot invariant, then this symbol coincides with the totality of indices of V, corresponding to all classes of A-configurations. The invariance of these indices, mentioned in Proposition 5, follows from the fact, that the 0-dimensional cocycle of a local system is just a global section of this system and thus is invariant under its monodromy action.

5.5. **Multiplication.**

5.5.1. *Multiplication of knot invariants in* $\mathbb{R}^3$. We have two filtrations in the space of finite-order invariants of knots in $\mathbb{R}^3$: the order i and, for any fixed i, the degree α with respect to the reversed auxiliary filtration in $\sigma_i \setminus \sigma_{i-1}$. It is well-known that the order is multiplicative. In fact, the same is true also for the second number.

THEOREM 4. *Let V', V'' be two knot invariants of orders i', i''. Then for any multiindex $A = (a_1, \dots, a_{\#A})$ of complexity $i = i' + i''$ and any equivalence class $\mathbf{J}$ of A-configurations, the index $\langle V' \cdot V'' | \mathbf{J} \rangle \in \Xi_*(\mathbf{J})$ is equal to*

$$\sum_{J' \cup J'' = J} \langle V' | J' \rangle \otimes \langle V'' | J'' \rangle, \qquad (13)$$

where summation is taken over all decompositions of any A-configuration J representing $\mathbf{J}$ *into disjoint unions of an A'-configuration J' and A''-configuration J'' of complexities i' and i'', such that any group of J belongs to either J' or J'.*

Proof. For $A = (2^{\times i})$ this is just the Kontsevich's decomposition formula for the index of $V' \cdot V''$ on an i-chord diagram: for any $(2, \dots, 2)$-configuration τ,

$$\langle (V' \cdot V''), (\tau) \rangle = \sum_{\tau' \cup \tau'' = \tau} V'(\tau')V''(\tau''), \tag{14}$$

summation over all $\binom{i}{i'}$ ordered decompositions of the i-chord diagram τ into an i'-chord diagram τ' and an i''-chord diagram τ''; see e.g. [BN2].

For an arbitrary multiindex A of the same complexity i the formula (13) follows by induction over the stickiness, while the induction step is the formula (12). $\square$

Corollary 4. *The bi-order (i, α) is multiplicative: if V' and V'' are two invariants of bi-orders (i', α') and (i'', α'') respectively, then $V' \cdot V''$ is an invariant of bi-order $(i' + i'', \alpha' + \alpha'')$.*

Indeed, in this case the formula (13) gives one and the same answer for all A-configurations J of complexity $i' + i''$ and stickiness $\alpha' + \alpha''$, thus our Corollary follows from Proposition 5. $\square$

5.5.2. *Multiplication conjecture for higher cohomology of spaces of knots.* For any miltiindices A', A'' and $A \equiv A' \cup A''$, there are natural operations

$$H^*(C(A'), \Xi(A')) \otimes H^*(C(A''), \Xi(A'')) \to H^*(C(A), \Xi(A)), \tag{15}$$

$$H^*(C(A'), \tilde{\Xi}(A')) \otimes H^*(C(A''), \tilde{\Xi}(A'')) \to H^*(C(A), \tilde{\Xi}(A)) \tag{16}$$

such that for any cohomology classes $V', V'' \in H^*(\mathcal{K} \setminus \Sigma)$ of bi-orders (i', α') and (i'', α'') respectively, the class $V' \cdot V''$ has bi-order $(i' + i'', \alpha' + \alpha'')$, and its principal symbol $s(A, V' \cdot V'')$ can be expressed through the symbols of V' and V'' by a formula similar to (13), in which the operation of the tensor multiplication is replaced by the operation (15) if n is odd and by operation (16) if n is even, cf. [Fu], § 8.

5.6. **Configurations, containing groups of 2 points, do not contribute to the cohomology of lowest possible dimension.** Consider the *reversed spectral sequence* $\mathcal{E}^r_{p,q}$, calculating the group $\bar{H}_*(\sigma_i \setminus \sigma_{i-1})$ and generated by the reversed filtration $\{\Phi_\alpha\}$. By definition, the term $\mathcal{E}^1_{\alpha,q}$ of this spectral sequence is isomorphic to $\bar{H}_{\alpha+q}(\Phi_\alpha \setminus \Phi_{\alpha-1})$ and thus splits into the direct sum of similar groups $\bar{H}_{\alpha+q}(S(A))$ over all indices A of the form (9) with $\#A = i - \alpha$.

In particular, by Corollary 3 it is trivial if $\alpha + q > \omega - 1 - (n-3)i$.

For instance, for any i the term Φ_0 of our filtration consists of unique stratum $S(A)$ with $A = (2, \dots, 2)$.

PROPOSITION 6. *For any i, two top terms $\mathcal{E}^1_{0,\omega-1-(n-3)i}$ and $\mathcal{E}^1_{0,\omega-2-(n-3)i}$ of the column $\mathcal{E}^1_{0,*} \equiv \mathcal{E}^1_{0,*}(i)$ of the reversed auxiliary spectral sequence, calculating $\bar{H}_*(\sigma_i \setminus \sigma_{i-1})$, are trivial.*

This proposition is a special case of the following one.

PROPOSITION 7. *For any multiindex A of the form (9), such that at least one (respectively, at least two) of numbers a_i are equal to 2, the group $\bar{H}_{\omega-1-(n-3)i}(S(A))$ (respectively, both groups $\bar{H}_{\omega-1-(n-3)i}(S(A))$, $\bar{H}_{\omega-2-(n-3)i}(S(A))$) is trivial.*

Example. Let be $n = 3$, then, calculating the group $\bar{H}_{\omega-1}(\sigma_i)$ of knot invariants of order i, we can take into account only strata $S(A)$ with at most one character 2 in the multiindex A. Moreover, such strata with exactly one 2 in A can provide only relations in this group, and not generators.

LEMMA 3. *For any natural k, $\bar{H}_*(B(\Psi, k)) \simeq 0$ and $\bar{H}_*(B(\Psi, k), \pm\mathbb{Z}) \simeq 0$.*

(For the definition of the local system $\pm Z$, see § 5.1 above.)

Proof of Lemma 3. We use the decomposition of $B(\Psi, k)$ similar to the cell decomposition of the space $B(\mathbb{R}^2, k)$ used in [Fu]. Namely, to any decomposition $k = k_1 + \cdots + k_m$ of the number k into natural numbers k_i we assign the set of all k-subsets of Ψ, consisting of points $((t_1 \leq t'_1), \dots, (t_k \leq t'_k))$, such that the smallest value of numbers t_l appears in exactly k_1 pairs $(t_l \leq t'_l)$, the next small value in k_2 pairs etc. Denote this set by $e(k_1, \dots, k_m)$. Filter the space $B(\Psi, k)$, assigning to the l-th term of the filtration the union of all such sets of dimension $\leq l$. It is easy to see that any such set is diffeomorphic to the direct product of a closed octant in $\mathbb{R}^m$ and an open k-dimensional cell, thus its Borel–Moore homology group is trivial, as well as the similar group with coefficients in (the restriction on this set of) the local system $\pm\mathbb{Z}$. Therefore the spectral sequences, calculating both groups $\bar{H}_*(B(\Psi, k))$, $\bar{H}_*(B(\Psi, k), \pm\mathbb{Z})$ and generated by our filtration, vanish in the first term. □

Proof of Proposition 6. If $A = (2^{\times i})$, then $C(A)$ is an open subset in the configuration space $B(\Psi, i)$, whose complement $B(\Psi, i) \setminus C(A)$ has codimension 3. In particular, by Lemma 3 $\bar{H}_j(C(A)) = 0$ for $j = 2i$ or $2i - 1$. It is easy to calculate that for odd n the fiber bundle $S(A) \to C(A)$ is orientable, and for even n it changes its orientation together with the local system $\pm\mathbb{Z}$. Thus we have the Thom isomorphism $\bar{H}_{*+\omega-(n-2)i-1}(S(A)) \simeq \bar{H}_*(C(A))$ (for odd n) and $\simeq \bar{H}_*(C(A), \pm\mathbb{Z})$ (for even n). □

LEMMA 4. *For any finite subset $\theta \subset \Psi$, both groups $\bar{H}_l(B(\Psi \setminus \theta, k))$, $\bar{H}_l(B(\Psi \setminus \theta, k), \pm\mathbb{Z})$ are trivial for $l \neq k$, and for $l = k$ they are free Abelian of rank equal to the number of functions $\chi : \theta \to \mathbb{Z}_+$ such that $\sum_{z \in \theta} \chi(z) = k$.*

Proof. Consider any direction in the plane $\mathbb{R}^2 \supset \Psi$, transversal to the diagonal $\partial\Psi = \{t = t'\}$ and such that none two points of θ are connected by a vector of this direction. Let $\pi : \Psi \to \partial\Psi$ be the projection along this direction. To any point $Z = (z_1, \dots, z_k) \subset \Psi \setminus \theta$ of the space $B(\Psi \setminus \theta, k)$ the following data are assigned:

(1) the topological type of the configuration in $\partial\Psi$, formed by the points $\pi(z_j)$ (counted with multiplicities) and the set $\pi(\theta)$;

(2) for any point $w \in \theta$, the number of points $z_j \in Z$ such that $\pi(z_j) = \pi(w)$ and z_j is separated by w from $\partial\Psi$ in the line $\pi^{-1}(\pi(w))$.

For any such collection of data, the subset in $B(\Psi \setminus \theta, k)$, formed by configurations Z with these data, is homeomorphic to $\mathbb{R}^l_+ \times \mathbb{R}^s$, where l is the number of lines $\pi^{-1}(\cdot)$ containing points of Z, not separated from $\partial\Psi$ by the points of θ, and $l+s$ is equal to the dimension of this subset, i.e. to $k+$(the number of geometrically distinct points of $\pi(Z)$ for any Z from this subset). In particular, the Borel–Moore homology group of such a subset is trivial if $l > 0$, and the set of configurations with $l = 0$ consists of several k-dimensional cells. □

Proof of Proposition 7. If exactly one of numbers a_j is equal to 2, then $S(A)$ is a connected manifold with non-empty boundary, in particular its Borel–Moore homology group of top dimension is trivial.

Now suppose that there are at least two such numbers. Let $A_{>2}$ be the same multiindex A, from which all numbers a_j equal to 2 are removed. Consider the projection

$$C(A) \to C(A_{>2}), \tag{17}$$

erasing from any A-set all its groups of cardinality 2. For any point $\nu \in C(A_{>2})$ denote by $\theta(\nu)$ the set of all pairs (x, y) lying in some of groups of ν.

By the Leray spectral sequence of the composite fiber bundle $S(A) \to C(A) \to C(A_{>2})$, we need only to prove the following lemma.

LEMMA 5. *If the number r of twos in A is greater than 1, then for any $\nu \in C(A_{>2})$ the group $\bar{H}_*$ of the fiber of the projection (17) over the point ν is trivial in dimensions $2r$ and $2r-1$.*

Proof. Any such fiber consists of all configurations $Z \in B(\Psi, r)$, not containing the points of the finite subset $\theta(\nu) \subset \Psi$ and satisfying some additional restrictions, which forbid certain subvariety of codimension ≥ 3 in $B(\Psi, r)$. Thus our lemma follows from Lemma 4. □

Recall that the greatest possible value of the reversed filtration of a cohomology class (in particular, of an knot invariant in $\mathbb{R}^3$) of order i is equal to $i - 1$.

PROPOSITION 8. *For any i, the number of linearly independent knot invariants of bi-order $(i, i-1)$ (modulo invariants of lower bi-orders) is estimated from above by the number $(i-1)! \equiv \dim \bar{H}_{2i-2}(\Delta^2(i+1))$.*

This follows from Theorem 2 and the fact that unique multiindex A of the form (9) with given i and stickiness $i-1$ consists of one number $(i+1)$. $\square$

This estimate is not realistic. Indeed, the group $\mathbb{Z}_{i+1}$ of cyclic permutations of vertices acts naturally on the complex $\Delta(i+1)$, hence also on the group $\bar{H}_*(\Delta^2(i+1))$. It is easy to see, that an element of this group, corresponding to a knot invariant of bi-order $(i, i-1)$, should be invariant under this action, and we obtain the following improvement of Proposition 8.

PROPOSITION 8′. *The number of linearly independent knot invariants of bi-order $(i, i-1)$ (modulo invariants of lower bi-orders) does not exceed the dimension of the subgroup in $\bar{H}_{2i-2}(\Delta^2(i+1))$, consisting of $\mathbb{Z}_{i+1}$-invariant elements.* $\square$

The exact formula for this dimension was found in [BBLSW], see Corollary 4.7 there.

6. Order 3 cohomology of spaces of non-compact knots

Here we calculate the column $E^\infty_{3,*}$ of the main spectral sequence, converging to the finite-order cohomology of the space of non-compact knots $\mathbb{R}^1 \to \mathbb{R}^n$. In fact, it is sufficient to calculate the column $E^1_{3,*}$ of its initial term E^1, i.e. the group $\bar{H}_*(\sigma_3 \setminus \sigma_2)$ of the corresponding discriminant variety.

This group is described in the following statement.

THEOREM 5. *For any $n \geq 3$, all groups $\bar{H}_j(\sigma_3 \setminus \sigma_2)$ are trivial, except for such groups with $j = \omega - 1 - 3(n-3)$ and $\omega - 2 - 3(n-3)$, which are isomorphic to $\mathbb{Z}$.*

This statement for $n = 3$ is not new: the group $\bar{H}_{\omega-1}(\sigma_3 \setminus \sigma_2) \simeq \mathbb{Z}$ (of knot invariants of order 3) was calculated in [V2], and all other groups $\bar{H}_j(\sigma_3 \setminus \sigma_2)$ in a non-published work of D. M. Teiblum and V. E. Turchin, see § 4.4.2 above. Their calculation is based on the cellular decomposition of $\sigma_3 \setminus \sigma_2$, constructed in [V2], and is non-trivial even for a computer.

COROLLARY 5. *The column $E^\infty_{3,*}$ of the main spectral sequence coincides with $E^1_{3,*}$, namely, it consists of exactly two nontrivial terms $E^\infty_{3,\omega-1-3(n-2)} \simeq \mathbb{Z}$ and $E^\infty_{3,\omega-2-3(n-2)} \simeq \mathbb{Z}$.*

Proof of Corollary 5. The fact that all differentials $d^r : E^r_{3,*} \to E^r_{3-r,*+r-1}$, $r \geq 1$, are trivial, follows immediately from the construction of columns $E^1_{2,*}$ and $E^1_{1,*}$, see Examples 3 and 4 in § 5.1.

On the other hand, the inequality (3) implies that also there are no non-trivial differentials d^r acting *into* the cell $E^r_{3,\omega-1-3(n-2)}$; if $n > 3$ then the same is true also for the cell $E^r_{3,\omega-2-3(n-2)}$. Finally, if $n = 3$, then the similar triviality of the homomorphisms $d^r : E^r_{3+r,\omega-4-r} \to E^r_{3,\omega-5}$ follows from the Kontsevich's realization theorem. $\square$

6.1. **Term E^1 of the reversed spectral sequence.** The term $\sigma_3 \setminus \sigma_2$ of the main filtration consists of three terms of the reversed auxiliary filtration: $\Phi_0 \subset \Phi_1 \subset \Phi_2 \equiv \sigma_3 \setminus \sigma_2$.

The sets Φ_0, $\Phi_1 \setminus \Phi_0$ and $\Phi_2 \setminus \Phi_1$ consist of one component $S(A)$ each, with A equal to $(2,2,2)$, $(3,2)$ and (4), respectively.

Lemma 6. *For any n,*
1. $\bar{H}_j(\Phi_0) = 0$ for all $j \neq \omega - 3n + 6$, and $\bar{H}_{\omega-3n+6}(\Phi_0) = \mathbb{Z}$.
2. $\bar{H}_j(\Phi_1 \setminus \Phi_0) = 0$ for all $j \neq \omega - 3n + 7$, and $\bar{H}_{\omega-3n+7}(\Phi_1 \setminus \Phi_0) = \mathbb{Z}^3$.
3. $\bar{H}_j(\Phi_2 \setminus \Phi_1) = 0$ for all $j \neq \omega - 3n + 8$, and $\bar{H}_{\omega-3n+8}(\Phi_2 \setminus \Phi_1) = \mathbb{Z}^2$.

Proof. 1. The space $C(2,2,2)$ of all $(2,2,2)$-sets is an open subset in the configuration space $B(\Psi, 3)$, and their difference $\mathbf{T} \equiv B(\Psi,3) \setminus C(2,2,2)$ is the set of all triples of the form $(x,y),(x,z),(y,z) \subset \Psi$ with $x < y < z$. This set is obviously a closed subset in $B(\Psi,3)$ diffeomorphic to a 3-dimensional cell. Thus by Lemma 3 and the exact sequence of the pair $(B(\Psi,3), \mathbf{T})$, both groups $\bar{H}_l(C(2,2,2))$ and $\bar{H}_l(C(2,2,2), \pm\mathbb{Z})$ are trivial for any l other than 4 and are isomorphic to $\mathbb{Z}$ in dimension 4. Fibres of the bundle $S(2,2,2) \to C(2,2,2)$ are products of the open triangle $\Lambda(2,2,2)$ and an affine space of dimension $\omega - 3n$. It is easy to calculate that for odd n this bundle is orientable, and for even n its orientation bundle coincides with $\pm\mathbb{Z}$, thus statement 1 follows from the Thom isomorphism.

2. The space $C(3,2)$ can be considered as the space of a fiber bundle, whose base is the configuration space $B(\mathbb{R}^1, 3)$, and the fiber over the triple $(x < y < z) \subset \mathbb{R}^1$ is the half-plane Ψ with three interior points (x,y), (x,z) and (y,z) removed. The Borel–Moore homology group of this punctured half-plane is concentrated in dimension 1 and is generated by three rays connecting these three removed points to infinity, say, by rays shown in Fig. 5. The complex $\Lambda(3,2)$ consists of unique 3-dimensional open face. The bundle $S(3,2) \to C(3,2)$ is orientable, thus statement 2 also follows from the Thom isomorphism.

3. The space $C(4) \sim B(\mathbb{R}^1, 4)$ is obviously diffeomorphic to a 4-cell, thus $S((4))$ is homeomorphic to the direct product $B(\mathbb{R}^1,4) \times \mathbb{R}^{\omega-3n} \times \Delta^2(n)$, and statement 3 follows immediately from Theorem 2. For the realization of generators of the group $\bar{H}_4(\Delta^2(4)) \simeq \bar{H}_{\omega-3n+8}(\Phi_2 \setminus \Phi_1)$ see Example 2 in § 2. □

Remark. To specify the last isomorphism, we need to choose an orientation of $B(\mathbb{R}^1, 4)$ and a (co)orientation of the fiber $\mathbb{R}^{\omega-3n}$. We do it as follows.

The standard orientation of $B(\mathbb{R}^1, 4)$ is defined by local coordinate systems, whose coordinate functions are coordinates of four points taken in their increasing order.

If $\xi_1, \dots, \xi_n$ are coordinates in $\mathbb{R}^n$, so that any non-compact knot is given parametrically by n real functions $\xi_k(t)$, $k = 1, \dots, n$, then the fiber $\mathbb{R}^{\omega-3n}$ over the configuration $(r < s < t < u)$ is distinguished by the equations $\xi_1(r) = \xi_1(s) = \xi_1(t) = \xi_1(u)$, $\dots$, $\xi_n(r) = \xi_n(s) = \xi_n(t) = \xi_n(u)$. Then the transversal orientation of this fiber in

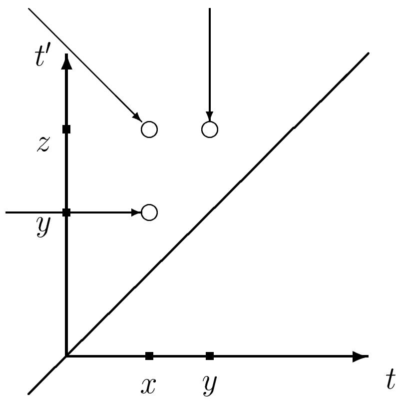


FIGURE 5. Basic cycles in the space $C(3, 2)$

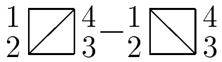
the space $K \sim \mathbb{R}^\omega$ will be specified by the skew form

$$\begin{gathered} d(\xi_1(s) - \xi_1(r)) \wedge d(\xi_2(s) - \xi_2(r)) \wedge \ldots \wedge d(\xi_n(s) - \xi_n(r)) \wedge \\ \wedge\, d(\xi_1(t) - \xi_1(s)) \wedge d(\xi_2(t) - \xi_2(s)) \wedge \ldots \wedge d(\xi_n(t) - \xi_n(s)) \wedge \\ \wedge\, d(\xi_1(u) - \xi_1(t)) \wedge d(\xi_2(u) - \xi_2(t)) \wedge \ldots \wedge d(\xi_n(u) - \xi_n(t)). \end{gathered} \tag{18}$$

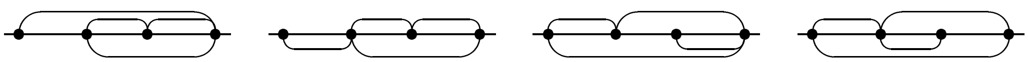
Finally, the calculation of groups $\bar{H}_j(\sigma_3 \setminus \sigma_2)$ in dimensions $j = \omega - 3n + 8, \omega - 3n + 7$ and $\omega - 3n + 6$ is reduced to the calculation of a certain complex of the form $0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$, whose boundary operator is the (horizontal) differential d_1 of our reversed spectral sequence.

6.2. **The differential $d^1 : \bar{H}_{\omega-3n+8}(\Phi_2 \setminus \Phi_1) \to \bar{H}_{\omega-3n+7}(\Phi_1 \setminus \Phi_0)$.** First we calculate this operator for the generator of the group $\bar{H}_{\omega-3n+8}(\Phi_2 \setminus \Phi_1)$, corresponding to the generator

(19)



of $\bar{H}_4(\Delta^2(4))$, where numbers of vertices correspond to the order of four points in $\mathbb{R}^1$. The boundary of any of these two graphs is the sum of 5 graphs with 4 edges. One of them (obtained by removing the diagonal edge) appears in both sums and vanishes. Remaining boundary graphs (and the signs with which they participate in the algebraic boundary of the cycle (19)) are:



(20) A(−) B(+) C(−) D(−)

for the first graph in (19), and

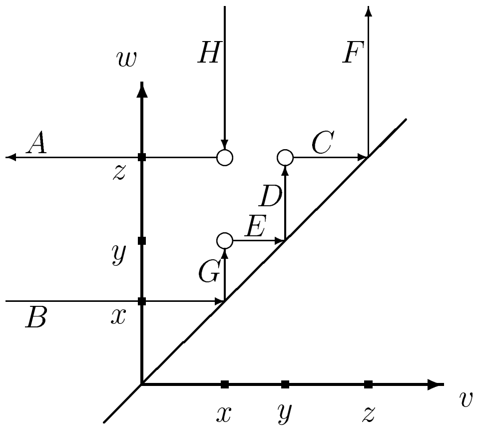
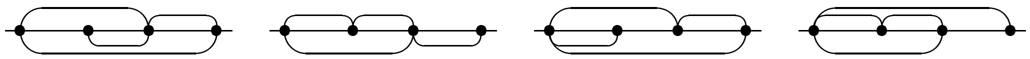


FIGURE 6. Boundary of the chain (19)



(21) E(+) F(+) G(−) H(+)

for the second. The term $\Phi_1 \setminus \Phi_0$ (in which the boundary lies) can be considered as a fiber bundle (we will call it the *former bundle*), whose base is the set of triples $\{x < y < z\} \subset \mathbb{R}^1$, and the fiber over such a point is the space of the *latter fiber bundle*, whose base is the set of all pairs $\{u \leq v\} \in \Psi$ not coinciding with (x, y), (x, z) or (y, z), and the fiber is the direct product of an open tetrahedron with vertices called

$$(x, y), (x, z), (y, z), (u, v) \tag{22}$$

and an affine subspace of codimension $3n$ in K (consisting of all maps $\phi : \mathbb{R}^1 \to \mathbb{R}^n$ such that $\phi(x) = \phi(y) = \phi(z)$ and $\phi(u) = \phi(v)$).

For an arbitrary basepoint (x, y, z) of the former bundle, consider the intersection set of the fiber over it with the parts of the boundary of the cycle (19), corresponding to 8 graphs (20), (21). It is easy to calculate that these intersections are complete pre-images of the latter bundle over the open segments in $\Psi \setminus ((x, y) \cup (x, z) \cup (y, z))$ labeled in Fig. 6 by characters $A, \dots, H$, corresponding to the notation of these graphs in pictures (20) and (21).

For instance, for any (4)-configuration $(r < s < t < u) \in C((4)) = B(\mathbb{R}^1, 4)$ the part of the boundary of the fiber in $\Phi_2 \setminus \Phi_1$ over this configuration, corresponding to the graph E in (21), belongs to the fiber of the former bundle in $\Phi_1 \setminus \Phi_0$ over the point $(x < y < z) = (r < t < u)$, and coordinates (v, w) in the base of the latter bundle satisfy the relations $w = t \equiv y$, $v = s \in (x, y)$.

The part (E) of the boundary of the generator (19) of $\bar{H}_{\omega-3n+8}(\Phi_2 \setminus \Phi_1)$ coincides thus with the complete pre-image in $\Phi_1 \setminus \Phi_0$ of the 4-dimensional submanifold in $C(3,2)$ formed by *all* configurations $((x<y<z),(v,w))$ such that $w=y, x<v<y$.

To calculate the coefficients, with which this and other similar pre-images, corresponding to other letters $A,\dots,H$, appear in the *algebraic* boundary, we need to fix their orientations. Again, any such orientation consist of orientations of three objects:

a) the 4-dimensional base manifold in the space of all configurations (x,y,z,u,v),

b) the tetrahedron (22), and

c) for any point of this 4-dimensional manifold, the subspace in $\mathbb{R}^\omega$, consisting of maps $\phi : \mathbb{R}^1 \to \mathbb{R}^n$, such that $\phi(x)=\phi(y)=\phi(z)$ and $\phi(u)=\phi(v)$.

We fix these orientations as follows.

The manifold in the configuration space will be oriented by the pair of orientations, the first of which is lifted from the orientation $dx \wedge dy \wedge dz$ of the base of the former bundle, and the second is given by increase of the unique "free" number u or v, i.e. all horizontal segments in Fig. 6 should be oriented to the right, and all vertical segments should be oriented to the top.

The tetrahedron (22) over any point $((x<y<z),(v<w))$ of the space $C(3,2)$ will be oriented by the order (22) of its vertices.

The (co)orientation of the fiber $\mathbb{R}^{\omega-3n}$ over the same point $((x<y<z),(v<w))$ will be specified by the skew form

$$\begin{aligned} & d(\xi_1(y)-\xi_1(x)) \wedge d(\xi_2(y)-\xi_2(x)) \wedge \dots \wedge d(\xi_n(y)-\xi_n(x)) \wedge \\ & \wedge\, d(\xi_1(z)-\xi_1(y)) \wedge d(\xi_2(z)-\xi_2(y)) \wedge \dots \wedge d(\xi_n(z)-\xi_n(y)) \wedge \\ & \wedge\, d(\xi_1(w)-\xi_1(v)) \wedge d(\xi_2(w)-\xi_2(v)) \wedge \dots \wedge d(\xi_n(w)-\xi_n(v)) \end{aligned} \tag{23}$$

on its normal bundle.

Now, for any of 8 components (20), (21) of the boundary of the generator (19), we need to compare these orientations with ones induced from the canonical orientations of similar objects in $\Phi_2 \setminus \Phi_1$. We present here these comparisons only for one component (E).

Compare orientations of configurations. In the base $C((4))$ of $\Phi_2 \setminus \Phi_1$ the orientation is given by $dr \wedge ds \wedge dt \wedge du$, and in $\Phi_1 \setminus \Phi_0$ the orientation of the corresponding component of the boundary is given by $dx \wedge dy \wedge dz \wedge dv$. For the component E we have $x=r, y=t, z=u, v=s$, hence these orientations coincide.

Compare orientations of tetrahedra. The lexicographic order of vertices of the graph E in (21) is as follows: $((13),(14),(23),(34))$, or, in notation used in (22), $((x,y),(x,z),(u,v),(y,z))$. This ordering is of opposite sign with (22).

Compare coorientations of subspaces $\mathbb{R}^{\omega-3n}$. If n is even, then all our coorientations of all such fibers canonically coincide. For odd n it is not more so, e.g. it is easy to check that for the component E of the boundary orientations (18) and (23) are opposite.

	A	B	C	D	E	F	G	H
Coefficient	−	+	−	−	+	+	−	+
Compare Δ	−	−	+	−	−	+	−	−
Configurations	−	−	−	−	+	+	+	+
Subspaces	+	+	+	−	−	+	+	+
Odd n	−	+	+	+	+	+	+	−
Even n	−	+	+	−	−	+	+	−

TABLE 1

	A	B	C	D	E	F	G	H		A	B	C	D	E	F	G	H
Coefficient	−	+	+	−	+	−	−	+		+	−	+	−	+	−	+	−
Compare Δ	−	+	−	+	−	−	+	+		−	+	+	−	−	−	−	−
Configs	+	−	+	−	−	−	+	+		−	+	−	+	−	−	+	+
Subspaces	−	+	+	−	+	+	+	+		−	+	−	+	+	+	−	−
Odd n	−	−	−	−	+	−	−	+		−	−	+	+	+	−	+	−
Even n	+	−	−	+	+	−	−	+		+	−	−	+	+	−	−	+

TABLE 2

Doing the same calculations for all other components $A, \dots, H$, we obtain Table 1, in which the second line is the sign indicated in (20) or (21), and the next three lines are results of comparisons or orientations of tetrahedra, configuration spaces and subspaces in K (in the case of odd n) respectively. The 6-th line contains the final sign, with which the corresponding oriented component, depicted by a segment in Fig. 6, participates in the boundary of the generator (19) in the case of odd n; this sign is just the product of previous four signs. The last line indicates a similar sign in the case of even n; it is equal to the product of three first signs in the column.

It follows from these calculations, that in the case of odd n the boundary in $\Phi_1 \setminus \Phi_0$ of the element (19) is homologous to zero. Indeed, we can make all signs in the 6-th line equal to $+$, if defining the orientation of the 4-submanifold in $C(3,2)$ we orient segments A and H as shown in Fig. 6. Therefore our boundary chain coincides with the boundary of the complete pre-image under the projection $S(3,2) \to C(3,2)$ of the domain, consisting of all configurations $((x < y < z), (v < w))$, such that the point (v, w) belongs to the domain in $\mathbb{R}^2$, bounded from left and above by segments H and A (see Fig. 6), and from right and below by the union of segments B, G, E, D, C and F, depending on x, y and z.

In the case of even n the boundary of the element (19) is not a cycle in $\Phi_1 \setminus \Phi_0$. However, making all the same calculations for two other generators ${}^{1}_{2}\boxtimes^{4}_{3} - {}^{1}_{2}\boxtimes^{4}_{3}$

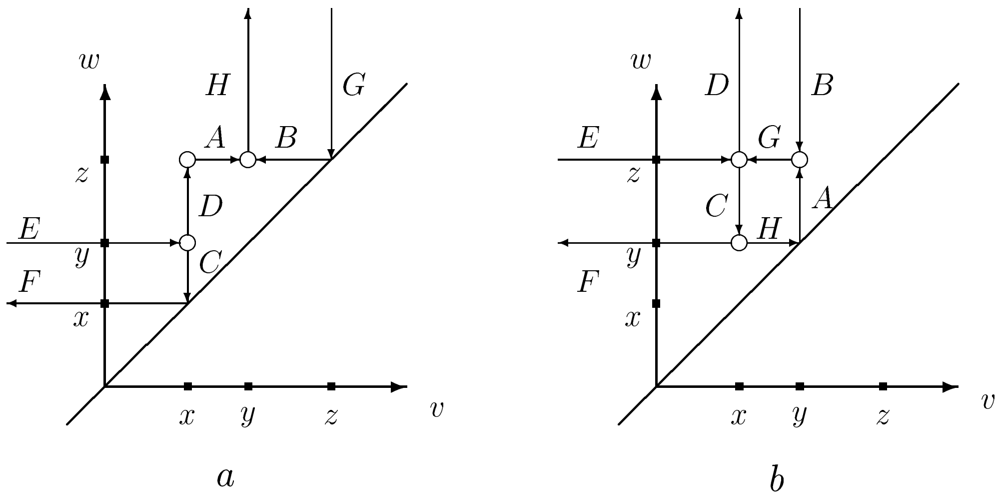


FIGURE 7. Boundaries of non-invariant 4-chains

and $\begin{smallmatrix}1\\2\end{smallmatrix}\boxtimes\begin{smallmatrix}4\\3\end{smallmatrix}-\begin{smallmatrix}1\\2\end{smallmatrix}\boxtimes\begin{smallmatrix}4\\3\end{smallmatrix}$ of the group $\bar{H}_4(\Delta^2(4))$, we obtain that the *difference* of corresponding chains in $\Phi_2 \setminus \Phi_1$ has in $\Phi_1 \setminus \Phi_0$ boundary homologous to zero. These calculations are represented in Fig. 7 and Table 2.

Finally, we get the following statement.

PROPOSITION 9. *For any n, the group $\bar{H}_{\omega-3n+8}(\sigma_3 \setminus \sigma_2)$ is isomorphic to $\mathbb{Z}$. Its basis element coincides in $\Phi_2 \setminus \Phi_1$ with the sum (if n is odd) or difference (for even n) of elements, corresponding to generators $\begin{smallmatrix}1\\2\end{smallmatrix}\boxtimes\begin{smallmatrix}4\\3\end{smallmatrix}-\begin{smallmatrix}1\\2\end{smallmatrix}\boxtimes\begin{smallmatrix}4\\3\end{smallmatrix}$ and $\begin{smallmatrix}1\\2\end{smallmatrix}\boxtimes\begin{smallmatrix}4\\3\end{smallmatrix}-\begin{smallmatrix}1\\2\end{smallmatrix}\boxtimes\begin{smallmatrix}4\\3\end{smallmatrix}$ of the group $\bar{H}_4(\Delta^2(4))$.*

For any n, the image of the operator $d^1 : \bar{H}_{\omega-3n+8}(\Phi_2 \setminus \Phi_1) \to \bar{H}_{\omega-3n+7}(\Phi_1 \setminus \Phi_0)$ is isomorphic to $\mathbb{Z}$ and the quotient group $\bar{H}_{\omega-3n+7}(\Phi_1 \setminus \Phi_0)/d^1(\bar{H}_{\omega-3n+8}(\Phi_2 \setminus \Phi_1))$ is isomorphic to $\mathbb{Z}^2$. $\square$

6.3. **The differential** $d^1 : \bar{H}_{\omega-3n+7}(\Phi_1 \setminus \Phi_0) \to \bar{H}_{\omega-3n+6}(\Phi_0)$.

PROPOSITION 10. *The boundary operator d^1 of our spectral sequence maps any of three generators of the group $\bar{H}_{\omega-3n+7}(\Phi_1 \setminus \Phi_0)$, indicated by rays in Fig. 5, into a generator of the group $\bar{H}_{\omega-3n+6}(\Phi_0)$.*

We prove this statement for one generator of the former group, depicted by the horizontal ray in Fig. 5, consisting of points (λ, y) with $\lambda < x$. The chain, representing this generator, is the fiber bundle, whose base is the configuration space $B(\mathbb{R}^1, 4)$, and the fiber over its point $\{\lambda < x < y < z\}$ is the direct product of a tetrahedron (whose vertices are called (λ, y), (x, y), (x, z) and (y, z), see Fig. 8a) and an affine space of dimension $\omega - 3n$ (consisting of all maps $\phi : \mathbb{R}^1 \to \mathbb{R}^n$ such that $\phi(\lambda) = \phi(x) = \phi(y) = \phi(z)$).

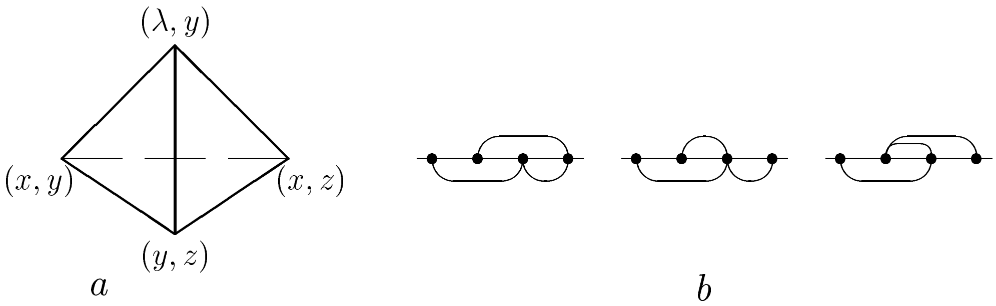


FIGURE 8. The boundary of the first generator of $E^1_{1,\omega-3n+6}$

One face of the simplex from Fig. 8a) (opposite to the vertex (λ, y)) is a non-connected graph, and the corresponding stratum belongs to σ_2. The strata, swept out by three other faces (opposite to vertices (x, y), (x, z) and (y, z) respectively), belong to the term $\Phi_0 \subset \sigma_3 \setminus \sigma_2$, namely, they are complete pre-images in $S(2, 2, 2)$ of 4-dimensional cycles in $C(2, 2, 2)$, consisting of all triples of the form

$$\{(\lambda, y), (x, z), (y, z)\}, \quad \{(\lambda, y), (x, y), (y, z)\}, \quad \{(\lambda, y), (x, y), (x, z)\} \tag{24}$$

respectively with all possible $\lambda < x < y < z$, see Fig. 8b). By the Thom isomorphism for the bundle $S(2, 2, 2) \to C(2, 2, 2)$, we need only to prove that the union of these three cycles in $C(2, 2, 2)$ defines a generator of the group $\bar{H}_4(C(2, 2, 2)) \equiv \bar{H}_4(B(\Psi, 3), \mathbf{T})$ (if n is odd) or $\bar{H}_4(C(2, 2, 2), \pm\mathbb{Z}) \equiv \bar{H}_4(B(\Psi, 3), \mathbf{T}; \pm\mathbb{Z})$ (if n is even); or equivalently, that its boundary in $\bar{H}_3(\mathbf{T})$ is the fundamental cycle of $\mathbf{T} \sim B(\mathbb{R}^1, 3)$. But among three manifolds (24) only the first approaches $\mathbf{T}$ and its boundary obviously coincides with $\mathbf{T}$.

The calculation for two other generators of the group $\bar{H}_{\omega-3n+7}(\Phi_1 \setminus \Phi_0)$ is exactly the same. □

Theorem 5 is a direct corollary of Lemma 6 and Propositions 9 and 10. □

7. COHOMOLOGY CLASSES OF COMPACT KNOTS IN $\mathbb{R}^n$.

7.1. **Classes of order 1.** It is well-known that there are no first-order knot invariants in $\mathbb{R}^3$, see [V2]. However, the subgroup $F^*_{1,\mathbb{Z}_2} \subset H^*(\mathcal{K} \setminus \Sigma, \mathbb{Z}_2)$ of all $\mathbb{Z}_2$-valued first-order cohomology classes of the space of compact knots is non-trivial. For instance, the generator of the group $F^1_{1,\mathbb{Z}_2} \subset H^1(\mathcal{K} \setminus \Sigma, \mathbb{Z}_2)$ proves that already the component of unknots in $\mathbb{R}^3$ is not simple-connected.

More generally, the following statements hold.

THEOREM 6. *A. For any $n \geq 3$, the subgroup $F^*_{1,\mathbb{Z}_2} \subset H^*(\mathcal{K} \setminus \Sigma, \mathbb{Z}_2)$ of first-order cohomology classes of the space of knots in $\mathbb{R}^n$ contains exactly two non-trivial homogeneous components $F^{n-2}_{1,\mathbb{Z}_2} \sim F^{n-1}_{1,\mathbb{Z}_2} \sim \mathbb{Z}_2$.*

B. If n is even, then both these cohomology classes give rise to integer cohomology classes, i.e. $F^{n-2}_{1,\mathbb{Z}} \sim F^{n-1}_{1,\mathbb{Z}} \sim \mathbb{Z}$, and there are no other non-trivial integer cohomology groups $F^d_{1,\mathbb{Z}}$, $d \neq n-2, n-1$.

C. If n is odd, then the generator of the group $F^{n-1}_{1,\mathbb{Z}_2}$ is equal to the first Steenrod operation of the generator of $F^{n-2}_{1,\mathbb{Z}_2}$.

D. The generator of the group $F^{n-2}_{1,\mathbb{Z}_2}$ can be defined as the linking number with the $\mathbb{Z}_2$-fundamental cycle of the variety $\Gamma \subset \Sigma$, formed by all maps $\phi : S^1 \to \mathbb{R}^n$, gluing together some two opposite points of S^1; the generator of the group $F^{n-1}_{1,\mathbb{Z}_2}$ is the linking number with the $\mathbb{Z}_2$-fundamental cycle of the subvariety $\Gamma! \subset \Gamma$, formed by all maps $\phi : S^1 \to \mathbb{R}^n$, gluing together some two **fixed** *opposite points, say, the points 0 and π. Moreover, if n is even, then these two varieties are orientable, and the groups $F^{n-2}_{1,\mathbb{Z}}$, $F^{n-1}_{1,\mathbb{Z}}$ are generated by the linking numbers with the corresponding $\mathbb{Z}$-fundamental cycles.*

E. If $n = 3$, then the cycles, generating the groups $F^1_{1,\mathbb{Z}_2}$ and $F^2_{1,\mathbb{Z}_2}$ are non-trivial already in the restriction on the component of the unknot in $\mathbb{R}^3$.

Proof. The term σ_1 of the main filtration of σ is the space of an affine fiber bundle of dimension $\omega - n$, whose base is the configuration space Ψ; in the case of compact knots this base space is diffeomorphic to the closed Möbius band. It is easy to check that this affine bundle is orientable if and only if n is even. Thus the term $E_1^{-1,q}(\mathbb{Z}_2)$ of the main spectral sequence with coefficients in $\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_2$ for $q = n-1$ or n and is trivial for all other q. Moreover, if n is even, then also the terms $E_1^{-1,q}(\mathbb{Z})$ are isomorphic to $\mathbb{Z}$ if $q = n-1$ or n and are trivial for all other q. The basic cycles in σ_1, generating these groups, are the manifolds $\tilde{\Gamma}$ (respectively, $\tilde{\Gamma}!$), consisting of all pairs $(\phi, (x, y))$ such that $\phi(x) = \phi(y)$ and $y = x + \pi$ (respectively, $x = 0, y = \pi$). Thus the direct images of these cycles in Σ are exactly the fundamental classes of varieties Γ, $\Gamma!$, mentioned in statement D of Theorem 6. We need to prove that the homology classes in Σ of these cycles are non-trivial, i.e. the linking numbers with them (which we denote by $\{\Gamma\}$ and $\{\Gamma!\}$ respectively) are non-trivial cohomology classes in $\mathcal{K} \setminus \Sigma$.

For $n > 4$ this follows from the dimensional reasons. Indeed, for any $n > 3$ our spectral sequence converges to the entire group $H^*(\mathcal{K} \setminus \Sigma)$, and by (3), (5) all groups $E^{p,q}_{\geq 1}$, $p < -1$, with $p + q < n - 1$ are trivial, hence all differentials d^r, acting into the cells $E_r^{-1,q}$, also are trivial. For $n = 4$ there is unique non-trivial group $E_1^{-2,4}$, from which, in principle, there could act a non-trivial homomorphism $d^1 : E_1^{-2,4} \to E_1^{-1,4} \sim \mathbb{Z}$. We will see in the next subsection, that the group $E_1^{-2,4}(G)$ is isomorphic to the basic coefficient group G. Thus for $G = \mathbb{Z}$ and $n = 4$ the differential $d^1(E_1^{-2,4})$ is trivial by the Kontsevich's realization theorem (6), and for $G = \mathbb{Z}_2$ the same follows from the functoriality of spectral sequences under the coefficient homomorphisms. Thus statements A, B, D of Theorem 6 are proved for all $n > 3$.

The statement C for odd $n > 3$ will be proved as follows. First, we prove the equality $Sq^1(\{\Gamma\}) = \{\Gamma!\}$ in the group $H^*(\mathcal{K} \setminus \Gamma)$, then the desired similar equality in the cohomology group of the subspace $\mathcal{K} \setminus \Sigma \subset \mathcal{K} \setminus \Gamma$ will follow from the naturality of Steenrod operations.

For any $n \geq 3$, the group $H^*(\mathcal{K} \setminus \Gamma)$ can be calculated by a cohomological spectral sequence similar to (but much more easy than) our main spectral sequence. Namely, it is "Alexander dual" (in the sense of formula (4)) to the homological sequence associated to the standard filtration of the simplicial resolution γ of Γ. The term $\gamma_i \setminus \gamma_{i-1}$ of this filtration is the space of a fiber bundle with base $B(S^1, i)$ (where S^1 is the space of all diametral pairs of points of the original circle) and the fiber equal to the direct product of the open simplex $\dot{\Delta}^{i-1}$ and the space $\mathbb{R}^{\omega - ni}$. In particular $E_1^{p,q} = 0$ for $p > 0$ or $q + p(n-1) < 0$. For any $n > 3$ the group $H^{n-1}(\mathcal{K} \setminus \Gamma, \mathbb{Z}_2)$, containing the element $Sq^1(\{\Gamma\})$, is one-dimensional and is generated by the linking number with the cycle $\Gamma!$. Consider any loop $l \subset \Gamma$, lying in the set of regular points of Γ and such that going along l we permute two points $x, y = x + \pi \in S^1$, glued together by the corresponding maps. (Such l exist for any $n \geq 3$, because the codimension in the manifold $\tilde{\Gamma} \simeq \gamma_1$ of the preimage of the set of singular points of Γ is equal to $n - 1$, and we can realize l as the projection into Γ of almost any loop in $\tilde{\Gamma}$, permuting the points x and y.) Let L be the "tube" around l in $\mathcal{K} \setminus \Gamma$, i.e. the union of boundaries of small $(n-1)$-dimensional discs transversal to Γ with centers at the points of l. The fibration $L \to l$ with fiber S^{n-2} is non-orientable if n is odd, thus already in the restriction to L $Sq^1(\{\Gamma\})$ is non-trivial and is equal to the $\mathbb{Z}_2$-fundamental cocycle of L. The union of these transversal discs spans L in $\mathcal{K}$, and its $\mathbb{Z}_2$-intersection number with $\Gamma!$ is equal to 1, in particular the fundamental cycle of L is nontrivial in $H_{n-1}(\mathcal{K} \setminus \Gamma, \mathbb{Z}_2)$ for any $n \geq 3$ and generates this group for $n > 3$. Also, we get that for any odd $n \geq 3$ in restriction to L $Sq^1(\{\Gamma\}) = \{\Gamma!\}$. Since for $n > 3$ the inclusion homomorphism $H^{n-1}(\mathcal{K} \setminus \Gamma, \mathbb{Z}_2) \to H^{n-1}(L, \mathbb{Z}_2)$ is an isomorphism, statement C of Theorem 6 is proved for all odd $n > 3$.

Finally, let be $n = 3$. Consider the loop $\Lambda : S^1 \to (\mathcal{K} \setminus \Sigma)$, some whose eight points are shown in Fig. 9. Note that any two (un)knots of this family, placed in this picture one over the other, have the same projection to $\mathbb{R}^2$. Let us connect any such two unknots by a segment in $\mathcal{K}$, along which the projection to $\mathbb{R}^2$ also is preserved. The union of these segments is a disc in $\mathcal{K}$, spanning the loop Λ; it is obvious that the $\mathbb{Z}_2$-intersection number of this disc with the variety Γ is equal to 1, in particular the class $\{\Gamma\} \in H^1(\mathcal{K} \setminus \Sigma, \mathbb{Z}_2)$ is non-trivial.

It is easy to see that this loop Λ is homotopic to the loop Λ', consisting of knots, obtained from the standard embedding $\phi : S^1 \to \mathbb{R}^2 \subset \mathbb{R}^3$ by rotations by all angles $\alpha \in [0, 2\pi]$ around any diagonal of $\phi(S^1)$, and also to the loop Λ'', consisting of

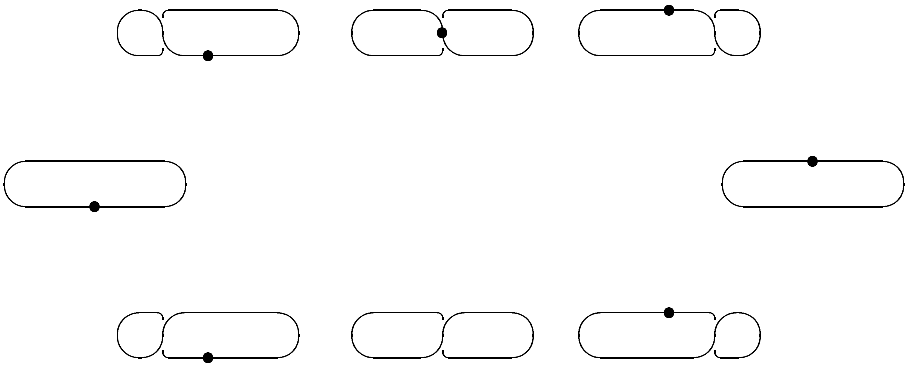


FIGURE 9. Non-trivial 1-cycle in the space of unknots

all knots ϕ_τ, $\tau \in [0, 2\pi]$, having the same image as ϕ and given by the formula $\phi_\tau(\alpha) = \phi(\alpha + \tau)$.

Let us fix some sphere $S^2 \subset \mathbb{R}^3$ and consider the space GC of all naturally parametrized great circles in it. This space is obviously diffeomorphic to $SO(3) \sim \mathbb{R}P^3$, and its group $H_1(GC, \mathbb{Z}_2)$ is generated by our loop $\Lambda' \sim \Lambda$. To complete the proof of statements A, C, D for $n = 3$ and of statement E, it remains to show that the linking number of the generator of $H_2(GC, \mathbb{Z}_2)$ with the variety $\Gamma!$ is equal to 1.

If $n = 3$, then the group $H^2(\mathcal{K} \setminus \Gamma)$, containing the element $Sq^1(\{\Gamma\})$, is two-dimensional: besides the class $\{\Gamma!\}$ it contains a basic element of second order, coming from the cell $E^{-2,4}$ of the canonical spectral sequence calculating $H^*(\mathcal{K} \setminus \Gamma, \mathbb{Z}_2)$. The reduction mod γ_1 of its dual class in $\bar{H}_{\omega-3}(\Gamma)$ is the fundamental cycle of $\gamma_2 \setminus \gamma_1$. As we will see in the next subsection, this fundamental cycle is homologous to zero in the space $\sigma_2 \setminus \sigma_1 \supset \gamma_2 \setminus \gamma_1$. Thus the element $Sq^1(\{\Gamma\}) \in H^2(\mathcal{K} \setminus \Sigma, \mathbb{Z}_2)$ belongs to the group of elements of order 1, which is generated by the class $\{\Gamma!\}$. On the other hand, $Sq^1(\{\Gamma\}) \equiv \{\Gamma\}^2$ is non-trivial already in restriction to the submanifold $GC \subset \mathcal{K} \setminus \Sigma$ (because $\{\Gamma\}$ is), thus it coincides with $\{\Gamma!\}$. Theorem 6 is completely proved. □

7.2. **Classes of order 2.** THEOREM 7. *For any $n \geq 3$, the group $F^*_{2,\mathbb{Z}}/F^*_{1,\mathbb{Z}}$ of order 2 cohomology classes of the space of compact knots, reduced modulo classes of order 1, contains at most two non-trivial homogeneous components in dimensions $2n - 6$ and $2n - 3$. The component in dimension $2n - 6$ is always isomorphic to $\mathbb{Z}$, the component in dimension $2n - 3$ is isomorphic to $\mathbb{Z}$ if $n > 3$ and is a cyclic group if $n = 3$. The generator of the nontrivial group in dimension $2n - 3$ has bi-order* $(2, 0)$, *and the generator in dimension $2n - 6$ has bi-order* $(2, 1)$.

Proof. The auxiliary filtration of the space $\sigma_2 \setminus \sigma_1$ consists of two terms $\Phi_0 \subset \Phi_1$. Φ_0 is the space of a fiber bundle over $B(\Psi, 2)$, whose fiber is the direct product of a vector subspace of codimension $2n$ in $\mathcal{K}$ and an open interval. It is easy to calculate that the Borel–Moore homology group $\bar{H}_*(\Phi_0)$ of the space of this bundle coincides with that of its restriction on the subset $B(S^1,2) \subset B(\Psi,2)$, where S^1 is the *equator* of the Möbius band Ψ, consisting of all pairs of the form $(\alpha, \alpha+\pi)$. This subset $B(S^1,2)$ is homeomorphic to the open Möbius band. The generator of its fundamental group $\pi_1(B(S^1,2)) \simeq \mathbb{Z}$ preserves the orientation of the bundle of $(\omega-2n)$-dimensional subspaces in $\mathcal{K}$, and destroys the orientations of both the bundle of open intervals and the base $B(S^1,2)$ itself. Therefore $\bar{H}_*(\Phi_0)$ coincides with the homology group of a circle up to the shift of dimensions by $\omega-2n+2$, namely, it has only two non-trivial homology groups $\bar{H}_{\omega-2n+3}$ and $\bar{H}_{\omega-2n+2}$, which are isomorphic to $\mathbb{Z}$. The first (respectively, second) of them is generated by the fundamental cycle of the preimage of the set $B(S^1,2) \subset B(\Psi,2)$ (respectively, by that of the circle in $B(S^1,2)$ formed by all pairs of pairs of the form $((\alpha,\alpha+\pi),(\alpha+\pi/2,\alpha+3\pi/2))$, where α is defined up to addition of a multiple of $\pi/2$).

The term $\Phi_1 \setminus \Phi_0$ is the space of a fiber bundle over $B(S^1,3)$, whose fiber is the direct product of a vector subspace of codimension $2n$ in $\mathcal{K}$ and an open triangle. Both these bundles (of vector subspaces and triangles) over $B(S^1,3)$ are orientable, thus $\bar{H}_*(\Phi_1 \setminus \Phi_0) \simeq \bar{H}_{*-(\omega-2n+2)}(B(S^1,3))$, i.e., this group is equal to $\mathbb{Z}$ in dimensions $\omega-2n+5$ and $\omega-2n+4$ and is trivial in all other dimensions. The $(\omega-2n+5)$-dimensional component is generated by the fundamental cycle of $\Phi_1 \setminus \Phi_0$, and the $(\omega-2n+4)$-dimensional one by the preimage of the cycle in $B(S^1,3)$, formed by all triples $(\alpha,\beta,\gamma) \subset S^1$ such that $\alpha+\beta+\gamma \equiv 0 \,(\mathrm{mod}\ 2\pi)$.

The unique nontrivial differential $d^1 : \mathcal{E}^1_{1,*} \to \mathcal{E}^1_{0,*}$ of the reversed auxiliary spectral sequence sends the $(\omega-2n+4)$-dimensional generator of the group $\bar{H}_*(\Phi_1 \setminus \Phi_0)$ to a multiple of the $(\omega-2n+3)$-dimensional generator of $\bar{H}_*(\Phi_0)$. Now we prove that the coefficient in this multiple is equal to ± 1.

By construction, the image of this differential is the fundamental cycle of the preimage in Φ_0 of a certain 2-dimensional cycle $\Delta \subset B(\Psi,2)$. Namely, this cycle is the space of a fiber bundle over S^1, whose fiber over the point $\alpha \in S^1$ is homeomorphic to an open interval and consists of all unordered pairs $\{(\alpha,\beta);(\alpha,\gamma)\}$ such that $\beta \neq \gamma$ and $\alpha+\beta+\gamma \equiv 0 \,(\mathrm{mod}\ 2\pi)$ (although either β or γ can coincide with α).

Lemma 7. *The submanifolds Δ and $B(S^1,2)$ of $B(\Psi,2)$ are homeomorphic, and there is a proper homotopy between this homeomorphism $\Delta \to B(S^1,2)$ and the identical embedding of Δ into $B(\Psi,2)$.*

Proof. We will consider all points α,β etc. as points of the unit circle in the complex line $\mathbb{C}^1$. Any point $\{(\alpha,\beta);(\alpha,\gamma)\} \in \Delta$ can be deformed to a point of $B(S^1,2)$ by a homotopy h_t, $t \in [0,1]$ (see Fig. 10), along which the chords $[\alpha'(t),\beta(t)]$ and $[\alpha''(t),\gamma(t)]$ preserve their directions. The easiest possible "direct and uniform" realization of such a deformation depends continuously on the initial point of Δ; it is easy to calculate that any point of $B(S^1,2)$ can be obtained by such a homotopy from exactly one point of Δ. □

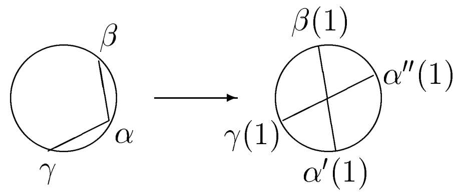


FIGURE 10. The homotopy $\Delta \to B(S^1, 2)$

So, the differential $d^1 : \mathcal{E}^1_{1,*} \to \mathcal{E}^1_{0,*}$ kills both groups $\bar{H}_{\omega-2n+4}(\Phi_1 \setminus \Phi_0)$ and $\bar{H}_{\omega-2n+3}(\Phi_0)$, and the column $E_1^{-2,*}$ of the main cohomological spectral sequence contains exactly two non-trivial terms $E_1^{-2,2n-4} \sim E_1^{-2,2n-1} \sim \mathbb{Z}$. By the Kontsevich's realization theorem (see § 4.2), they survive after the action of all differentials of the spectral sequence and define certain cohomology classes of $\mathcal{K} \setminus \Sigma$ in dimensions $2n-6$ and $2n-3$ respectively. However, in the case $n = 3$ we cannot be sure that the $(2n-3)$-dimensional class or some its multiple will not be killed by some unstable cycle in Σ, not counted by our spectral sequence. The $(2n-6)$-dimensional class in this case is a well-known knot invariant, and Theorem 7 is proved.

Note however, that for $n = 4$ its statement, concerning the group in dimension $2n-3$, depends on a non-published theorem of Kontsevich. (For $n > 5$ the fact that nothing acts into the corresponding cell $E^{-2,2n-1}$ follows from the dimensional reasons, see (5), and for $n = 5$ from the simplest version (6) of the Kontsevich's theorem, whose proof essentially coincides with that for the classical case $n = 3$, see [K1], [BN2]. □

8. Three problems

PROBLEM 1. To establish a direct correspondence between the algebra of two-connected graphs and that of Chinese Character Diagrams, see [BN2]. Perhaps this correspondence will give us the most natural proof of Theorem 2. The starting point of our construction was the space of all smooth maps $\mathbb{R}^1 \to \mathbb{R}^n$. Is there some analog of it behind the Chinese Character Diagrams?

PROBLEM 2. To present a precise (and economical) description of generators of groups $\bar{H}_*(\Delta^2(k))$ The absence of such a description is now almost unique objective to the calculation of order 4 cohomology classes of spaces of non-compact knots.

PROBLEM 3. The multiplication conjecture of § 5.5.2.

References

[A1] V. I. Arnold, *On some topological invariants of algebraic functions,* Trudy Moskov. Mat. Obshch. 1970, 21, 27–46. Engl. transl.: Transact. Moscow Math. Soc., 1970, 21, 30–52.
[A2] V. I. Arnold, *The spaces of functions with mild singularities*, Funct. Anal. Appl. 23:3 (1989).

[BBLSW] E. Babson, A. Björner, S. Linusson, J. Shareshian, V. Welker, *The complexes of not i-connected graphs,* preprint, 1997.

[BN] D. Bar-Natan, *Some computations related to Vassiliev invariants,* preprint, 1994.

[BN2] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology, 34 (1995), p. 423–472.

[BL] Joan S. Birman and X.-S. Lin, *Knot polynomials and Vassiliev's invariants,* Invent. Math. 111 (1993), p. 225–270.

[B] A. Björner, *Topological methods,* In: R. Graham, M. Grötschel, L. Lovász, editors, *Handbook of combinatorics,* North–Holland, Amsterdam, 1995, pp. 1819–1872.

[Fo] J. Folkman, *The homology group of a lattice,* J. Math. Mech. 15 (1966), p. 631–636.

[Fu] D. B. Fuchs, *Cohomology of the braid group mod 2*, Funct. Anal. and its Appl., 24:2 (1970), 62–73.

[K1] M. Kontsevich, *Vassiliev's knot invariants,* Adv. in Sov. Math., vol.16, part 2, Amer. Math. Soc., Providence, RI, 1993, p. 137–150.

[K2] M. Kontsevich, *Private communication, April 1994, Texel Island.*

[T] V. E. Turchin, *Homology of complexes of two-connected graphs*, Russian Math. Surveys (Uspekhy), 1997, to appear.

[V1] V. A. Vassiliev, *Stable cohomology of complements of discriminants of singularities of smooth functions*, Current Problems of Math., v. 33, Moscow, VINITI, 1988, p. 3–29; Engl. transl.: J. Soviet Math. 52:4 (1990), p. 3217–3230.

[V2] V. A. Vassiliev, *Cohomology of knot spaces.* In: Adv. in Sov. Math.; Theory of Singularities and its Appl. (ed. V. I. Arnold). AMS, Providence, R.I., 1990, p. 23–69.

[V3] V. A. Vassiliev, *Complexes of connected graphs*, The Gelfand's mathematical seminars 1990–1992 (L. Corwin, I. Gelfand, J. Lepovsky, eds.), Birkhäuser, Boston, 1993, 223–235.

[V4] V. A. Vassiliev, *Invariants of ornaments,* Advances in Soviet Math., Vol. 21, AMS, Providence RI, 1994, 225–262.

[V5] V. A. Vassiliev, *Complements of discriminants of smooth maps: topology and applications,* revised edition, vol. 98 of *Transl. of Math. Monographs,* AMS, Providence, RI, 1994.

[V6] V. A. Vassiliev, *Topology of complements of discriminants,* Moscow, Phasis (in Russian, to appear in 1997).

[V7] V. A. Vassiliev, *On invariants and homology of spaces of knots in arbitrary manifolds*, Preprint MSRI, March 1997.

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