SOME ORTHOGONAL VERY-WELL-POISED $_8\varphi_7$-FUNCTIONS THAT GENERALIZE ASKEY–WILSON POLYNOMIALS

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ABSTRACT. In a recent paper Ismail, Masson, and Suslov [16] have established a continuous orthogonality relation and some other properties of a $_2\varphi_1$-Bessel function on a q-quadratic grid. Dick Askey [3] suggested that the "Bessel-type orthogonality" found in [16] at the $_2\varphi_1$-level has really a general character and can be extended up to the $_8\varphi_7$-level. Very-well-poised $_8\varphi_7$-functions are known as a nonterminating version of the classical Askey–Wilson polynomials [5], [6]. Askey's congecture has been proved in [33]. In the present paper which is an extended version of [33] we discuss in details some properties of the orthogonal $_8\varphi_7$-functions. Another type of the orthogonality relation for a very-well-poised $_8\varphi_7$-function was recently found by Askey, Rahman, and Suslov [4].

1. INTRODUCTION

The Askey–Wilson polynomials [6] are

$$\begin{aligned} p_n(x) &= p_n(x; a, b, c, d) \\ &= a^{-n}\ (ab,\ ac,\ ad; q)_n\ {}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\ ;\ q,\ q\right), \end{aligned} \tag{1.1}$$

where $x = \cos\theta$. These polynomials are the most general known classical orthogonal polynomials (see [1], [6], [7], [8], [12], [23], and [25]).

The symbol $_4\varphi_3$ in (1.1) is a special case of basic hypergeometric series [12],

$$\begin{aligned} {}_r\varphi_s(t) &:= {}_r\varphi_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}\ ;\ q,\ t\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} t^n. \end{aligned} \tag{1.2}$$

The standard notations for the q-shifted factorials are

$$(a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}), \tag{1.3}$$

$$(a_1, a_2, \dots, a_r; q)_n := \prod_{k=1}^{r} (a_k; q)_n, \tag{1.4}$$

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and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \tag{1.5}$$

$$(a_1,a_2,\dots,a_r;q)_\infty := \prod_{k=1}^{r}(a_k;q)_\infty \tag{1.6}$$

provided $|q|<1$. For an excellent account on the theory of basic hypergeometric series see [12]. We shall just mention here that the ${}_{s+1}\varphi_s$-series is called balanced if $qa_1a_2\cdots a_{s+1}=b_1b_2\cdots b_s$ and $t=q$; it is a very-well-poised if $a_2b_1=a_3b_2=\ldots=a_{s+1}b_s=qa_1$, and $a_2=q\sqrt{a_1}$, $a_3=-q\sqrt{a_1}$.

Askey and Wilson found the orthogonality relation

$$\begin{aligned}\int_0^\pi & \frac{p_n(\cos\theta;a,b,c,d)\,p_m(\cos\theta;a,b,c,d)\,\left(e^{2i\theta},e^{-2i\theta};q\right)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty}\,d\theta \\ &= \delta_{nm}\,\frac{(q,ab,ac,ad,bc,bd,cd;q)_\infty}{2\pi\,(abcd;q)_\infty} \\ &\quad\times\frac{(1-abcdq^{-1})\,(q,ab,ac,ad,bc,bd,cd;q)_n}{(1-abcdq^{2n-1})\,(abcdq^{-1};q)_n}.\end{aligned} \tag{1.7}$$

In the fundamental memoir [6], they studied in details many other properties of these polynomials.

As is well known, the Askey–Wilson polynomials and their special and limiting cases are the simplest and the most important orthogonal solutions of a difference equation of hypergeometric type on nonuniform lattices (see, for example, [6], [7], [8], [9], [12], [25], and [31]). Recently Ismail, Masson, and Suslov have found another type of orthogonal solutions of this difference equation [16]–[17]. They considered the ${}_2\varphi_1$-function,

$$\begin{aligned}J_\nu(z,r) &= \widetilde{J}_\nu(x(z),r|q) \\ &:= \left(\frac{r}{2}\right)^\nu \frac{(q^{\nu+1},\,-r^2/4;q)_\infty}{(q;q)_\infty}\;{}_2\varphi_1\left(\begin{matrix}q^{(\nu+1)/2}e^{i\theta},\;q^{(\nu+1)/2}e^{-i\theta}\\ q^{\nu+1}\end{matrix}\;;\;q,\,-\frac{r^2}{4}\right),\end{aligned} \tag{1.8}$$

as a q-analog on a q-quadratic grid of the Bessel function [34],

$$J_\nu(x)=\left(\frac{x}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{\left(-x^2/4\right)^n}{n!\,\Gamma(\nu+n+1)}. \tag{1.9}$$

Ismail, Masson, and Suslov established the following orthogonality property for the q-Bessel function,

$$\begin{aligned}\int_0^\pi &\widetilde{J}_\nu(\cos\theta,r)\,\widetilde{J}_\nu(\cos\theta,r')\,\frac{\left(e^{2i\theta},\,e^{-2i\theta};q\right)_\infty}{(q^\alpha e^{i\theta},q^\alpha e^{-i\theta},q^{1-\alpha}e^{i\theta},q^{1-\alpha}e^{-i\theta};q)_\infty} \\ &\times\left(q^{(\nu+1)/2}e^{i\theta},q^{(\nu+1)/2}e^{-i\theta},q^{(\nu+1)/2}e^{i\theta},q^{(\nu+1)/2}e^{-i\theta};q\right)_\infty^{-1}\,d\theta=0\end{aligned} \tag{1.10}$$

if $r\neq r'$,

$$\begin{aligned}\int_0^\pi &\left(\widetilde{J}_\nu(\cos\theta,r)\right)^2\,\frac{\left(e^{2i\theta},\,e^{-2i\theta};q\right)_\infty}{(q^\alpha e^{i\theta},q^\alpha e^{-i\theta},q^{1-\alpha}e^{i\theta},q^{1-\alpha}e^{-i\theta};q)_\infty} \\ &\times\left(q^{(\nu+1)/2}e^{i\theta},q^{(\nu+1)/2}e^{-i\theta},q^{(\nu+1)/2}e^{i\theta},q^{(\nu+1)/2}e^{-i\theta};q\right)_\infty^{-1}\,d\theta\end{aligned} \tag{1.11}$$

$$= \frac{-4\pi(1-q)q^{-(\nu+1)/2}}{\left(q, q^{(\nu+1)/2+\alpha}, q^{(\nu+1)/2-\alpha+1}; q\right)_\infty^2} \frac{\partial \widetilde{J}_\nu(x(\alpha), r)}{\partial r^2} \frac{\nabla \widetilde{J}_\nu(x(\alpha), r)}{\nabla x(\alpha)}$$

if $r = r'$. Here r and r' are two roots of the equation

$$J_\nu(\alpha, r) = J_\nu(\alpha, r') = 0, \tag{1.12}$$

and $J_\nu(z, r) = \widetilde{J}_\nu(x(z), r)$, $x(z) = \frac{1}{2}\left(q^z + q^{-z}\right)$, ($x = \cos\theta$, if $q^z = e^{i\theta}$), Re $\nu > -1$, and $0 <$ Re $\alpha <$ 1. (See [16]–[17] for more details.) This is a q-version of the orthogonality relation for the classical Bessel function

$$\int_0^1 x\, J_\nu(rx)\, J_\nu(r'x)\, dx = \begin{cases} 0 & \text{if } r \neq r', \\ \frac{1}{2}\, \left(J_{\nu+1}(r)\right)^2 & \text{if } r = r', \end{cases} \tag{1.13}$$

under the conditions $J_\nu(r) = J_\nu(r') = 0$ [34].

Another example of an orthogonality relation of this type has been discovered recently. The following functions

$$\begin{aligned} C(x) :=& C_q(x;\omega) \\ =& \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty}\ {}_2\varphi_1\left(\begin{matrix} -qe^{2i\theta},\ -qe^{-2i\theta} \\ q \end{matrix}\ ;\ q^2,\ -\omega^2\right) \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} S(x) :=& S_q(x;\omega) \\ =& \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty}\ \frac{2q^{1/4}}{1-q}\,\omega \\ &\times \cos\theta\ {}_2\varphi_1\left(\begin{matrix} -q^2e^{2i\theta},\ -q^2e^{-2i\theta} \\ q^3 \end{matrix}\ ;\ q^2,\ -\omega^2\right) \end{aligned} \tag{1.15}$$

were discussed in [8] and [22] as analogs of $\cos\omega x$ and $\sin\omega x$ on a q-quadratic lattice, respectively. Ismail and Zhang [22] expanded the corresponding basic exponential function in terms of "q-spherical harmonics" and later, together with Rahman [20], they extended this q-analog of the expansion formula of the plane wave from q-ultraspherical polynomials to continuous q-Jacobi polynomials. "Addition" theorems for the basic trigonometric functions (1.14)–(1.15) where found in [32]. Bustoz and Suslov [11] have established an orthogonality property,

$$\int_0^\pi C(\cos\theta;\omega)\, C(\cos\theta;\omega')\, \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{\left(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q\right)_\infty}\, d\theta = 0, \tag{1.16}$$

$$\int_0^\pi S(\cos\theta;\omega)\, S(\cos\theta;\omega')\, \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{\left(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q\right)_\infty}\, d\theta = 0, \tag{1.17}$$

$$\int_0^\pi C(\cos\theta;\omega)\, S(\cos\theta;\omega')\, \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{\left(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta}; q\right)_\infty}\, d\theta = 0, \tag{1.18}$$

and

$$\int_0^\pi C^2(\cos\theta;\omega)\,\frac{\left(e^{2i\theta},\,e^{-2i\theta};q\right)_\infty}{(q^{1/2}e^{2i\theta},q^{1/2}e^{-2i\theta};q)_\infty}\,d\theta \tag{1.19}$$
$$= \int_0^\pi S^2(\cos\theta;\omega)\,\frac{\left(e^{2i\theta},\,e^{-2i\theta};q\right)_\infty}{(q^{1/2}e^{2i\theta},q^{1/2}e^{-2i\theta};q)_\infty}\,d\theta$$
$$= \pi\,\frac{\left(q^{1/2},-q^{1/2}\omega^2;q\right)_\infty}{(q,-\omega^2;q)_\infty}\,\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}$$
$$\times\ {}_2\varphi_1\left(\begin{matrix} -q^{1/2},\,-\omega^2 \\ \\ -q^{1/2}\omega^2\end{matrix}\ ;\ q,\,q\right).$$

Here ω and ω' are different solutions of the equation

$$S\left(\frac{1}{2}\left(q^{1/4}+q^{-1/4}\right);\omega\right)=0. \tag{1.20}$$

Bustoz and Suslov introduced the corresponding q-Fourier series and established several important facts about the basic trigonometric system and the q-Fourier series.

Dick Askey [3] has suggested that the orthogonality relation (1.10)–(1.11) can be extended to the level of very-well-poised ${}_8\varphi_7$-functions. The author was able to prove his conjecture in [33]. Our main objective in this paper is to study the new orthogonality property for ${}_8\varphi_7$-function in details. This article is an extended version of the short paper [33], originally submitted as a Letter, we include proofs of the main results etablished in [33] to make this work as self-contained as possible.

The paper is organized as follows. In Section 2 we consider difference equation of hypergeometric type on a q-quadratic grid and discuss a solution of this equation in terms of a very-well-poised ${}_8\varphi_7$-function. In the next two sections we derive a continuous orthogonality property for this ${}_8\varphi_7$-function. Section 5 is devoited to the investigation of zeros and asymptotics of this function and in Section 6 we evaluate the normalization constants in the orthogonality relation for the ${}_8\varphi_7$. In Section 7 we find an analog of the Wronskian determinant for solutions of difference equation of hypergeometric type. Some special and limiting cases of our orthogonality relation are discussed in Section 8. We close this paper by estimating the number of zeros on the basis of Jensen's theorem in Section 9.

2. Difference Equation and Its ${}_8\varphi_7$-Solutions

Let us consider a *difference equation of hypergeometric type*

$$\sigma(z)\,\frac{\Delta}{\nabla x_1(z)}\left(\frac{\nabla u(z)}{\nabla x(z)}\right)+\tau(z)\,\frac{\Delta u(z)}{\Delta x(z)}+\lambda\,u(z)=0, \tag{2.1}$$

on a q-quadratic lattice $x(z)=\dfrac{1}{2}\left(q^z+q^{-z}\right)$ with $x_1(z)=x\left(z+\frac{1}{2}\right)$ and $\Delta f(z)=\nabla f(z+1)=f(z+1)-f(z)$. Here, in the most general case,

$$\sigma(z)=q^{-2z}\,(q^z-a)\,(q^z-b)\,(q^z-c)\,(q^z-d)\,, \tag{2.2}$$

$$
\begin{aligned}
\tau(z) &= \frac{\sigma(-z)-\sigma(z)}{\nabla x_1(z)} \\
&= \frac{2q^{1/2}}{1-q}\left(abc+abd+acd+bcd-a-b-c-d+2(1-abcd)\,x\right), \qquad (2.3)\\
\lambda = \lambda_\nu &= \frac{4q^{3/2}}{(1-q)^2}\,\left(1-q^{-\nu}\right)\left(1-abcd\,q^{\nu-1}\right). \qquad (2.4)
\end{aligned}
$$

Equation (2.1) can also be rewritten in self-adjoint form,

$$\frac{\Delta}{\nabla x_1(z)}\left(\sigma(z)\,\rho(z)\,\frac{\nabla u(z)}{\nabla x(z)}\right)+\lambda\,\rho(z)\,u(z)=0, \tag{2.5}$$

where $\rho(z)$ is a solution of the *Pearson equation*,

$$\Delta\left(\sigma(z)\,\rho(z)\right)=\tau(z)\,\nabla x_1(z). \tag{2.6}$$

See [25] and [31] for details.

Methods of solving of the difference equation of hypergeometric type (2.1) were discussed in [8], [13], and [31]. As is well known, there are different kinds of solutions of this equation. For integer values of the parameter $\nu = n = 0, 1, 2, \ldots$, the famous solutions of (2.1) are the Askey–Wilson ${}_4\varphi_3$-polynomials (see, for example, [5], [6], [9], and [12]). For arbitrary values of this parameter solutions of (2.1) can be written in terms of ${}_8\varphi_7$-functions [8], [18], and [31]. Let us choose the following solution, $u_\nu(z) = u_\nu\left(x(z); a, b, c; d\right)$, such that

$$
\begin{aligned}
&u_\nu(x;a,b,c;d)\\
&= \frac{(qa/d,\, bcq^\nu,\, q^{1-\nu+z}/d,\, q^{1-\nu-z}/d;\, q)_\infty}{(q^{1-\nu}a/d,\, bc,\, q^{1+z}/d,\, q^{1-z}/d;\, q)_\infty} \qquad (2.7)\\
&\quad\times\ {}_8\varphi_7\left(\begin{matrix} \dfrac{aq^{-\nu}}{d},\, q\sqrt{\dfrac{aq^{-\nu}}{d}},\, -q\sqrt{\dfrac{aq^{-\nu}}{d}},\, q^{-\nu},\, \dfrac{q^{1-\nu}}{bd},\, \dfrac{q^{1-\nu}}{cd},\, aq^z,\, aq^{-z} \\ \sqrt{\dfrac{aq^{-\nu}}{d}},\, -\sqrt{\dfrac{aq^{-\nu}}{d}},\, \dfrac{aq}{d},\, ab,\, ac,\, \dfrac{q^{1-\nu+z}}{d},\, \dfrac{q^{1-\nu-z}}{d}\end{matrix}\,;\ q,\ bcq^\nu\right)\\
&= \frac{(bcq^\nu,\, q^{1-\nu}/ad;\, q)_\infty}{(bc,\, q/ad;\, q)_\infty}\ {}_4\varphi_3\left(\begin{matrix} q^{-\nu},\, abcdq^{\nu-1},\, aq^z,\, aq^{-z} \\ ab,\, ac,\, ad\end{matrix}\,;\ q,\ q\right) \qquad (2.8)\\
&\quad+ \frac{(q^{-\nu},\, abcdq^{\nu-1},\, qb/d,\, qc/d;\, q)_\infty}{(ab,\, ac,\, bc,\, ad/q;\, q)_\infty}\\
&\quad\times \frac{(aq^z,\, aq^{-z};\, q)_\infty}{(q^{1+z}/d,\, q^{1-z}/d;\, q)_\infty}\ {}_4\varphi_3\left(\begin{matrix} q^{1-\nu}/ad,\, bcq^\nu,\, q^{1+z}/d,\, q^{1-z}/d \\ qb/d,\, qc/d,\, q^2/ad\end{matrix}\,;\ q,\ q\right).
\end{aligned}
$$

We have used Bailey's formula (III.36) of [12] to transform (2.7) to (2.8). Both ${}_4\varphi_3$-functions in (2.8) are balanced and converge when $|q| < 1$. The ${}_8\varphi_7$-function in (2.7) has a very-well-poised structure, it converges when $|bcq^\nu| < 1$; (2.8) provides an analytic continuation of (2.7). This ${}_8\varphi_7$ can be transformed to another equivalent form by (III.23)–(III.24) of [12].

A similar function was earlier discussed by Rahman [27] and in a recent paper [28] he has found a q-extension of a product formula of Watson involving this function. Rahman has also shown that

$$\lim_{q\to 1^-} u_\nu(x; q^{\alpha+1/2}, q^{\alpha+3/4}, -q^{\beta+1/2}; -q^{\beta+3/4}) = {}_2F_1\left(-\nu, \alpha+\beta+\nu+1; \alpha+1; \frac{1-x}{2}\right). \tag{2.9}$$

Thus, the ${}_8\varphi_7$-function in (2.7) can be thought as a q-extension of the hypergeometric function of Gauss.

It is easy to see that $u_\nu(x; a, b, c; d)$ defined by (2.7)–(2.8) is a function in $x = (q^z + q^{-z})/2$. Indeed, by (1.3) and (1.4),

$$\left(\xi q^z, \xi q^{-z}\right)_n = \prod_{k=0}^{n-1}\left(1 - 2\xi x q^k + \xi^2 q^{2k}\right),$$

where $\xi = a,\ q/d,\ q^{1-\nu}/d$ and $n = 1, 2, 3, \dots$ or ∞. It is also important to mention that

$$\begin{aligned}
&\left(q^{-\nu}, abcdq^{\nu-1}; q\right)_n \\
&= \prod_{k=0}^{n-1}\left(1 - \left(q^{-\nu} + abcdq^{\nu-1}\right) q^k + abcd\, q^{2k-1}\right) \\
&= \prod_{k=0}^{n-1}\left(1 - \left(1 + abcdq^{-1} - \frac{(1-q)^2}{4q^{3/2}}\,\lambda_\nu\right) q^k + abcd\, q^{2k-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(bcq^\nu, q^{1-\nu}/ad; q\right)_n \\
&= \prod_{k=0}^{n-1}\left(1 - \frac{1}{ad}\left(q^{-\nu} + abcdq^{\nu-1}\right) q^{k+1} + \frac{bc}{ad}\, q^{2k+1}\right) \\
&= \prod_{k=0}^{n-1}\left(1 - \frac{1}{ad}\left(1 + abcdq^{-1} - \frac{(1-q)^2}{4q^{3/2}}\,\lambda_\nu\right) q^{k+1} + \frac{bc}{ad}\, q^{2k+1}\right)
\end{aligned}$$

by (1.3)–(1.4) and (2.4), where $n = 1, 2, 3, \dots$ or ∞. Therefore, $u_\nu(x; a, b, c; d)$ is, really, a function of λ_ν as well.

The definition of the u_ν in this paper is the same as in [27] and [28], but different from one in [33]. This definition emphasizes the symmetry properties. Function $u_\nu(x; a, b, c; d)$ in (2.7)–(2.8) is obviously symmetric in b and c. Applying (III.36) and Bailey's transform (III.36) of [12] once again we obtain

$$\begin{aligned}
&u_\nu(x; a, b, c; d) \\
&= \frac{(abcq^{\nu+z}, q^{1-\nu+z}/d; q)_\infty}{(abcq^z, q^{1+z}/d; q)_\infty} \qquad (2.10)\\
&\quad \times {}_8W_7\left(abcq^{z-1}; q^{-\nu}, abcdq^{\nu-1}, aq^z, bq^z, cq^z;\ q,\ \frac{q^{1-z}}{d}\right) \\
&= \frac{(abq^\nu, acq^\nu, bcq^\nu, abcdq^{\nu-1}, q^{1-\nu+z}/d, q^{1-\nu-z}/d; q)_\infty}{(ab, ac, bc, abcdq^{2\nu-1}, q^{1+z}/d, q^{1-z}/d; q)_\infty} \qquad (2.11)
\end{aligned}$$

$$\times\ {}_4\varphi_3\left(\begin{matrix} q^{-\nu}, q^{1-\nu}/ad, q^{1-\nu}/bd, q^{1-\nu}/cd \\ q^{2-2\nu}/abcd, q^{1-\nu+z}/d, q^{1-\nu-z}/d \end{matrix}\ ;\ q,\ q\right)$$
$$+\ \frac{(q^{-\nu}, q^{1-\nu}/ad, q^{1-\nu}/bd, q^{1-\nu}/cd, abcq^{\nu+z}, abcq^{\nu-z}; q)_\infty}{(ab, ac, bc, q^{1-2\nu}/abcd, q^{1+z}/d, q^{1-z}/d; q)_\infty}$$
$$\times\ {}_4\varphi_3\left(\begin{matrix} abcdq^{\nu-1}, abq^{\nu}, acq^{\nu}, bcq^{\nu} \\ abcdq^{2\nu}, abcq^{\nu+z}, abcq^{\nu-z} \end{matrix}\ ;\ q,\ q\right).$$

These representations show that $u_\nu(x; a, b, c; d)$ is actually symmetric in a, b, and c. Here we have used the standard notation,

$$\begin{aligned} &{}_{2r+2}W_{2r+1}\left(a; a_1, a_2, \dots, a_{2r-1}; q, t\right) \\ &:=\ {}_{2r+2}\varphi_{2r+1}\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a_1, a_2, \dots, a_{2r-1} \\ \sqrt{a}, -\sqrt{a}, qa/a_1, qa/a_2, \dots, qa/a_{2r-1} \end{matrix}\ ;\ q,\ t\right), \end{aligned} \tag{2.12}$$

for a very-well-poised basic hypergeometric series defined before.

Let us discuss analyticity properties of the function $u_\nu(z)$ defined in (2.7)–(2.8). One can easily see, that for integers $\nu = n = 0, 1, 2, \dots$ this function is just a multiple of the Askey–Wilson polynomial (1.1) of the n-th degree,

$$u_n(x; a, b, c; d) = \frac{(-1)^n q^{-n(n-1)/2}}{(ab, ac, bc; q)_n}\, d^{-n}\ p_n(x; a, b, c, d), \tag{2.13}$$

where $x = \left(q^z + q^{-z}\right)/2$. In this case the Askey–Wilson polynomials $p_n(x; a, b, c, d)$ are symmetric with respect to a permutation of all four parameters a, b, c, and d due to Sears' transformation [6].

On the other hand, if ν is not an integer, the essential poles of the ${}_8\varphi_7$-solution in (2.7) coincide with the simple poles of the infinite product

$$\left(q^{1+z}/d, q^{1-z}/d; q\right)_\infty^{-1}.$$

Therefore, the function

$$v_\nu(z) = v_\nu(x(z); a, b, c; d) := \left(q^{1+z}/d, q^{1-z}/d; q\right)_\infty u_\nu(x(z); a, b, c; d), \tag{2.14}$$

where $u_\nu(x; a, b, c; d)$ is defined in (2.7)–(2.8), is an entire function in the complex z-plane.

Let us mention also that function $u_\nu(x; a, b, c; d)$ satisfies the simple difference-differentiation formula

$$\begin{aligned} \frac{\delta}{\delta x(z)}\, u_\nu(x(z); a, b, c; d) =& \frac{2q}{(1-q)d}\, \frac{\left(1-q^{-\nu}\right)\left(1-abcdq^{\nu-1}\right)}{(1-ab)(1-ac)(1-bc)} \\ &\times u_{\nu-1}\left(x(z); aq^{1/2}, bq^{1/2}, cq^{1/2}; dq^{1/2}\right), \end{aligned} \tag{2.15}$$

where $\delta f(z) = f(z+1/2) - f(z-1/2)$ and $x(z) = \left(q^z + q^{-z}\right)/2$.

3. Solutions of Pearson Equation

In order to rewrite equation (2.1) for the function (2.7)–(2.8) in self-adjoint form (2.5), we have to find a solution of the Pearson-type equation (2.6). In the case of the q-quaratic grid $x = \frac{1}{2}\left(q^z + q^{-z}\right)$ this equation can be rewritten in the form

$$\begin{aligned}\frac{\rho(z+1)}{\rho(z)} &= \frac{\sigma(-z)}{\sigma(z+1)} \\ &= q^{-4z-2}\, q^{2z+1}\, \frac{(1-aq^z)(1-bq^z)(1-cq^z)(1-q^{-z}/d)}{(1-aq^{-z-1})(1-bq^{-z-1})(1-cq^{-z-1})(1-q^{z+1}/d)}.\end{aligned} \tag{3.1}$$

It is easy to check that

$$\frac{\rho_0(z+1)}{\rho_0(z)} = q^{-4z-2} \qquad \text{for} \quad \rho_0(z) = \frac{(q^{2z}, q^{-2z}; q)_\infty}{q^z - q^{-z}}, \tag{3.2}$$

$$\frac{\rho_\alpha(z+1)}{\rho_\alpha(z)} = q^{-2z-1} \qquad \text{for} \quad \rho_\alpha(z) = (\alpha q^z, \alpha q^{-z}, q^{1+z}/\alpha, q^{1-z}/\alpha; q)_\infty, \tag{3.3}$$

$$\frac{\rho_a(z+1)}{\rho_a(z)} = \frac{1-aq^{-z-1}}{1-aq^z} \qquad \text{for} \quad \rho_a(z) = (aq^z, aq^{-z}; q)_\infty. \tag{3.4}$$

(See, for example, [25], [29], and [30] for methods of solving the Pearson equation.) Therefore, one can choose the following solution of (3.1),

$$\rho(z) = \frac{(q^z - q^{-z})^{-1}(q^{2z}, q^{-2z}, q^{1+z}/d, q^{1-z}/d; q)_\infty}{(\alpha q^z, \alpha q^{-z}, q^{1+z}/\alpha, q^{1-z}/\alpha, aq^z, aq^{-z}, bq^z, bq^{-z}, cq^z, cq^{-z}; q)_\infty}, \tag{3.5}$$

where α is an arbitrary additional parameter. It was shown in [33] that this solution satisfies the correct boundary conditions for the second order divided-difference Askey–Wilson operator (2.5) for certain values of the parameter α.

Special cases $\alpha = d$ or $\alpha = q/d$ of (3.5) result in the weight function for the Askey–Wilson polynomials (cf. [6], [9]).

One can rewrite (3.1) as

$$\frac{\rho(z+1)}{\rho(z)} = \frac{(1-aq^z)(1-bq^z)(1-q^{-z}/c)(1-q^{-z}/d)}{(1-aq^{-z-1})(1-bq^{-z-1})(1-q^{z+1}/c)(1-q^{z+1}/d)}. \tag{3.6}$$

We shall use the corresponding solution,

$$\rho(z) = \frac{(q^{1+z}/c, q^{1-z}/c, q^{1+z}/d, q^{1-z}/d; q)_\infty}{(aq^z, aq^{-z}, bq^z, bq^{-z}; q)_\infty}, \tag{3.7}$$

in Section 7.

4. Orthogonality Property

Now we can prove the orthogonality relation of the ${}_8\varphi_7$-functions (2.7) with respect to the weight function (3.5) established in [33]. Let us apply the following q-version of the Sturm–Liouville procedure (cf. [9], [10], and [25]). Consider the difference equations for the functions $u_\nu(z) = u_\nu(x(z); a, b, c; d)$ and $u_\mu(z) = u_\mu(x(z); a, b, c; d)$ in self-adjoint form,

$$\frac{\Delta}{\nabla x_1(z)} \left(\sigma(z)\,\rho(z)\,\frac{\nabla u_\mu(z)}{\nabla x(z)}\right) + \lambda_\mu\,\rho(z)\,u_\mu(z) = 0, \tag{4.1}$$

$$\frac{\Delta}{\nabla x_1(z)} \left(\sigma(z)\,\rho(z)\,\frac{\nabla u_\nu(z)}{\nabla x(z)}\right) + \lambda_\nu\,\rho(z)\,u_\nu(z) = 0, \tag{4.2}$$

where the eigenvalues $\lambda = \lambda_\nu$ and $\lambda' = \lambda_\mu$ are defined by (2.4). Let us multiply the first equation by $u_\nu(z)$, the second one by $u_\mu(z)$, and subtract the second equality from the first one. As a result we get

$$\begin{aligned}(\lambda_\mu - \lambda_\nu)\;& u_\mu(z)\,u_\nu(z)\,\rho(z)\,\nabla x_1(z)\\ &= \Delta\left[\sigma(z)\,\rho(z)\,W\left(u_\mu(z), u_\nu(z)\right)\right],\end{aligned} \tag{4.3}$$

where

$$\begin{aligned} W\left(u_\mu(z),\ u_\nu(z)\right) &= \begin{vmatrix} u_\mu(z) & u_\nu(z) \\ \dfrac{\nabla u_\mu(z)}{\nabla x(z)} & \dfrac{\nabla u_\nu(z)}{\nabla x(z)} \end{vmatrix} \\ &= u_\mu(z)\,\frac{\nabla u_\nu(z)}{\nabla x(z)} - u_\nu(z)\,\frac{\nabla u_\mu(z)}{\nabla x(z)} \\ &= \frac{u_\nu(z)\,u_\mu(z-1) - u_\mu(z)\,u_\nu(z-1)}{x(z) - x(z-1)} \end{aligned} \tag{4.4}$$

is the analog of the Wronskian [25].

We need to know the pole structure of the analog of the Wronskian $W(u_\mu, u_\nu)$ in (4.4). Let us transform the u's to the entire functions v's by (2.14),

$$u_\varepsilon(z) = \varphi(z)\;v_\varepsilon(z), \tag{4.5}$$

where $\varepsilon = \mu, \nu$ and

$$\varphi(z) = \left(q^{1+z}/d, q^{1-z}/d; q\right)_\infty^{-1}. \tag{4.6}$$

Thus,

$$W(u_\mu(z), u_\nu(z)) = \varphi(z)\;\varphi(z-1)\;W(v_\mu(z), v_\nu(z)), \tag{4.7}$$

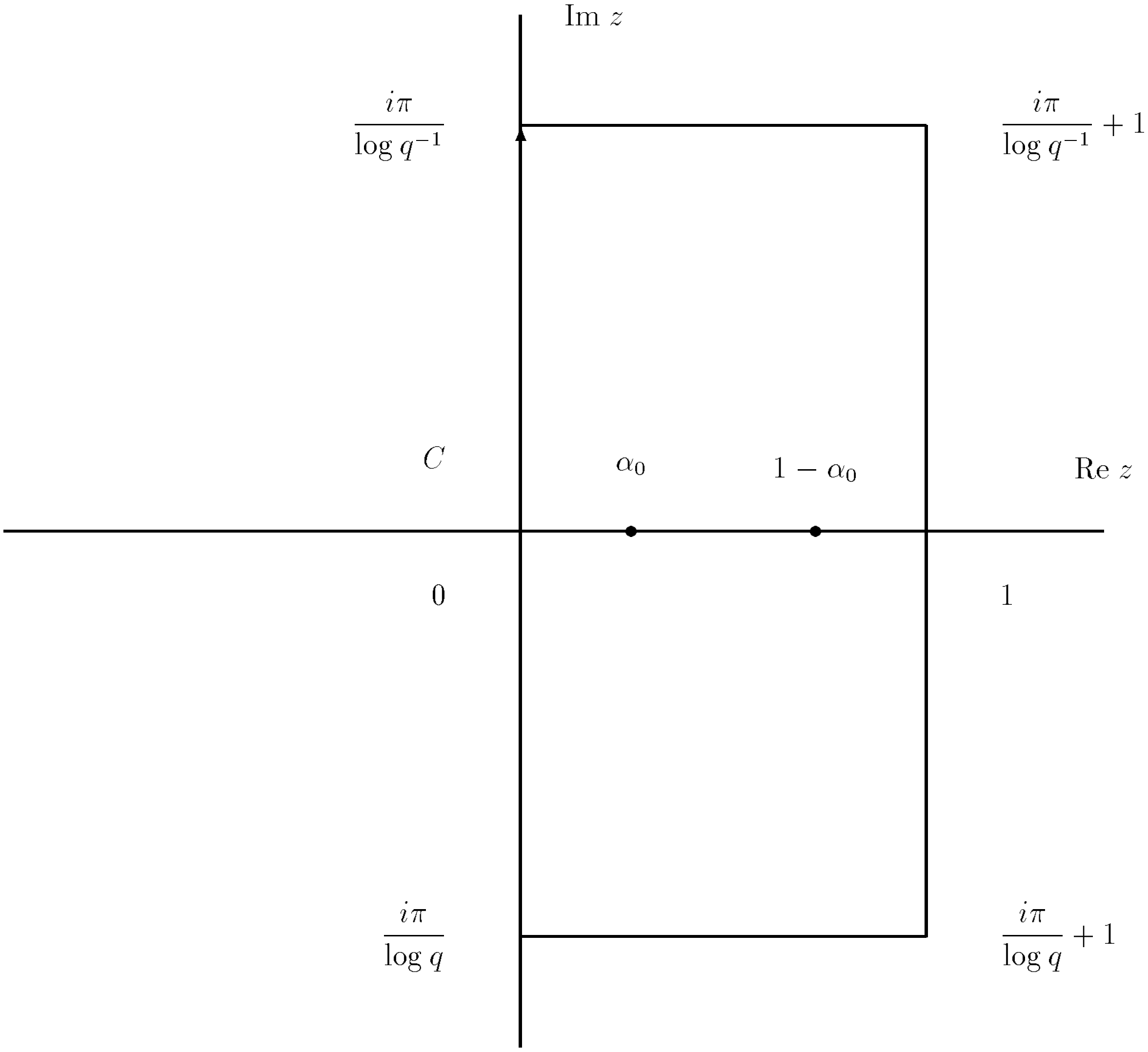
where the new "Wronskian", $W(v_\mu(z), v_\nu(z))$, is clearly an entire function in z.

Integrating (4.3) over the contour C indicated in the Figure; where the variable z is such that $z = i\theta/\log q^{-1}$ and $-\pi \le \theta \le \pi$ (we shall assume that $0 < q < 1$ throughout this work); gives

$$\begin{aligned}(\lambda_\mu - \lambda_\nu)\;&\int_C u_\mu(z)\,u_\nu(z)\;\rho(z)\,\nabla x_1(z)\,dz\\ &= \int_C \Delta\left[\sigma(z)\,\rho(z)\,\varphi(z)\,\varphi(z-1)\;W\left(v_\mu(z), v_\nu(z)\right)\right]\,dz.\end{aligned} \tag{4.8}$$

All poles of the integrand in the right side of (4.8) coincide with the simple poles of the function

$$\begin{aligned}&\sigma(z)\,\rho(z)\,\varphi(z)\,\varphi(z-1)\\ &= -\frac{d\,(q^{2z}, q^{1-2z}; q)_\infty}{(\alpha q^z, \alpha q^{-z}, q^{1+z}/\alpha, q^{1-z}/\alpha; q)_\infty}\end{aligned} \tag{4.9}$$



Figure

$$\times \left(aq^z, aq^{1-z}, bq^z, bq^{1-z}, cq^z, cq^{1-z}, q^{1+z}/d, q^{2-z}/d; q\right)_\infty^{-1}.$$

The integrand in the right side of (4.8) has the natural purely imaginary period $T = 2\pi i/\log q$ when $0 < q < 1$, so this integral is equal to

$$\int_D \left[\sigma(z)\,\varphi(z)\,\varphi(z-1)\,\rho(z)\;W\left(v_\mu(z), v_\nu(z)\right)\right]\,dz, \tag{4.10}$$

where D is the boundary of the rectangle on the Figure oriented counterclockwise.

The analog of the Wronskian $W(v_\mu, v_\nu)$ is an entire function. Thus, when $\max\left(|a|, |b|, |c|, |q/d|\right) < 1$, the poles of the integrand in (4.10) inside the rectangle in the Figure are just the simple poles of $\rho(z)$ at $z = \alpha_0$ and $z = 1-\alpha_0$, where $q^{\alpha_0} = \alpha$ and $0 < \mathrm{Re}\ \alpha_0 < 1/2$. By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_D \left[\sigma(z)\,\varphi(z)\,\varphi(z-1)\,\rho(z)\;W\left(v_\mu(z), v_\nu(z)\right)\right]\,dz \tag{4.11}$$

$$= \text{Res } f(z)\big|_{z=\alpha_0} + \text{Res } f(z)\big|_{z=1-\alpha_0} ,$$

where

$$\begin{aligned} f(z) =& \sigma(z)\, \rho(z)\, \varphi(z)\, \varphi(z-1)\ W\left(v_\mu(z), v_\nu(z)\right) \\ =& -\frac{d\left(q^{2z}, q^{1-2z}; q\right)_\infty}{\left(\alpha q^z, \alpha q^{-z}, q^{1+z}/\alpha, q^{1-z}/\alpha; q\right)_\infty} \\ & \times \frac{W\left(v_\mu(z), v_\nu(z)\right)}{\left(aq^z, aq^{1-z}, bq^z, bq^{1-z}, cq^z, cq^{1-z}, q^{1+z}/d, q^{2-z}/d; q\right)_\infty}. \end{aligned} \tag{4.12}$$

Evaluation of the residues at these simple poles gives

$$\begin{aligned} \text{Res } f(z)\big|_{z=\alpha_0} &= \lim_{z\to\alpha_0} (z-\alpha_0)\ f(z) \\ &= -\frac{d\, W\left(v_\mu(z), v_\nu(z)\right)\big|_{z=\alpha_0}}{\log q\ \left(q, q, \alpha a, qa/\alpha, \alpha b, qb/\alpha, \alpha c, qc/\alpha, q\alpha/d, q^2/\alpha d; q\right)_\infty} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \text{Res } f(z)\big|_{z=1-\alpha_0} &= \lim_{z\to 1-\alpha_0} (z-1+\alpha_0)\ f(z) \\ &= -\frac{d\, W\left(v_\mu(z), v_\nu(z)\right)\big|_{z=1-\alpha_0}}{\log q\ \left(q, q, \alpha a, qa/\alpha, \alpha b, qb/\alpha, \alpha c, qc/\alpha, q\alpha/d, q^2/\alpha d; q\right)_\infty}. \end{aligned} \tag{4.14}$$

But

$$W\left(v_\mu(z), v_\nu(z)\right)\big|_{z=\alpha_0} = W\left(v_\mu(z), v_\nu(z)\right)\big|_{z=1-\alpha_0} \tag{4.15}$$

due to the last line in (4.4) and the symmetry $v_\varepsilon(z) = v_\varepsilon(-z)$, $x(z) = x(-z)$. Thus, the residues are equal and as a result we evaluate the integral in the right side of (4.8),

$$\begin{aligned} & \int_C \left[\sigma(z)\, \varphi(z)\, \varphi(z-1)\, \rho(z)\ W\left(v_\mu(z), v_\nu(z)\right)\right]\, dz \\ &= -\frac{4\pi i\, d\ W\left(v_\mu(\alpha_0), v_\nu(\alpha_0)\right)}{\log q\ \left(q, q, \alpha a, qa/\alpha, \alpha b, qb/\alpha, \alpha c, qc/\alpha, q\alpha/d, q^2/\alpha d; q\right)_\infty} \\ &= -\frac{4\pi i\, d\ \left(\alpha/d, q/\alpha d; q\right)_\infty\ W\left(u_\mu(\alpha_0), u_\nu(\alpha_0)\right)}{\log q\ \left(q, q, \alpha a, qa/\alpha, \alpha b, qb/\alpha, \alpha c, qc/\alpha; q\right)_\infty} \end{aligned} \tag{4.16}$$

by (4.7).

Combining (4.8) and (4.16), we, finally, arrive at the main equation,

$$\begin{aligned} & (\lambda_\mu - \lambda_\nu) \int_C u_\mu(z)\, u_\nu(z)\ \rho(z)\, \delta x(z)\, dz \\ &= -\frac{4\pi i\, d\ \left(\alpha/d, q/\alpha d; q\right)_\infty}{\log q\ \left(q, q, \alpha a, qa/\alpha, \alpha b, qb/\alpha, \alpha c, qc/\alpha; q\right)_\infty} \\ &\quad \times W\left(u_\mu(\alpha_0), u_\nu(\alpha_0)\right), \end{aligned} \tag{4.17}$$

where $\max\left(|a|, |b|, |c|, |q/d|\right) < 1$ and $0 < \text{Re } \alpha_0 < 1/2$, $q^{\alpha_0} = \alpha$.

It is worth mentioning the important special case first. If both of the "degree" parameters μ and ν are nonnegative integers: $\mu = m = 0, 1, 2, \dots$ and $\nu = n = 0, 1, 2, \dots$; equations (4.17) and (2.13) result in the following real integral

$$\int_0^\pi \frac{p_m(\cos\theta;a,b,c;d)\;p_n(\cos\theta;a,b,c;d)}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta};q)_\infty} \times \frac{\left(e^{2i\theta},\;e^{-2i\theta},qe^{i\theta}/d,qe^{-i\theta}/d;q\right)_\infty}{(\alpha e^{i\theta},\alpha e^{-i\theta},qe^{i\theta}/\alpha,qe^{-i\theta}/\alpha;q)_\infty}\,d\theta = \frac{(\alpha/d,q/\alpha d;q)_\infty}{(q,q,\alpha a,qa/\alpha,\alpha b,qb/\alpha,\alpha c,qc/\alpha;q)_\infty} \times \frac{-4\pi q^{1/2}d}{1-q}\;\frac{W\left(p_m\left(\eta;a,b,c;d\right),\;p_n\left(\eta;a,b,c;d\right)\right)}{\lambda_m-\lambda_n} \tag{4.18}$$

involving the Askey–Wilson polynomials. Here we use the notation

$$\eta = x(\alpha_0) = \frac{1}{2}\left(\alpha+\alpha^{-1}\right). \tag{4.19}$$

One can easily see that when $\alpha = d$ or $\alpha = q/d$ our equation (4.18) implies the orthogonality relation for the Askey–Wilson polynomials (1.7).

The new orthogonality property for the ${}_8\phi_7$-function appears when both of the parameters μ and ν are not nonnegative integers. It is convenient in this case to rewrite (4.17) in terms of the entire functions v's as follows

$$\int_0^\pi \frac{v_\mu(\cos\theta;a,b,c;d)\;v_\nu(\cos\theta;a,b,c;d)}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta},qe^{i\theta}/d,qe^{-i\theta}/d;q)_\infty} \times \frac{\left(e^{2i\theta},\;e^{-2i\theta};q\right)_\infty}{(\alpha e^{i\theta},\alpha e^{-i\theta},qe^{i\theta}/\alpha,qe^{-i\theta}/\alpha;q)_\infty}\,d\theta = \left(q,q,\alpha a,qa/\alpha,\alpha b,qb/\alpha,\alpha c,qc/\alpha,q\alpha/d,q^2/\alpha d;q\right)_\infty^{-1} \times \frac{-4\pi q^{1/2}d}{1-q}\;\frac{W\left(v_\mu\left(\eta;a,b,c;d\right),\;v_\nu\left(\eta;a,b,c;d\right)\right)}{\lambda_\mu-\lambda_\nu}. \tag{4.20}$$

The limiting case $\mu\to\nu$ of (4.20) is also of interest. From (4.4),

$$\lim_{\mu\to\nu}\frac{W\left(v_\mu(z),\;v_\nu(z)\right)}{\lambda_\mu-\lambda_\nu} = \lim_{\mu\to\nu}\frac{v_\mu(z)\,\dfrac{\nabla v_\nu(z)}{\nabla x(z)} - v_\nu(z)\,\dfrac{\nabla v_\mu(z)}{\nabla x(z)}}{\lambda_\mu-\lambda_\nu} = \lim_{\mu\to\nu}\frac{\dfrac{\partial v_\mu(z)}{\partial\mu}\,\dfrac{\nabla v_\nu(z)}{\nabla x(z)} - v_\nu(z)\,\dfrac{\partial}{\partial\mu}\left(\dfrac{\nabla v_\mu(z)}{\nabla x(z)}\right)}{\partial\lambda_\mu/\partial\mu} = \frac{\dfrac{\partial v_\nu(z)}{\partial\nu}\,\dfrac{\nabla v_\nu(z)}{\nabla x(z)} - v_\nu(z)\,\dfrac{\partial}{\partial\nu}\left(\dfrac{\nabla v_\nu(z)}{\nabla x(z)}\right)}{\partial\lambda_\nu/\partial\nu}.$$

Therefore,

$$\int_0^{\pi} \frac{\left(v_\nu(\cos\theta; a, b, c; d)\right)^2}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, qe^{i\theta}/d, qe^{-i\theta}/d; q)_\infty} \times \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha; q)_\infty}\, d\theta$$
$$= \left(q, q, \alpha a, qa/\alpha, \alpha b, qb/\alpha, \alpha c, qc/\alpha, q\alpha/d, q^2/\alpha d; q\right)_\infty^{-1} \times \frac{-4\pi q^{1/2} d}{1-q} \left[\frac{\partial}{\partial\lambda_\nu} v_\nu(\eta; a, b, c; d) \left(\frac{\nabla}{\nabla x} v_\nu(x; a, b, c; d)\right)\bigg|_{x=\eta} - v_\nu(\eta; a, b, c; d) \frac{\partial}{\partial\lambda_\nu} \left(\frac{\nabla}{\nabla x} v_\nu(x; a, b, c; d)\right)\bigg|_{x=\eta}\right]. \tag{4.21}$$

Finally, choosing the parameters μ and ν as ε-solutions of the equation

$$v_\varepsilon\left(\frac{1}{2}\left(\alpha + \alpha^{-1}\right); a, b, c; d\right) = 0, \tag{4.22}$$

we arrive from (4.20) and (4.21) at the *orthogonality relation* of the ${}_8\varphi_7$-functions under consideration,

$$\int_0^{\pi} \frac{v_\mu(\cos\theta; a, b, c; d)\, v_\nu(\cos\theta; a, b, c; d)}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, qe^{i\theta}/d, qe^{-i\theta}/d; q)_\infty} \times \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha; q)_\infty}\, d\theta = 0 \tag{4.23}$$

if $\mu \neq \nu$, and

$$\int_0^{\pi} \frac{\left(v_\nu(\cos\theta; a, b, c; d)\right)^2}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, qe^{i\theta}/d, qe^{-i\theta}/d; q)_\infty} \times \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha; q)_\infty}\, d\theta$$
$$= \left(q, q, \alpha a, qa/\alpha, \alpha b, qb/\alpha, \alpha c, qc/\alpha, q\alpha/d, q^2/\alpha d; q\right)_\infty^{-1} \times \frac{-4\pi q^{1/2} d}{1-q}\ \frac{\partial}{\partial\lambda_\nu} v_\nu(\eta; a, b, c; d) \left(\frac{\nabla}{\nabla x} v_\nu(x; a, b, c; d)\right)\bigg|_{x=\eta} \tag{4.24}$$

if $\mu = \nu$, respectively. We remind the reader that $\max(|a|, |b|, |c|, |q/d|) < 1$. This will be assumed throughout this work.

One can easily see that $v_\varepsilon(x; a, b, c; d)$ with $\varepsilon = \mu, \nu$ in (4.23)–(4.24) are real-valued functions of x for real ε which are orthogonal with respect to a positive weight function when all parameters a, b, c, d, and α are real, or when any two of a, b, c are complex conjugate and all other parameters are real. We shall usually assume that as well.

5. Properties of Zeros and Asymptotics

In the previous section we have proved that the ${}_8\varphi_7$-functions (2.7)–(2.8) are orthogonal if the "boundary" condition (4.22) is satisfied. Let us discuss now some properties of ν-zeros of the corresponding "boundary" function,

$$
\begin{aligned}
&v_\nu\left(\frac{1}{2}\left(\alpha+\alpha^{-1}\right); a, b, c; d\right)\\
&= \frac{(qa/d, bcq^\nu, \alpha q^{1-\nu}/d, q^{1-\nu}/\alpha d; q)_\infty}{(q^{1-\nu}a/d, bc; q)_\infty} \qquad (5.1)\\
&\quad\times\ {}_8\varphi_7\left(\begin{matrix} \dfrac{aq^{-\nu}}{d}, q\sqrt{\dfrac{aq^{-\nu}}{d}}, -q\sqrt{\dfrac{aq^{-\nu}}{d}}, q^{-\nu}, \dfrac{q^{1-\nu}}{bd}, \dfrac{q^{1-\nu}}{cd}, \alpha a, a/\alpha \\ \sqrt{\dfrac{aq^{-\nu}}{d}}, -\sqrt{\dfrac{aq^{-\nu}}{d}}, \dfrac{aq}{d}, ab, ac, \dfrac{q^{1-\nu}\alpha}{d}, \dfrac{q^{1-\nu}}{\alpha d}\end{matrix}\ ;\ q,\ bcq^\nu\right)\\
&= \frac{(q\alpha/d, q^{1-\nu}/\alpha d, abcq^\nu/\alpha; q)_\infty}{(abc/\alpha; q)_\infty} \qquad (5.2)\\
&\quad\times\ {}_8\varphi_7\left(\begin{matrix} \dfrac{abc}{q\alpha}, \sqrt{\dfrac{abc}{q^{1/2}\alpha}}, -\sqrt{\dfrac{abc}{q^{1/2}\alpha}}, q^{-\nu}, abcdq^{\nu-1}, \dfrac{a}{\alpha}, \dfrac{b}{\alpha}, \dfrac{c}{\alpha} \\ \sqrt{\dfrac{abc}{q\alpha}}, -\sqrt{\dfrac{abc}{q\alpha}}, \dfrac{q^\nu abc}{\alpha}, \dfrac{q^{1-\nu}}{\alpha d}, ab, ac, bc\end{matrix}\ ;\ q,\ q\,\dfrac{\alpha}{d}\right).
\end{aligned}
$$

We have used (III.23) of [12] to transform (5.1) to (5.2), compare also (2.10). Again, $v_\nu(\eta; a, b, c; d)$ is real-valued function of ν when all parameters a, b, c, d, and α are real, or when any two of a, b, c are complex conjugate and all other parameters are real.

Main properties of zeros of the function (5.1)–(5.2) can be investigated by using the same methods as in [11], [14], [16] and [17]. The first property is that under certain conditions the real-valued function $v_\nu(\eta; a, b, c; d)$ has an infinity of positive ν-zeros. In order to establish this fact, one can consider the large ν-asymptotics of the ${}_8\varphi_7$-function in (5.2),

$$
\begin{aligned}
&{}_8\varphi_7\left(\begin{matrix} \dfrac{abc}{q\alpha}, \sqrt{\dfrac{abc}{q^{1/2}\alpha}}, -\sqrt{\dfrac{abc}{q^{1/2}\alpha}}, q^{-\nu}, abcdq^{\nu-1}, \dfrac{a}{\alpha}, \dfrac{b}{\alpha}, \dfrac{c}{\alpha} \\ \sqrt{\dfrac{abc}{q\alpha}}, -\sqrt{\dfrac{abc}{q\alpha}}, \dfrac{q^\nu abc}{\alpha}, \dfrac{q^{1-\nu}}{\alpha d}, ab, ac, bc\end{matrix}\ ;\ q,\ q\,\dfrac{\alpha}{d}\right) \qquad (5.3)\\
&\xrightarrow[\nu\to\infty]{}\ {}_6\varphi_5\left(\begin{matrix} \dfrac{abc}{q\alpha}, \sqrt{\dfrac{abc}{q^{1/2}\alpha}}, -\sqrt{\dfrac{abc}{q^{1/2}\alpha}}, \dfrac{a}{\alpha}, \dfrac{b}{\alpha}, \dfrac{c}{\alpha} \\ \sqrt{\dfrac{abc}{q\alpha}}, -\sqrt{\dfrac{abc}{q\alpha}}, ab, ac, bc\end{matrix}\ ;\ q,\ \alpha^2\right)\\
&= \frac{(\alpha a, \alpha b, \alpha c, abc/\alpha; q)_\infty}{(ab, ac, bc, \alpha^2; q)_\infty}
\end{aligned}
$$

by (II.20) of [12]. Therefore,

$$v_\nu\left(\frac{1}{2}\left(\alpha+\alpha^{-1}\right);a,b,c;d\right)=\frac{(\alpha a,\alpha b,\alpha c,q\alpha/d;q)_\infty}{(ab,ac,bc,\alpha^2;q)_\infty}\times\left(q^{1-\nu}/\alpha d;q\right)_\infty[1+\mathrm{o}(1)], \tag{5.4}$$

as $\nu\to\infty$. But for the positive values of q and αd the function

$$\left(q^{1-\nu}/\alpha d;q\right)_\infty$$

oscillates and has an infinity of real zeros as ν approaches infinity (see [14] and [16]). Indeed, consider the points $\nu=\omega_n=\omega_0+n$, such that $q^{\omega_0}=\beta$, where $n=0,1,2,\dots$ and $q<\beta<1$, as test points. Then, by using (I.8) of [12],

$$v_{\omega_0+n}(\eta;a,b,c;d)=\frac{(\alpha a,\alpha b,\alpha c,q\alpha/d;q)_\infty}{(ab,ac,bc,\alpha^2;q)_\infty}\times(-1)^n q^{-n(n+1)/2}\frac{(\alpha\beta dq;q)_n}{(\alpha\beta d)^n}[1+\mathrm{o}(1)], \tag{5.5}$$

as $n\to\infty$, and one can see that the right side of (5.5) changes sign infinitely many times at the test points $\nu=\omega_n$ as ν approaches infinity.

Thus we have established the following theorem.

Theorem 5.1. *The real-valued function $v_\nu(\eta;a,b,c;d)$ defined by (5.2) has an infinity of positive ν-zeros when $\alpha q<d$, $\alpha^2<1$, $\alpha d>0$, and $0<q<1$. Also, all parameters a, b, c are real, or when any of two of a, b, c are complex conjugate and the remainder parameter is real.*

Now we can prove the next result.

Theorem 5.2. *Function $v_\nu(\eta;a,b,c;d)$ defined in (5.2) has only real ν-zeros under the following conditions:*

1. *parameters α and d are real; parameters a, b, c are all real, or, if any two of them are complex conjugate, the third parameter is real;*
2. *the following inequalities holds:*
   (a) $max(|a|,|b|,|c|,|q/d|)<1,\quad q^{1/2}<\alpha<1;$
   (b) $\alpha d>q,\quad \alpha abc<q.$

Proof. Suppose that ν_0 is a zero of function (5.2) which is not real. Let ν_1 be the complex number conjugate to ν_0, so that ν_1 is also a zero of (5.2) because this function is real under the hypotheses of the theorem.

Consider equation (4.20) with $\mu=\nu_0$ and $\nu=\nu_1$. The integral on the left does not equal zero due to the positivity of the integrand, but the analog of the Wronskian on the right is zero in view of (4.22). Therefore,

$$\lambda_{\nu_0}=\lambda_{\nu_1}, \tag{5.6}$$

the eigenvalues defined by (2.4) are real. The last equation can be rewritten as

$$\left(1-q^{\nu_1-\nu_0}\right)\left(1-abcdq^{\nu_0+\nu_1-1}\right)=0. \tag{5.7}$$

The first solution is $\nu_0 = \nu_1$, so the roots are real in this case.

The second solution of (5.7) is

$$q^{\nu_0} = \sqrt{\frac{q}{abcd}}\, e^{i\chi}, \qquad q^{\nu_1} = \sqrt{\frac{q}{abcd}}\, e^{-i\chi}, \tag{5.8}$$

where $abc > 0$ and χ is an arbitrary real number. But in this case our function can be represented as a multiple of a positive function. Indeed, when $\nu = \nu_0$ or $\nu = \nu_1$,

$$\begin{aligned} & v_\nu(\eta; a, b, c; d) \qquad\qquad (5.9) \\ &= \frac{(q/\alpha d, q^{1-\nu}\alpha/d, \alpha abcq^\nu; q)_\infty}{(\alpha abc; q)_\infty} \\ &\quad \times {}_8\varphi_7 \left(\begin{matrix} \dfrac{\alpha abc}{q}, \sqrt{\dfrac{\alpha abc}{q^{1/2}}}, -\sqrt{\dfrac{\alpha abc}{q^{1/2}}}, q^{-\nu}, abcdq^{\nu-1}, \alpha a, \alpha b, \alpha c \\ \sqrt{\dfrac{\alpha abc}{q}}, -\sqrt{\dfrac{\alpha abc}{q}}, q^\nu \alpha abc, q^{1-\nu}\alpha/d, ab, ac, bc \end{matrix} ; q, \frac{q}{\alpha d} \right) \\ &= \frac{(q/\alpha d, q^{1-\nu}\alpha/d, \alpha abcq^\nu; q)_\infty}{(\alpha abc; q)_\infty} \\ &\quad \times \left[1 + \sum_{n=1}^\infty \frac{(1-\alpha abcq^{2n-1})\,(\alpha abcq^{-1}, \alpha a, \alpha b, \alpha c; q)_n}{(1-\alpha abcq^{-1})\,(q, ab, ac, bc; q)_n} \right. \\ &\qquad\qquad \left. \times \frac{(q^{-\nu}, abcdq^{\nu-1}; q)_n}{(q^\nu \alpha abc, q^{1-\nu}\alpha/d; q)_n} \left(\frac{q}{\alpha d}\right)^n \right] \end{aligned}$$

by (5.2), (III.23) and (I.31) of [12]. One can easily see that under the hypotheses of the theorem all products in the last but one line of (5.9) are positive. From (1.3)–(1.4) and (5.8),

$$\begin{aligned} \left(q^{-\nu}, abcdq^{\nu-1}; q\right)_n &= \prod_{k=0}^{n-1} \left(1 - 2\sqrt{\frac{abcd}{q}} \cos\chi\; q^k + \frac{abcd}{q}\, q^{2k}\right) \\ &= \prod_{k=0}^{n-1} \left(\left(1 - \sqrt{\frac{abcd}{q}} \cos\chi\; q^k\right)^2 + \frac{abcd}{q} \sin^2\chi\; q^{2k} \right) \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} \left(q^\nu \alpha abc, q^{1-\nu}\alpha/d; q\right)_n &= \prod_{k=0}^{n-1} \left(1 - 2\alpha\sqrt{\frac{abcq}{d}} \cos\chi\; q^k + \alpha^2 \frac{abcq}{d}\, q^{2k}\right) \\ &= \prod_{k=0}^{n-1} \left(\left(1 - \alpha\sqrt{\frac{abcq}{d}} \cos\chi\; q^k\right)^2 + \alpha^2 \frac{abcq}{d} \sin^2\chi\; q^{2k} \right), \end{aligned} \tag{5.11}$$

where $n = 1, 2, 3, \ldots, \infty$. Thus, these products are positive too, which proves the positivity of the "boundary" function in (5.9). So we have obtained a contradiction and the complex zeros (5.8) cannot exist. This completes the proof of the theorem. □

Theorem 5.3. *The real ν-zeros of $v_\nu(\eta; a, b, c; d)$ are simple under the hypotheses 1 and 2(a) of Theorem 5.2.*

Proof. This follows directly from the relation (4.21). Indeed, the integral on the left side is positive under the conditions of the theorem, which means that $\dfrac{\partial}{\partial\nu}\, v_\nu(\eta; a, b, c; d) \neq 0$ when $v_\nu(\eta; a, b, c; d) = 0$. □

Let us consider two functions,

$$f(\nu) = v_\nu(\eta; a, b, c; d), \tag{5.12}$$

$$g(\nu) = \frac{\nabla}{\nabla x}\, v_\nu(x; a, b, c; d)\Big|_{x=\eta}. \tag{5.13}$$

Our next property is that positive zeros of $f(\nu)$ and $g(\nu)$ are interlaced.

Theorem 5.4. *If $\nu_1, \nu_2, \nu_3, \ldots$ are the positive zeros of $f(\nu)$ arranged in ascending order of magnitude, and $\mu_1, \mu_2, \mu_3, \ldots$ are those of $g(\nu)$, then*

$$0 < \mu_1 < \nu_1 < \mu_2 < \nu_2 < \mu_3 < \nu_3 < \ldots\,, \tag{5.14}$$

if the conditions of Theorem 5.3 are satisfied.

Proof. Suppose that ν_k and ν_{k+1} are two successive zeros of $f(\nu)$. Then the derivative $\dfrac{\partial}{\partial\nu} f(\nu)$ has different signs at $\nu = \nu_k$ and $\nu = \nu_{k+1}$. This means, in view of the positivity of the integral on the left side of (4.21), that $g(\nu)$ changes its sign between ν_k and ν_{k+1} and, therefore, has at least one zero on each interval (ν_k, ν_{k+1}).

To complete the proof of the theorem, we have to show that $g(\nu)$ changes its sign on each interval (ν_k, ν_{k+1}) only once. Suppose that $g(\mu_k) = g(\mu_{k+1}) = 0$ and $\nu_k < \mu_k < \mu_{k+1} < \nu_{k+1}$. Then, by (4.21), function $f(\nu)$ has different signs at $\nu = \mu_k$ and $\nu = \mu_{k+1}$ and, therefore, this function has at least one more zero on (ν_k, ν_{k+1}). So, we have obtained a contradiction, and, therefore, function $g(\nu)$ has exactly one zero between any two successive zeros of $f(\nu)$.

In order to finish the proof of our theorem one has to show that $\mu_1 < \nu_1$. We would like to leave the proof of this fact as a congecture for the reader. □

Let us discuss an asymptotic behavior of $u_\nu(x; a, b, c; d)$ for large values of ν applying the methods used in [21], [26], and [12] at the level of the Askey–Wilson polynomials. In a similar manner, the ${}_8\varphi_7$-function in (2.10) can be transformed by (III.37) of [12],

$$\begin{aligned}
&{}_8W_7\left(abcq^{z-1}; q^{-\nu}, abcdq^{\nu-1}, aq^{z}, bq^{z}, cq^{z}; q,\ \frac{q^{1-z}}{d}\right) \\
&= \frac{(aq^{-z}, bq^{-z}, cq^{-z}, bq^{1+\nu+z}, cq^{1+\nu+z}, dq^{\nu-z}, abcq^{z}, bcdq^{\nu+z}; q)_\infty}{(q^{-2z}, ab, ac, bc, q^{1+\nu}, bdq^{\nu}, cdq^{\nu}, bcq^{1+\nu+2z}; q)_\infty} \\
&\quad \times\ {}_8W_7\left(bcq^{\nu+2z}; bcq^{\nu}, bq^{z}, cq^{z}, q^{1+z}/a, q^{1+z}/d; q,\ adq^{\nu}\right) \\
&+ \frac{(aq^{z}, bq^{z}, cq^{z}, bq^{1+\nu-z}, cq^{1+\nu-z}, dq^{\nu+z}, abcq^{z}, bcdq^{\nu-z}; q)_\infty}{(q^{2z}, ab, ac, bc, q^{1+\nu}, bdq^{\nu}, cdq^{\nu}, bcq^{1+\nu-2z}; q)_\infty} \\
&\quad \times \frac{(q^{1-\nu-z}/d, q^{1+z}/d, abcq^{\nu-z}; q)_\infty}{(q^{1-z}/d, q^{1-\nu+z}/d, abcq^{\nu+z}; q)_\infty} \\
&\qquad \times\ {}_8W_7\left(bcq^{\nu-2z}; bcq^{\nu}, bq^{-z}, cq^{-z}, q^{1-z}/a, q^{1-z}/d; q,\ adq^{\nu}\right).
\end{aligned} \tag{5.15}$$

Therefore,

$$\begin{aligned} & u_\nu(x;a,b,c;d) \qquad (5.16)\\ &= \frac{(bq^{1+\nu+z}, cq^{1+\nu+z}, dq^{\nu-z}, abcq^{\nu+z}, bcdq^{\nu+z};q)_\infty}{(q^{1+\nu}, bdq^\nu, cdq^\nu, bcq^{1+\nu+2z};q)_\infty}\\ &\quad\times \frac{(aq^{-z}, bq^{-z}, cq^{-z}, q^{1-\nu+z}/d;q)_\infty}{(ab,ac,bc,q^{-2z},q^{1+z}/d;q)_\infty}\\ &\quad\times\ {}_8W_7\left(bcq^{\nu+2z}; bcq^\nu, bq^z, cq^z, q^{1+z}/a, q^{1+z}/d; q,\ adq^\nu\right)\\ &+ \frac{(bq^{1+\nu-z}, cq^{1+\nu-z}, dq^{\nu+z}, abcq^{\nu-z}, bcdq^{\nu-z};q)_\infty}{(q^{1+\nu}, bdq^\nu, cdq^\nu, bcq^{1+\nu-2z};q)_\infty}\\ &\quad\times \frac{(aq^{z}, bq^{z}, cq^{z}, q^{1-\nu-z}/d;q)_\infty}{(ab,ac,bc,q^{2z},q^{1-z}/d;q)_\infty}\\ &\quad\times\ {}_8W_7\left(bcq^{\nu-2z}; bcq^\nu, bq^{-z}, cq^{-z}, q^{1-z}/a, q^{1-z}/d; q,\ adq^\nu\right). \end{aligned}$$

We shall use the last expression to determine the large ν asymptotic of our orthogonal ${}_8\varphi_7$-function later.

For the "boundary" function (5.2) equation (5.16) takes the form

$$\begin{aligned} & v_\nu\left(\frac{1}{2}\left(\alpha+\alpha^{-1}\right); a,b,c;d\right) \qquad (5.17)\\ &= \frac{(\alpha bq^{1+\nu}, \alpha cq^{1+\nu}, dq^\nu/\alpha, \alpha abcq^\nu, \alpha bcdq^\nu;q)_\infty}{(q^{1+\nu}, bdq^\nu, cdq^\nu, \alpha^2 bcq^{1+\nu};q)_\infty}\\ &\quad\times \frac{(a/\alpha, b/\alpha, c/\alpha, q/\alpha d, \alpha q^{1-\nu}/d;q)_\infty}{(ab,ac,bc,\alpha^{-2};q)_\infty}\\ &\quad\times\ {}_8W_7\left(\alpha^2 bcq^\nu; bcq^\nu, \alpha b, \alpha c, q\alpha/a, q\alpha/d; q,\ adq^\nu\right)\\ &+ \frac{(bq^{1+\nu}/\alpha, cq^{1+\nu}/\alpha, \alpha dq^\nu, abcq^\nu/\alpha, bcdq^\nu/\alpha;q)_\infty}{(q^{1+\nu}, bdq^\nu, cdq^\nu, bcq^{1+\nu}/\alpha^2;q)_\infty}\\ &\quad\times \frac{(\alpha a, \alpha b, \alpha c, q\alpha/d, q^{1-\nu}/\alpha d;q)_\infty}{(ab,ac,bc,\alpha^2;q)_\infty}\\ &\quad\times\ {}_8W_7\left(bcq^\nu/\alpha^2; bcq^\nu, b/\alpha, c/\alpha, q/\alpha a, q/\alpha d; q,\ adq^\nu\right). \end{aligned}$$

The last term dominates here as ν approaches infinity (cf. (5.4)).

The proof of Theorem 5.1 has strongly indicated that asymptotically the large positive ν-zeros of $v_\nu(\eta;a,b,c;d)$ are

$$\nu_n = n + \epsilon_n, \qquad 0 \le \epsilon_n < 1 \qquad (5.18)$$

as $n\to\infty$. The same consideration as in [14], [17], and [11] shows that this function changes sign only once between any two successive test points $\nu=\omega_n$ and $\nu=\omega_{n+1}$ defined on the page 15 for sufficiently large values of n. We include details of this proof in Section 9 to make this work as self-contained as possible.

Our next theorem provides a more accurate estimate for the distribution of the large positive zeros of the "boundary" function (5.2).

Theorem 5.5. *If $\nu_1, \nu_2, \nu_3, \ldots$ are the positive zeros of $v_\nu(\eta; a, b, c; d)$ arranged in ascending order of magnitude, then*

$$\nu_n = n - \frac{\log(\alpha d)}{\log q} + o(1), \tag{5.19}$$

as $n \to \infty$.

Proof. All zeros of our function (5.2) coincide with the zeros of a new function,

$$w_\nu(\eta; a, b, c; d) := \left(-q^{1-\nu}/\alpha d; q\right)_\infty^{-1} \ v_\nu(\eta; a, b, c; d), \tag{5.20}$$

because $(-q^{1-\nu}/\alpha d; q)_\infty$ is positive for real ν. Equations (5.17) and (5.20) give us the large ν-asymptotic of $w_\nu(\eta; a, b, c; d)$. When $\nu = \gamma_n$ such that $q^{\gamma_n} = q^n/\alpha d$ and $n = 1, 2, 3, \ldots$, the second term in (5.17) vanishes and we get

$$\begin{aligned} w_{\gamma_n}(\eta; a, b, c; d) = & \frac{(q\alpha^2, a/\alpha, bc, qb/d, qc/d, abc/d; q)_\infty}{(-q, ab, ac, bc, \alpha bcq/d; q)_\infty} \\ & \times \frac{(q/\alpha d, b/\alpha, c/\alpha, \alpha bcq/d; q)_n}{(-1, bc, b/d, qc/d, abc/d; q)_n} \left(-\alpha^2\right)^n \\ & \times {}_8W_7\left(\frac{\alpha bc}{d}\, q^n; \frac{b}{\alpha}\, q^n, \alpha b, \alpha c, q\,\frac{\alpha}{a}, q\,\frac{\alpha}{d}; q,\ \frac{a}{\alpha}\, q^n\right) \end{aligned} \tag{5.21}$$

with the help of (I.9) of [12]. Thus,

$$\lim_{n\to\infty} w_{\gamma_n}(\eta; a, b, c; d) = 0, \tag{5.22}$$

which proves our theorem. $\square$

Let us discuss also the large positive ν-asymptotic of $u_\nu(x; a, b, c; d)$ when $x = \cos\theta$ belongs to the interval of orthogonality $-1 < x < 1$. It is clear from (5.16) that the leading terms in the asymptotic expansion of this function are given by

$$\begin{aligned} u_\nu(\cos\theta; a, b, c; d) \sim & \frac{\left(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, q^{1-\nu}e^{-i\theta}/d; q\right)_\infty}{(ab, ac, bc, e^{2i\theta}, qe^{-i\theta}/d; q)_\infty} \\ & + \frac{\left(ae^{-i\theta}, be^{-i\theta}, ce^{-i\theta}, q^{1-\nu}e^{i\theta}/d; q\right)_\infty}{(ab, ac, bc, e^{-2i\theta}, qe^{i\theta}/d; q)_\infty}. \end{aligned} \tag{5.23}$$

In particular, when $\nu = \nu_n$ are large zeros of $u_\nu(\eta; a, b, c; d)$, we can estimate

$$u_{\nu_n}(\cos\theta; a, b, c; d) \sim u_{\gamma_n}(\cos\theta; a, b, c; d), \tag{5.24}$$

where $q^{\gamma_n} = q^n/\alpha d$, due to (5.19) as $n \to \infty$. Relations (5.23)–(5.24) lead to the following theorem.

Theorem 5.6. *For $-1 < x = \cos\theta < 1$ and $|q| < 1$ the leading term in the asymptotic expansion of $u_{\gamma_n}(\cos\theta; a, b, c; d)$ as $n \to \infty$ is given by*

$$\begin{aligned} u_{\gamma_n}(\cos\theta; a, b, c; d) \sim & \frac{q^{-(n-1)(n-2)/2}}{(ab, ac, bc; q)_\infty} \\ & \times \left|A\left(e^{i\theta}\right)\right| \ \cos\left((n-1)\theta - \chi\right), \end{aligned} \tag{5.25}$$

where

$$A\left(e^{i\theta}\right)=\frac{\left(ae^{i\theta},be^{i\theta},ce^{i\theta},\alpha e^{-i\theta},qe^{i\theta}/\alpha;q\right)_\infty}{\left(e^{2i\theta},qe^{-i\theta}/d;q\right)_\infty}, \tag{5.26}$$

$$\begin{aligned}\left|A\left(e^{i\theta}\right)\right|^{-2}=&\frac{\left(e^{2i\theta},\ e^{-2i\theta};q\right)_\infty}{\left(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta};q\right)_\infty}\\ &\times\frac{\left(qe^{i\theta}/d,\ qe^{-i\theta}/d;q\right)_\infty}{\left(\alpha e^{i\theta},\alpha e^{-i\theta},qe^{i\theta}/\alpha,qe^{-i\theta}/\alpha;q\right)_\infty},\end{aligned} \tag{5.27}$$

and

$$\chi=\arg A\left(e^{i\theta}\right). \tag{5.28}$$

It is worth mentioning that (5.27) coinsides with the weight function in the orthogonality relation for $u_\nu(\cos\theta;a,b,c;d)$.

In a similar fashion, one can consider some other properties of zeros of the ${}_8\varphi_7$-function (5.1)–(5.2) close to those established in [11], [16] and [17] at the level of the basic trigonometric functions and the q-Bessel function, respectively.

6. Evaluation of Some Constants

In this section we shall find an explicit expression for the squared norm,

$$\begin{aligned}d_\nu^2=\int_0^\pi&\frac{u_\nu^2(\cos\theta;a,b,c;d)}{\left(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta};q\right)_\infty}\\ &\times\frac{\left(e^{2i\theta},\ e^{-2i\theta},qe^{i\theta}/d,qe^{-i\theta}/d;q\right)_\infty}{\left(\alpha e^{i\theta},\alpha e^{-i\theta},qe^{i\theta}/\alpha,qe^{-i\theta}/\alpha;q\right)_\infty}\,d\theta,\end{aligned} \tag{6.1}$$

on the right side of (4.21). It is more convinient to write the u_ν's instead of v_ν's here, cf. (2.14), (4.21), and (6.1). Using (2.8) with a and c interchanged we get

$$\begin{aligned}d_\nu^2=&\frac{\left(abq^\nu,q^{1-\nu}/cd;q\right)_\infty}{\left(ab,q/cd;q\right)_\infty}\\ &\times\sum_{n=0}^\infty q^n\,\frac{\left(q^{-\nu},abcdq^{\nu-1};q\right)_n}{\left(q,ac,bc,cd;q\right)_n}\\ &\times\int_0^\pi\frac{u_\nu(\cos\theta;a,b,c;d)}{\left(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},cq^ne^{i\theta},cq^ne^{-i\theta};q\right)_\infty}\\ &\qquad\times\frac{\left(e^{2i\theta},\ e^{-2i\theta},qe^{i\theta}/d,qe^{-i\theta}/d;q\right)_\infty}{\left(\alpha e^{i\theta},\alpha e^{-i\theta},qe^{i\theta}/\alpha,qe^{-i\theta}/\alpha;q\right)_\infty}\,d\theta\\ &+\frac{\left(q^{-\nu},abcdq^{\nu-1},qa/d,qb/d;q\right)_\infty}{\left(ab,ac,bc,cd/q;q\right)_\infty}\\ &\times\sum_{n=0}^\infty q^n\,\frac{\left(abq^\nu,q^{1-\nu}/cd;q\right)_n}{\left(q,qa/d,qb/d,q^2/cd;q\right)_n}\end{aligned} \tag{6.2}$$

$$\times \int_0^\pi \frac{u_\nu(\cos\theta; a,b,c;d)}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, q^{1+n}e^{i\theta}/d, q^{1+n}e^{-i\theta}/d;q)_\infty} \times \frac{\left(e^{2i\theta},\ e^{-2i\theta}, qe^{i\theta}/d, qe^{-i\theta}/d;q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha;q)_\infty}\, d\theta.$$

Both integrals on the right side have the same structure and can be evaluated in a similar way. Let

$$\begin{aligned} I_n(\gamma) := &\int_0^\pi \frac{u_\nu(\cos\theta; a,b,c;d)}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, \gamma q^n e^{i\theta}, \gamma q^n e^{-i\theta};q)_\infty} \\ &\quad\times \frac{\left(e^{2i\theta},\ e^{-2i\theta}, qe^{i\theta}/d, qe^{-i\theta}/d;q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha;q)_\infty}\, d\theta \\ = &\frac{(qa/d, bcq^\nu;q)_\infty}{(q^{1-\nu}a/d, bc;q)_\infty} \\ &\times \int_0^\pi {}_8W_7\left(\frac{a}{d}q^{-\nu}; q^{-\nu}, q^{1-\nu}/bd, q^{1-\nu}/cd, ae^{i\theta}, ae^{-i\theta}; q,\ bcq^\nu\right) \\ &\quad\times \frac{\left(e^{2i\theta},\ e^{-2i\theta};q\right)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, \gamma q^n e^{i\theta}, \gamma q^n e^{-i\theta};q)_\infty} \\ &\quad\times \frac{\left(q^{1-\nu}e^{i\theta}/d,\ q^{1-\nu}e^{-i\theta}/d;q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha;q)_\infty}\, d\theta \end{aligned} \tag{6.3}$$

by (2.7) with $\gamma = c$ and $\gamma = q/d$. The last integral can be evaluated in terms of two balanced ${}_5\varphi_4$'s by (6.4) of [19],

$$\begin{aligned} &\int_0^\pi {}_8W_7\left(dg/q; h, r, g/f, de^{i\theta}, de^{-i\theta}; q, gf/hr\right) \\ &\quad\times \frac{\left(e^{2i\theta}, e^{-2i\theta};q\right)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta};q)_\infty} \times \frac{\left(ge^{i\theta}, ge^{-i\theta};q\right)_\infty}{(fe^{i\theta}, fe^{-i\theta};q)_\infty}\, d\theta \\ &= \frac{2\pi\,(abcd, dg, g/d;q)_\infty}{(q, ab, ac, ad, bc, bd, cd, df, f/d;q)_\infty} \\ &\quad\times {}_5\phi_4\left(\begin{matrix} ad, bd, cd, g/f, dg/hr \\ abcd, qd/f, dg/h, dg/r \end{matrix}; q,\ q\right) \\ &+ \frac{2\pi\,(abcf, dg, g/f;q)_\infty}{(q, ab, ac, af, bc, bf, cf, df, d/f;q)_\infty} \\ &\quad\times \frac{(fg/h, fg/r, dg/hr;q)_\infty}{(dg/h, dg/r, fg/hr;q)_\infty} \\ &\quad\times {}_5\phi_4\left(\begin{matrix} af, bf, cf, g/d, gf/hr \\ abcf, qf/d, gf/h, gf/r \end{matrix}; q,\ q\right), \end{aligned} \tag{6.4}$$

this formula appears in a straightforward manner if one uses Bailey's transform (III.36) of [12] and the Askey–Wilson integral. Thus,

$$\begin{aligned} I_n(\gamma) = & \frac{2\pi\,(qa/d, q^{1-\nu}/ad, bcq^{\nu}, a\gamma q; q)_\infty}{(q, q, ab, bc, b/a, \alpha a, qa/\alpha, \alpha\gamma, q\gamma/\alpha, a\gamma; q)_\infty} \\ & \times \frac{(\alpha\gamma, q\gamma/\alpha, a\gamma; q)_n}{(a\gamma q; q)_n} \\ & \times {}_5\phi_4\left(\begin{matrix} a\gamma q^n, acq^{\nu}, q^{1-\nu}/bd, \alpha a, qa/\alpha \\ a\gamma q^{1+n}, ac, qa/b, qa/d \end{matrix} ; q, q \right) \\ & + \frac{2\pi\,(qb/d, q^{1-\nu}/bd, acq^{\nu}, b\gamma q; q)_\infty}{(q, q, ab, ac, a/b, \alpha b, qb/\alpha, \alpha\gamma, q\gamma/\alpha, b\gamma; q)_\infty} \\ & \times \frac{(\alpha\gamma, q\gamma/\alpha, b\gamma; q)_n}{(b\gamma q; q)_n} \\ & \times {}_5\phi_4\left(\begin{matrix} b\gamma q^n, bcq^{\nu}, q^{1-\nu}/ad, \alpha b, qb/\alpha \\ b\gamma q^{1+n}, bc, qb/a, qb/d \end{matrix} ; q, q \right), \end{aligned} \tag{6.5}$$

where $\gamma = c$ and $\gamma = q/d$. One can see that the second term here equals the first one with a and b interchanged.

Combining (6.2), (6.3), and (6.5), we obtain

$$\begin{aligned} d_\nu^2 = & \int_0^\pi \frac{v_\nu^2(\cos\theta; a, b, c; d)}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, qe^{i\theta}/d, qe^{-i\theta}/d; q)_\infty} \\ & \times \frac{\left(e^{2i\theta}, e^{-2i\theta}; q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha; q)_\infty}\, d\theta \\ = & \frac{2\pi}{(1-ac)(q, ab; q)^2} \frac{(qa/d, abq^{\nu}, bcq^{\nu}, q^{1-\nu}/ad, q^{1-\nu}/cd; q)_\infty}{(bc, b/a, q/cd, \alpha a, qa/\alpha, \alpha c, qc/\alpha; q)_\infty} \\ & \times \sum_{n=0}^{\infty} q^n \frac{(q^{-\nu}, abcdq^{\nu-1}, \alpha c, qc/\alpha; q)_n}{(q, \alpha cq, bc, cd; q)_n} \\ & \times {}_5\phi_4\left(\begin{matrix} acq^n, acq^{\nu}, q^{1-\nu}/bd, \alpha a, qa/\alpha \\ acq^{n+1}, ac, qa/b, qa/d \end{matrix} ; q, q \right) \\ & + \frac{2\pi}{(1-bc)(q, ab; q)^2} \frac{(qb/d, abq^{\nu}, acq^{\nu}, q^{1-\nu}/bd, q^{1-\nu}/cd; q)_\infty}{(ac, a/b, q/cd, \alpha b, qb/\alpha, \alpha c, qc/\alpha; q)_\infty} \\ & \times \sum_{n=0}^{\infty} q^n \frac{(q^{-\nu}, abcdq^{\nu-1}, \alpha c, qc/\alpha; q)_n}{(q, q\alpha c, ac, cd; q)_n} \\ & \times {}_5\phi_4\left(\begin{matrix} bcq^n, bcq^{\nu}, q^{1-\nu}/ad, \alpha b, qb/\alpha \\ bcq^{n+1}, bc, qb/a, qb/d \end{matrix} ; q, q \right) \\ & + \frac{2\pi\,(qa/d; q)_\infty^2}{(1-qa/d)(q, ab, bc; q)^2} \frac{(q^{-\nu}, abcdq^{\nu-1}, q^{1-\nu}/ad, bcq^{\nu}, qb/d; q)_\infty}{(ac, b/a, cd/q, \alpha a, qa/\alpha, q\alpha/d, q^2/\alpha d; q)_\infty} \end{aligned} \tag{6.6}$$

$$\times \sum_{n=0}^{\infty} q^n \frac{(abq^\nu, q^{1-\nu}/cd, q\alpha/d, q^2/\alpha d; q)_n}{(q, qb/d, q^2a/d, q^2/cd; q)_n}$$
$$\times \; {}_5\phi_4 \left(\begin{matrix} aq^{n+1}/d, acq^\nu, q^{1-\nu}/bd, \alpha a, qa/\alpha \\ aq^{n+2}/d, ac, qa/b, qa/d \end{matrix} ; q, q \right)$$
$$+ \frac{2\pi\,(qb/d;q)_\infty^2}{(1-qb/d)(q,ab,ac;q)^2} \frac{(q^{-\nu}, abcdq^{\nu-1}, q^{1-\nu}/bd, acq^\nu, qa/d; q)_\infty}{(bc, a/b, cd/q, \alpha b, qb/\alpha, q\alpha/d, q^2/\alpha d; q)_\infty}$$
$$\times \sum_{n=0}^{\infty} q^n \frac{(abq^\nu, q^{1-\nu}/cd, q\alpha/d, q^2/\alpha d; q)_n}{(q, qa/d, q^2b/d, q^2/cd; q)_n}$$
$$\times \; {}_5\phi_4 \left(\begin{matrix} bq^{n+1}/d, bcq^\nu, q^{1-\nu}/ad, \alpha b, qb/\alpha \\ bq^{n+2}/d, bc, qb/a, qb/d \end{matrix} ; q, q \right).$$

One can see again that the second and the forth terms in this formula are equal to the first and the third ones, respectively, with a and b interchanged. When ν satisfies the boundary condition (4.22) the last integral gives the values of the normalization constants in the orthogonality relation (4.23)–(4.24).

7. Some Identity

In this section we shall derive an interesting relation involving a determinant of four ${}_8\varphi_7$-functions of type (2.7). Let

$$u(z) = u_\nu(x(z); a, b, c; d), \tag{7.1}$$
$$v(z) = u_\nu(x(z); a, b, d; c) = u(z)|_{c\leftrightarrow d} \tag{7.2}$$

be two solutions of equation (2.1) corresponding to the same eigenvalue (2.4). Then, due to (4.3),

$$\Delta\left[\sigma(z)\,\rho(z)\,W\left(u(z),\ v(z)\right)\right] = 0, \tag{7.3}$$

where $W(u,v)$ is the analog of the Wronskian defined in (4.4) and $\rho(z)$ is the appropriate solution of the Pearson equation (3.7). Using the difference-differentiation formula (2.15) we can rewrite the "Wronskian" as

$$\begin{aligned} W\left(u(z),\ v(z)\right) = & \frac{2q}{(1-q)c} \frac{(1-q^{-\nu})(1-abcdq^{\nu-1})}{(1-ab)(1-ad)(1-bd)} \\ & \times u_{\nu-1}\left(x(z-1/2); aq^{1/2}, bq^{1/2}, dq^{1/2}; cq^{1/2}\right) \\ & \times u_\nu\left(x(z); a, b, c; d\right) \\ & - \frac{2q}{(1-q)d} \frac{(1-q^{-\nu})(1-abcdq^{\nu-1})}{(1-ab)(1-ac)(1-bc)} \\ & \times u_{\nu-1}\left(x(z-1/2); aq^{1/2}, bq^{1/2}, cq^{1/2}; dq^{1/2}\right) \\ & \times u_\nu\left(x(z); a, b, d; c\right). \end{aligned} \tag{7.4}$$

One can easily see that the function

$$g(z) = \sigma(z)\,\rho(z)\,W\left(u(z),\ v(z)\right) \tag{7.5}$$

$$= cd\,\frac{(q^z/c, q^{1-z}/c, q^z/d, q^{1-z}/d; q)_\infty}{(aq^z, aq^{1-z}, bq^z, bq^{1-z}; q)_\infty} \times W\left(u(z),\ v(z)\right)$$

in (7.3) is doubly periodic function in z without poles in the rectangle on the Figure. Therefore, this function is just a constant by Liouville's theorem,

$$\sigma(z)\,\rho(z)\,W\left(u(z),\ v(z)\right) = C. \tag{7.6}$$

To find the value of this constant we can choose here $z = z_0$ such that $q^{z_0} = a$. From (7.4) and (2.7) one gets

$$\begin{aligned} &W\left(u(z_0),\ v(z_0)\right) \\ &= \frac{2q\,(1-q^{-\nu})\,(1-abcdq^{\nu-1})}{(1-q)} \\ &\times \left[\frac{(q, qbc, qb/d, qc/d, bdq^\nu, q^{1-\nu}/ac, q^{1-\nu}a/d, a^2bcq^\nu; q)_\infty}{acd\,(ab, qac, bc, bd, 1/ac, q/ad, a/d, qabc/d; q)_\infty} \right. \\ &\qquad \times\ {}_8W_7\left(abc/d; ab, ac, a/d, q^{1-\nu}/ad, bcq^\nu; q,\ q\right) \\ &\quad - \frac{(q, qbd, qb/c, qd/c, bcq^\nu, q^{1-\nu}/ad, q^{1-\nu}a/c, a^2bdq^\nu; q)_\infty}{acd\,(ab, qad, bd, bc, 1/ad, q/ac, a/c, qabd/c; q)_\infty} \\ &\qquad \left. \times\ {}_8W_7\left(abd/c; ab, ad, a/c, q^{1-\nu}/ac, bdq^\nu; q,\ q\right) \right] \\ &\quad = \frac{2q\,(q, a^2, qb/a, d/c, qc/d, q^{-\nu}, abcdq^{\nu-1}, abq^\nu, q^{1-\nu}/cd; q)_\infty}{(1-q)acd\,(ab, qac, ad, bc, bd, 1/ac, q/ad, a/c, a/d; q)_\infty} \end{aligned} \tag{7.7}$$

by (III.24) and (II.25) of [12].

As a result, from (7.6) and (7.7) we find the value of the "Wronskian" of the ${}_8\varphi_7$-functions (7.1) and (7.2),

$$\begin{aligned} W\left(u(z),\ v(z)\right) = &\frac{2q\,(c/d, qd/c, q^{-\nu}, abcdq^{\nu-1}, abq^\nu, q^{1-\nu}/cd; q)_\infty}{(1-q)c\,(ab, ab, ac, ad, bc, bd; q)_\infty} \\ &\times \frac{(aq^z, aq^{1-z}, bq^z, bq^{1-z}; q)_\infty}{(q^z/c, q^{1-z}/c, q^z/d, q^{1-z}/d; q)_\infty}. \end{aligned} \tag{7.8}$$

Due to (7.4) and (2.7) the last equation can also be rewritten in a more explicit form,

$$\begin{aligned} &\frac{d\,(ac, adq; q)_\infty}{(q^{2-\nu}a/c, q^{1-\nu}a/d; q)_\infty} \\ &\times \frac{(q^{1-\nu+z}/c, q^{2-\nu-z}/c, q^{1-\nu+z}/d, q^{1-\nu-z}/d; q)_\infty}{(q^z/c, q^{1-z}/c, q^{1+z}/d, q^{1-z}/d; q)_\infty} \\ &\times\ {}_8W_7\left(q^{1-\nu}a/c; q^{1-\nu}, q^{1-\nu}/bc, q^{1-\nu}/cd, aq^z, aq^{1-z}; q,\ q\right) \\ &\times\ {}_8W_7\left(q^{-\nu}a/d; q^{-\nu}, q^{1-\nu}/bd, q^{1-\nu}/cd, aq^z, aq^{-z}; q,\ q\right) \\ &- \frac{c\,(ad, acq; q)_\infty}{(q^{2-\nu}a/d, q^{1-\nu}a/c; q)_\infty} \end{aligned} \tag{7.9}$$

$$
\begin{aligned}
&\times \frac{(q^{1-\nu+z}/d, q^{2-\nu-z}/d, q^{1-\nu+z}/c, q^{1-\nu-z}/c; q)_\infty}{(q^z/d, q^{1-z}/d, q^{1+z}/c, q^{1-z}/c; q)_\infty} \\
&\times\ {}_8W_7\left(q^{1-\nu}a/d; q^{1-\nu}, q^{1-\nu}/bd, q^{1-\nu}/cd, aq^z, aq^{1-z}; q,\ q\right) \\
&\times\ {}_8W_7\left(q^{-\nu}a/c; q^{-\nu}, q^{1-\nu}/bc, q^{1-\nu}/cd, aq^z, aq^{-z}; q,\ q\right) \\
= &\frac{(c/d, qd/c, q^{1-\nu}, abcdq^\nu, abq^\nu, q^{1-\nu}/cd; q)_\infty}{c\,(ab, abq, qa/c, qa/d, bcq^\nu, bdq^\nu; q)_\infty} \\
&\times \frac{d\,(aq^z, aq^{1-z}, bq^z, bq^{1-z}; q)_\infty}{(q^z/c, q^{1-z}/c, q^z/d, q^{1-z}/d; q)_\infty}.
\end{aligned}
$$

The second term on the left side here is the same as the first one with c and d interchanged.

One can see from (7.8) that two solutions $u(z)$ and $v(z)$ are linear dependent when ν is an integer. In this case due to (2.13) both solutions are the Askey–Wilson polynomials, up to a factor, which are related by Sears' transformation. On the other hand, equation (7.8) shows that there is no analog of Sears's transformation at the level of very-well-poised ${}_8\phi_7$-functions.

8. Some Special and Limiting Cases

The Askey–Wilson polynomials are known as the most general system of classical orthogonal polynomials. They include all other classical orthogonal polynomials as special and/or limiting cases [1], [6], [12], [23], and [25]. Let us discuss in a similar manner a few interesting special cases of the orthogonal ${}_8\varphi_7$-functions (2.7).

8.1. **Extension of continuous dual q-Hahn polynomials.** Letting $c \to 0$ in (2.7) and then changing d by c we get

$$
\begin{aligned}
u_\nu(x; a, b; c) = &\frac{(qa/c, q^{1-\nu+z}/c, q^{1-\nu-z}/c; q)_\infty}{(q^{1-\nu}a/c, q^{1+z}/c, q^{1-z}/c; q)_\infty} \\
&\times\ {}_7\varphi_7\left(\begin{matrix} \dfrac{aq^{-\nu}}{c}, q\sqrt{\dfrac{aq^{-\nu}}{c}}, -q\sqrt{\dfrac{aq^{-\nu}}{c}}, q^{-\nu}, \dfrac{q^{1-\nu}}{bc}, aq^z, aq^{-z} \\ \sqrt{\dfrac{aq^{-\nu}}{c}}, -\sqrt{\dfrac{aq^{-\nu}}{c}}, \dfrac{aq}{c}, ab, 0, \dfrac{q^{1-\nu+z}}{c}, \dfrac{q^{1-\nu-z}}{c} \end{matrix}\ ;\ q,\ q\,\frac{b}{c}\right) \\
= &\frac{(q^{1-\nu}/ac; q)_\infty}{(q/ac; q)_\infty}\ {}_3\varphi_2\left(\begin{matrix} q^{-\nu}, aq^z, aq^{-z} \\ ab, ac \end{matrix}\ ;\ q,\ q\right) \\
&+ \frac{(q^{-\nu}, qb/c, aq^z, aq^{-z}; q)_\infty}{(ab, ac/q, q^{1+z}/c, q^{1-z}/c; q)_\infty} \\
&\times\ {}_3\varphi_2\left(\begin{matrix} q^{1-\nu}/ac, q^{1+z}/c, q^{1-z}/c \\ qb/c, q^2/ac \end{matrix}\ ;\ q,\ q\right).
\end{aligned}
\tag{8.1}
$$

The ${}_7\varphi_7$-function here can also be transformed to a ${}_3\varphi_2$ by (3.2.11) of [12],

$$
u_\nu(x; a, b; c) = \frac{(abq^\nu, q^{1-\nu+z}/c, q^{1-\nu-z}/c; q)_\infty}{(ab, q^{1+z}/c, q^{1-z}/c; q)_\infty}
\tag{8.2}
$$

$$\times \ {}_3\varphi_2\left(\begin{matrix} q^{-\nu}, q^{1-\nu}/ac, q^{1-\nu}/bc \\ \\ q^{1-\nu+z}/c, q^{1-\nu-z}/c \end{matrix} ; \ q, \ abq^{\nu}\right).$$

One can easily see that for an integer ν our function (8.1)–(8.2) is just a multiple of the continuous dual q-Hahn polynomial [1], [6], [23], and [25].

The orthogonality relation for the corresponding entire function,

$$v_\nu(x;a,b;c) = \left(q^{1+z}/c, q^{1-z}/c; q\right)_\infty \ u_\nu(x;a,b;c), \tag{8.3}$$

takes the form

$$\begin{aligned}\int_0^\pi & \frac{v_\mu(\cos\theta;a,b;c)\ v_\nu(\cos\theta;a,b;c)}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, qe^{i\theta}/c, qe^{-i\theta}/c; q)_\infty} \\ & \times \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha; q)_\infty}\, d\theta = 0\end{aligned} \tag{8.4}$$

if $\mu \neq \nu$, and

$$\begin{aligned}\int_0^\pi & \frac{\left(v_\nu(\cos\theta;a,b;c)\right)^2}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, qe^{i\theta}/c, qe^{-i\theta}/c; q)_\infty} \\ & \times \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha; q)_\infty}\, d\theta \\ & = \left(q, q, \alpha a, qa/\alpha, \alpha b, qb/\alpha, q\alpha/c, q^2/\alpha c; q\right)_\infty^{-1} \\ & \times \frac{-4\pi q^{1/2} c}{1-q}\ \frac{\partial}{\partial \lambda_\nu}\, v_\nu(\eta;a,b;c)\ \left(\frac{\nabla}{\nabla x}\, v_\nu(x;a,b;c)\right)\Bigg|_{x=\eta}\end{aligned} \tag{8.5}$$

if $\mu = \nu$, respectively. The "degree" parameters μ and ν here satisfy the "boundary" condition

$$\begin{aligned} & v_\varepsilon\left(\frac{1}{2}\left(\alpha + \alpha^{-1}\right); a, b; c\right) \\ & = \left(q\alpha/c, q^{1-\varepsilon}/\alpha c; q\right)_\infty \ {}_3\varphi_2\left(\begin{matrix} q^{-\varepsilon}, a/\alpha, b/\alpha \\ \\ q^{1-\varepsilon}/\alpha c, ab \end{matrix} ; \ q, \ q\,\frac{\alpha}{c}\right) = 0.\end{aligned} \tag{8.6}$$

Properties of zeros of this function follow as a special case of the results of Section 5.

8.2. **Extension of Al-Salam and Chihara polynomials.** Letting $b \to 0$ in (8.1)–(8.2) and then changing c by b one gets

$$\begin{aligned} u_\nu(x;a;b) = & \frac{(q^{1-\nu+z}/b, q^{1-\nu-z}/b; q)_\infty}{(q^{1+z}/b, q^{1-z}/b; q)_\infty} \\ & \times \ {}_2\varphi_2\left(\begin{matrix} q^{-\nu}, q^{1-\nu}/ab \\ \\ q^{1+z}/b, q^{1-z}/b \end{matrix} ; \ q, \ q\,\frac{a}{b}\right) \\ = & \frac{(q^{1-\nu}/ab; q)_\infty}{(q/ab; q)_\infty}\ {}_3\varphi_2\left(\begin{matrix} q^{-\nu}, aq^{z}, aq^{-z} \\ \\ ab, 0 \end{matrix} ; \ q, \ q\right)\end{aligned} \tag{8.7}$$

$$+\frac{(q^{-\nu}, aq^{z}, aq^{-z}; q)_\infty}{(ab/q, q^{1+z}/b, q^{1-z}/b; q)_\infty} \times {}_3\varphi_2\left(\begin{matrix} q^{1-\nu}/ab, q^{1+z}/b, q^{1-z}/b \\ q^2/ab, 0 \end{matrix} ;\ q,\ q\right).$$

For an integer ν this function is a multiple of the Al-Salam and Chihara polynomial [1], [4], [6], and [23].

The orthogonality relation of the entire function,

$$v_\nu(x;a;b) = \left(q^{1+z}/b, q^{1-z}/b; q\right)_\infty\ u_\nu(x;a;b), \tag{8.8}$$

is

$$\int_0^\pi \frac{v_\mu(\cos\theta;a;b)\ v_\nu(\cos\theta;a;b)}{(ae^{i\theta}, ae^{-i\theta}, qe^{i\theta}/b, qe^{-i\theta}/b; q)_\infty} \times \frac{\left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha; q)_\infty}\, d\theta = 0, \tag{8.9}$$

where $\mu \neq \nu$ are solutions of

$$v_\varepsilon\left(\frac{1}{2}\left(\alpha+\alpha^{-1}\right); a; b\right) = \left(q\alpha/b, q^{1-\varepsilon}/\alpha b; q\right)_\infty\ {}_2\varphi_1\left(\begin{matrix} q^{-\varepsilon}, a/\alpha \\ q^{1-\varepsilon}/\alpha b \end{matrix} ;\ q,\ q\frac{\alpha}{b}\right) = 0. \tag{8.10}$$

For properties of zeros see Section 5.

8.3. **Extension of continuous big q-Hermite polynomials.** Letting $a \to 0$ in (8.7) and then changing b by a we have

$$u_\nu(x;a) = \frac{(q^{1-\nu+z}/a, q^{1-\nu-z}/a; q)_\infty}{(q^{1+z}/a, q^{1-z}/a; q)_\infty} \times {}_1\varphi_2\left(\begin{matrix} q^{-\nu} \\ q^{1-\nu+z}/a, q^{1-\nu-z}/a \end{matrix} ;\ q,\ q^{2-\nu}/a^2\right). \tag{8.11}$$

For an integer ν this function is a multiple of the continuous big q-Hermite polynomial [23].

The orthogonality relation has the form

$$\int_0^\pi \frac{v_\mu(\cos\theta;a)\ v_\nu(\cos\theta;a)\ \left(e^{2i\theta},\ e^{-2i\theta}; q\right)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, qe^{i\theta}/\alpha, qe^{-i\theta}/\alpha, qe^{i\theta}/a, qe^{-i\theta}/a; q)_\infty}\, d\theta = 0, \tag{8.12}$$

where $\mu \neq \nu$ satisfy

$$v_\varepsilon\left(\frac{1}{2}\left(\alpha+\alpha^{-1}\right); a\right) = \left(q^{1-\varepsilon}/\alpha a; q\right)_\infty\ {}_1\varphi_1\left(\begin{matrix} q/\alpha a \\ q^{1-\varepsilon}/\alpha a \end{matrix} ;\ q,\ q^{1-\varepsilon}\alpha/a\right) = 0 \tag{8.13}$$

and $v_\varepsilon(x;a) = (q^{1+z}/a, q^{1-z}/a;q)_\infty \; u_\varepsilon(x;a)$.

8.4. **Extension of continuous q-Hermite polynomials.** The continuous q-Hermite polynomials $H_n(x|q)$ are the simplest special case $a=b=c=d=0$ of the Askey–Wilson polynomials $p_n(x;a,b,c,d)$ or the special case $a=0$ of the continuous big q-Hermite polynomials $p_n(x;a)$ [6], [12], [23], and [25]. See [2] and [4] for the proof of

$$\lim_{a\to 0} p_n(x;a) = H_n(x|q)$$

directly from the series representation of the continuous big q-Hermite polynomials.

Let us consider the difference equation (2.1) with $a=b=c=d=0$ and let us choose the following solution, $u_\nu(z) = H_\nu\left(x(z)|q\right)$, such that

$$\begin{aligned} H_\nu(x|q) = & \frac{(-q^{1-\nu+2z}, -q^{1-\nu-2z};q^2)_\infty}{(-q^{1+2z}, -q^{1-2z};q^2)_\infty} \\ & \times\ {}_2\varphi_2\left(\begin{matrix} q^{-\nu}, \quad q^{1-\nu} \\ \\ -q^{1-\nu+2z}, \ -q^{1-\nu-2z} \end{matrix}\ ;\ q^2,\ q\right) \end{aligned} \tag{8.14}$$

as a nonterminating extension of the continuous q-Hermite polynomials. This solution differs from the corresponding one in [8] by a periodic factor. For an integer ν function (8.14) coinsides with $H_n(x|q)$ up to a constant.

Comparing (8.14) with the equation (2.8) of [11] one can see that function $H_\nu(x|q)$ is a multiple of the basic cosine function $C(x;\omega)$ for certain values of parameter ν. Therefore, this function satisfies the orthogonality relation (1.16) under the "boundary" conditions (1.20). We would like to leave the details to the reader.

In a similar fashion one can consider some other special and limiting cases of the orthogonal ${}_8\varphi_7$-functions.

9. Appendix: Estimate of Number of Zeros

In this section we give an estimate for number of zeros of the "boundary" function $v_\nu(\eta;a,b,c;d)$ on the basis of Jensen's theorem (see, for example, [24]). We shall apply the method proposed by Mourad Ismail at the level of the third Jackson q-Bessel functions [14] (see also [17] and [11] for an extension of his idea to q-Bessel functions on a q-quadratic grid and q-trigonometric functions, respectively).

Let us consider the entire function

$$\begin{aligned} f(\zeta) &= v_\nu\left(\frac{1}{2}\left(\alpha+\alpha^{-1}\right);a,b,c;d\right) \\ &= \frac{(q\alpha/d;q)_\infty}{(abc/\alpha;q)_\infty} \\ &\quad\times \prod_{k=0}^{\infty}\left(1-\zeta q^{k+1}/\alpha d + abcq^{2k+1}/\alpha^2 d\right) \\ &\quad\times \sum_{m=0}^{\infty}\left(q\frac{\alpha}{d}\right)^m \frac{(1-abcq^{2m-1}/\alpha)\,(abcq^{-1}/\alpha, a/\alpha, b/\alpha, c/\alpha;q)_m}{(1-abcq^{-1}/\alpha)\,(q,ab,ac,bc;q)_m} \end{aligned} \tag{9.1}$$

$$\times \prod_{k=0}^{m-1} \frac{1 - \zeta q^k + abcdq^{2k-1}}{1 - \zeta q^{k+1}/\alpha d + abcq^{2k+1}/\alpha^2 d}$$

in a complex variable

$$\zeta = q^{-\nu} + abcdq^{\nu-1}, \tag{9.2}$$

where $|q\alpha/d| < 1$ (cf. (5.2)).

Let $n_f(r)$ be the number of of zeros of $f(\zeta)$ in the circle $|\zeta| < r$. Consider also circles of radius $R = R_n = q^{-n}/\beta + \beta abcdq^{n-1}$, $q < \beta < 1$ with $n = 1, 2, 3, \ldots$ in the complex ζ-plane. Since $n_f(r)$ is nondecreasing with r one can write

$$n_f(R_n) \le n_f(r) \le n_f(R_{n+1}) \tag{9.3}$$

if $R_n \le r \le R_{n+1}$, and, therefore,

$$n_f(R_n) \int_{R_n}^{R_{n+1}} \frac{dr}{r} \le \int_{R_n}^{R_{n+1}} \frac{n_f(r)}{r}\, dr \le n_f(R_{n+1}) \int_{R_n}^{R_{n+1}} \frac{dr}{r}. \tag{9.4}$$

But

$$\int_{R_n}^{R_{n+1}} \frac{dr}{r} = \log r \Big|_{R_n}^{R_{n+1}} = \log \left(\frac{q^{-1} + \beta^2 abcdq^{2n}}{1 + \beta^2 abcdq^{2n-1}} \right) = \log q^{-1} + \mathrm{o}(1) \tag{9.5}$$

as $n \to \infty$, and, finally, one gets

$$\log q^{-1}\ n_f(R_n) < \int_{R_n}^{R_{n+1}} \frac{n_f(r)}{r}\, dr < \log q^{-1}\ n_f(R_{n+1}) \tag{9.6}$$

for sufficiently large n.

The next step is to estimate the integral in (9.6). By Jensen's theorem [24]

$$\begin{aligned} \int_{R_n}^{R_{n+1}} \frac{n_f(r)}{r}\, dr &= \int_0^{R_{n+1}} \frac{n_f(r)}{r}\, dr - \int_0^{R_n} \frac{n_f(r)}{r}\, dr \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f\left(R_{n+1}\, e^{i\vartheta}\right)}{f\left(R_n\, e^{i\vartheta}\right)} \right|\, d\vartheta. \end{aligned} \tag{9.7}$$

For large values of n we have $R_n \sim q^{-n}/\beta$ and, in view of (5.3) and (9.1),

$$\begin{aligned} \frac{f\left(R_{n+1}\, e^{i\vartheta}\right)}{f\left(R_n\, e^{i\vartheta}\right)} &\sim \prod_{k=0}^{\infty} \frac{1 - e^{i\vartheta}\, q^{k-n}/\alpha\beta d + abcq^{2k+1}/\alpha^2 d}{1 - e^{\vartheta}\, q^{k-n+1}/\alpha\beta d + abcq^{2k+1}/\alpha^2 d} \\ &\sim -e^{i\vartheta} q^{-n}/\alpha\beta d, \end{aligned} \tag{9.8}$$

where we have used the formula

$$\begin{aligned} &\prod_{k=0}^{\infty} \left(1 - hq^{k-n} + gq^{2k}\right) \\ &= \prod_{k=0}^{n-1} \left(1 - hq^{k-n} + gq^{2k}\right) \prod_{k=n}^{\infty} \left(1 - hq^{k-n} + gq^{2k}\right) \\ &= (-h)^n\, q^{-n(n+1)/2} \prod_{k=0}^{n-1} \left(1 - \frac{1}{h}\, q^{k+1} - \frac{g}{h}\, q^{2n-k-1}\right) \end{aligned} \tag{9.9}$$

$$\times \prod_{k=0}^{\infty} \left(1 - hq^k + gq^{2n+2k}\right)$$

with $h = qe^{i\vartheta}/\alpha\beta d$ and $g = qabc/\alpha^2 d$. Thus,

$$\log \left| \frac{f\left(R_{n+1}\, e^{i\vartheta}\right)}{f\left(R_n\, e^{i\vartheta}\right)} \right| \sim n\ \log q^{-1} - \log \gamma,$$

where $\gamma = |\alpha\beta d|$, and

$$\int_{R_n}^{R_{n+1}} \frac{n_f\left(r\right)}{r}\, dr = n \log q^{-1} - \log \gamma + \mathrm{o}\left(1\right) \tag{9.10}$$

as $n \to \infty$.

From (9.6) and (9.10),

$$1 - \frac{\log \gamma / \log q^{-1}}{n} - \frac{1}{n} < \frac{n_f\left(R_n\right)}{n} < 1 - \frac{\log \gamma / \log q^{-1}}{n} \tag{9.11}$$

and, therefore,

$$\lim_{n\to\infty} \frac{n_f\left(R_n\right)}{n} = 1. \tag{9.12}$$

On the other hand, from (9.11),

$$n - 1 - \log \gamma / \log q^{-1} < n_f\left(R_n\right) < n - \log \gamma / \log q^{-1}. \tag{9.13}$$

The difference between the upper and the lower bounds here is 1 which means that there is only one positive root of $v_\nu(\eta; a, b, c; d)$ between the test points $\nu = \omega_n$ and $\nu = \omega_{n+1}$ defined on the page 15 during the proof of Theorem 5.1 for large values of n.

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