VERTEX THEOREMS FOR CAPILLARY DROPS ON SUPPORT PLANES

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ABSTRACT. We consider a capillary drop that contacts several planar bounding walls so as to produce singularities (vertices) in the boundary of its free surface. It is shown under various conditions that when the number of vertices is less than or equal to three, then the free surface must be a portion of a sphere. These results extend the classical theorem of H. Hopf on constant mean curvature immersions of the sphere. The conclusion of sphericity cannot be extended to more than three vertices, as we show by examples.

1. OVERVIEW

We consider in this paper liquid drops in $\mathbb{R}^3$ resting in mechanical equilibrium in the absence of gravity on rigid support surfaces that consist of a finite number of intersecting planes Π_j. We restrict attention to the physically familiar configurations for which the surface interface $\mathcal{S}$ is topologically a disk. Typical configurations with four and eight vertices are indicated in Figure 1.

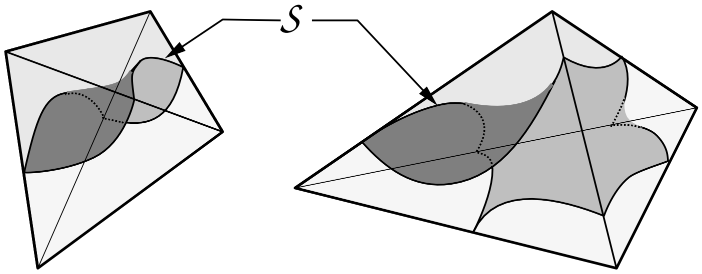

FIGURE 1. Typical Configurations

Under the indicated conditions, $\mathcal{S}$ will have constant mean curvature H, and it is natural to ask to what extent a theorem of H. Hopf [Hop51] on closed immersions of constant H will apply to the configurations we consider. That is, we wish to determine criteria under which $\mathcal{S}$ will necessarily be part of a metric sphere. We will

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present such conditions, and we will provide examples to show that our criteria are reasonably sharp.

Our results relate closely to (and in fact were inspired by) recent discoveries on tubular liquid bridges [McC] and on capillary surfaces in cylindrical tubes with protruding edges [CF96, Fin96]; these papers form the background for the perspective that we adopt. The support surfaces $\mathcal{P}$ considered here consist of a finite collection of planes, no three of which intersect in a common line, and which together contain the boundary of an open connected region $\mathcal{I}$ in $\mathbb{R}^3$. We assume $\mathcal{S}$ to be a constant mean curvature surface, lying in the closure of $\mathcal{I}$, whose trace on each supporting plane Π_j is a smooth *contact line* $\mathcal{C}_j$ along which $\mathcal{S}$ meets Π_j in a constant *contact angle* γ_j. Formally, $\mathcal{S}$ satisfies the variational condition

$$\delta\left(|\mathcal{S}| + \sum \beta_j \mathcal{S}_j + 2HV\right) = 0,$$

with $\beta_j = \cos\gamma_j$, $\mathcal{S}_j$ the wetted area on the plane Π_j, and V the drop volume, see, e.g., [Fin86, Chapter 1]. However, much of our formal analysis will encompass immersions $\mathcal{S}$ for which the enclosed volume may not be defined, and we focus attention on solutions of the system of (Euler-Lagrange) equations

$$\Delta\vec{x} = 2H\vec{N}$$

under the indicated boundary conditions. Here, the Laplacian is the intrinsic operator on the surface, and $\vec{N}$ is a unit normal to $\mathcal{S}$.

Physically, the constancy of the γ_j means that the liquid and each of the planes are assumed to be of homogeneous materials. The materials (and hence the angles γ_j) may differ from plane to plane, but the condition is introduced that the angle pair (γ_j, γ_k) on any two adjacent planes in contact with the drop, and for which the intersection line $\mathcal{L}_{jk}$ passes through $\mathcal{S}$, lies interior to a certain rectangle $\mathcal{Q}$, see Section 2 below. We assume also that the angles between any two intersecting support planes, measured in an appropriate sense, are less than π. There is strong heuristic evidence that both these conditions are (in general) necessary. No boundary condition is introduced on the lines $\mathcal{L}_{jk}$ themselves. With regard to behavior near these lines, our conceptual point of view will be that any contact point of $\mathcal{S}$ with an $\mathcal{L}_{jk}$ is a *vertex* $\mathcal{V}$. Vertices will turn out to be uniquely determined points, but it is possible (and we think it desirable) to start off in some cases with weaker hypotheses, which do not initially require $\mathcal{S}$ to be defined on the $\mathcal{L}_{jk}$.

A case of primary interest is the dihedral angle consisting of two planes Π_1 and Π_2 that intersect in a line $\mathcal{L}$. We discuss this case heuristically in Section 2 below. We enumerate in Hypothesis A physically natural conditions applying to a drop in a dihedral angle that serve as motivation for the weaker conditions of Hypothesis B below.

Hypothesis A. *$\mathcal{S}$ is globally embedded, and together with portions of the supporting planes bounds a drop volume topologically a ball. Each vertex $\mathcal{V}$ is a unique point on $\mathcal{L}$, and $\mathcal{S}$ can be parametrized locally up to $\mathcal{V}$ by continuous functions. There exists a plane Π orthogonal to $\mathcal{L}$, cutting off a portion $\mathcal{S}_{\mathcal{V}}$ of $\mathcal{S}$ containing $\mathcal{V}$, such that the reflection of $\mathcal{S}_{\mathcal{V}}$ in that plane lies interior to the drop volume.*

We will now state precisely several more general assumptions that are sufficient to imply our main results and are used (sometimes without explicit mention) throughout the paper. We have not attempted to find the weakest possible assumptions except for those directly related to the main contribution of the paper, vis. the singular behavior of the boundary curve. The following four conditions collectively generalize Hypothesis A.

Hypothesis B.

Let D be the closed unit disk in $\mathbb{R}^2$ and v_i, $i = 1, \ldots, V$ a finite clockwise-ordered collection of points in ∂D. For $i = 1, \ldots, V$, let A_i be the arc on ∂D between v_i and v_{i+1}; $A = \cup A_i$.

1. Topological Condition: *There is a local homeomorphism ϕ of $D_v = D \backslash \{v_i\}$ onto $\mathcal{S}$, i.e., for each $a \in D_v$ there is some neighborhood $B(a)$ of a in D_v such that ϕ restricted to $B(a)$ is a homeomorphism.*

The following subsets of $\mathcal{P}$ have special significance: If Π_j and Π_k intersect in a line $\mathcal{L}_{jk}$, then $\mathcal{E}_{jk} = \mathcal{L}_{jk} \cap \bar{\mathcal{I}}$ is the *jk-edge.* The *edge of $\mathcal{P}$* is $\mathcal{E} = \cup \mathcal{E}_{jk}$. The *faces* of $\mathcal{P}$ are $\mathcal{F} = \bar{\mathcal{I}} \backslash \mathcal{E}$. No confusion results if we refer to a connected component of $\mathcal{F}$ as Π_j or Π_k.

2. Smoothness Condition: *$\phi_{|A}$ can be made locally smooth, i.e., for each $a \in A$ there is a homeomorphism $\psi : (-1, 1) \to A$ such that $\psi(0) = a$ and $X^\partial = \phi \circ \psi : (-1, 1) \to \mathcal{F}$ represents a differentiable embedded curve.*

Also, $\phi_{|D_v}$ can be made locally smooth, i.e., for each $a \in \operatorname{int} D$ there is a homeomorphism $\psi : B_1(0) \to \operatorname{int} D$ such that $X = \phi \circ \psi : B_1(0) \to \mathcal{I}$ represents a C^2 (open) embedded surface, and for $a \in A$ a similar statement holds with ψ defined on a half neighborhood $B_1^+(0) = \{(a_1, a_2) \in B_1(0) : a_2 \geq 0\}$.

Notice that we have required each boundary component $\mathcal{C}_i = \phi(A_i)$ to lie in some face Π_j; it is not required that this association be one-to-one.

For each pair of planes Π_j, Π_k in $\mathcal{P}$ that intersect in a line $\mathcal{L}_{jk}$, denote by $\Pi = \Pi_{jk}$ a plane orthogonal to Π_j and Π_k. The heart of Hypothesis B is the following

3. Vertex Condition: *To each $v_i \in \partial D$, there is associated a pair of intersecting planes Π_j, $\Pi_k \in \mathcal{P}$ and two neighborhoods: B, a neighborhood of v_i in D, and $\mathcal{N}$, a neighborhood of $O = \mathcal{L}_{ij} \cap \Pi$ in $\Pi \cap \bar{\mathcal{I}}$, such that $\phi(B \backslash v_i)$ is a graph $u(p)$ over $\mathcal{N} \backslash O$.*

In the case of Hypothesis B, each such triple $\{$planepair, neighborhood, function$\}$ = $\{(\Pi_j, \Pi_k), \mathcal{N}, u\}$ will be said to determine a vertex $\mathcal{V}$, and we write $S_{\mathcal{V}} = \phi(B \backslash v_i)$, $\mathcal{N}_{\mathcal{V}} = \mathcal{N}$, $u_{\mathcal{V}} = u$, etc.. As noted above, vertices need not in this case be initially defined as points; we shall however prove the existence of the vertex points and the smoothness of $u_{\mathcal{V}}(p)$ up to those points, and in fact we shall do so without growth hypotheses on the graph. Once that is done, the local homeomorphism ϕ that parameterizes $\mathcal{S}$ can be extended continuously to D.

In general however, we require an additional condition in order to separate the vertices one from another. Though not the weakest possible condition, in order to simplify the proofs of Theorem 2 and of Section 5 below we impose the following

4. Separation Condition: *For each vertex $\mathcal{V}$, $\mathcal{S}_{\mathcal{V}}$ can be chosen so that $\phi : \phi^{-1}(\mathcal{S}_{\mathcal{V}}) \to \mathcal{S}_{\mathcal{V}}$ is a local homeomorphism, and if there is another vertex $\mathcal{V}'$ corresponding to the same plane pair, then there exists a sequence $p_j \to \mathcal{N}_{\mathcal{V}} \cap \mathcal{L}_{jk}$ on Π, on which $|u_{\mathcal{V}} - u_{\mathcal{V}'}|$ is bounded away from zero.*

If ϕ extends continuously to the boundary, then the Separation Condition becomes simply: *For each vertex $\mathcal{V}$, $\phi^{-1}(\mathcal{V}) = v_i$*

Under either of the above hypotheses, and under the above conditions on the opening angles and the adjacent contact angle pairs, we intend to prove that if V is the number of vertices and if $V \leq 2$ then $\mathcal{S}$ is a portion of a metric sphere. We will obtain the same result when $V = 3$, under an additional condition on the orientation of $\mathcal{S}$. Remarkably, the orientation required is the reverse of the (uniquely determined) orientation that occurs when $V \leq 2$. The interest in these results is underscored by the fact that if $V \geq 4$ then $\mathcal{S}$ need not be spherical, as we show by example.

If we restrict attention to a dihedral angle and the case $V = 2$ (drop in a wedge), then the Vertex and Separation Conditions can be proved, as is shown in Theorem 1. We emphasize that in the case $V \leq 2$ and Hypothesis B, $\mathcal{S}$ is not assumed to be embedded. In this context, the requirement that $\mathcal{S}$ be disk-type is necessary; in fact, Wente [Wen95] has given an example of an immersed constant mean curvature bridge joining the two faces of a wedge and meeting those faces in the contact angles $\gamma_1 = \gamma_2 = \pi/2$. Such a surface cannot lie on a metric sphere.

In the case $V = 2$, the underlying idea for our work consists simply of adjoining a theorem on intersecting surfaces proved by Joachimsthal [Joa46] in 1846, to the method that H. Hopf [Hop51] used in 1951 to prove that every immersed closed surface of genus zero and constant mean curvature is a sphere. Some technical effort will be needed to prove our results without superfluous smoothness requirements. We present those details in later sections of the paper; in order not to obscure the ideas, we outline the proof for the case $V \leq 2$ in the following section, under some assumptions on smoothness of $\mathcal{S}$ and of certain conformal mappings of $\mathcal{S}$. In the next following section we provide counterexamples for the case $V = 4$. The underlying assumptions used in Section 2 will be justified in the ensuing Section 4.

In the final Section 5, we present our results for the case $V = 3$. This material is based entirely on comparison procedures, and requires an additional hypothesis, although in another sense our requirements are weaker (data on a part of the boundary set of $\mathcal{Q}$ are allowed whereas for $V = 2$ they must be interior to $\mathcal{Q}$). It is perhaps worth observing that the difficulty in extending the $V = 2$ proof to $V = 3$ lies in the singular behavior at the third vertex, of the conformal mapping that takes $\mathcal{S}$ to an infinite strip, with two of the vertices going to infinity. That the difficulty is essential can be seen from the fact that it is exactly the reason the theorem fails when $V \geq 4$. This kind of behavior underscores the importance we attach to the material of Section 4, in which the assumptions made in Section 2, on asymptotic structure of the mapping at the vertices, are justified on the basis of more primitive (and seemingly reasonable) hypotheses.

There is also an essential difficulty extending the $V = 3$ proof to $V = 2$, as the additional hypothesis just referred to is violated by spheres in the $V = 2$ case, see the comments at the end of Section 5.

We note finally that our method provides as corollary a conceptual simplification of a proof given earlier by Nitsche [Nit85], that a disk-type free surface interface of a connected drop, that rests in the absence of gravity on a spherical support surface which it meets in a constant angle, is necessarily a spherical cap. We indicate our improvement to that result in the Appendix to this paper.

2. Outline of proof: case $V \leq 2$.

We outline here the structure of our proof, assuming to fix the ideas that all functions appearing are as smooth as required by the context. It is intuitively clear that the case $V = 1$ does not occur; the case $V = 0$ will be encompassed in the procedure for $V = 2$. In this case the drop can meet only two planes; we may ignore any other planes and consider a drop in a dihedral angle, of opening 2α. We adopt the intersection line $\mathcal{L}$ as z-axis, and assume representations $z = u^{\pm}(x, y)$ for $\mathcal{S}$ at the two vertices $\mathcal{V}^+$ and $\mathcal{V}^-$, in a neighborhood of the origin interior to the wedge. The condition for $\mathcal{S}$ to have a tangent plane at the origin is precisely the condition that the pair (γ_j, γ_k) lie in the closed rectangle $\mathcal{Q}$ introduced in [CF96], see Figure 3. We assume this condition satisfied, and assume further that (γ_j, γ_k) lies interior to $\mathcal{Q}$, so that the tangent plane is not vertical. The contact lines then meet each other on $\mathcal{L}$ in the same positive angle $2\beta < \pi$ at both vertices, determined entirely by (γ_j, γ_k) and by α.

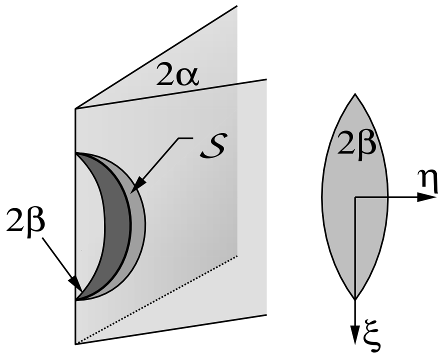

Figure 2. Mapping to lens

We map S conformally onto a convex lens (Figure 2) with vertex angles 2β at the points $P_{-1} = (-1, 0)$ and $P_{+1} = (+1, 0)$ in the $\zeta = \xi + i\eta$ plane, in such a way that P_{-1}, P_{+1} are the images of the vertices on $\mathcal{S}$. We assume this mapping to be sufficiently smooth that the second derivatives of the representation $\vec{\mathbf{x}}(\xi, \eta)$ of $\mathcal{S}$ have at worst a singularity admitting the estimate

$$|D^2\vec{\mathbf{x}}| = \circ(\rho^{-2}) \tag{1}$$

at P_{-1} and P_{+1}, ρ being distance to the respective point. The mapping

$$Z = \ln\left(\frac{\zeta - 1}{\zeta + 1}\right) \tag{2}$$

takes the lens region to a horizontal infinite strip. The boundary of this domain consists of coordinate lines. According to a theorem of Joachimsthal [Joa46], *if two surfaces intersect in a constant angle, and if the intersection curve is a curvature line on one of the surfaces, then it is a curvature line on both surfaces.* In the present case, every curve is a curvature line on the support planes Π_j; since both planes meet $\mathcal{S}$ in the respective constant angles γ_j, the contact lines are curvature lines on $\mathcal{S}$. They are also coordinate lines in the Z plane; thus, if we denote by L, M, N the coefficients of the second fundamental form in the coordinates X, Y of the Z-plane, we will have $M = 0$ on the boundary of the strip. Considering the corresponding coefficients l, m, n in the ζ - plane, we note that they can be expressed as scalar products of the (assumed continuous) normal vector and the derivatives $D^2\vec{\mathbf{x}}(\xi, \eta)$, and thus we have by (1)

$$|l|, |m|, |n| = \circ(\rho^{-2}). \tag{3}$$

Further, we have from (2)

$$\left|\frac{d\zeta}{dZ}\right| = \bigcirc(\rho). \tag{4}$$

It follows from the Codazzi equations that the expression

$$\Phi \equiv ((l - n) - 2im)d\zeta^2 \tag{5}$$

is a holomorphic quadratic differential on $\mathcal{S}$ (see [Hop89, Chapter 6]). We thus have

$$((L - N) - 2iM) = ((l - n) - 2im)\left(\frac{d\zeta}{dZ}\right)^2 \tag{6}$$

from which we conclude from (3) and (4) that $M \to 0$ uniformly at each end of the strip. Since M is harmonic in the strip and vanishes on the entire finite boundary, it follows from the maximum principle that $M \equiv 0$, and thus that its conjugate $(L-N)$ is identically constant. But $(L - N) \to 0$ at the ends for the same reason that M does, and hence $(L - N) \equiv 0$. We conclude that $\mathcal{S}$ is totally umbilic, and must therefore be part of a metric sphere in $\mathbb{R}^3$, as was to be proved. □

3. The case $V = 4$

It is proved in [Fin86, Sec. 6.4] that if $\pi/4 < \gamma < \pi/2$, then there exists a solution surface $\mathcal{S} : u(x, y)$ of the nonparametric constant mean curvature equation

$$\operatorname{div} Tu = 2\frac{a+b}{ab}\cos\gamma \tag{7}$$

in a rectangle $\mathcal{R}$ of arbitrary side lengths a and b, such that the solution surface meets all four vertical walls over the sides in the constant contact angle γ. The solution is uniquely determined up to an additive constant. There are four vertices on $\mathcal{S}$, on the vertical lines through the four vertices of $\mathcal{R}$. If $a = b$, then $\mathcal{S}$ is known explicitly as a lower spherical cap. However, if $a \neq b$, although the solution continues to exist, it cannot be spherical.

Henry Wente pointed out to us that if γ is allowed to differ on adjacent walls, then an example can be given explicitly. Choose $\gamma = 0$ on the two opposite walls of length a, and $\gamma = \pi/2$ on the two other walls. Then the lower half of a horizontal cylinder of radius $b/2$ provides an explicit surface of constant mean curvature $1/b$, meeting the walls in the respective angles indicated and having four vertices.

We devote most of the remainder of this paper to justifying the hypotheses we introduced in Section 2 above. In the final section we will discuss the case of three vertices.

4. The drop configuration; $V \leq 2$

4.1. **Preliminary lemmas.** We consider a connected drop supported by a finite number of planes, with free surface $\mathcal{S}$ topologically a disk. We suppose $\mathcal{S}$ to have constant mean curvature and to be differentiable up to the (interiors of the) contact lines, where it cuts the planes Π_j transversally in the respective angles γ_j interior to the drop. We will make two kinds of hypotheses as to the behavior of $\mathcal{S}$ near the intersection lines; both of them will lead to identical further conditions, under which the drop must be spherical. The first of them is relevant only to the case of a drop in a dihedral angle formed by two planes Π_1, Π_2, with two vertices on the single intersection line $\mathcal{L}$ (edge of the wedge). We may suppose that $\mathcal{L}$ is oriented vertically.

Lemma 4.1. *Under the conditions just stated, assume Hypothesis A with respect to two distinct points $\mathcal{V}^{\pm}$ on $\mathcal{L}$. Then $\mathcal{S}$ is symmetric about a horizontal plane Π, and each half of $\mathcal{S}$ is a graph over Π.*

Proof: We use the planar reflection method, as introduced by Alexandrov [Ale58] and developed for drops on planar surfaces by Wente [Wen80]. Let $\mathcal{V}_0$ be the point on $\mathcal{L}$ midway between $\mathcal{V}^+$ and $\mathcal{V}^-$ below it. We start with a horizontal plane Π^+ as indicated in Hypothesis A. This plane clearly separates $\mathcal{V}_0$ and $\mathcal{V}^+$. We lower the plane continuously, reflecting in it the part of $\mathcal{S}$ that lies above it. If a point of tangency of the reflected with the original surface is attained, one can show as in [McC] via appropriate versions of the maximum principle that the lower surface is a reflection of the upper one. Thus the procedure can be continued until the plane reaches $\mathcal{V}_0$. If no point of tangency (other than at $\mathcal{V}_0$) has been reached till then, it follows that on the intersection of the plane Π_0 through $\mathcal{V}_0$ with $\mathcal{S}$, the derivative with respect to height of each horizontal distance from $\mathcal{L}$ to the intersection curve $\mathcal{C}$ of a generic Π with $\mathcal{S}$ must be negative. But if that situation occurs then the same procedure, starting with a plane Π^- and moving upward, would have to yield an earlier point of tangency. Thus $\mathcal{S}$ is symmetric under reflection in Π_0. If the upper and lower parts of $\mathcal{S}$ were not graphs over Π_0 then a point of tangency would be obtained during the procedure, which is not possible since $\mathcal{V}^+$ must reflect onto $\mathcal{V}^-$. $\square$

Lemma 4.1 reduces the further discussion to the following case, which leads independently to conditions under which the existence of the vertices $\mathcal{V}^{\pm}$ as uniquely determined points can be proved. In what follows we continue to assume the smoothness of $\mathcal{S}$ in a deleted neighborhood of each intersection line $\mathcal{L}$, however $\mathcal{S}$ is no longer assumed to be embedded, or in any sense defined on $\mathcal{L}$, nor is any growth hypothesis introduced with regard to behavior of $\mathcal{S}$ near $\mathcal{L}$.

Lemma 4.2. *Suppose there is a neighborhood $\mathcal{N}$ on a plane Π orthogonal to $\mathcal{L}$ as in Hypothesis B, such that a subset $\mathcal{S}^* \subset \mathcal{S}$ appears as a graph $u^*(x,y)$ over $\mathcal{N}\backslash(\mathcal{N}\cap\mathcal{L})$. Then $u^*(x,y)$ is bounded in $\mathcal{N}\backslash(\mathcal{N}\cap\mathcal{L})$.*

Proof: The set of all admissible boundary data (γ_j, γ_k), constant on each side of the wedge domain determined by the intersecting planes Π_1, Π_2, can be restricted to a square of side length π. In [CF96] it was shown that the set of all data that can lead to constant mean curvature graphs over such an $\mathcal{N}$ with tangent planes at $\mathcal{L}$ lie in an inscribed rectangle $\mathcal{Q}$. The complement of $\mathcal{Q}$ in the square consists of two diagonally opposite domains $\mathcal{D}_1^\pm$ and two diagonally opposite domains $\mathcal{D}_2^\pm$, see Figure 3. In $\mathcal{D}_1^\pm$ there can be no such graph over $\mathcal{N}$, regardless of growth conditions. It was shown in [Fin96] that for data in $\mathcal{D}_2^\pm$ graphs meeting Π_1, Π_2 in the prescribed angles can under some conditions exist, although they cannot admit tangent planes at $\mathcal{L}$, and in [CFM97] it is shown that such graphs must be discontinuous at $\mathcal{L}$ but are nevertheless bounded there.

In the present case, the data (γ_j, γ_k) cannot lie in a $\mathcal{D}_1^\pm$ domain as in that event there could be no constant mean curvature surface as a graph in $\mathcal{N}\backslash(\mathcal{N}\cap\mathcal{L})$ ([CF96, Theorem 3]). The boundedness for other data follows from Proposition 1 of [LS], or alternatively from the material of [CFM97, Section 7]. □

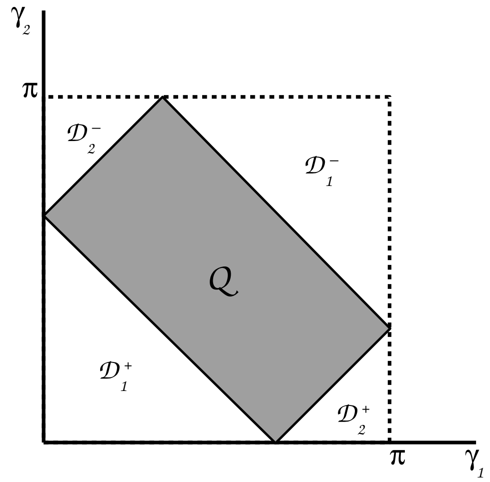

FIGURE 3. Limit configurations for given data

Lemma 4.3. *Under the hypotheses of Lemma 4.2, suppose additionally that the data arise from an interior point of $\mathcal{Q}$. Then $u^*(x,y)$ can be defined at $\mathcal{L}$ (determining a vertex $\mathcal{V}^*$ as a point on $\mathcal{L}$) so as to have uniformly Hölder continuous derivatives up* *to $\mathcal{L}$, and the data are achieved exactly near $\mathcal{L}$ by the plane Π tangent to $\mathcal{S}$ at the vertex.*

The result is obtained by adapting procedures used by Simon [Sim80] and by Tam [Tam86] to prove differentiability in the case $\gamma_1 = \gamma_2$, and then by adapting the methods of Lieberman [Lie88] or of Miersemann [Mie89] to prove the Hölder continuity of the derivatives. For details of the initial step, see [CFM97]. The methods extend without essential change to all data interior to $\mathcal{Q}$. For data on the boundary of $\mathcal{Q}$ we still have

Lemma 4.4. *Under the conditions of Lemma 4.2, if the data lie on the interior of the segments $\partial\mathcal{Q} \cap \partial\mathcal{D}_1^{\pm}$, then $u^*(x,y)$ can be defined at $\mathcal{L}$ (determining a vertex $\mathcal{V}^*$ as a point on $\mathcal{L}$) so as to be continuous and have continuous unit normal vector.*

In this case the first derivatives necessarily become infinite as $\mathcal{L}$ is approached. The proof follows from the procedure of Tam [Tam86], see [CFM97].

Lemma 4.5. *Under the hypotheses of Lemma 4.3, the two contact lines $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ intersect on $\mathcal{L}$ in an angle 2β, with $0 < 2\beta < \pi$. In the (single angle) case $\gamma_1 = \gamma_2$ there holds additionally $0 < 2\beta \leq 2\alpha$.*

Proof: According to Lemma 4.3, $\mathcal{S}$ has a non-vertical tangent plane T at the vertex, meeting the walls in the constant angles (γ_1, γ_2). The angle made by $\mathcal{C}_1^*, \mathcal{C}_2^*$ with each other is the same as the angle between the intersection lines of the tangent plane with the wedge planes. It thus suffices to determine the angle between these lines. Set $B_1 = \cos\gamma_1$, $B_2 = \cos\gamma_2$. A calculation yields

$$\sin^2 2\beta = \frac{\sin^2 2\alpha - (B_1^2 + B_2^2 + 2B_1B_2\cos 2\alpha)}{(1-B_1^2)(1-B_2^2)}. \tag{8}$$

It was shown in [CF96] that data arise from an interior point of $\mathcal{Q}$ if and only if the numerator in (8) is positive, and thus the first assertion follows. To prove the second statement, we note that (8) can be written in the form

$$\cos 2\beta = \frac{B_1B_2 + \cos 2\alpha}{\sqrt{1-B_1^2}\sqrt{1-B_2^2}} \tag{9}$$

and thus if $B_1 = B_2 = B$ then

$$\cos 2\beta = \frac{B^2 + \cos 2\alpha}{1-B^2}. \tag{10}$$

The right side of (10) is increasing in B^2 and reduces to $\cos 2\alpha$ when $B^2 = 0$; the assertion follows. $\square$

Lemma 4.6. *Under the hypotheses of Lemma 4.3, the second derivatives of u^* are Hölder continuous to the wedge walls, and satisfy an estimate $|D^2u^*| < Cr^{-\alpha}$ in terms of distance r to the vertex, with $0 < \alpha < 1$.*

Proof: We observe first that by adjoining work of Siegel [Sie80] to that of Ural'tseva [Ura73] and of Gerhardt [Ger76, Ger79], we obtain that locally $u^*(x,y) \in C^{2+\epsilon}$ to the wedge walls $(\Pi_1 \cap \Pi_2)\backslash\mathcal{V}^*$, with $0 < \epsilon < 1$. By Lemma 4.3, $u^*(x,y) \in C^{1+\epsilon}$ in the closed wedge domain. We next observe that $\mathcal{S}$ can be represented locally near $\mathcal{V}^*$ as a graph $u(x,y)$ over its tangent plane. Since $u^*(x,y) \in C^{1+\epsilon}$, there follows $|u| < Cr^{1+\epsilon}$ in polar coordinates centered at $\mathcal{V}^*$. We have also that $u(x,y)$ satisfies a uniformly elliptic equation of the form

$$a_{ij}(x)u_{x_ix_j} = 2H \tag{11}$$

near $\mathcal{V}^*$, with coefficients and Dirichlet boundary data that are Hölder continuous in the closed corner domain. Such solutions were studied by Azzam [Azz79], who obtained the stated growth estimate on second derivatives. □

Lemma 4.7. *Under the hypotheses of Lemma 4.3, a neighborhood of $\mathcal{V}^*$ on $\mathcal{S}$ can be mapped 1-1conformally onto a corresponding neighborhood of a rectilinear angle, of opening 2β, such that the inverse representation of $\mathcal{S}$ over the angular neighborhood is locally of class $C^{2+\alpha}$ up to the rectilinear sides, and of class $C^{1+\alpha}$ to the vertex image $\mathcal{V}'$. For the second derivatives of the position vector $\vec{\mathbf{x}}$ in this representation, there holds $|D^2\vec{\mathbf{x}}| < Cr^{\epsilon-1}$, with $0 < \epsilon < 1$.*

Proof: We consider $\mathcal{S}$ in local representation $u(x,y)$ over its tangent plane Π at $\mathcal{V}^*$. In view of Lemma 4.3, the quantities

$$E = 1 + u_x^2,\ F = u_x u_y,\ G = 1 + u_y^2 \tag{12}$$

are defined in a wedge region $\mathcal{W}$ determined by the projections of $\mathcal{C}_1^*$, $\mathcal{C}_2^*$ onto Π; in view of Lemma 4.6, they are in class C^α on $\mathcal{W}$, with $E = G = 1$, $F = 0$ at $\mathcal{V}^*$, and in class $C^{1+\alpha}$ in $\mathcal{W}\backslash\mathcal{V}^*$. We extend these functions to functions with the same

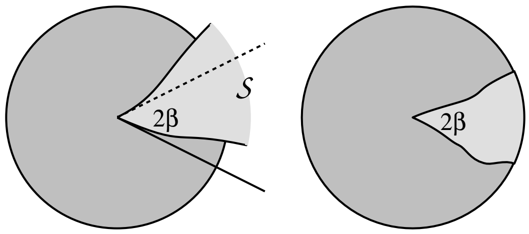

FIGURE 4. Local mapping at vertex

smoothness properties defined in a (small) disk about $\mathcal{V}^*$ (Figure 4), and observe that non-singular solutions $\zeta \in \mathbb{C}$ of the *Beltrami Equations*

$$\frac{d\zeta}{d\bar{z}} = \lambda \frac{d\zeta}{dz} \tag{13}$$

where

$$\lambda = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}$$

in the disk determine conformal maps of the portion of $\mathcal{S}$ that projects onto $\mathcal{W}$. Since E, F, G are Hölder continuous, there exists a local solution $\zeta = \xi + i\eta$ about $\mathcal{V}^*$ with Hölder continuous derivatives and non-vanishing Jacobian determinant [CH62], mapping $\mathcal{W} \leftrightarrow \mathcal{W}'$. Since at $\mathcal{V}^*$, $E = G = 1$, $F = 0$, the mapping is conformal between the planar domains at $\mathcal{V}^*$ and thus the vertex angle 2β remains unchanged in the $\mathcal{W}'$ coordinates. We may assume the angles oriented as in Figure 4.

The mapping $\Xi = \zeta^{\pi/2\beta}$ opens the wedge to a domain bounded locally by a Hölder differentiable curve $\mathcal{C}$. A further mapping $Z = F(\Xi)$, again Hölder differentiable and invertible to the image of $\mathcal{V}^*$, takes $\mathcal{C}$ onto a linear segment (Figure 5). Finally,

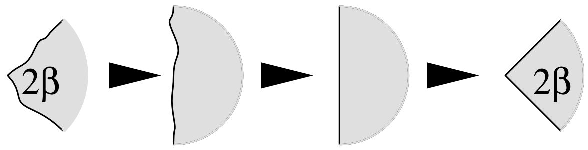

FIGURE 5. Successive mappings

$\Psi = Z^{2\beta/\pi}$ completes the mapping to a rectilinear wedge domain of opening 2β.

According to Lemma 4.3, the surface representation $\vec{\mathbf{x}}(x, y)$ is Hölder differentiable to $\mathcal{V}^*$. Since the mapping $\zeta(x, y)$ is Hölder differentiable to $\mathcal{V}^*$, it follows that $\vec{\mathbf{x}}$ has the same property in the (ξ, η) variables. The asserted properties of the first derivatives of $\vec{\mathbf{x}}$ in the Ψ variables follow by tracing through the mappings, each of which is either smooth at the vertex or is an explicitly known power mapping. The singularities in the power mappings cancel each other.

To obtain the growth estimate on second derivatives, we observe that in the Ψ-variables, the representation $\vec{\mathbf{x}}$ satisfies the equation

$$\Delta \vec{\mathbf{x}} = 2H\vec{N} \tag{14}$$

where $\vec{N}$ is a unit normal to $\mathcal{S}$. By the material just proved, $\vec{N}$ is Hölder continuous to the vertex, while $\vec{\mathbf{x}}$ is in $C^{2+\alpha}$ locally to the wedge walls, and in $C^{1+\epsilon}$ to the vertex. The result then follows as in the second part of Lemma 4.6. □

Lemma 4.8. *Under the hypotheses of Lemma 4.3, every conformal map of a neighborhood of $\mathcal{V}^*$ on $\mathcal{S}$ onto a rectilinear wedge neighborhood of opening 2β, with vertex going to vertex, leads to a representation for $\mathcal{S}$ with the smoothness properties described in Lemma 4.7.*

Proof: A given mapping onto the $\hat{\zeta}$ plane, in conjunction with the particular mapping $\hat{\Xi} = \hat{\zeta}^{\pi/2\beta}$ applied in the proof of Lemma 4.7, leads to a conformal mapping into itself of a half disk of which the image of a diameter lies on a diameter, with origin O going into itself. This map can be extended by reflection into a univalent conformal map of a disk containing O, and is therefore analytic with non-zero Jacobian at O. The mappings to the wedge are effected by an identical mapping for the two functions, with the requisite smoothness properties, and hence the Hölder differentiability of the constructed function leads to the same property for the given one. Similarly, the second derivatives of the two functions have the same Hölder growth exponent at the respective images of O. □

As a consequence of Lemma 4.8, we obtain immediately, since l, m, n are scalar products of the unit normal to $\mathcal{S}$ with second derivatives of the position vector:

Corollary 4.9. *In terms of the conformal parameters introduced in Lemma 4.7, the coefficients l, m, n of the second fundamental form satisfy*

$$\sqrt{l^2+m^2+n^2} < Cr^{\epsilon-1} \tag{15}$$

near the vertex image.

4.2. Main theorems, $V \leq 2$.

Theorem 1. *Under either of the hypotheses A or B, if $V = 2$ and $\mathcal{S}$ is topologically a disk, and if the contact angles γ_1, γ_2 lie interior to the rectangle $\mathcal{Q}$, then $\mathcal{S}$ is metrically spherical.*

Proof: If Hypothesis A holds, we conclude by Lemmas 4.1 to 4.5 that $\mathcal{S}$ is symmetric about a plane Π orthogonal to the intersection line $\mathcal{L}$, can be represented globally by functions with Hölder continuous first derivatives, and forms at each vertex an angle 2β given by (10). We can therefore map $\mathcal{S}$ conformally onto a lens domain (Figure 2) bounded by circular arcs meeting at angle 2β, with vertices going into vertices at the points $\zeta = \pm 1$. This configuration is locally related to a rectilinear angle via a linear fractional transformation; we may thus conclude from Corollary 4.9 that in the lens coordinate ζ, the coefficients l, m, n satisfy (15). We now apply the mapping (2) and the invariance of the form Φ as in Section 2, arriving at the desired conclusion by the identical reasoning.

If Hypothesis B holds, we are assured directly of the hypotheses of Lemma 4.2; the remainder of the reasoning then proceeds without change. □

If we assume additionally the Vertex Condition, then we can provide a more inclusive formulation of Theorem 1. We observe that the variational condition characterizing the mechanical equilibrium of the drop surfaces is not affected by the presence or absence of support surfaces that do not meet $\mathcal{S}$. Thus, *if a property of $\mathcal{S}$ has been determined by its interaction with certain support planes which it contacts, the removal of planes which do not contact $\mathcal{S}$ will not affect that result.* Using this observation, we are able to characterize equivalence classes of configurations in terms of a few particular cases. Specifically, we find:

Lemma 4.10. *Assume the Vertex Condition and that one of the Hypotheses A or B holds at each vertex, with data arising from the interior of $\mathcal{Q}$ or from the interior of $\partial\mathcal{Q} \cap \partial\mathcal{D}_1^{\pm}$. Then if $V = 0$, every drop configuration is either a closed surface without boundary or else it can be realized by a drop on a single plane. The case $V = 1$ does not occur. If $V = 2$ then the configuration can be realized by a drop in a dihedral angle (wedge). If $V = 3$ then the configuration is equivalent either to a drop covering the vertex in a trihedral angle, or else to a drop covering the (planar) base of a cylindrical container, whose side walls consist of three planes, no two of which are parallel.*

Proof: We have assumed the boundary $\mathcal{C}$ of $\mathcal{S}$ to be locally smooth on each face; by Lemmas 4.3, 4.4 and 4.5 it is piecewise smooth at each vertex, that is, continuous with a jump in unit tangent vector. Since the entire configuration is compact and since $\mathcal{S}$ is disk type, $\mathcal{C}$ is globally a piecewise smooth closed curve (which may conceivably have self-intersections). We may start with any point on $\mathcal{C}$, and traverse $\mathcal{C}$ in any chosen direction.

If $V = 0$ and $\mathcal{C}$ is the null set, then $\mathcal{S}$ is closed and without boundary. If $V = 0$ and $\mathcal{C}$ is non-null, we start with an arbitrary point of $\mathcal{C}$, which will be on a support plane Π, and traverse $\mathcal{C}$ in one of the two possible directions. $\mathcal{C}$ cannot enter a plane distinct from Π across an intersection line with Π, as, by the Vertex Condition, that would create a vertex. Thus all planes distinct from Π can be removed without affecting the variational conditions determining $\mathcal{S}$. We are left with a drop with disk-type free surface resting on a single plane.

Suppose $V = 1$. We traverse $\mathcal{C}$ beginning at its single vertex $\mathcal{V}$, along one of the two intersecting planes Π_1 and Π_2. $\mathcal{C}$ is a closed curve and on each of Π_1 and Π_2 it contains points distinct from the intersection line $\mathcal{L}$. $\mathcal{C}$ cannot meet other planes, as by the Vertex Condition that would create a new vertex. Therefore it cannot close without crossing again over $\mathcal{L}$ at a point distinct from $\mathcal{V}$, which cannot occur by the Vertex Condition. We conclude that this case does not occur.

The same reasoning shows that if $V = 2$, the second vertex must be a (distinct) point on the same intersection line $\mathcal{L}$. Again no other planes can be contacted, and we may thus discard all planes distinct from Π_1 and Π_2.

Suppose finally that $V = 3$. Starting with a given vertex $\mathcal{V}_{12}$ on the intersection line $\mathcal{L}_{12}$ joining planes Π_1 and Π_2, we follow $\mathcal{C}$ from Π_1 across $\mathcal{L}_{12}$ onto Π_2 until the vertex $\mathcal{V}_{23}$ appears on the intersection line $\mathcal{L}_{23}$ between Π_2 and Π_3. We claim that $\mathcal{L}_{23}$ is distinct from $\mathcal{L}_{12}$. Otherwise, since no three planes can intersect along $\mathcal{L}_{12}$, $\mathcal{C}$ must continue back onto Π_1 and then to the third vertex $\mathcal{V}_{31}$ on an intersection line $\mathcal{L}_{31}$. If $\mathcal{L}_{31}$ coincides again with $\mathcal{L}_{12}$, then $\mathcal{C}$ must continue back onto Π_2 and could not join the initially chosen points on Π_1 without crossing a fourth vertex. Also, if $\mathcal{L}_{31}$ is distinct from $\mathcal{L}_{12}$ the same contradiction arises.

Thus, we may assume that $\mathcal{L}_{23}$ is distinct from $\mathcal{L}_{12}$, and that $\mathcal{C}$ crosses $\mathcal{L}_{23}$ at the vertex $\mathcal{V}_{23}$ onto a plane Π_3 distinct from Π_1 and Π_2 (and proceeds to the third vertex $\mathcal{V}_{31}$ on an intersection line $\mathcal{L}_{31}$ distinct from $\mathcal{L}_{12}$ and $\mathcal{L}_{23}$, by the same argument).

We observe that $\mathcal{V}_{12}$ and $\mathcal{V}_{23}$ share Π_2 as a common plane serving as one of the intersecting planes for both vertices. Similarly Π_3 is shared by $\mathcal{V}_{23}$ and $\mathcal{V}_{31}$.

We assert that Π_1 is shared by $\mathcal{V}_{31}$ and by $\mathcal{V}_{12}$. For by construction, it is one of the sides for $\mathcal{V}_{12}$. If $\mathcal{C}$ were to continue through $\mathcal{V}_{31}$ onto a plane distinct from Π_1, it would have to pass through still another vertex before returning to $\mathcal{V}_{12}$, contrary to hypothesis. Thus, $\mathcal{C}$ encounters only the three planes Π_1, Π_2, Π_3 of the supporting family, of which no two can be parallel, and we conclude also that $\mathcal{S}$ can encounter only those planes, as otherwise it would not be disk-type. All other planes can be deleted without affecting the variational conditions or the configuration.

If Π_1, Π_2, Π_3 share a common point $\mathcal{O}$, then we have a drop covering the vertex of a trihedral angle (see Figure 6). The other possibility is that the normals of Π_1, Π_2, Π_3 lie in a common plane, or equivalently that the three planes share a common (generating) direction. In this case we can replace all other planes of the supporting family with a single plane, situated far enough along the generating direction so as not to meet $\mathcal{S}$. We are done. $\square$

We are now prepared to prove:

Theorem 2. *Suppose $\mathcal{S}$ has $V \leq 2$ vertices, that the Vertex Condition holds, and that the data on interior points of any two adjacent support planes come from interior points of $\mathcal{Q}$. We assume further either that the hypotheses of Lemma 4.1 are fulfilled, or else that the hypotheses of Lemma 4.2 hold, with respect to each vertex. Then $\mathcal{S}$ is metrically spherical.*

Proof: We use Lemma 4.10. If $V = 0$ then the configuration is equivalent to a drop on a single plane, which it meets in a constant contact angle γ, $0 < \gamma < \pi$. If the contact set $\mathcal{C}$ is null, then the statement is equivalent to Hopf's theorem [Hop51]. If $\mathcal{C}$ is non-null, then in suitable local coordinates near any of its points, $\mathcal{S}$ can be represented as a graph $u(x, y)$ meeting a vertical wall in angle γ. As in the proof of Lemma 4.6 above, we find that u is twice Hölder differentiable to the boundary. It follows that $\mathcal{S}$ can be mapped conformally to the unit disk $|\zeta| < 1$, with l, m, n Hölder continuous to the boundary. As noted above, $\Phi \equiv ((l - n) - 2im)\, d\zeta^2$ is a holomorphic quadratic differential in the disk.

The mapping (2) takes the disk (and hence $\mathcal{S}$) to an infinite horizontal strip, so that the contact line $\mathcal{C}$ goes into the two bounding coordinate lines, with $l, m, n \to L, M, N$. By Joachimsthal's theorem [Joa46] these lines are curvature lines on $\mathcal{S}$, so that $M = 0$ on them. Following the discussion in Section 2, we see that L, M, N all tend uniformly to zero at the ends of the strip. Since M and $L - N$ are harmonic in the strip, we conclude from the maximum principle first that $M \equiv 0$, and then that $L \equiv N$. $\mathcal{S}$ is therefore totally umbilic and hence must be spherical.

The case $V = 1$ is vacuous by Lemma 4.10, and $V = 2$ reduces to a dihedral angle, which case is covered by Theorem 1. □

5. The Case $V = 3$

By Lemma 4.10, we may assume the configuration to be a trihedral angle bounded by three planes Π_1, Π_2, Π_3, with the angle vertex O covered by the liquid, or else a cylinder with sides Π_1, Π_2, Π_3 and closed at one end by a base (as illustrated conceptually in Figure 6).

We consider first the trihedral angle. In this configuration an explicit spherical cap solution Σ can be given (Figure 6) corresponding to any prescribed mean curvature $|H| > 0$, and contact angles γ_1, γ_2, γ_3, each pair of which lies in $\mathcal{Q}$. Specifically, the condition that Σ cuts Π_1 in angle γ_1 and Π_2 in angle γ_2 and encloses a segment of the edge $\mathcal{L}_{12}$ is precisely that $(\gamma_1, \gamma_2) \in \mathcal{Q}$. The centers of all such spheres of radius $1/|H|$ lie on a line $\mathcal{L}'_{12}$ parallel to $\mathcal{L}_{12}$. (For a proof that $\mathcal{L}'_{12}$ is uniquely determined, see [Fin96]). Similarly, the condition that Σ cuts Π_2 in angle γ_2 and Π_3 in angle γ_3 and encloses a portion of $\mathcal{L}_{23}$ is that $(\gamma_2, \gamma_3) \in \mathcal{Q}$. The line $\mathcal{L}'_{23}$ of centers of such spheres is not parallel to $\mathcal{L}'_{12}$, and it lies in a plane of centers of spheres of the given radius that meet Π_2 in angle γ_2. This plane contains $\mathcal{L}'_{12}$. Thus $\mathcal{L}'_{23}$ intersects $\mathcal{L}'_{12}$ in a unique point P_{123}, which provides the center of a sphere of radius $1/|H|$ with the stated property. We note that the procedure provides exactly two spherical caps, each part of the same sphere, for which the mean curvature vectors are directed

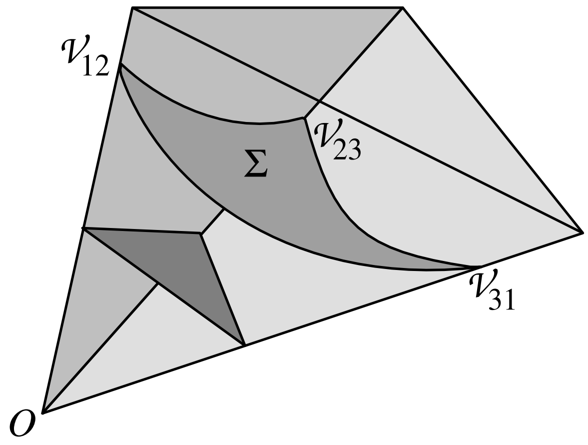

FIGURE 6. Equivalent spherical cap configuration; the dark triangle indicates a conceivable support plane, which would not affect the shape of Σ. The spherical cap solution Σ shown is by Theorem 3 the unique solution.

respectively into or out of the region cut off by the planes and the spherical surface Σ. A corresponding sphere of any radius meeting the walls in the same angles can then be obtained by a similarity transformation. An examination of the procedure shows that it applies also for data in the interior of $\partial\mathcal{Q} \cap \partial\mathcal{D}_1^{\pm}$. In that case the sphere will meet the corresponding line or lines in a single point, rather than enclose a segment as above. On the other hand, if any data pair lie exterior to the sets considered, then we find from the results of [CF96] that no spherical solution can exist. We conclude for trihedral angles that *if $V = 3$, then every surface of mean curvature $|H| > 0$ that meets the bounding planes in the given angles is spherical if and only if any spherical solution is uniquely determined among solutions of the same mean curvature.*

We now introduce a condition under which the spherical surface is the only possibility. We model our discussion on the procedure used by Vogel in [Vog93], who restricted attention to embedded surfaces bounded on a *smooth* supporting cone, in the sense that the support surface is assumed to be conical and to cut a unit sphere centered at the vertex in a curve with continuously turning tangent and lying in a hemisphere. Under the convention that $H > 0$ if the mean curvature vector points exterior to the region bounded between $\mathcal{S}$ and the vertex, he was able to prove that if $H > 0$ then $\mathcal{S}$ is uniquely determined by the boundary angle, which he assumed constant. His proof as given in [Vog93] does not apply to the (non-smooth) configurations and discontinuous boundary angles studied in the present paper. However, we are able to extend the result.

Theorem 3. *Assume the hypotheses of Lemma 4.2, that $V = 3$, and that the Vertex Condition holds. Given data that arise from the interior of $\mathcal{Q}$ or from the interior of $\partial\mathcal{Q} \cap \partial\mathcal{D}_1^{\pm}$, if the support surface given by Lemma 4.10 is a trihedral angle, then for any constant $H \geq 0$ any embedded disk-type surface $\mathcal{S}$ of mean curvature H that* *lies interior to the angle and meets the three plane pairs, in whose intersections the vertices lie, in the prescribed angles, is metrically spherical.*

Proof: As noted just above, if $H > 0$ then it suffices to show the uniqueness of a spherical cap solution in a trihedral angle as in Figure 6. More generally, we will prove the uniqueness of any surface of the type considered. By Lemma 4.2, if we choose any of the three intersection lines $\mathcal{L}$ as vertical axis in a Euclidean frame, then the height $u(x)$ of any such $\mathcal{S}$ is bounded near $\mathcal{L}$. By Tam's theorem [Tam86], see also [CFM97], $u(x)$ has first derivatives continuous to $\mathcal{L}$, and hence is itself continuous in the closure of a neighborhood $\mathcal{N}$ in a base plane. Thus, the closure of $\mathcal{S}$ can be represented by continuous functions.

We follow in outline Vogel's reasoning. If there were two distinct surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ with the same H and the same boundary conditions, there would be a point on one of the surfaces, say $\mathcal{S}_1$, that is exterior to the region $\mathcal{I}_2$ bounded between $\mathcal{S}_2$ and O (see Figure 6). Now scale $\mathcal{S}_1$ by a factor $\lambda < 1$, so that the scaled surface $\lambda\mathcal{S}_1$ lies in the closure of $\mathcal{I}_2$, and that there is at least one point on the closure of $\lambda\mathcal{S}_1$ that lies on the closure of $\mathcal{S}_2$. If any such point lies at a point of that closure distinct from the vertices, we can proceed as in [Vog93] and derive a contradiction from a touching principle. We may thus assume that all contact points lie at the vertices. Let $\mathcal{V}$ denote such a vertex, and $\mathcal{L}$ the corresponding intersection line, which we adopt as z-axis. We may then introduce a segment Γ in $\mathcal{N}$ as in Figure 7, cutting off with the projections $\mathcal{C}_\alpha^*$, $\mathcal{C}_\beta^*$ of the contact lines through $\mathcal{V}$ a closed triangle T over which $u_1(x,y) \le u_2(x,y)$, equality holding only at the projection P of $\mathcal{V}$.

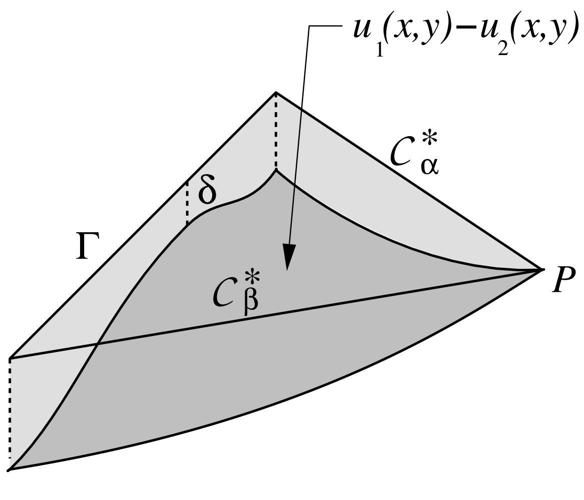

FIGURE 7. Construction for comparison proof

On Γ, $u_1(x,y) \le u_2(x,y) - \delta$, with $\delta > 0$. Also, $\lambda\mathcal{S}_1$ and $\mathcal{S}_2$ meet the vertical planes over $\mathcal{C}_\alpha^*$, $\mathcal{C}_\beta^*$ in identical angles γ_α, γ_β. In T, the functions $u_2(x,y)$ and $u_1^\delta(x,y) \equiv$

$u_1(x, y) + \delta$ satisfy respectively the equations

$$\operatorname{div} Tu = 2H \tag{16}$$

$$\operatorname{div} Tu = \frac{2}{\lambda} H > 2H, \tag{17}$$

where $Tu \equiv \nabla u / \sqrt{1 + |\nabla u|^2}$.

On Γ, $u_2 \geq u_1^\delta$. On $\mathcal{C}_\alpha^*$, $\mathcal{C}_\beta^*$ we have $\nu \cdot Tu_2 =$ cosine of boundary angle $= \nu \cdot Tu_1^\delta$. The point P is a set of linear Hausdorff measure zero. By the comparison principle Theorem 5.1 of [Fin86], there follows $u_2 \geq u_1^\delta$ throughout T. But by the construction, $u_1^\delta > u_2$ near P. This contradiction establishes the theorem when $H > 0$.

If $H = 0$ the discussion requires some changes in detail; notably the configuration is no longer uniquely determined, as scaling of any given solution leads to a continuum of further solutions. Also a planar solution is determined by data on only two of the three support planes; the data on the third plane lead to an overdetermined problem. On the other hand, scaling does not affect the curvature, thus permitting freedom in the direction in which the surface is scaled. We proceed as follows:

We suppose given an embedded surface $\mathcal{S}$ of mean curvature $H = 0$ in the trihedral angle formed by the planes Π_1, Π_2, Π_3, meeting the planes in the angles γ_1, γ_2, γ_3, and having tangent planes continuous to the three vertices. Denote by Π a plane tangent to $\mathcal{S}$ at the vertex $\mathcal{V}_{12}$. We intend to show that Π and $\mathcal{S}$ are identical.

According to the construction, Π meets the planes Π_1 and Π_2 in the angles γ_1, γ_2. It meets Π_3 in a constant angle $\hat{\gamma}_3$ (if Π is parallel to Π_3 then $\hat{\gamma}_3 = 0$), and we suppose initially that $\hat{\gamma}_3 \leq \gamma_3$. We move Π rigidly away from O,keeping its unit normal vector constant, until $\mathcal{S}$ is contained strictly in the region bounded between Π and O, and then move Π back toward O until a first point of contact appears.

If $\hat{\gamma}_3 < \gamma_3$, then such a point cannot appear on Π_3. In this case, the reasoning of Vogel excludes the appearance of other contact points distinct from the vertices unless $\mathcal{S}$ and Π coincide. If $\hat{\gamma}_3 = \gamma_3$, then Vogel's reasoning shows directly that any such contact point must be a vertex.

We may thus suppose as before that all initial contact points occur at vertices. If such a point appears at $\mathcal{V}_{12}$ then we may complete the reasoning as above. We therefore suppose an initial contact point at one of the other vertices, which we denote by $\mathcal{V}$; this implies in particular that Π is not parallel to Π_3, and thus that we can adopt the intersection line $\mathcal{L}$ through $\mathcal{V}$ as z-axis, with local representations $u_1(x, y)$, $u_2(x, y)$ for $\mathcal{S}$ and for Π. The configuration is again illustrated by Figure 7, however both functions $u_1(x, y)$ and $u_2(x, y)$ satisfy the same equation $\operatorname{div} Tu = 0$, and the relation $\nu \cdot Tu_2 = \nu \cdot Tu_1^\delta$ is now replaced by $\nu \cdot Tu_2 \geq \nu \cdot Tu_1^\delta$. Since $u_2 \geq u_1^\delta$ on Γ, the comparison principle again yields $u_2 \geq u_1^\delta$ near P, a contradiction.

There remains the possibility $\hat{\gamma}_3 > \gamma_3$. In this event Π cannot be parallel to Π_3, and we can move Π inwards toward O and then outward till an initial contact point with $\mathcal{S}$ appears. An analogous reasoning then applies, and we conclude finally that $\mathcal{S}$ and Π coincide, so that $\mathcal{S}$ is planar. □

As a particular consequence of the above reasoning, we obtain:

Corollary 5.1. *If a disk-type minimal surface $\mathcal{M}$ lies in a trihedral angle, meets the sides of that angle in constant angles γ_1, γ_2, γ_3 and has a tangent plane continuous to the edges, then $\mathcal{M}$ is a plane.*

Remark: In spherical coordinates, a surface $u(\theta, \phi)$ of mean curvature H satisfies the equation

$$\frac{\partial}{\partial \theta}\frac{u_\theta}{W} + \frac{\partial}{\partial \phi}\frac{u_\phi \sin^2\phi}{W} = 2\left(\frac{\sin\phi}{W} + H\right) u \sin\phi, \tag{18}$$

where $W = \sqrt{(u^2 + u_\phi^2)\sin^2\phi + u_\theta^2}$. The method of Ambrazevičius [Amb81, Amb82] applied to such an equation when $H \geq 0$ yields a comparison principle analogous to that of [Fin86, Theorem 5.1], and shows directly the uniqueness of any solution with mixed Dirichlet and boundary angle data. For surfaces that admit such a representation, this result could have replaced the procedure we used to prove Theorem 3. The results in [Amb81, Amb82] have independent interest, and assure the existence of solutions in conical regions, with prescribed boundary data and contact angle. □

In the case of a cylinder, the sign of H is irrelevant:

Theorem 4. *Assume the hypotheses of Lemma 4.2, that $V = 3$, and that the Vertex Condition holds. Given data that arise from the interior of $\mathcal{Q}$ or from the interior of $\partial\mathcal{Q} \cap \partial\mathcal{D}_1^\pm$, if the support surface given by Lemma 4.10 is a cylinder, then any embedded disk-type surface $\mathcal{S}$ of constant mean curvature that meets the three plane pairs, in whose intersections the vertices lie, in the prescribed angles, is metrically spherical.*

Proof: A discussion analogous to that proceeding Theorem 3 shows that *any angle data of the type described in the theorem determines a unique (up to translation) spherical solution* Σ. Denote the mean curvature of this spherical solution with respect to the normal pointing out of the enclosed volume by H_0.

Let $\mathcal{S}$ be any non-spherical solution satisfying the same boundary conditions and with constant mean curvature H. If $H \leq H_0$, then translate $\mathcal{S}$ along the generating direction (see Figure 8) until the drop volume determined by Σ is contained in that determined by $\mathcal{S}$ and yet $\mathcal{S} \cap \Sigma \neq \phi$. Proceeding as in the proof of Theorem 3, we obtain a contradiction by the comparison principle. If $H \geq H_0$, then we translate Σ to obtain the same contradiction. □

While we have extended Theorem 3 to the (limiting) case in which the three planes have coplanar normals, so that the supporting configuration is cylindrical, we note that Theorem 4 is false for polyhedral angles of more than three sides; in fact, the spherical cap solution is determined by any three sides of the angle. Thus, given any three such sides, the remaining configuration is uniquely determined by the requirement that a spherical cap be a solution.

It should be noted again that our demonstrations for the case $V = 3$ differ in essential ways from those we present when $V = 2$: the latter rely chiefly on properties of conformal mappings, while the former are based entirely on particular forms of the

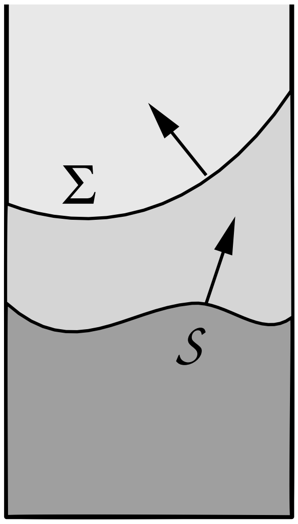

FIGURE 8. Construction for comparison proof, cylindrical case

comparison principle. Our discussion above for a trihedral angle applies only to the case $H \geq 0$. But Theorem 1 shows that under the hypotheses of that theorem there can be no drop with $H \geq 0$ in a wedge. Consider as an example a trihedral angle formed by three orthogonal planes, with the same contact angle $\gamma = \cos^{-1}(\sqrt{3}/3)$ on all planes. A symmetrically placed planar surface $\mathcal{S}$ then determines a drop that wets all planes, and by Theorem 3 it is the unique such drop for which $H = 0$. If we now rotate one of the planar sides Π into the angle about its intersection line with one of the other planes, then a surface $\mathcal{S}$ with $H > 0$ can be found as a spherical cap, and according to Theorem 3 it is the unique disk-type surface with the given γ and that value of H. On the other hand, if we rotate Π an amount less than $\pi/2$ in the other direction then $H < 0$ for the spherical cap, and no uniqueness theorem is available. But if we continue the rotation back until Π coincides with the other plane and the trihedral angle becomes a wedge, then Theorem 1 guarantees the uniqueness of the spherical cap among all competing disk-type surfaces. On the basis of these remarks, we formulate

Conjecture 1. *Theorem 3 holds without the hypothesis $H \geq 0$.*

Appendix

The method of this paper (for the case $V = 2$) yields as corollary a conceptually simpler proof of a theorem of Nitsche [Nit85], that an immersed disk $\mathcal{S}$ of constant mean curvature that intersects a sphere Σ at constant angle along a smooth closed curve $\mathcal{C}$ is necessarily a spherical cap. In fact, every curve is a curvature line on Σ, thus $\mathcal{C}$ has that property and thus by [Joa46] $\mathcal{C}$ is a curvature line on $\mathcal{S}$. We map $\mathcal{S}$ conformally to a unit disk in the plane, and the disk to a strip by (2). We find immediately using (6) that the second fundamental form vanishes identically in the strip, from which follows that $\mathcal{S}$ is totally umbilic and hence a metric sphere.

We wish to thank Erich Miersemann for a number of helpful discussions, and notably for informing us of the paper of Azzam [Azz79].

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