BASES IN SYSTEMS OF SIMPLICES AND CHAMBERS

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ABSTRACT. We consider a finite set E of points in the n-dimensional affine space and two sets of objects that are generated by the set E: the system Σ of n-dimensional simplices with vertices in E and the system Γ of chambers. The incidence matrix $A = \| a_{\sigma,\gamma} \|$, $\sigma \in \Sigma$, $\gamma \in \Gamma$, induces the notion of linear independence among simplices (and among chambers). We present an algorithm of construction of bases of simplices (and bases of chambers). For the case $n = 2$ such an algorithm was described in [1]. However, the case of n-dimensional space required a different technique. It is also proved that the constructed bases of simplices are geometrical (according to [1]).

1. INTRODUCTION.

Let $E = (e_1, e_2, \ldots, e_N)$, $N > n$, be a finite set of points in an n-dimensional affine space V. Let $P = conv(E)$ be the convex hull of E. Let $\sigma = \sigma(e_{i_1}, \ldots, e_{i_{n+1}})$ be the n-dimensional simplex with the vertices $e_{i_1}, \ldots, e_{i_{n+1}} \in E$. Denote by Σ the set of all such simplices σ. All the simplices σ (as a rule overlapping) cover the polytope P. The simplices σ divide the polytope P into a finite number of chambers γ (see Definition (1.1)). Denote by Γ the set of all chambers γ in P.

Definition 1.1. *Let $\sigma \in \Sigma$ and $\tilde{\sigma}$ be the boundary of σ. Let $\tilde{\Sigma} = \bigcup_{\sigma \in \Sigma} \tilde{\sigma}$ and $\bar{P} = P \setminus \tilde{\Sigma}$. Let $\bar{\gamma}$ be a connected component of $\bar{P}$ and γ be closure of $\bar{\gamma}$. We call γ a chamber and $\bar{\gamma}$ an open chamber.*

Let A be the incidence matrix between simplices and chambers, i.e.

$$\| a_{\sigma,\gamma} \| = 1 \text{ iff } \gamma \subset \sigma.$$

Consider the linear space V_Σ generated by the rows of A and the linear space V_Γ generated by the columns of A over some field of characteristic 0. Due to one-to-one correspondence between the rows of A and the simplices $\sigma \in \Sigma$, we can speak about a linear combination of simplices instead of a linear combination of the corresponding rows of A. An important question is to construct bases of simplices and bases of chambers, i.e. bases in V_Σ (or in V_Γ) that consists of simplices (or chambers) and not of their linear combinations.

A basis of simplices can be also defined as follows. Let $\phi_\sigma(x)$ be the characteristic function of a simplex σ, i.e.

$$\phi_\sigma(x) = 1, \; x \in \sigma \;\; and \;\; \phi_\sigma(x) = 0, \quad x \notin \sigma.$$

Date: October 6, 1997.

Supported by the Gabriella and Paul Rosenbaum Foundation; part of these results was obtained at MSRI supported by NSF grant DMS 9022140.

A basis of simplices is a maximal subset of simplices such that their characteristic functions $\phi_\sigma(x)$ are linearly independent.

In this paper we will describe (Section 2) the inductive algorithm of constructing bases of simplices and bases of chambers in the n-dimensional affine space; the algorithm uses the case $n = 2$ (see [1]) as the first step of induction. In Sections 3, 4 we prove that the set B of simplices and the set B' of chambers constructed by the algorithm are indeed bases in V_Σ and in V_Γ respectively.

2. Construction of a basis of simplices and a basis of chambers.

2.1. **A special ordering of points $e \in E$ and related polytopes.** Let $E = \{e_1, \dots , e_N\}$ be a set of points in an n-dimensional affine space V^n, Σ the set of n-dimensional simplices σ with the vertices in E and Γ the set of chambers γ (defined in Introduction.)

We will define an ordering of points $e_i \in E$ which is essential in the construction.

Lemma 2.1. *Let $E = \{e_1, e_2, \dots , e_N\}$ be a finite set of points in the n-dimensional affine space. There exists an ordering $e_{i_1}, e_{i_2}, \dots , e_{i_N}$, $e_{i_k} \in E$, such that for $k = 1, \dots , N$*

$$conv(e_{i_1}, \dots , e_{i_k}) \cap conv(e_{i_{k+1}}, \dots , e_{i_N}) = \emptyset, \tag{1}$$

and there exists a hyperplane H_k which separates the polytopes

$$F_k = conv(e_{i_1}, \dots , e_{i_k}) \tag{2}$$

and

$$P_k = conv(e_{i_{k+1}}, \dots , e_{i_N}) = conv(E \setminus (e_{i_1}, \dots , e_{i_k})). \tag{3}$$

We assume also $F_0 = \emptyset, P_N = \emptyset, P_0 = conv(E) = P, F_N = conv(E) = P$.

This lemma is proved, for example, in [1]. The ordering $e_1, \dots , e_N$ satisfying Lemma 2.1 yields a shelling of the polytope P, see [4].

Let the ordering $e_1, \dots , e_N$ satisfy (1). Consider the sequence of polytopes $P_0 = P, P_1, \dots , P_N = \emptyset$ defined by formula (3). We have $P_0 \supset P_1 \supset \dots \supset P_N$. Let us denote

$$S_k = \overline{P_{k-1} \setminus P_k}, \tag{4}$$

where $\overline{A}$ means the closure of the set A. Let $int(S)$ be the interior of S.

It is easy to check that the following statements are true:

1. Each ordering $e_1, \dots , e_N$ determines a decomposition of the polytope P:

$$P = \bigcup_{i=1}^{N-n} S_i, \tag{5}$$

where $int(S_i) \cap int(S_j) = \emptyset$ for $i \neq j$.

2. $int(S_k) \cap E = \emptyset$ for $k = 1, \dots , N - n$.

3. The polytope S_k is part of the convex cone with the vertex e_k and bounded by some part $\mathcal{L}_k$ of the boundary of P_{k-1}, (where $\mathcal{L}_k = S_k \cap P_{k-1}$). Note that the polytope S_k is not necessarily convex[1].
4.

$$P = S_1 \cup \ldots \cup S_{k-1} \cup P_{k-1} \tag{6}$$

2.2. **A map from the polytope P to the hyperplane H_k.** Let an ordering of points $e_1, \ldots, e_N$ satisfy (1). Consider the point $e_k \in P_{k-1}$ and the corresponding hyperplane H_k. For a segment (e_k, e_i), $i = k+1, \ldots, N$ let us denote

$$e_i^k = (e_k, e_i) \cap H_k. \tag{7}$$

Thus, on the hyperplane H_k we obtained the set E_k of points $e_1^k, e_2^k, \ldots, e_{N_k}^k$. (Note that $N_k \leq N - k$ since the points $e \in E$ are not necessarily in general position.) In the hyperplane H_k we use the points $e^k \in E_k$ to construct simplices σ^k and chambers γ^k in the same way as it was done for the set E in V^n . Let Σ_k be the set of all these simplices σ^k and Γ_k the set of all chambers γ^k.

Let $\sigma^k = \sigma(e_{i_1}^k, \ldots, e_{i_n}^k) \in \Sigma_k$. We denote by $\overrightarrow{(e_k, e_{i_j}^k)}$ the ray starting at e_k and passing through the point $(e_{i_j}^k)$. Consider the following map:

$$\mu : \sigma^k \mapsto \sigma, \quad where\ \sigma = \sigma(e_k, e_{i_1}, \ldots, e_{i_n}), \tag{8}$$

and where $e_{i_j} \in E$ is the nearest point to the point e_k on the ray $\overrightarrow{(e_k, e_{i_j}^k)}$. It is clear that $\sigma = \mu(\sigma^k) \in \Sigma$. The map μ is an injection and has the following easy to check property.

Proposition 2.2. *Let $\sigma^k, \sigma_0^k \in \Sigma_k$ be open simplices such that $\sigma^k \cap \sigma_0^k = \emptyset$. Then $\mu(\sigma^k) \cap \mu(\sigma_0^k) = \emptyset$.*

Consider the set of points E_k in the hyperplane H_k. Let us reorder $e^k \in E_k$ according to (1). Let $e_1^k, e_2^k, \ldots, e_{N_k}^k$ be such an ordering. Similarly to formulas (3) and (4) we denote

$$P^k = P_0^k = conv(E_k), \;\; P_j^k = conv(e_{j+1}^k, \ldots, e_{N_k}^k), \;\; S_j^k = P_{j-1}^k \setminus P_j^k \tag{9}$$

where $j = 1, \ldots, N_k$. For these $(n-1)$-dimensional polytopes the formulas analogous to (5) and (6) hold:

$$P^k = \bigcup_j S_j^k, \tag{10}$$

where $int(S_i^k) \cap int(S_j^k) = \emptyset$ for $i \neq j$, and

$$P^k = S_1^k \cup \ldots \cup S_{j-1}^k \cup P_{j-1}^k. \tag{11}$$

On the polytopes S_j^k and P_j^k let us define the following map τ: $\tau(S_j^k)$ is an n-dimensional cone with the vertex e_k and generated by the rays $\overrightarrow{(e_k, x)}$, where $x \in S_j^k$. The map $\tau(P^k)$ is defined similarly. Clearly, the following decomposition holds:

[1] The polytopes S_k are considered in more detail in [1].

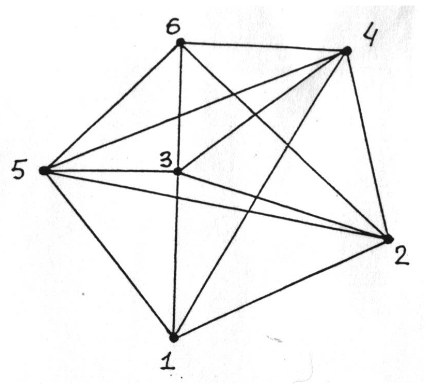

FIGURE 1

$$\tau(P^k) = \bigcup_i \tau(S_i^k), \tag{12}$$

where $int(\tau(S_i^k)) \cap int(\tau(S_j^k)) = \emptyset$ for $i \neq j$.

2.3. **Algorithm of construction of the set B of simplices and the set B' of chambers.** We will construct the set $B \subset \Sigma$ of simplices and the set B' of chambers and will prove in Sections 3 and 4 that the set B is a basis in V_Σ and the set B' is a basis in V_Γ. The algorithm of construction of B and B' is inductive on the dimension n of the affine space V^n.

Let an ordering of points $e_1, \dots, e_N$ satisfy (1). For each point e_k we construct a set B_k of simplices $\sigma \in \Sigma$ and a set B'_k of chambers $\gamma \in \Gamma$. Then we define $B = \bigcup B_k$ and $B' = \bigcup B'_k$.

First step ($n = 2$). The points $e_1, \dots, e_N$ lie on the affine plane V^2. Let us denote by q an edge of a simplex $\sigma \in \Sigma$ and by Q the set of all edges of all simplices $\sigma \in \Sigma$. Consider the point $e_k \in P_{k-1}$. In P_{k-1} from the point e_k there are following edges $q_i = (e_k, e_i)$, where $i \in (k+1, \dots, N)$. Note that since the points $e \in E$ are not necessarily in general position, some of these edges may coincide and several points e_i, $i \in (k+1, \dots, N)$ may lie on the same edge.

Let $q_i = (e_k, e_i)$ and $q_j = (e_k, e_j)$, where $i, j \in (k+1, \dots, N)$, be two neighbor edges with the vertex e_k (i.e. there is no other edge $q = (e_k, e_m)$, $m \in (k+1, \dots, N)$ which lies between q_i and q_j). Let the point e_i be the nearest point of E to the point e_k on the edge q_i and, respectively, e_j the nearest point to the point e_k on the edge q_j. Let $\sigma = \sigma(e_k, e_i, e_j)$. We define B_k as the set of all such simplices σ. We define then

$$B = \bigcup_k B_k.$$

In Figure 1 there is an example of a set B of simplices constructed according to this algorithm.

For this example we have: $B_1 = (153, 134, 142)$; $B_2 = (253, 236, 264)$; $B_3 = (356, 364)$; $B_4 = (456)$.

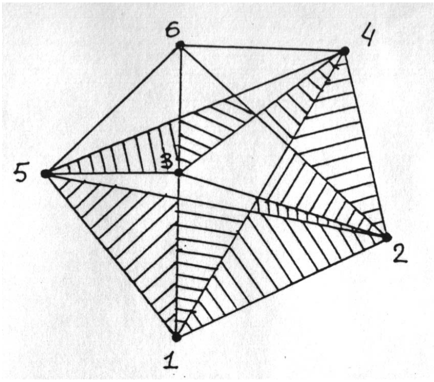

FIGURE 2. (There is a mistake in the figure. One shaded chamber should be in a different place.)

With the point e_k we also associate the following set of chambers B'_k. In each simplex σ let us choose one chamber[2] adjacent to the point e_k. We define the set B'_k as the set of all such chambers and define

$$B' = \bigcup_k B'_k.$$

In Figure 2 the chambers from the set B' are shaded.

Suppose that we have described the construction in V^{n-1}. Let us describe it in V^n.

Let the points $e_i \in E$ be in V^n and let $e_1, \dots, e_N$ be an ordeing satisfying (1). Consider the point e_k and the hyperplane H_k from Lemma 2.1. In the hyperplane H_k we have the set of points E_k (see (7)), the set of simplices Σ_k and the set of chambers Γ_k. Let us reorder the points e^k_i so that the ordering $e^k_1, e^k_2, \dots, e^k_{N_k}$ satisfies (1).

By the induction hypothesis, we can construct a set of simplices in H_k, i.e. the set $\tilde{B}_k \subset \Sigma_k$ and the set of chambers in H_k, i.e. the set $\tilde{B}'_k \subset \Gamma_k$. Then we define the set B_k of simplices in V^n as $B_k = \{\mu(\sigma^k), \forall \sigma_k \in \Sigma_k\}$, where the map μ is defined by (8). Finally, we define $B = \bigcup_k B_k$.

We have already constructed the set $\tilde{B}'_k$ of chambers in the hyperplane H_k. Consider $\gamma^k \in \tilde{B}'_k$. According to the algorithm in H_k the chamber γ^k was chosen at a certain step j, $j \in (1, \dots, N_k)$ and the point e^k_j is a vertex of γ^k. Besides, there is one simplex $\sigma^k \in \tilde{B}_k$ such that e^k_j is a vertex of σ^k and $\gamma^k \subset \sigma^k$. Let us choose a chamber $\gamma \in \Gamma$ such that

1) $\gamma \subset \mu(\sigma^k)$;

2) γ is adjacent to the point e_k and to the edge (e_k, e_j).

[2]In case of an affine plane in each simplex σ there is only one chamber adjacent to the point e_k since every edge of a chamber $\gamma \in \Gamma$ necessarily lies on some edge $q \in Q$.

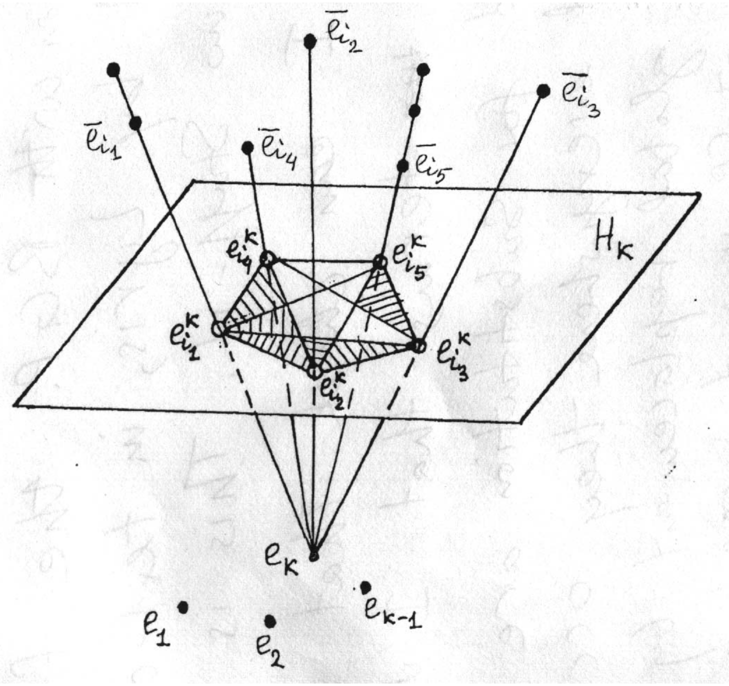

FIGURE 3

Thus, with each chamber $\gamma^k \in \tilde{B}'_k$ we associate a chamber $\gamma \in \Gamma$. Denote by B'_k the set of all such chambers γ corresponding to $\gamma^k \in \tilde{B}'_k$ and define

$$B' = \bigcup_k B'_k.$$

In Figure 3 there is a fragment of a configuration of points in the 3-dimensional space. In the plane H_k, separating the points $e_1, \dots, e_k$ and $e_{k+1}, \dots, e_N$, there are five points which are reordered according to condition (1). For this ordering in the plane H_k we construct the basis

$$\tilde{B}_k = \{(e_{i_1}e_{i_4}e_{i_5}), (e_{i_1}e_{i_5}e_{i_3}), (e_{i_1}e_{i_3}e_{i_2}); (e_{i_2}e_{i_4}e_{i_5}), (e_{i_2}e_{i_5}e_{i_3}); (e_{i_3}e_{i_4}e_{i_5})\}$$

of simplices and the basis $\tilde{B}'_k$ of chambers (are shaded). The set B_k consists of the following simplices:

$$B_k = \{(e_k\bar{e}_{i_1}\bar{e}_{i_4}\bar{e}_{i_5}),\ (e_k\bar{e}_{i_1}\bar{e}_{i_5}\bar{e}_{i_3}),\ (e_k\bar{e}_{i_1}\bar{e}_{i_3}\bar{e}_{i_2});\ (e_k\bar{e}_{i_2}\bar{e}_{i_4}\bar{e}_{i_5}),\ (e_k\bar{e}_{i_2}\bar{e}_{i_5}\bar{e}_{i_3});\ (e_k\bar{e}_{i_3}\bar{e}_{i_4}\bar{e}_{i_5})\},$$

where $\bar{e}_j = \mu(e^k_j)$.

Note that a chamber from B'_k cannot be seen in Figure 3 since we need to take into account the points $e_1, \dots, e_{k-1}$ which are below the plane H_k.

3. Linear independence of simplices $\sigma \in B$ and linear independence of chambers $\gamma \in B'$.

Let $B \subset \Sigma$ and $B' \subset \Gamma$ be the sets constructed in Section 2.

Theorem 3.1. *The simplices $\sigma \in B$ are linearly independent in V_Σ and the chambers $\gamma \in B'$ are linearly independent in V_Γ.*

First let us prove the following proposition.

Proposition 3.2. *Let $\tilde{A}$ be a submatrix of the incidence matrix A corresponding to the rows $\sigma \in B$ and the columns $\gamma \in B'$. The columns and rows of the matrix $\tilde{A}$ can be ordered in such a way that: 1) $\tilde{A}$ is a block matrix $\| \mathcal{A}_{ik} \|$, where $\mathcal{A}_{ik} = 0$ for $i > k$, and 2) each diagonal element $a_{\sigma,\gamma}$ of the matrix $\tilde{A}$ equals 1, i.e. $a_{\sigma,\gamma} = 1$.*

Proof. Let us recall that by the algorithm the set B of simplices was constructed as $B = \cup B_k$ and the set B' of chambers as $B' = \cup B'_k$, where B_k, B'_k correspond to the point e_k in the ordering $e_1, \dots, e_N$. Thus, the matrix $\tilde{A}$ is a block-matrix $\| \mathcal{A}_{ik} \|$, where

$$\mathcal{A}_{ik} = \{a_{\sigma,\gamma},\ \sigma \in B_i,\ \gamma \in B'_k\}. \tag{13}$$

Consider a diagonal block $\mathcal{A}_{kk}$ ($k = 1, \dots, N-n$). By the construction, to each simplex $\sigma \in B_k$ there corresponds a chamber $\gamma \in B'_k$ such that $\gamma \subseteq \sigma$, therefore, $a_{\sigma,\gamma} = 1$. Thus, if we choose the corresponding orderings of columns in B'_k and rows in B_k, we obtain "1" on the main diagonal in the block $\mathcal{A}_{kk}$ and, therefore, any diagonal element $a_{\sigma,\gamma}$ of the matrix $\tilde{A}$ is such that $a_{\sigma,\gamma} = 1$.

Consider a block $\mathcal{A}_{ik}$, where $i > k$. We need to prove that $a_{\sigma,\gamma} = 0$, where $\sigma \in B_i$ and $\gamma \in B'_k$. Let $\sigma \in B_i$ and $\gamma \in B'_1 \cup \ldots \cup B'_{i-1}$. Then $\sigma \in P_{i-1} = conv(e_i, \dots, e_N)$. ¿From the algorithm for the construction of the set B' it is easy to see that $\gamma \in S_1 \cup \ldots \cup S_{i-1}$. Due to formulas (5) and (6) we conclude that $\gamma \not\subset \sigma$ and, therefore, $a_{\sigma,\gamma} = 0$. Thus, the matrix $\tilde{A}$ is upper triangular as a block matrix. □

Proof of Theorem 3.1. We will show that the submatrix $\tilde{A}$ defined above is an upper triangular matrix. Due to Proposition 3.2 it is sufficient to prove that a block $\mathcal{A}_{kk}$ (where $k = 1, \dots, N-n$) of the matrix $\tilde{A}$ is upper triangular.

Let us consider the case $n = 2$. Let $e_1, \dots, e_N$ be an ordering satisfying (1). By the construction, the simplices $\sigma \in B_k$ lie between the neighbor edges starting at the point e_k, therefore, the open simplices $\sigma \in B_k$ are disjoint. Besides, there is exactly one chamber $\gamma \in B'_k$ (i.e. with the vertex e_k) such that $\gamma \in \sigma$, therefore, the block $\mathcal{A}_{kk}$ is the identity matrix.

Since for a general n the notations are cumbersome we consider in detail the case $n = 3$ which already contains all the technique. Let $e_1, \dots, e_N$ be an ordering satisfying (1). Consider a point e_k and the block $\mathcal{A}_{kk}$ of the matrix $\tilde{A}$. We recall (see Section 2.3) that a simplex $\sigma \in B_k$ is defined as $\mu(\sigma^k)$, where the 2-dimensional simplex $\sigma^k \in \tilde{B}_k$ lies in the plane H_k separating the points $e_1, \dots, e_k$ and $e_{k+1}, \dots, e_N$.

The set $\tilde{B}_k$ of simplices is constructed according to the algorithm in H_k. For this we order the points e^k_i (see (7)) according to condition (1). Let $e^k_1, \dots, e^k_{N_k}$ be such an ordering. Then $\tilde{B}_k = \bigcup_i \tilde{B}_{k,i}$, where $\tilde{B}_{k,i}$ is the set of 2-dimensional simplices which were chosen in the algorithm for the point e^k_i. We have a similar equality for chambers: $\tilde{B}'_k = \bigcup_i \tilde{B}'_{k,i}$, where $\tilde{B}'_{k,i}$ is the set of 2-dimensional chambers chosen in the algorithm for the point e^k_i.

Due to the inductive construction of simplices $\sigma \in B_k$ and chambers $\gamma \in B'_k$ we obtain also the following formulas:

$$B_k = \bigcup_i B_{k,i}$$

and

$$B'_k = \bigcup_i B'_{k,i}.$$

This means that the block $\mathcal{A}_{kk}$ consists, in turn, of blocks $\mathcal{B}_{i_1,k_1}$:

$$\mathcal{B}_{i_1,k_1} = \{a_{\sigma,\gamma} : \sigma^k \in \tilde{B}_{k,i_1}, \gamma^k \in \tilde{B}'_{k,k_1}\},$$

where $\sigma = \mu(\sigma^k)$ and the chamber γ corresponds [3] to the chamber $\gamma^k \subset H_k$.

1^0. Consider a diagonal block $\mathcal{B}_{k_1,k_1}$ of the block $\mathcal{A}_{kk}$, i.e. all $a_{\sigma,\gamma}$, where $\sigma \in B_{k,k_1}$ and $\gamma \in B'_{k,k_1}$.

Let $\sigma, \sigma_0 \in B_{k,k_1}$. Then the corresponding simplices σ^k and σ^k_0 were constructed on the plane H_k from the same point $e^k_{k_1}$. By the algorithm the open simplices σ^k, σ^k_0 are disjoint. Then (see Proposition 2.2) $\mu(\sigma^k) \cap \mu(\sigma^k_0) = \emptyset$, where μ is defined by formula (8). But $\sigma = \mu(\sigma^k)$, and $\sigma_0 = \mu(\sigma^k_0)$. Obviously, for any chamber γ such that $\gamma \subset \sigma$, we have $\gamma \not\subset \sigma_0$. This means that the diagonal block $\mathcal{B}_{k_1,k_1}$ is the identity matrix.

2^0. Let us show that $\mathcal{B}_{i_1,k_1} = 0$ for $i_1 > k_1$. Indeed, let $\sigma \in B_{k,i_1}$ and $\gamma \in B'_{k,k_1}$. Then from the construction we have $\sigma^k \in \tilde{B}_{k,i_1}$ and $\sigma^k \in P^k_{i_1-1} = conv(e^k_{i_1}, \dots, e^k_{N_k})$ (see (9)).

Concerning γ we know that there is a simplex $\sigma_0 \in B_{k,k_1}$ such that:

1) $\gamma \in \sigma_0$;
2) γ is adjacent to the point e_k;
3) γ is adjacent to the edge $(e_k, e^k_{k_1})$.

Since $\sigma_0 \in B_{k,k_1}$ we have $\sigma^k_0 \in \tilde{B}_{k,k_1}$ and $\sigma^k_0 \subset P^k_{k_1-1}$.

For $i_1 > k_1$ we can rewrite the formula (10) as follows:

$$P^k = S^k_1 \cup \ldots \cup S^k_{k_1} \cup S^k_{k_1+1} \cup \ldots \cup S^k_{i_1-1} \cup P^k_{i_1-1},$$

where all the open polytopes S^k_j and $P^k_{i_1-1}$ are disjoint. Note that if $i_1 = k_1 + 1$ then $S^k_{k_1} = S^k_{i_1-1}$.

Due to decomposition (12) we obtain

$$\tau(P^k) = \tau(S^k_1) \cup \ldots \cup \tau(S^k_{k_1}) \cup \tau(S^k_{k_1+1}) \cup \ldots \cup \tau(S^k_{i_1-1}) \cup \tau(P^k_{i_1-1}),$$

where all the corresponding open cones are disjoint. We have $\sigma = \mu(\sigma^k) \subset \tau(P^k_{i_1-1})$. Let us show that $\gamma \subset \tau(S^k_{k_1})$. Since $\gamma \subset \sigma_0$, then $\gamma \subset \mu(\sigma^k_0)$, i.e. $\gamma \subset \tau(P^k_{k_1-1})$.

[3]Note that in the algorithm there is no direct correspondence between the chambers γ and γ^k; given a chamber $\gamma^k \subset H_k$ we find the corresponding simplex $\sigma^k \subset H_k$, then in the simplex $\sigma = \mu(\sigma^k)$ we choose a certain chamber (see Section 2.3).

Note that $P^k_{k_1-1} = S^k_{k_1} \cup S^k_{k_1+1} \cup \dots$ (see formulas (9) and (10)). Then $\gamma \subset \tau(S^k_{k_1}) \cup \tau(S^k_{k_1+1}) \cup \dots$. Any chamber $\gamma \in \Gamma$ may lie in only one of these cones. Since γ is adjacent to the edge $(e_k, e^k_{k_1})$ we obtain $\gamma \subset \tau(S^k_{k_1})$. Then $\gamma \not\subset \tau(P^k_{i_1-1})$, i.e. $\gamma \not\subset \sigma$. We have proved that $a_{\sigma,\gamma} = 0$ for $\sigma \in B_{k,i_1}$ and $\gamma \in B'_{k,k_1}$ if $i_1 > k_1$.

This completes the proof (for $n = 3$) that the matrix $\tilde{A}$ is an upper triangular matrix.

In case of n-dimensional space an element $a_{\sigma,\gamma}$ of the matrix $\tilde{A}$ belongs to a sequence of enclosed blocks which can be denoted as $\mathcal{A}_{i_1,k_1}, \mathcal{B}_{i_2,k_2}, \mathcal{C}_{i_3,k_3}, \dots$. (One can check that the "depth" of the enclosed blocks, i.e. the length of the sequence, is $n-1$.)

1^0. Let us show that a diagonal block of maximal depth is the identity matrix. Let $\mathcal{Z}_{k_{n-2},k_{n-2}}$ be such a block. Then $\mathcal{Z}_{k_{n-2},k_{n-2}}$ lies inside all diagonal blocks $\mathcal{A}_{k_0,k_0}, \mathcal{B}_{k_1,k_1}, \mathcal{C}_{k_2,k_2}, \dots$ of the matrix $\tilde{A}$. Let σ, σ_0 be two simplices from the block $\mathcal{Z}_{k_{n-2},k_{n-2}}$. This means that σ and σ_0 have the same sequence of corresponding points in the inductive construction. Let us denote these points by $e_{k_0}, e^{(1)}_{k_1}, e^{(2)}_{k_2}, \dots, e^{(n-2)}_{k_{n-2}}$, where $e_{k_0} \in E$, $e^{(1)}_{k_1} \in H_{k_0}$, $e^{(2)}_{k_2}$ lies in the $(n-2)$-dimensional plane which corresponded to the point $e^{(1)}_{k_1}$ in the algorithm, and so on; finally, the point $e^{(n-2)}_{k_{n-2}}$ lies in the 2-dimensional plane which corresponded to the point $e^{(n-3)}_{k_{n-3}}$. On this latter plane there are two simplices $\tilde{\sigma}, \tilde{\sigma_0}$ which were both chosen for the point $e^{(n-2)}_{k_{n-2}}$.

According to the algorithm for the plane the open simplices $\tilde{\sigma}, \tilde{\sigma_0}$ are disjoint. Therefore (see Proposition 2.2), for the 3-dimensional open simplices we have $\mu(\sigma) \cap \mu(\sigma_0) = \emptyset$, where μ is a map defined by formula (8) for the point $e^{(n-3)}_{k_{n-3}}$ instead of e_k. Then in order to obtain 4-dimensional simplices, the map μ for the point $e^{(n-4)}_{k_{n-4}}$ was applied to the simplices $\mu(\sigma), \mu(\sigma_0)$ and so on. Finally, the map μ for the point e_{k_0} gives the simplices $\sigma, \sigma_0 \in \Sigma$. Clearly, we have $\sigma \cap \sigma_0 = \emptyset$. Therefore, for any chamber $\gamma \subset \sigma$ we have $\gamma \not\subset \sigma_0$, i.e. the block $\mathcal{Z}_{k_{n-2},k_{n-2}}$ is the identity matrix.

2^0. In order to prove that the matrix $\tilde{A}$ is upper triangular it suffices to check that for $i > j$ any block $\mathcal{X}_{i,j} \in (\mathcal{B}_{i_1,k_1}, \mathcal{C}_{i_2,k_2}, \dots)$ satisfies the condition $\mathcal{X}_{i,j} = 0$. In Proposition 3.2 we have proved that for $i > k$ a block $\mathcal{A}_{ik} = 0$. By applying the arguments from 2^0 of $n = 3$ case, we can prove that in a diagonal block $\mathcal{A}_{kk}$ we have $\mathcal{B}_{i_1,k_1} = 0$ for $i_1 > k_1$. Similarly, $\mathcal{C}_{i_2,k_2} = 0$ for $i_2 > k_2$, etc.

It follows from 1^0 and 2^0 that the submatrix $\tilde{A}$ of the incidence matrix A is an upper triangular matrix. □

4. The set B is a geometrical basis of simplices.

In this section we will prove that the set B constructed by the algorithm is a geometrical basis. First we repeat some of the definitions and theorems from [3] and [1].

Let $E = (e_1, \dots, e_N)$ be a set of points in the n-dimensional affine space V^n, Σ the set of simplices and Γ the set of chambers (defined in the Introduction).

Consider a subset $S \subseteq E$ consisting of $n+2$ points and such that S contains at least $n+1$ points in general position. Denote

$$f = \{\sigma : \sigma(e_{i_1}, \dots, e_{i_{n+1}}) \in \Sigma \text{ and } e_{i_k} \in S\} \tag{14}$$

Thus, with each S we associate a subset $f \subset \Sigma$ (clearly, $f \neq \emptyset$). Let F be the set of all such f corresponding to all possible $S \subseteq E$.

Definition 4.1. *We say that a point p of affine space is visible from a point e, $e \neq p$, with respect to a simplex σ if the open segment (e,p) ia disjoint with σ, i.e. $(e,p) \cap \sigma = \emptyset$.*

We say also that a subset S of points is visible from the point e if every point of this subset is visible from e.

Theorem 4.2. [4] *Let $\sigma \in \Sigma$ be a simplex and $e \in E$ a point that is not a vertex of σ.*

There is the following linear relation in V_Σ among simplices:

$$\sigma = \sum_{q_i \in Q^+} \sigma(q_i, e) - \sum_{q_i \in Q^-} \sigma(q_i, e), \tag{15}$$

where

Q^+ is the set of all facets (i.e. $(n-1)$-dimensional faces) q_i of the simplex σ that are not visible from e (with respect to σ);

Q^- is the set of all facets q_i of the simplex σ that are visible from e (with respect to σ);

$\sigma(q_i, e)$ is the n-dimensional simplex spanned by the facet q_i of the simplex σ and the point e.

Definition 4.3. *Let $B \subset \Sigma$. We say that an element $\sigma \notin B$ is expressed in one step "in terms of the set B using F" if there exists $f \in F$ such that $\sigma \in f$, and $f \setminus \sigma \subseteq B$.*

We say that an element $\sigma \notin B$ can be expressed in k steps in terms of the set B using F if there exists a sequence $\sigma_1, \dots, \sigma_k$, $\sigma_i \in \Sigma$, such that $\sigma_k = \sigma$ and σ_1 is expressed in one step in terms of the set B (using F), σ_2 is expressed in one step in terms of the set $B \cup \sigma_1, \dots,$ σ_k is expressed in one step in terms of the set $B \cup \sigma_1 \ldots \cup \sigma_{k-1}$.

Definition 4.4. *A subset $B \subset \Sigma$ is a geometrical basis in V_Σ with respect to F if it satisfies the following conditions:*

1) B is a basis in V_Σ;

2) for any $\sigma \in \Sigma$, $\sigma \notin B$ there exists k such that σ can be expressed in k steps in terms of B using F.

Let $B \subset \Sigma$ be the set of simplices constructed by the algorithm of Section 1.

Theorem 4.5. *The set B is a geometrical basis in V_Σ with respect to the system F defined by formula (14).*

[4] This theorem is stated in [3] in another form. Here we use the important geometric notion of visibility.

Proof. Due to Theorem 3.1 the simplices $\sigma \in B$ are linearly independent, therefore it is sufficient to prove that any simplex $\sigma \in \Sigma$ can be expressed in a finite number of steps in terms of B with respect to the system F. This will be proved by induction on the dimension n of the space V^n.

Induction on n. *First step* $(n = 2)$. The theorem is proved in [2].

Passing from $n-1$ *to* n. Suppose that the statement is true for V^{n-1}. Let us prove it for V^n.

Let the points $e_i \in E$ be in V^n and an ordering $e_1, \dots, e_N$ satisfy (1). Let us denote by Σ^i the set of simplices $\sigma \in \Sigma$ such that σ has the point e_i as the vertex with the minimal number. We have

$$\Sigma = \Sigma^1 \cup \ldots \cup \Sigma^{N-n} = \bigcup_{i=n}^{N-1} \Sigma^{N-i}, \tag{16}$$

where $\Sigma^i \cap \Sigma^j = \emptyset,\ i \neq j$. We also have $\Sigma^N = \Sigma^{N-1} = \ldots = \Sigma^{N-n+1} = \emptyset$. The set Σ^{N-n} either contains only one simplex $\sigma(e_N, e_{N-1}, \dots, e_{N-n})$ or $\Sigma^{N-n} = \emptyset$.

We need to prove that any simplex $\sigma \in \Sigma$ can be expressed in terms of B using F. Due to the partition (16) we can prove this statement by induction on i considering cases when $\sigma \in \Sigma^{N-i}$, where $i = n, \dots, N-1$.

Induction on i. *First step.* Let us check the first nontrivial step of induction. Let Σ^{N-i_0} (where $i_0 \geq n$) be the first nonempty set, i.e.

$$\Sigma^{N-i_0} = \{\sigma(e_N, \dots, e_{N-i_0+1}, e_{N-i_0})\},$$

where the points $e_N, \dots, e_{N-i_0+1}$ lie in an $(n-1)$-dimensional hyperplane, while the point e_{N-i_0} does not lie in this hyperplane. It is clear from the algorithm that the simplex $\sigma(e_N, \dots, e_{N-i_0+1}, e_{N-i_0})$ belongs to the set $B_{N-i_0} \subset B$ and, therefore, can be expressed in terms of B in 0 steps.

Passing from $i-1$ *to* i. Suppose that any simplex $\sigma \in \Sigma^{N-(i-1)}$ can be expressed in terms of B using F. Let us show that any simplex $\sigma_0 \in \Sigma^{N-i}$ can also be expressed in terms of B using F.

Consider the point e_{N-i} and the hyperplane $H = H_{N-i}$ of the algorithm. According to the algorithm we mark in H the set E_{N-i} of points $e_1^{N-i}, e_2^{N-i}, \dots, e_{N_i}^{N-i}$, where $e_j^{N-i} = \overrightarrow{(e_{N-i}, e_j)} \cap H$. To any vertex of σ_0 other than e_{N-i}, there corresponds a point from E_{N-i}, therefore, to the simplex σ_0 there corresponds one $(n-1)$-dimensional simplex σ_0^{N-i} in H.

Let Σ_{N-i} be the set of all $(n-1)$-dimensional simplices with the vertices in E_{N-i}. We have $\sigma_0^{N-i} \in \Sigma_{N-i}$. Applying the algorithm in the hyperplane H, we construct the set $\tilde{B}_{N-i}$ of simplices, $\tilde{B}_{N-i} \subset \Sigma_{N-i}$. Due to the assumption of the induction on n, the simplex $\sigma_0^{N-i} \in \Sigma_{N-i}$ can be expressed in terms of $\tilde{B}_{N-i}$ using F' (where F' is the set of all subsets f' defined in H by formula (14)).

First, let us show that the simplex $\mu(\sigma_0^{N-i})$, where μ is the map defined by (8) for $k = N-i$, can be expressed in terms of B using F.

As we have proved, the simplex σ_0^{N-i} can be expressed in terms of $\tilde{B}_{N-i}$ using F' in a finite number of steps, for example, in k steps. This means that there exists a sequence of simplices $\sigma_1^{N-i},\dots,\sigma_k^{N-i}$, where $\sigma_k^{N-i}=\sigma_0^{N-i}$ and a sequence of $f_1',\dots,f_k',\quad f_i'\in F'$, such that

$$f_1'\setminus\sigma_1^{N-i}\subseteq\tilde{B}_{N-i}, \tag{17}$$

$$f_2'\setminus\sigma_2^{N-i}\subseteq(\tilde{B}_{N-i}\cup\sigma_1^{N-i}), \tag{18}$$

$$\dots$$

$$f_k'\setminus\sigma_k^{N-i}\subseteq(\tilde{B}_{N-i}\cup\sigma_1^{N-i}\cup\ldots\cup\sigma_{k-1}^{N-i}). \tag{19}$$

Consider formula (17). It means that the simplex σ_1^{N-i} can be expressed in terms of $\tilde{B}_{N-i}$ using F'. Let us show that this implies that the simplex $\mu(\sigma_1^{N-i})$ can be expressed in terms of B_{N-i} using F. Indeed, according to the definition, the set f_1' contains all the simplices with vertices in some $n+1$ points $e_{i_1}^{N-i},\dots,e_{i_{n+1}}^{N-i}\in E_{N-i}$. One of these simplices is σ_1^{N-i}. For simplicity of notations let us assume that $\sigma_1^{N-i}=\sigma(\widehat{e_{i_1}^{N-i}},e_{i_2}^{N-i},\dots,e_{i_{n+1}}^{N-i})$, where $\widehat{e}$ means that the point e is not a vertex of the simplex. Thus, f_1' consists of the following simplices:

$$\sigma_1^{N-i}=\sigma(\widehat{e_{i_1}^{N-i}},e_{i_2}^{N-i},\dots,e_{i_{n+1}}^{N-i}),$$
$$\sigma(e_{i_1}^{N-i},\widehat{e_{i_2}^{N-i}},\dots,e_{i_{n+1}}^{N-i}),$$
$$\dots$$
$$\sigma(e_{i_1}^{N-i},e_{i_2}^{N-i},\dots,\widehat{e_{i_{n+1}}^{N-i}}),$$

where each simplex except σ_1^{N-i} belongs to $\tilde{B}_{N-i}$ due to formula (17).

To $f_1'\in F'$ let us associate an element $f_1\in F$. For this consider $n+2$ points $e_{N-i},e_{i_1}^{\mu},\dots,e_{i_{n+1}}^{\mu}\in E$, where e_j^{μ} is the point of E which is the closest to the point the point e_{N-i} among all points of E lying on the ray $\overrightarrow{(e_{N-i},e_j^{N-i})}$. Then f_1 is the set of all simplices with the vertices in these $n+2$ points. Thus, f_1 consists of the following simplices:

$$\sigma(e_{i_1}^{\mu},e_{i_2}^{\mu},\dots,e_{i_{n+1}}^{\mu}),$$
$$\sigma(e_{N-i},\widehat{e_{i_1}^{\mu}},e_{i_2}^{\mu},\dots,e_{i_{n+1}}^{\mu}),$$
$$\sigma(e_{N-i},e_{i_1}^{\mu},\widehat{e_{i_2}^{\mu}},\dots,e_{i_{n+1}}^{\mu}),$$
$$\dots$$
$$\sigma(e_{N-i},e_{i_1}^{\mu},e_{i_2}^{\mu},\dots,\widehat{e_{i_{n+1}}^{\mu}}).$$

The point e_{N-i} is not the vertex of the simplex $\sigma(e_{i_1}^{\mu},\dots,e_{i_{n+1}}^{\mu})$. Therefore, $\sigma(e_{i_1}^{\mu},\dots,e_{i_{n+1}}^{\mu})\in\Sigma^{N-(i-1)}$. By induction hypothesis for i this simplex can be expressed in terms of B using F.

Note that according to the algorithm,

$$\sigma(e_{N-i},e_{i_1}^{\mu},\dots,\widehat{e_{i_j}^{\mu}},\dots,e_{i_{n+1}}^{\mu})=\mu(\sigma(e_{i_1}^{N-i},\dots,\widehat{e_{i_j}^{N-i}},\dots,e_{i_{n+1}}^{N-i}))$$

for any $j = 1, \dots , n + 1$. Therefore, we have

$$\sigma(e_{N-i}, \widehat{e^{\mu}_{i_1}}, e^{\mu}_{i_2}, \dots , e^{\mu}_{i_{n+1}}) = \mu(\sigma_1^{N-i}).$$

Since each simplex $\sigma(e^{N-i}_{i_1}, \dots , \widehat{e^{N-i}_{i_j}}, \dots , e^{N-i}_{i_{n+1}})$, $j = 2, \dots , n+1$, belongs to $\tilde{B}_{N-i}$, each simplex $\sigma(e_{N-i}, e^{\mu}_{i_1}, \dots , \widehat{e^{\mu}_{i_j}}, \dots , e^{\mu}_{i_{n+1}})$ belongs to B_{N-i} according to the construction of B_{N-i}. Thus, $f_1 \setminus \mu(\sigma_1^{N-i}) \subseteq (B_{N-i} \cup \sigma(e^{\mu}_{i_1}, \dots , e^{\mu}_{i_{n+1}})$, where, as we have already mentioned, the simplex $\sigma(e^{\mu}_{i_1}, \dots , e^{\mu}_{i_{n+1}})$ can be expressed in terms of B using F. We have proved that $\mu(\sigma_1^{N-i})$ can be expressed in terms of B using F.

Considering consecutively formulas (18) – (19) we can prove similarly that the simplex $\mu(\sigma_0^{N-i})$ can be expressed in terms of B using F.

Consider the simplex σ_0. The following cases are possible:
1) $\sigma_0 = \mu(\sigma_0^{N-i})$; in this case the proof is already finished.
2) $\sigma_0 \neq \mu(\sigma_0^{N-i})$. Let $\sigma_0^{N-i} = \sigma(e^{N-i}_{j_1}, \dots , e^{N-i}_{j_n})$. This case can only occur if there is a vertex of σ_0 which lies on some ray $\overrightarrow{(e_{N-i}, e^{N-i}_{j_k})}$, where $k \in (1, \dots , n)$, and which is not the closest point from E to the point e_{N-i}.

Let us show that since the simplex $\mu(\sigma_0^{N-i})$ can be expressed in terms of B using F then the simplex σ_0 can be also expressed in terms of B using F.

For simplicity of notations let us denote here vertices of the simplices σ_0 and $\mu(\sigma_0^{N-i})$ as follows:

$$\sigma_0 = \sigma(e_{N-i}, e^0_1, \dots , e^0_n),$$
$$\mu(\sigma_0^{N-i}) = \sigma(e_{N-i}, e^{\mu}_1, \dots , e^{\mu}_n).$$

We can also assume that in this notation the points e^0_k and e^{μ}_k lie on the same ray $\overrightarrow{(e_{N-i}, e^{N-i}_{j_k})}$.

First, let us consider the case when the simplices σ_0 and $\mu(\sigma_0^{N-i})$ differ only by one point, i.e. $e^0_k \neq e^{\mu}_k$ for some k and we have

$$\sigma_0 = \sigma(e_{N-i}, e^{\mu}_1, \dots , e^{\mu}_{k-1}, e^0_k, e^{\mu}_{k+1}, \dots , e^{\mu}_n),$$
$$\mu(\sigma_0^{N-i}) = \sigma(e_{N-i}, e^{\mu}_1, \dots , e^{\mu}_{k-1}, e^{\mu}_k, e^{\mu}_{k+1}, \dots , e^{\mu}_n).$$

Then for the $n + 2$ points $e_{N-i}, e^{\mu}_1, \dots , e^{\mu}_{k-1}, e^{\mu}_k, e^{\mu}_{k+1}, \dots , e^{\mu}_n, e^0_k \in E$ there exists $f \in F$ which consists of the following simplices:

$$\sigma(e_{N-i}, e^{\mu}_1, \dots , e^{\mu}_{k-1}, e^{\mu}_k, e^{\mu}_{k+1}, \dots , e^{\mu}_n) = \mu(\sigma_0^{N-i}),$$
$$\sigma(e_{N-i}, e^{\mu}_1, \dots , e^{\mu}_{k-1}, e^{\mu}_k, e^{\mu}_{k+1}, \dots , e^{\mu}_{n-1}, e^0_k),$$
$$\dots$$
$$\sigma(e_{N-i}, e^{\mu}_1, \dots , e^{\mu}_{k-1}, e^{\mu}_k, e^0_k, \dots , e^{\mu}_n)$$
$$\sigma(e_{N-i}, e^{\mu}_1, \dots , e^{\mu}_{k-1}, e^0_k, e^{\mu}_{k+1}, \dots , e^{\mu}_n) = \sigma_0,$$
$$\sigma(e_{N-i}, e^{\mu}_1, \dots , e^0_k, e^{\mu}_k, e^{\mu}_{k+1}, \dots , e^{\mu}_n),$$
$$\dots$$
$$\sigma(e_{N-i}, e^0_k, \dots , e^{\mu}_{k-1}, e^{\mu}_k, e^{\mu}_{k+1}, \dots , e^{\mu}_n),$$
$$\sigma(e^0_k, e^{\mu}_1, \dots , e^{\mu}_{k-1}, e^{\mu}_k, e^{\mu}_{k+1}, \dots , e^{\mu}_n).$$

Since the points e_{N-i}, e_k^0, e_k^μ lie on the same ray $\overrightarrow{(e_{N-i}, e_{j_k})}$, all the simplices except σ_0 and $\mu(\sigma_0^{N-i})$ are not n-dimensional and do not belong to Σ. This implies that the simplex σ_0 can be expressed in terms of B using $f \in F$.

If the simplices σ_0 and $\mu(\sigma_0^{N-i})$ differ by two points, then we consider first a simplex σ' which differs from the simplex $\mu(\sigma_0^{N-i})$ only by one point and express it in terms of B using F. Then by applying the above arguments to the simplices σ' and σ_0 we can express the simplex σ_0 in terms of B using F. By repeating this process we prove that the simplex σ_0 can be expressed in terms of B using F. Thus, the set B constructed by the algorithm is a geometrical basis in V_Σ. $\square$

A pair of a basis $e_1, \dots, e_n$ in the space V and a basis $f_1, \dots, f_n$ in the dual space V' is called a *triangular pair* if $(e_i, f_k) = 0$ for $i > k$ and $(e_i, f_i) = 1$.

Theorems 4.5 and 3.1 imply

Theorem 4.6. *The set $B' \subset \Gamma$ constructed by the algorithm is a basis in V_Γ. The basis of simplices B and the basis of chambers B' form a triangular pair.*

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