COMBINATORIAL REPRESENTATION THEORY

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ABSTRACT. We attempt to survey the field of combinatorial representation theory, describe the main results and main questions and give an update of its current status. We give a personal viewpoint on the field, while remaining aware that there is much important and beautiful work that we have not been able to mention.

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Introduction

In January 1997, during the special year in combinatorics at MSRI, at a dessert party at Hélène's house, Gil Kalai, in his usual fashion, began asking very pointed questions about exactly what all the combinatorial representation theorists were doing their research on. After several unsuccessful attempts at giving answers that Gil would find satisfactory, it was decided that some talks should be given in order to explain to other combinatorialists what the specialty is about and what its main questions are.

In the end, Arun gave two talks at MSRI in which he tried to clear up the situation. After the talks several people suggested that it would be helpful if someone would write a survey article containing what had been covered in the two talks and including further interesting details. After some arm twisting it was agreed that Arun and Hélène would write such a paper on combinatorial representation theory. What follows is our attempt to define the field of combinatorial representation theory, describe the main results and main questions and give an update of its current status.

Of course this is wholly impossible. Everybody in the field has their own point of view and their own preferences of questions and answers. Furthermore, there is much too much material in the field to possibly collect it all in a single article (even conceptually). We therefore feel that we must stress the obvious; in this article we give a personal viewpoint on the field while remaining aware that there is much important and beautiful work that we have not been able to mention.

On the other hand, we have tried very hard to give a focused approach and to make something that will be useful to both specialists and non specialists, for understanding what we do, for learning the concepts of the field, and for tracking down history and references. We have chosen to write in an informal style in the hope that this way we can better convey the conceptual aspects of the field. Readers should keep this in mind and refer to the notes and references and the appendices when there are questions about the precision in definitions and statements of results. We have included a table of contents at the end of the paper which should help with navigation. Having made these points, and put in a lot of work, we leave it to you the reader, with the earnest hope that you find it useful.

We would like to thank the many people in residence at the special year 1996-97 in combinatorics at MSRI, for their interest, their suggestions, and for continually encouraging us to explain and write about the things that we enjoy doing. We both are extremely indebted to our graduate advisors, A. Garsia and H. Wenzl, who (already many years ago) introduced us to and taught us this wonderful field.

Part I

1. What is Combinatorial Representation Theory?

What do we mean by "combinatorial representation theory"? First and foremost, combinatorial representation theory is representation theory. The adjective "combinatorial" will refer to the way in which we answer representation theoretic questions. We will discuss this more fully later. For the moment let us begin with, what is representation theory?

What is representation theory?

If representation theory is a black box, or a machine, then the input is an algebra A. The output of the machine is information about the modules for A.

An *algebra* is a vector space A over $\mathbb{C}$ with a multiplication.

An important example is the case of the group algebra.

Define the *group algebra* of a group G to be

$$A = \mathbb{C}G = \mathbb{C}\text{-span } \{g \in G\},$$

so that the elements of G form a basis of A. The multiplication in the group algebra is inherited from the multiplication in the group.

We want to study the algebra A via its actions on vector spaces.

An *A-module* is a finite dimensional vector space M over $\mathbb{C}$ with an A action.

See Appendix A1 for a complete definition. We shall use the words module and *representation* interchangeably. Representation theorists are always trying to break up modules into pieces.

An A-module M is *indecomposable* if $M \ncong M_1 \oplus M_2$ where M_1 and M_2 are nonzero A-modules.

An A-module M is *irreducible* or *simple* if it has no submodules.

In reference to modules the words "irreducible" and "simple" are used completely interchangeably.

The algebra A is *semisimple* if

$$\textbf{indecomposable} = \textbf{irreducible}$$

for A-modules.

The "non-semisimple" case, i.e. where indecomposable is not the same as irreducible, is called modular representation theory. We will not consider this case much in these notes. However, before we banish it completely let us describe the flavor of modular representation theory.

A *composition series* for M

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k = 0$$

is a sequence of submodules of M such that each M_i/M_{i+1} is simple.

The Jordan-Hölder theorem says that two different composition series of M will always produce the same multiset $\{M_i/M_{i+1}\}$ of simple modules. Modular representation theorists are always trying to determine this multiset of *composition factors of M*.

Remarks.

(1) We shall not make life difficult in this article but one should note that it is common to work over general fields rather than just using the field $\mathbb{C}$.
(2) If one is bold one can relax things in the definition of module and let M be infinite dimensional.
(3) Of course the definition of irreducible modules is not correct since 0 and M are always submodules of M. So we are ignoring these two submodules in this definition. But conceptually the definition is the right one, we want a simple module to be something that has no submodules.
(4) The definition of semisimple above is not a technically correct definition. Look in Appendix A1 for the proper definition. However, the power of semisimplicity is exactly that it makes all indecomposable modules irreducible. So "indecomposable = irreducible" is really the right way to think of semisimplicity.
(5) A good reference for the basics of representation theory is [CR1]. The book [Bou2] contains a completely general and comprehensive treatment of the theory of semisimple algebras. The appendix, §A1, to this article also contains a brief (and technically correct) introduction with more specific references.

Main questions in representation theory

I. What are the irreducible A-modules?

What do we mean by this question? We would like to be able to give some kind of answers to the following more specific questions.

(a) How do we index/count them?

(b) What are their dimensions?

The dimension of a module is just its dimension as a vector space.

(c) What are their characters?

The character of a module M is the function $\chi_M : A \to \mathbb{C}$, where $\chi_M(a)$ is the trace of the linear transformation determined by the action of a on M. More precisely,

$$\chi_M(a) = \sum_{b_i \in B} ab_i\big|_{b_i},$$

where the sum is over a basis B of the module M and $ab_i\big|_{b_i}$ denotes the coefficient of b_i in ab_i when we expand in terms of the basis B.

C. How do we construct the irreducible modules?

S. Special/Interesting representations M

(a) How does M decompose into irreducibles?

If we are in the semisimple case then M will always be a direct sum of irreducible modules. If we group the irreducibles of the same type together we can write

$$M \cong \bigoplus_{\lambda} (V^\lambda)^{\oplus c_\lambda},$$

where the modules V^λ are the irreducible A-modules and c_λ is the number of times an irreducible of type V^λ appears as a summand in M. It is common to abuse notation and write

$$M = \sum_{\lambda} c_\lambda V^\lambda.$$

(b) What is the character of M?

Special modules often have particularly nice formulas describing their characters. It is important to note that having a nice character formula for M does not necessarily mean that it is easy to see how M decomposes into irreducibles. Thus this question really is different from the previous one.

(c) How do we find interesting representations?

Sometimes special representations turn up by themselves and other times one has to work hard to construct the right representation with the right properties. Often very interesting representations come from other fields.

(d) Are they useful?

A representation may be particularly interesting just because of its structure while other times it is a special representation that helps to prove some particularly elusive theorem. Sometimes these representations lead to a completely new understanding of previously known facts. A famous example (which unfortunately we won't have space to discuss, see [Hu1]) is the Verma

module, which was discovered in the mid 1960s and completely changed representation theory.

M. The modular case

In the modular case we have the following important question in addition to those above.

(a) What are the indecomposable representations?

(b) What are the structures of their composition series?

For each indecomposable module M there is a multiset of irreducibles $\{M_i/M_{i+1}\}$ determined by a composition series of M. One would like to determine this multiset.

Even better (especially for combinatorialists), the submodules of M form a lattice under inclusion of submodules and one would like to understand this lattice. This lattice is always a modular lattice and we may imagine that each edge of the Hasse diagram is labeled by the simple module N_1/N_2 where N_1 and N_2 are the modules on the ends of the edge. With this point of view the various compositions series of M are the maximal chains in this lattice of modules. The Jordan-Hölder theorem says that every maximal chain in the lattice of submodules of M has the same multiset of labels on its edges. What modular representation theorists try to do is determine the set of labels on a maximal chain.

Remark. The abuse of notation which allows us to write $M = \sum_\lambda c_\lambda V^\lambda$ has been given a formal setting which is called the *Grothendieck ring*. In other words, the formal object which allows us to write such identities has been defined carefully. See [Se1] for precise definitions of the Grothendieck ring.

Answers should be of the form ...

Now we come to the adjective "Combinatorial." It refers to the way in which we give the answers to the main questions of representation theory.

I. What are the irreducible A-modules?

(a) How do we index/count them?

We want to answer with **a** *bijection* **between**

Nice combinatorial objects λ $\quad\overset{1-1}{\longleftrightarrow}\quad$ **Irreducible representations** V^λ**.**

(b) What are their dimensions?

We should answer with **a formula of the form**

$$\dim(V^\lambda) = \#\text{ of nice combinatorial objects.}$$

(c) What are their characters?

We want **a character formula of the type**

$$\chi(a) = \sum_T \mathbf{wt}^a(T),$$

where the sum runs over all T in a set of nice combinatorial objects and wt^a is a weight on these objects which depends on the element $a \in A$ where we are evaluating the character.

C. How do we construct the irreducible modules?

We want to give **constructions that have a very explicit and very combinatorial flavor.** What we mean by this will be more clear from the examples, see **(C1-2)** of Section 2.

S. Special/Interesting representations M

(a) How does M decompose into irreducibles?

If M is an interesting representation we want to **determine the positive integers** c_λ in the decomposition

$$M \cong \bigoplus_\lambda (V^\lambda)^{\oplus c_\lambda}$$

in the form

$$c_\lambda = \#\ \textbf{of nice combinatorial objects.}$$

In the formula for the decomposition of M the sum is over all λ which are objects indexing the irreducible representations of A.

(b) What is the character of M?

As in the case I(c) we want **a character formula of the type**

$$\chi_M(a) = \sum_T \mathbf{wt}^a(T)$$

where the sum runs over all T in some set of nice combinatorial objects and wt^a is a weight on these objects which depends on the element $a \in A$ where we are evaluating the character.

(c) How do we find interesting representations?

It is particularly pleasing when interesting representations arise in **other parts of combinatorics!** One such example is a representation on the homology of the partition lattice which also, miraculously, appears as a representation on the free Lie algebra. We won't have space to discuss this here, see the original references [Hn],[Jy], [Kl], [Sta2], the article [Gar] for some further basics, and [Ba2] for a study of how it can be that this representation appears in two completely different places.

(d) Are they useful?

How about for solving combinatorial problems? Or making new combinatorial problems? Sometimes a representation is exactly what is most helpful for solving a combinatorial problem. One example of this is in the recent solution of the the last few plane partition conjectures. See [Sta1], [Mac] I §5 Ex. 13-18, for the statement of the problem and [Ku1-3] and [Ste5-7] for the solutions. These solutions were motivated by the method of Proctor [Prc].

The main point of all this is that a combinatorialist thinks in a special way (nice objects, bijections, weighted objects, etc.) and this method of thinking should be an integral part of the form of the solution to the problem.

2. Answers for S_n, the symmetric group

Most people in the field of combinatorial representation theory agree that the field begins with the fundamental results for the symmetric group S_n. Let us give the answers to the main questions for the case of S_n. The precise definitions of all the objects used below can be found in Appendix A2. As always, by a representation of the symmetric group we mean a representation of its group algebra $A = \mathbb{C}S_n$.

I. What are the irreducible S_n-modules?

(a) How do we index/count them?

There is a bijection

$$\textbf{Partitions } \lambda \textbf{ of } n \quad \overset{1-1}{\longleftrightarrow} \quad \textbf{Irreducible representations } S^\lambda.$$

(b) What are their dimensions?

The dimension of the irreducible representation S^λ is given by

$$\begin{aligned}\dim(S^\lambda) &= \#\text{ of standard tableaux of shape } \lambda \\ &= \frac{n!}{\prod_{x\in\lambda} h_x},\end{aligned}$$

where h_x is the hook length at the box x in λ, see Appendix A2.

(c) What are their characters?

Let $\chi^\lambda(\mu)$ be the character of the irreducible representation S^λ evaluated at a permutation of cycle type $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$. Then **the character $\chi^\lambda(\mu)$ is given by**

$$\chi^\lambda(\mu) = \sum_T \mathbf{wt}^\mu(T),$$

where the sum is over all standard tableaux T of shape λ and

$$\mathbf{wt}^\mu(T) = \prod_{i=1}^{n} f(i, T),$$

where

$$f(i,T) = \begin{cases} -1, & \textbf{if } i \notin B(\mu) \textbf{ and } i+1 \textbf{ is sw of } i, \\ 0, & \textbf{if } i, i+1 \notin B(\mu), \ i+1 \textbf{ is ne of } i, \textbf{ and } i+2 \textbf{ is sw of } i+1, \\ 1, & \textbf{otherwise,} \end{cases}$$

and $B(\mu) = \{\mu_1 + \mu_2 + \cdots + \mu_k | 1 \le k \le \ell\}$**.** In the formula for $f(i,T)$, **sw** means strictly south and weakly west and **ne** means strictly north and weakly east.

C. How do we construct the irreducible modules?

There are several interesting constructions of the irreducible S^λ.

(i) via **Young symmetrizers**.

Let T be a tableau. Let

$$R(T) = \text{ permutations which fix the rows of } T, \text{ as sets;}$$
$$C(T) = \text{ permutations which fix the columns of } T, \text{ as sets;}$$

$$P(T) = \sum_{w \in R(T)} w, \quad \text{and} \quad N(T) = \sum_{w \in C(T)} \varepsilon(w) w,$$

where $\varepsilon(w)$ is the sign of the permutation w. Then

$$S^\lambda \cong \mathbb{C}S_n P(T) N(T),$$

where the action of the symmetric group is by left multiplication.

(ii) **Young's seminormal construction**.

Let

$$S^\lambda = \mathbb{C}\textbf{-span-}\{v_T \mid T \textbf{ are standard tableaux of shape } \lambda\}$$

so that the vectors v_T are a basis of S^λ. **The action of** S_n **on** S^λ **is given by**

$$s_i v_T = (s_i)_{TT} v_T + (1 + (s_i)_{TT}) v_{s_i T}, \qquad \textbf{where} \qquad (s_i)_{TT} = \frac{1}{c(T(i+1)) - c(T(i))}.$$

and $s_i = (i, i+1)$**.** In this formula

$T(i)$ **denotes the box containing** i **in** T**;**
$c(b) = j - i$ **is the** *content* **of the box** b**, where** (i,j) **is the position of** b **in** λ**;**
$s_i T$ **is the same as** T **except that the entries** i **and** $i+1$ **are switched;**
$v_{s_i T} = 0$ **if** $s_i T$ **is not a standard tableau.**

There are other important constructions of the irreducible representations S^λ. We do not have room to discuss these constructions here, see the Notes and References (10-12) below and A3 in the appendix. The main ones are:

(iii) **Young's orthonormal construction**,
(iv) **The Kazhdan-Lusztig construction**,
(v) **The Springer construction**.

S. Particularly interesting representations

(S1) Let $k+\ell = n$. The module $S^\lambda \downarrow^{S_n}_{S_k \times S_\ell}$ is the same as S^λ except that we only look at the action of the subgroup $S_k \times S_\ell$. Then

$$S^\lambda \downarrow^{S_n}_{S_k \times S_\ell} = \bigoplus_{\mu \vdash k, \nu \vdash \ell} (S^\mu \otimes S^\nu)^{\oplus c^\lambda_{\mu\nu}} = \sum_{\mu,\nu} c^\lambda_{\mu\nu} (S^\mu \otimes S^\nu),$$

where $c^\lambda_{\mu\nu}$ **is the number of column strict fillings of** λ/μ **of content** ν **such that the word of the filling is a lattice permutation.** The positive integers $c^\lambda_{\mu\nu}$ are the *Littlewood-Richardson coefficients*. See Appendix A2.

(S2) Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a partition of n. Let $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_\ell}$. The module $\mathbf{1}\uparrow^{S_n}_{S_\mu}$ is the vector space

$$\mathbf{1}\uparrow^{S_n}_{S_\mu} = \mathbb{C}(S_n/S_\mu) = \mathbb{C}\text{-span}\{wS_\mu \mid w \in S_n\}$$

where the action of S_n on the cosets is by left multiplication. Then

$$\mathbf{1}\uparrow^{S_n}_{S_\mu} = \sum_\lambda K_{\lambda\mu} S^\lambda,$$

where

$$K_{\lambda\mu} = \#\ \textbf{of column strict tableaux of shape } \lambda \textbf{ and weight } \mu.$$

This representation also occurs in the following context:

$$\mathbf{1}\uparrow^{S_n}_{S_\mu} \cong H^*(\mathcal{B}_u),$$

where u is a unipotent element of $GL(n,\mathbb{C})$ with Jordan decomposition μ and $\mathcal{B}_u$ is the variety of Borel sugroups in $GL(n,\mathbb{C})$ containing u. This representation is related to the Springer construction mentioned in **C(v)** above. See Appendix A3 for further details.

(S3) If $\mu, \nu \vdash n$ then the tensor product S_n-module $S^\mu \otimes S^\nu$ is defined by $w(m \otimes n) = wm \otimes wn$, for all $w \in S_n$, $m \in S^\mu$ and $n \in S^\nu$. **There are positive integers** $\gamma_{mu\nu\lambda}$ **such that**

$$S^\mu \otimes S^\nu = \sum_{\lambda \vdash n} \gamma_{\mu\nu\lambda} S^\lambda.$$

Except for a few special cases the positive integers $\gamma_{\mu\nu\lambda}$ are still unknown. See [Rem] for a combinatorial description of the cases for which the coefficients $\gamma_{\mu\nu\lambda}$ are known.

Notes and references

(1) The bijection in **(Ia)**, between irreducible representations and partitions, is due to Frobenius [Fr]. Frobenius is the founder of representation theory and the symmetric group was one of the first examples that he worked out.

(2) The formula in **(Ib)** for the dimension of S^λ as the number of standard tableaux is immediate from the work of Frobenius, but it really came into the fore from the work of Young [Y]. The "hook formula" for $\dim(S^\lambda)$ is due to Frame-Robinson-Thrall [FRT].

(3) The formula for the characters of the symmetric group which is given in **(Ic)** is due to Fomin and Greene [FG]. For them, this formula arose by application of their theory of noncommutative symmetric functions. Roichman [Ro] discovered this formula independently in the more general case of the Iwahori-Hecke algebra. The formula for the Iwahori-Hecke algebra is exactly the same as the formula for the S_n case except that the 1 appearing in case 3 of the definition of $f(i,T)$ should be changed to a q.

(4) There is a different and more classical formula for the characters than the formula given in **(Ic)** which is called the Murnaghan-Nakayama rule [Mur] [Nak]. We have described the Murnaghan-Nakayama rule in the appendix, Theorem A2.2. Once the formula in **(Ic)** is given it is not hard to show combinatorially that it is equivalent to the Murnaghan-Nakayama rule but if one does not know the formula it is nontrivial to guess it from the Murnaghan-Nakayama rule.

(5) We do not know if anyone has compared the algorithmic complexity of the formula given in **(Ic)** with the algorithmic complexity of the Murnaghan-Nakayama rule. One would expect that they have the same complexity: the formula above is a sum over more objects than the sum in the Murnaghan-Nakayama rule but these objects are easier to create and many of them have zero weight.

(6) One of the beautiful things about the formula for the character of S^λ which is given in **(Ic)** is that it is a sum over the same set that we have used to describe the dimension of S^λ.

(7) The construction of S^λ by Young symmetrizers is due to Young [Y1-2] from 1900. It is used so often and has so many applications that it is considered classical. A review and generalization of this construction to skew shapes appears in [GW].

(8) The seminormal form construction of S^λ is also due to Young [Y4-5] although it was discovered some thirty years after the Young symmetrizer construction.

(9) Young's orthonormal construction differs from the seminormal construction only by multiplication of the basis vectors by certain constants. A comprehensive treatment of all three constructions of Young is given in the book by Rutherford [Ru].

(10) The Kazhdan-Lusztig construction uses the Iwahori-Hecke algebra in a crucial way. It is combinatorial but relies crucially on certain polynomials which seem to be impossible to compute in practice except for very small n, see [Bre] for further information. This construction has important connections to geometry and other parts of representation theory. The paper [GM] and the book [Hu2] give elementary treatments of the Kazhdan-Lusztig construction.

(11) Springer's construction is a geometric construction. In this construction the irreducible module S^λ is realized as the top cohomology group of a certain variety, see [Spr], [CG], and Appendix A3.

(12) There are many ways of constructing new representations from old ones. Among the common techniques are *restriction*, *induction*, and *tensoring*. The special representations **(S1)**, **(S2)**, and **(S3)** given above are particularly nice examples of these constructions. One should note that tensoring of representations works for group algebras (and Hopf algebras) but not for general algebras.

3. Answers for $GL(n,\mathbb{C})$, the general linear group

The results for the general linear group are just as beautiful and just as fundamental as those for the symmetric group. The results are surprisingly similar and yet different in many crucial ways. We shall see that the results for $GL(n,\mathbb{C})$ have been generalized to a very wide class of groups whereas the results for S_n have only been generalized successfully to groups that look very similar to symmetric groups. The representation theory of $GL(n,\mathbb{C})$ was put on a very firm footing from the fundamental work of Schur [Sc1-2] in 1901 and 1927.

I. What are the irreducible $GL(n,\mathbb{C})$-modules?

(a) How do we index/count them?

There is a bijection

$$\textbf{Partitions } \lambda \textbf{ with at most } n \textbf{ rows} \quad \stackrel{1-1}{\longleftrightarrow} \quad \textbf{Irreducible polynomial representations } V^\lambda.$$

See Appendix A4 for a definition and discussion of what it means to be a *polynomial* representation.

(b) What are their dimensions?

The **dimension of the irreducible representation V^λ is given by**

$$\begin{aligned}\textbf{dim}(V^\lambda) &= \#\textbf{ of column strict tableaux of shape } \lambda \\ &\qquad \textbf{filled with entries from } \{1,2,\dots,n\} \\ &= \prod_{x\in\lambda} \frac{n+c(x)}{h_x},\end{aligned}$$

where $c(x)$ is the content of the box x and h_x is the hook length at the box x.

(c) What are their characters?

Let $\chi^\lambda(g)$ be the character of the irreducible representation V^λ evaluated at an element $g \in GL(n,\mathbb{C})$. The **character $\chi^\lambda(g)$ is given by**

$$\begin{aligned}\chi^\lambda(g) &= \sum_T x^T \\ &= \frac{\sum_{w\in S_n} \varepsilon(w) w x^{\lambda+\delta}}{\sum_{w\in S_n} \varepsilon(w) w x^{\delta}} = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})},\end{aligned}$$

where the sum is over all column strict tableaux T of shape λ filled with entries from $\{1,2,\dots,n\}$ and

$$x^T = x^{\mu_1}x_2^{\mu_2}\cdots x_n^{\mu_n}, \qquad \textbf{where } \mu_i = \#\textbf{ of } i\textbf{'s in } T$$

and $x_1, x_2, \dots, x_n$ are the eigenvalues of the matrix g. Let us not worry about the first expression in the second line at the moment. Let us only say that it is routine to rewrite it as the second expression in that line which is one of the standard expressions for the Schur function, see [Mac] I §3.

C. How do we construct the irreducible modules?

There are several interesting constructions of the irreducible V^λ.

(C1) via **Young symmetrizers**.

Recall that the irreducible S^λ of the symmetric group S_k was constructed via Young symmetrizers in the form

$$S^\lambda \cong \mathbb{C}S_n P(T)N(T).$$

We can construct the irreducible $GL(n,\mathbb{C})$-module in a similar form. **If λ is a partition of k then**

$$V^\lambda \cong V^{\otimes k} P(T)N(T).$$

This important construction is detailed in Appendix A5.

(C2) Gelfan'd-Tsetlin bases

This construction of the irreducible $GL(n,\mathbb{C})$ representations V^λ is analogous to the Young's seminormal construction of the irreducible representations S^λ of the symmetric group. Let

$$V^\lambda = \textbf{span-}\{v_T \mid T \textbf{ are column strict tableaux of shape } \lambda \textbf{ filled with elements of } \{1,2,\dots,n\}\}$$

so that the vectors v_T are a basis of V^λ. Define an action of symbols $E_{k-1,k}$, $2 \le k \le n$, on the basis vectors v_T by

$$E_{k-1,k} v_T = \sum_{T^-} a_{T^- T}(k) v_{T^-},$$

where the sum is over all column strict tableaux T^- which are obtained from T by changing a k to a $k-1$ and the coefficients $a_{T^- T}(k)$ are given by

$$a_{T^- T}(k) = -\frac{\prod_{i=1}^{k} (T_{ik} - T_{j,k-1} + j - k)}{\prod_{\substack{i=1 \\ i \neq j}}^{k-1} (T_{i,k-1} - T_{j,k-1} + j - k)},$$

where j is the row number of the entry where T^- and T differ and T_{ik} is the position of the rightmost entry $\le k$ in row i of T. Similarly, define an action of symbols $E_{k,k-1}$, $2 \le k \le n$, on the basis vectors v_T by

$$E_{k,k-1} v_T = \sum_{T^+} b_{T^+ T}(k) v_{T^+},$$

where the sum is over all column strict tableaux T^+ which are obtained from T by changing a $k-1$ to a k and the coefficients $b_{T^+T}(k)$ are given by

$$b_{T^+T}(k) = \frac{\prod_{i=1}^{k-2}(T_{i,k-2} - T_{j,k-1} + j - k)}{\prod_{\substack{i=1 \\ i\neq j}}^{k-1}(T_{i,k-1} - T_{j,k-1} + j - k)},$$

where j is the row number of the entry where T^+ and T differ and T_{ik} is the position of the rightmost entry $\leq k$ in row i.
Since

$$g_i(z) = \begin{pmatrix} 1 & 0 & & \cdots & & & 0 \\ 0 & \ddots & & & & & \\ & & 1 & & & & \vdots \\ \vdots & & & z & & & \\ & & & & 1 & & \\ & & & & & \ddots & 0 \\ 0 & & & \cdots & & 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{C}^*,$$

$$g_{i-1,i}(z) = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & \ddots & & & & \\ & & 1 & z & & \vdots \\ \vdots & & 0 & 1 & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{C},$$

$$g_{i,i-1}(z) = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & \ddots & & & & \\ & & 1 & & & \vdots \\ \vdots & & z & 1 & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{C},$$

generate $GL(n,\mathbb{C})$, the action of these matrices on the basis vectors v_T will determine the action of all of $GL(n,\mathbb{C})$ on the space V^λ. The action of these generators is given by:

$$g_i(z)v_T = z^{(\#\text{ of } i\text{'s in } T)}v_T,$$

$$g_{i-1,i}(z)v_T = e^{zE_{i-1,i}}v_T = (1 + zE_{i-1,i} + \frac{1}{2!}z^2E_{i-1,i}^2 + \dots)v_T,$$

$$g_{i,i-1}(z)v_T = e^{zE_{i,i-1}}v_T = (1 + zE_{i,i-1} + \frac{1}{2!}z^2E_{i,i-1}^2 + \dots)v_T.$$

$b(S,T)$ is the box where S and T differ,

$r(b(S,T))$ is the row number of the box $b(S,T)$,
$p(b(S,T))$ is the position of the box $b(S,T)$ in its row,
$p(\leq k, i)$ is the position of the rightmost entry .

(C3) The Borel-Weil-Bott construction

Let λ be a partition. Then λ defines a character (one-dimensional representation) of the group T_n of diagonal matrices in $G = GL(n,\mathbb{C})$. This character can be extended to the group $B = B_n$ of upper triangular matrices in $G = GL(n,\mathbb{C})$ by letting it act trivially on U_n the group of upper unitriangular matrices in $G = GL(n,\mathbb{C})$. Then the fiber product

$$\mathcal{L}_\lambda = G \times_B \lambda$$

is a line bundle on G/B. Finally,

$$V^\lambda \cong H^0(G/B, \mathcal{L}_\lambda),$$

where $H^0(G/B,\mathcal{L}_\lambda)$ is the space of global sections of the line bundle $\mathcal{L}_\lambda$. More details on the construction of the character λ and the line bundle $\mathcal{L}_\lambda$ are given in Appendix A6.

S. Special/Interesting representations

(S1) Let

$$GL(k)\times GL(\ell) = \left(\begin{array}{cc} \left(\begin{array}{c} \\ GL(k,\mathbb{C}) \\ \\ \end{array} \right) & 0 \\ 0 & \left(\begin{array}{c} \\ GL(\ell,\mathbb{C}) \\ \\ \end{array} \right) \end{array} \right) \subseteq GL(n), \qquad \text{where } k+\ell = n.$$

Then

$$V^\lambda \downarrow^{GL(n)}_{GL(k)\times GL(\ell)} = \sum_{\mu,\nu} c^\lambda_{\mu\nu} (V^\mu \otimes V^\nu),$$

where $c^\lambda_{\mu\nu}$ **is the number of column strict fillings of λ/μ with content ν such that the word of the filling is a lattice permutation.** The positive integers $c^\lambda_{\mu\nu}$ are the *Littlewood-Richardson coefficients* that appeared earlier in the decomposition of $S^\lambda \downarrow^{S_n}_{S_k\times S_\ell}$ in terms of $S^\mu \otimes S^\nu$.

We may write this expansion in the form

$$V^\lambda \downarrow^{GL(n)}_{GL(k)\times GL(\ell)} = \sum_{F \text{ fillings}} V^{\mu(F)} \otimes V^{\nu(F)}.$$

We could do this precisely if we wanted. We won't do it now, but the point is that it may be nice to write this expansion as a sum over combinatorial objects. This will be the form in which this will be generalized later.

(S2) Let V^μ and V^ν be irreducible polynomial representations of $GL(n)$. Then

$$V^\mu \otimes V^\nu = \sum_\lambda c^\lambda_{\mu\nu} V^\lambda,$$

where $GL(n)$ acts on $V^\mu \otimes V^\nu$ by $g(m \otimes n) = gm \otimes gn$, for $g \in GL(n,\mathbb{C})$, $m \in V^\mu$ and $n \in V^\nu$. Amazingly, **the coefficients $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients again.** These are the same coefficients that appeared in the **(S1)** case above and in the **(S1)** case for the symmetric group.

Remarks

(1) There is a strong similarity between the results for the symmetric group and the results for $GL(n,\mathbb{C})$. One might wonder whether there is any connection between these two pictures.

There are TWO DISTINCT ways of making concrete connections between the representation theories of $GL(n,\mathbb{C})$ and the symmetric group. In fact these two are so different that **DIFFERENT SYMMETRIC GROUPS** are involved.

(a) *If λ is a partition of n* then the "zero weight space", or $(1,1,\dots,1)$ weight space, of the irreducible $GL(n,\mathbb{C})$-module V^λ is isomorphic to the irreducible module S^λ for the group S_n, where the action of S_n is determined by the fact that S_n is the subgroup of permutation matrices in $GL(n,\mathbb{C})$. This relationship is reflected in the combinatorics: the standard tableaux of shape λ are exactly the column strict tableaux of shape λ which are of weight $\nu = (1,1,\dots,1)$.

(b) Schur-Weyl duality, see §A5 in the appendix, says that the action of the symmetric group S_k on $V^{\otimes k}$ by permutation of the tensor factors generates the full centralizer of the $GL(n,\mathbb{C})$-action on $V^{\otimes k}$ where V is the standard n-dimensional representation of $GL(n,\mathbb{C})$. By double centralizer theory, this duality induces a correspondence between the irreducible representations of $GL(n,\mathbb{C})$ which appear in $V^{\otimes k}$ and the irreducible representations of S_k which appear in $V^{\otimes k}$. These representations are indexed by *partitions λ of k*.

(2) It is important to note that the word *character* has two different and commonly used meanings and the use of the word character in **(C3)** is different than in Section 1. In **(C3)** above the word character means *one dimensional representation*. This terminology is used particularly (but not exclusively) in reference to representations of abelian groups (like the group T_n in **(C3)**). In general one has to infer from the context which meaning is intended.

(3) The indexing and the formula for the characters of the irreducible representations is due to Schur [Sc1].

(4) The formula for the dimensions of the irreducibles as the number of column strict tableaux follows from the work of Kostka [Kk] and Schur [Sc1]. The "hook-content" formula appears in [Mac] I §3 Ex. 4, where the book of Littlewood [Lw] is quoted.

(5) The construction of the irreducibles by Young symmetrizers appeared in 1939 in the influential book [Wy1] of H. Weyl. It was generalized to the symplectic and orthogonal groups by H. Weyl in the same book. Further important information about this construction in the symplectic and orthogonal cases is found in [Be2] and [KWe]. It is not known how to generalize this construction to arbitrary complex semisimple Lie groups.
(6) The Gelfan'd-Tsetlin basis construction originates from 1950 [GT1]. A similar construction was given for the orthogonal group at the same time [GT2] and was generalized to the symplectic group by Zhelobenko, see [Zh1-2]. This construction does not generalize well to other complex semisimple groups since it depends crucially on a tower $G \supseteq G_1 \supseteq \cdots \supseteq G_k \supseteq \{1\}$ of "nice" Lie groups such that all the combinatorics is controllable.
(7) The Borel-Weil-Bott construction is not a combinatorial construction of the irreducible module V^λ. It is very important because it is a construction that generalizes well to all other compact connected real Lie groups.
(8) The facts about the special representations which we have given above are found in Littlewood's book [Lw].

4. Answers for finite dimensional complex semisimple Lie algebras $\mathfrak{g}$

Although the foundations for generalizing the $GL(n,\mathbb{C})$ results to all complex semisimple Lie groups and Lie algebras were laid in the fundamental work of Weyl [Wy2] in 1925, it is only recently that a complete generalization of the tableaux results for $GL(n,\mathbb{C})$ has been obtained by Littelmann [Li2]. The results which we state below are generalizations of those given for $GL(n,\mathbb{C})$ in the last section; partitions get replaced by points in a lattice called P^+, and column strict tableaux get replaced by paths. See the Appendix A7 for some basics on complex semisimple Lie algebras.

I. What are the irreducible $\mathfrak{g}$-modules?

(a) How do we index/count them?

There is a bijection

$$\lambda \in P^+ \qquad \overset{1-1}{\longleftrightarrow} \qquad \textbf{irreducible representations } V^\lambda,$$

where P^+ is the cone of dominant integral weights for $\mathfrak{g}$. The set P^+ is described in Appendix A8.

(Ib) What are their dimensions?

The dimension of the irreducible representation V^λ is given by

$$\begin{aligned}\dim(V^\lambda) &= \#\text{ of paths in } \mathcal{P}\pi_\lambda \\ &= \prod_{\alpha>0} \frac{\langle \lambda+\rho, \alpha\rangle}{\langle \rho, \alpha\rangle},\end{aligned}$$

where

$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$**, is the half sum of the positive roots,**

π_λ **is the straight line path from** 0 **to** λ**, and**

$\mathcal{P}\pi_\lambda = \{f_{i_1} \cdots f_{i_k} \pi_\lambda | 1 \le i_1, \ldots, i_k \le n\}$**, where**
$f_1, \ldots, f_n$ **are the path operators introduced in [Li2].**
We shall not define the operators f_i here (or in the appendix, see [Li2]), let us just say that they act on paths and they are partial permutations in the sense that if f_i acts on a path π then the result is either 0 or another path. See Appendix A8 for a few more details.

(c) What are their characters?

The character of the irreducible module V^λ is given by

$$\begin{aligned} \mathrm{char}(V^\lambda) &= \sum_{\eta \in \mathcal{P}\pi_\lambda} e^{\eta(1)} \\ &= \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \varepsilon(w) e^{w\rho}}, \end{aligned}$$

where $\eta(1)$ is the endpoint of the path η. These expressions live in the group algebra of the weight lattice P, $\mathbb{C}[P] = \mathrm{span}\{e^\mu \mid \mu \in P\}$, where e^μ is a formal variable indexed by μ and the multiplication is given by $e^\mu e^\nu = e^{\mu+\nu}$, for $\mu, \nu \in P$. See Appendix A7 for more details.

S. Special/Interesting representations

(S1) Let $\mathfrak{l} \subseteq \mathfrak{g}$ be a Levi subalgebra of $\mathfrak{g}$ (this is a Lie algebra corresponding to a subgraph of the Dynkin diagram which corresponds to $\mathfrak{g}$). The subalgebra $\mathfrak{l}$ corresponds to a subset J of the set $\{\alpha_1, \ldots, \alpha_n\}$ of simple roots. The **restriction rule from $\mathfrak{g}$ to $\mathfrak{l}$ is**

$$V^\lambda \downarrow_{\mathfrak{l}}^{\mathfrak{g}} = \sum_\eta V^{\eta(1)},$$

where
the sum is over all paths $\eta \in \mathcal{P}\pi_\lambda$ such that $\eta \in \overline{C}_{\mathfrak{l}}$,
$\eta \in \overline{C}_{\mathfrak{l}}$ means that $\langle \eta(t), \alpha_i \rangle \ge 0$, for all $t \in [0,1]$ and all $\alpha_i \in J$.

(S2) The tensor product of two irreducible modules is given by

$$V^\mu \otimes V^\nu = \sum_\eta V^{\mu + \eta(1)},$$

where the sum is over all paths $\eta \in \mathcal{P}\pi_\nu$ such that $\pi_\mu * \eta \in \overline{C}$,
π_μ and π_ν are straight line paths from 0 to μ and 0 to ν, respectively,
$\mathcal{P}\pi_\nu$ is as in (Ib),
$\pi_\mu * \eta$ is the path obtained by attaching η to the end of π_μ, and
$(\pi_\mu * \eta) \in \overline{C}$ means that $\langle (\pi_\mu * \eta)(t), \alpha_i \rangle \ge 0$, for all $t \in [0,1]$ and all simple roots α_i.

Notes and references

(1) The indexing of irreducible representations given in **(Ia)** is due to Cartan and Killing, the founders of the theory, from around the turn of the century. Introductory treatments of this result can be found in [FH] and [Hu1].

(2) The first equality in **(Ib)** is due to Littelmann [Li1], but his later article [Li2] has some improvements and can be read independently, so we recommend the later article. This formula for the dimension of the irreducible representation, the number of paths in a certain set, is exactly analogous to the formula in the $GL(n,\mathbb{C})$ case, the number of tableaux which satisfy a certain condition. The second equality is the Weyl dimension formula which was originally proved in [Wy2]. It can be proved easily from the Weyl character formula given in **(Ic)**, see [Hu1] and [Ste4] Lemma 2.5. This product formula is an analogue of the "hook-content" formula given in the $GL(n,\mathbb{C})$ case.

(3) A priori, it might be possible that the set $\mathcal{P}\pi_\lambda$ is an infinite set, at least the way that we have defined it. In fact, this set is always finite and there is a description of the paths that are contained in it. The paths in this set are called *Lakshmibai-Seshadri paths*, see [Li2]. The explicit description of these paths is a generalization of the types of indexings that were used in the "standard monomial theory" of Lakshmibai and Seshadri [LS].

(4) The first equality in **(Ic)** is due to Littelmann [Li2]. This formula, a weighted sum over paths, is an analogue of the formula for the irreducible character of $GL(n,\mathbb{C})$ as a weighted sum of column strict tableaux. The second equality in **(Ic)** is the celebrated Weyl character formula which was originally proved in [Wy2]. A modern treatment of this formula can be found in [BtD], [Hu1], and [Va].

(5) The general restriction formula **(S1)** is due to Littelmann [Li2]. This is an analogue of the rule given in **(S1)** of the $GL(n,\mathbb{C})$ results. In this case the formula is as a sum over paths which satisfy certain conditions whereas in the $GL(n,\mathbb{C})$ case the formula is a sum over column strict fillings which satisfy a certain condition.

(6) The general tensor product formula in **(S2)** is due to Littelmann [Li2]. This formula is an analogue of the formula given in **(S2)** of the $GL(n,\mathbb{C})$ results.

(7) The results of Littelmann given above are some of the most exciting results of combinatorial representation theory in recent years. They were very much inspired by some very explicit conjectures of Lakshmibai, see [LS], which arose out of the "standard monomial theory" developed by Lakshmibai and Seshadri. Although Littelmann's theory is actually much more general than we have stated above, the special set of paths $\mathcal{P}\pi_\lambda$ used in **(Ib-c)** is a modified description of the same set which appeared in Lakshmibai's conjecture. Another important influence on Littelmann in his work was Kashiwara's work on crystal bases [Ksh].

Part II

5. Generalizing the S_n results

Having the above results for the symmetric group in hand we would like to try to generalize as many of the S_n results to other similar groups and algebras as we can. Work along this line began almost immediately after the discovery of the S_n results and it continues today. In the current state of results this has been largely

(1) successful for the complex reflection groups $G(r,p,n)$ and their "Hecke algebras,"
(2) successful for tensor power centralizer algebras and their q-analogues, and
(3) unsuccessful for general Weyl groups and finite Coxeter groups.

Let us give a brief description of what the objects are in **(1)**, **(2)**, and **(3)**. For more precise definitions and discussion of everything below see Appendix B.

Definitions

(1) Complex reflection groups $G(r,p,n)$ and their Hecke algebras

The complex reflection groups $G(r,p,n)$

A finite Coxeter group is a finite group which is generated by reflections in $\mathbb{R}^n$. In other words, take a bunch of linear transformations of $\mathbb{R}^n$ which are reflections (in the sense of reflections and rotations in the orthogonal group) and see what group they generate. If the group is finite then it is a finite Coxeter group. Actually, this definition of finite Coxeter group is not the usual one (for that see Appendix B1), but since we have the following theorem we are not too far astray.

Theorem 5.1. *A group is a finite group generated by reflections if and only if it is a finite Coxeter group.*

The finite Coxeter groups have been classified completely and there is one group of each of the following "types"

$$A_n,\ B_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ H_3,\ H_4,\ \text{or}\ I_2(m).$$

The finite *crystallographic* reflection groups are called *Weyl groups* because of their connection with Lie theory. These are the finite Coxeter groups of types

$$A_n,\ B_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ \text{and}\ G_2 = I_2(6).$$

A complex reflection group is a group generated by complex reflections, i.e. invertible linear transformations of $\mathbb{C}^n$ which have finite order and which have exactly one eigenvalue that is not 1. Every finite Coxeter group is also a finite complex reflection group. The finite complex reflection groups have been classified by Shephard and Todd [ST] and each such group is one of the groups

(a) $G(r,p,n)$, where r,p,n are positive integers such that p divides r, or
(b) one of 34 "exceptional" finite complex reflection groups.

The groups $G(r,p,n)$ are very similar to the symmetric group S_n in many ways and this is probably why generalizing the S_n theory has been so successful for these groups. The symmetric groups and the finite Coxeter groups of types B_n, and D_n are all special cases of the groups $G(r,p,n)$.

The "Hecke algebras" of reflection groups

The Iwahori-Hecke algebra of a finite Coxeter group W is an algebra which is a q-analogue, or q-deformation, of the group algebra W. See Appendix B3

for a proper definition of this algebra. It has a basis T_w, $w \in W$, (so it is the same dimension as the group algebra of W) but the multiplication in this algebra depends on a particular number $q \in \mathbb{C}$, which can be chosen arbitrarily. These algebras are true Hecke algebras only when W is a finite Weyl group.

The "Hecke algebras" of the groups $G(r,p,n)$ are q-analogues of the group algebras of the groups $G(r,p,n)$. It is only recently (1990-1994) that they have been defined. It is important to note that these algebras are not true Hecke algebras. In group theory, a Hecke algebra is a very specific kind of double coset algebra and the "Hecke algebras" of the groups $G(r,p,n)$ do not fit this mold. See Appendix B3-4 for the proper definition of a Hecke algebra and some discussion of how the "Hecke algebras" for the groups $G(r,p,n)$ are defined.

(2) Tensor power centralizer algebras

A tensor power centralizer algebra is an algebra which is isomorphic to $\mathrm{End}_G(V^{\otimes k})$ for some group (or Hopf algebra) G and some representation V of G. In this definition

$$\mathrm{End}_G(V^{\otimes k}) = \{T \in \mathrm{End}(V^{\otimes k}) \mid Tgv = gTv, \text{ for all } g \in G \text{ and all } v \in V^{\otimes k}\}.$$

There are some examples of tensor power centralizer algebras that have been particularly important:

(a) The group algebras, $\mathbb{C}S_k$, of the symmetric groups S_k,
(b) The Iwahori-Hecke algebras, $H_k(q)$, of type A_{k-1},
(c) The Temperley-Lieb algebras, $TL_k(x)$,
(d) The Brauer algebras, $B_k(x)$,
(e) The Birman-Murakami-Wenzl algebras, $BMW_k(r,q)$.
(f) The spider algebras,
(f) The rook monoid algebras,
(g) The Solomon-Iwahori algebras,
(h) The wall algebras,
(i) The q-wall algebras,
(j) The partition algebras.

We certainly do not have space to discuss all of these objects in this paper, and thus we will limit ourselves to the cases (a)-(e) in our discussion below and in the Appendix, §B5-8. References for the remaining cases are as follows.

The spider algebras. These algebras were written down combinatorially and studied by G. Kuperberg [Ku4-5].

The rook monoid algebras. L. Solomon (unpublished work) recognized that this very natural monoid (the combinatorics of which has also been studied in [GR]) appears as a tensor power centralizer.

The Solomon-Iwahori algebras. This algebra was introduced in [So1]. The fact that it is a tensor power centralizer algebra is an unpublished result of L. Solomon, see [So2].

The wall algebras. These algebras were introduced in a nice combinatorial form in [BC] and in other forms in [Ko] and Procesi [Pr] and other older invariant theory works [Wy]. All of these works were related to tensor power centralizers and/or fundamental theorems of invariant theory.

The q-wall algebras. These algebras were introduced by Kosuda and Murakami [KM1-2] and studied subsequently in [Le] and [Ha1-2].

The partition algebras. These algebras were introduced by V. Jones in [Jo1] and have been studied subsequently by P. Martin [Ma].

Notes and references for answers to the main questions

Some partial results giving answers to the main questions for the complex reflection groups $G(r,p,n)$, their "Hecke algebras", the Temperley-Lieb algebras, the Brauer algebras, and the Birman-Murakami-Wenzl algebras can be found in Appendix B. The appropriate references are as follows.

(1) Complex reflection groups $G(r,p,n)$ and their Hecke algebras

The complex reflection groups $G(r,p,n)$

I. What are the irreducible modules?

The indexing, dimension formulas and character formulas for the representations of the groups $G(r,p,n)$ are originally due to

Young [Y1]	for finite Coxeter groups of types B_n and D_n, and
Specht [Spc]	for the group $G(r,1,n)$.

We do not know who first did the general $G(r,p,n)$ case but it is easy to generalize Young and Specht's results to this case. See [Ari] and [HR2] for recent accounts.

Essentially what one does to determine the indexing, dimensions and the characters of the irreducible modules is to use Clifford theory to reduce the $G(r,1,n)$ case to the case of the symmetric group S_n. Then one can use Clifford theory again to reduce the $G(r,p,n)$ case to the $G(r,1,n)$ case. The original reference for Clifford theory is [Cl] and the book by Curtis and Reiner [CR2] has a modern treatment. The articles [Ste3] and [HR2] explain how the reduction from $G(r,p,n)$ to $G(r,1,n)$ is done. The dimension and character theory for the case $G(r,1,n)$ has an excellent modern treatment in [Mac], Appendix B to Chapter I.

C. How do we construct the irreducible modules?

The construction of the irreducible representations by Young symmetrizers was extended to the finite Coxeter groups of types B_n and D_n by Young himself in his paper [Y3]. The authors don't know when the general case was first treated in the literature, but it is not difficult to extend Young's results to the general case $G(r,p,n)$. The $G(r,p,n)$ case does appear periodically in the literature, see for example [Al].

Young's seminormal construction was generalized to the "Hecke algebras" of $G(r,p,n)$ in the work of Ariki and Koike [AK] and Ariki [Ari]. One can easily set $q=1$ in the

constructions of Ariki and Koike and obtain the appropriate analogues for the groups $G(r,p,n)$. We do not know if the analogue of Young's seminormal construction for the groups $G(r,p,n)$ appeared in the literature previous to the work of Ariki and Koike on the "Hecke algebra" case.

S. Special/Interesting representations.

The authors do not know if the analogues of the S_n results, **(S1-3)** of Section 2, have explicitly appeared in the literature. It is easy to use symmetric functions and the character formulas of Specht, see [Mac] Chpt. I, App. B, to derive formulas for the $G(r,1,n)$ case in terms of the symmetric group results. Then one proceeds as described above to compute the necessary formulas for $G(r,p,n)$ in terms of the $G(r,1,n)$ results. See [Ste3] for how this is done.

The "Hecke algebras" of reflection groups

The definition.

The "Hecke algebras" corresponding to the groups $G(r,p,n)$ were defined by

Ariki and Koike [AK],	for the case $G(r,1,n)$, and
Broué and Malle [BM] and Ariki [Ari],	for the general case $G(r,p,n)$.

See Appendix B4 for a definition of these algebras and some partial answers to the main questions. Let us give references to the literature for the answers to the main questions for these algebras.

I. What are the irreducibles?

The results of Ariki-Koike [AK] and Ariki [Ari] say that the "Hecke algebras" of $G(r,p,n)$ are q-deformations of the group algebras of the groups $G(r,p,n)$. Thus, it follows from the Tits deformation theorem (see [Ca] Chapt 10, 11.2 and [CR2] §68.17) that the indexings and dimension formulas for the irreducible representations of these algebras must be the same as the indexings and dimension formulas for the groups $G(r,p,n)$. Finding analogues of the character formulas requires a bit more work and a Murnaghan-Nakayama type rule for the "Hecke algebras" of $G(r,p,n)$ was given by Halverson and Ram [HR2]. As far as we know, the formula for the irreducible characters of S_n as a weighted sum of standard tableaux which we gave in the symmetric group section has not yet been generalized to the case of $G(r,p,n)$ and its "Hecke algebras".

C. How do we construct the irreducible modules?

Analogues of Young's seminormal representations have been given by

Hoefsmit [Hfs] and Wenzl [Wz1], independently, for Iwahori-Hecke algebras of type A_{n-1},
Hoefsmit [Hfs], for Iwahori-Hecke algebras of types B_n and D_n,
Ariki and Koike [AK] for the "Hecke algebras" of $G(r,1,n)$, and
Ariki [Ari] for the general "Hecke algebras" of $G(r,p,n)$.

There seems to be more than one appropriate choice for the analogue of Young symmetrizers for Hecke algebras. The definitions in the literature are due to

Gyoja [Gy], for the Iwahori-Hecke algebras of type A_{n-1},
Dipper and James [DJ1] and Murphy [M1-2] for the Iwahori-Hecke algebras of type A_{n-1},
King and Wybourne [KW] and Duchamp, et al [DK] for the Iwahori-Hecke algebras of type A_{n-1},
Dipper, James, and Murphy [DJ2], [DJM] for the Iwahori-Hecke algebras of type B_n,
Pallikaros [Pa], for the Iwahori-Hecke algebras of type D_n.
Mathas [Mth] and Murphy [M3], for the "Hecke algebras" of $G(r, p, n)$.

The paper [GL] also contains important ideas in this direction.

S. Special/Interesting representations.

It follows from the Tits deformation theorem (or rather, an extension of it) that the results for the "Hecke algebras" of $G(r, p, n)$ must be the same as for the case of the groups $G(r, p, n)$.

(2) Tensor power centralizer algebras

The definitions.

The references for the combinatorial definitions of the various centralizer algebras are as follows.

Temperley-Lieb algebras. These algebras are due, independently, to many different people. Some of the discoverers were Rumer-Teller-Weyl [RTW], Penrose [P1-2], Temperley-Lieb [TL], Kaufmann [Ka] and Jones[Jo2]. The work of V. Jones was crucial in making them so important in combinatorial representation theory today.

The Iwahori-Hecke algebras of type A_{n-1}. Iwahori [Iw] introduced these algebras in 1964 in connection with $GL(n, \mathbb{F}_q)$. Jimbo [Ji] realized that they arise as tensor power centralizer algebras for quantum groups.

Brauer algebras. These algebras were defined by Brauer in 1937 [Br]. Brauer also proved that they are tensor power centralizers.

Birman-Murakami-Wenzl algebras. These algebras are due to Birman and Wenzl [BW] and Murakami [Mu1]. It was realized early [Re] [Wz3] that these arise as tensor power centralizers but there was no proof in the literature for some time. See the references in [CP] §10.2.

I. What are the irreducibles?

Indexing of the representations of tensor power centralizer algebras follows from double centralizer theory (see Weyl [Wy1]) and a good understanding of the indexings and tensor product rules for the group or algebra which it is centralizing (i.e. $GL(n, \mathbb{C})$, $O(n, \mathbb{C})$, $U_q\mathfrak{sl}(n)$, etc.). The references for resulting indexings and dimension formulas for the irreducible representations are as follows:

Temperley-Lieb algebras. These results are classical and can be found in the book by Goodman, de la Harpe, and Jones [GHJ].

Brauer algebras. These results were known to Brauer [Br] and Weyl [Wy]. An important combinatorial point of view was given by Berele [Be1-2] and further developed by Sundaram [Su1-3].

Iwahori-Hecke algebras of type A_{n-1}. These results follow from the Tits deformation theorem and the corresponding results for the symmetric group.

Birman-Murakami-Wenzl algebras. These results follow from the Tits deformation theorem and the corresponding results for the Brauer algebra.

The indexings and dimension formulas for the Temperley-Lieb and Brauer algebras also follow easily by using the techniques of the Jones basic construction, see [Wz2] and [HR1].

The references for the irreducible characters of the various tensor power centralizer algebras are as follows:

Temperley-Lieb algebras. Character formulas can be derived easily by using Jones Basic Construction techniques [HR1].

Iwahori-Hecke algebras of type A_{n-1}. The analogue of the formula for the irreducible characters of S_n as a weighted sum of standard tableaux was found by Roichman [Ro]. Murnaghan-Nakayama type formulas were found by several authors [KW], [vdJ], [VK], [SU], and [Ra2].

Brauer algebras and Birman-Murakami-Wenzl algebras. Murnaghan-Nakayama type formulas were derived in [Ra1] and [HR1], respectively.

Brauer algebra and Birman-Murakami-Wenzl algebra analogues of the formula for the irreducible characters of the symmetric groups as a weighted sum of standard tableaux have not appeared in the literature.

C. How do we construct the irreducibles?

Temperley-Lieb algebras. An application of the Jones Basic Construction (see [Wz2] and [HR1]) gives a construction of the irreducible representations of the Temperley-Lieb algebras. This construction is classical and has been rediscovered by many people. In this case the construction is an analogue of the Young symmetrizer construction. The analogue of the seminormal construction appears in [GHJ].

Iwahori-Hecke algebras of type A_{n-1}. The analogue of Young's seminormal construction for this case is due, independently, to Hoefsmit [Hfs] and Wenzl [Wz1]. Analogues of Young symmetrizers (different analogues) have been given by Gyoja [Gy] and Dipper, James, and Murphy [DJ1], [M1-2], King and Wybourne [KW] and Duchamp, et al. [DK].

Brauer algebras. Analogues of Young's seminormal representations have been given, independently, by Nazarov [Nz] and Leduc and Ram [LR]. An analogue of the Young symmetrizer construction can be obtained by applying the Jones Basic Construction to the classical Young symmetrizer construction and this is the one that has been used by many authors [BBL], [HW], [Ke], [GL]. The actual element of the algebra

which is the analogue of the Young symmetrizer involves a central idempotent for which there is no known explicit formula and this is the reason that most authors work with a quotient formulation of the appropriate module.

Birman-Murakami-Wenzl algebras. Analogues of Young's seminormal representations have been given by Murakami [Mu2] and Leduc and Ram [LR]. The methods in the two works are different, the work of Murakami uses the physical theory of Boltzmann weights and the work of Leduc and Ram uses the theory of ribbon Hopf algebras and quantum groups. Exactly in the same way as for the Brauer algebra, an analogue of the Young symmetrizer construction can be obtained by applying the Jones Basic Construction to the Young symmetrizer constructions for the Iwahori-Hecke algebra of type A_{n-1}. As in the Brauer algebra case one should work with a quotient formulation of the module to avoid using a central idempotent for which there is no known explicit formula.

(3) Reflection groups of exceptional type.

Generalizing the S_n theory to finite Coxeter groups of exceptional type, finite complex reflection groups of exceptional type and the corresponding Iwahori-Hecke algebras, has been largely unsuccessful. This is not to say that there haven't been some very nice partial results only that at the moment nobody has any understanding of how to make a good combinatorial theory to encompass all the classical and exceptional types at once. Two amazing partial results along these lines are

the Springer construction and the Kazhdan-Lusztig construction.

The Springer construction is a construction of the irreducible representations of the crystallographic reflection groups on cohomology of unipotent varieties [Spr]. It is a geometric construction and not a combinatorial construction. See Appendix A3 for more information in the symmetric group case. It is possible that this construction may be combinatorialized in the future, but to date no one has done this.

The Kazhdan-Lusztig construction [KL1] is a construction of certain representations called *cell representations* and it works for all finite Coxeter groups. The cell representations are almost irreducible but unfortunately not irreducible in general, and nobody understands how to break them up into irreducibles, except in a case by case fashion. The other problem with these representations is that they depend crucially on certain polynomials, the Kazhdan-Lusztig polynomials, which seem to be impossible to compute or understand well except in very small cases, see [Bre] for more information. See [Ca] for a summary and tables of the known facts about representation of finite Coxeter groups of exceptional type.

Remarks

(1) A Hecke algebra is a specific "double coset algebra" which depends on a group G and a subgroup B. Iwahori [Iw] studied these algebras in the case that G is a finite Chevalley group and B is a Borel subgroup of G and defined what

are now called *Iwahori-Hecke algebras*. These are q-analogues of the group algebras of finite Weyl groups. The work of Iwahori yields a presentation for these algebras which can easily be extended to define Iwahori-Hecke algebras for all Coxeter groups but, except for the original Weyl group case, these have never been realized as true Hecke algebras, i.e. double coset algebras corresponding to an appropriate G and B. The "Hecke algebras" corresponding to the groups $G(r,p,n)$ are q-analogues of the group algebras of $G(r,p,n)$. Although these algebras are not true Hecke algebras either, Broué and Malle [BM] have shown that many of these algebras arise in connection with non-defining characteristic representations of finite Chevalley groups and Deligne-Lusztig varieties.

(2) There is much current research on generalizing symmetric group results to affine Coxeter groups and affine Hecke algebras. The case of affine Coxeter groups was done by Kato [Kat] using Clifford theory ideas. The case of affine Hecke algebras has been intensely studied by Lusztig [Lu1-7], Kazhdan-Lusztig [KL2], and Ginzburg [G], [CG], but most of this work is very geometric and relies on intersection cohomology/K-theory methods. Hopefully some of their work will be made combinatorial in the near future.

(3) Wouldn't it be great if we had a nice combinatorial representation theory for finite simple groups!!

6. Generalizations of $GL(n,\mathbb{C})$ results

There have been successful generalizations of the $GL(n,\mathbb{C})$ results for the questions **(Ia-c), (S1), (S2)** to the following classes of groups and algebras.

(1) Connected complex semisimple Lie groups.

Examples: $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$, $PGL(n,\mathbb{C})$, $PSO(2n,\mathbb{C})$, $PSp(2n,\mathbb{C})$.

(2) Compact connected real Lie groups.

Examples: $SU(n,\mathbb{C})$, $SO(n,\mathbb{R})$, $Sp(n)$, where $Sp(n) = Sp(2n,\mathbb{C}) \cap U(2n,\mathbb{C})$.

(3) Finite dimensional complex semisimple Lie algebras.

Examples: $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{C})$. See Appendix A7 for the complete list of the finite dimensional complex semisimple Lie algebras.

(4) Quantum groups corresponding to complex semisimple Lie algebras.

The method of generalizing the $GL(n,\mathbb{C})$ results to the objects in **(1-4)** is to reduce them all to case **(3)** and then solve case **(3)**. The results for case **(3)** are given in Section 4. The reduction of cases **(1)** and **(2)** to case **(3)** are outlined in [Se2], and given in more detail in [Va] and [BtD]. The reduction of **(4)** to **(3)** is given in [CP] and in [Ja].

Partial results for further generalizations

Some partial results along the lines of the results **(Ia-c)** and **(S1-2)** for $GL(n,\mathbb{C})$ and complex semisimple Lie algebras have been obtained for the following groups and algebras.

(1) Kac-Moody Lie algebras and groups
(2) Yangians
(3) Simple Lie superalgebras

Other groups and algebras, for which the combinatorial representation theory is not understood very well, are

(4) Finite Chevalley groups
(5) $\mathfrak{p}$-adic Chevalley groups
(6) Real reductive Lie groups
(7) The Virasoro algebra

There are many many possible ways that we could extend this list but probably these four cases are the most fundamental cases where the combinatorial representation theory has not been formulated. There has been intense work on all of these cases, but hardly any by combinatorialists. Thus there are many beautiful results known but very few of them have been stated or interpreted through a combinatorialists eyes. The world is a gold mine, yet to be mined!

Notes and references

(1) An introductory reference to Kac-Moody Lie algebras is [Kc] . This book contains a good description of the basic representation theory of these algebras. We don't know of a good introductory reference for the Kac-Moody groups case, we would suggest beginning with the paper [KK] and following the references there.
(2) The basic introductory reference for Yangians and their basic representation theory is [CP], Chapter 12. See also the references given there.
(3) The best introductory reference for Lie superalgebras is Scheunert's book [Sch]. For an update on the combinatorial representation theory of these cases see the papers [Srg], [BR], [BRS], and [Snv].
(4) Finding a general combinatorial representation theory for finite Chevalley groups has been elusive for many years. After the fundamental work of J.A. Green [Gr] in 1955 which established a combinatorial representation theory for $GL(n,\mathbb{F}_q)$ there has been a concerted effort to extend these results to other finite Chevalley groups. G. Lusztig [Lu8-11] has made important contributions to this field; in particular, the results of Deligne-Lusztig [DL] are fundamental. However, this is a geometric approach rather than a combinatorial one and there is much work to be done for combinatorialists, even in interpreting the known results from combinatorial viewpoint. A good introductory treatment of this theory is the book by Digne and Michel [DM]. The original work of Green is treated in [Mac] Chapt. IV.

(5) The representation theory of $\mathfrak{p}$-adic Lie groups has been studied intensely by representation theorists but essentially not at all by combinatorialists. It is clear that there is a beautiful (although possibly very difficult) combinatorial representation theory lurking here. The best introductory reference to this work is the paper of R. Howe [Ho] on $\mathfrak{p}$-adic $GL(n)$. Recent results of G. Lusztig [Lu7] are a very important step in providing a general combinatorial representation theory for $\mathfrak{p}$-adic groups.
(6) The best place to read about the representation theory of real reductive groups is in the books of D. Vogan and N. Wallach [AV], [Vg1] , [Vg2], [Wa].
(7) The Virasoro algebra is a Lie algebra that seems to turn up in every back alley of representation theory. One can only surmise that it must have a beautiful combinatorial representation theory that is waiting to be clarified. A good place to read about the Virasoro algebra is in [FF].

Appendix A

A1. Basic Representation Theory

An *algebra* A is a vector space over $\mathbb{C}$ with a multiplication that is associative, distributive, has an identity and satisfies the following equation

$$(ca_1)a_2 = a_1(ca_2) = c(a_1a_2), \qquad \text{for all } a_1, a_2 \in A \text{ and } c \in \mathbb{C}.$$

An A-*module* is a vector space M over $\mathbb{C}$ with an A-action

$$\begin{array}{ccc} A \times M & \longrightarrow & M \\ (a, m) & \longmapsto & am, \end{array}$$

which satisfies

$$\begin{aligned} 1m &= m, \\ a_1(a_2m) &= (a_1a_2)m, \\ (a_1 + a_2)m &= a_1m + a_2m, \\ a(c_1m_1 + c_2m_2) &= c_1(am_1) + c_2(am_2). \end{aligned}$$

for all $a, a_1, a_2 \in A$, $m, m_1, m_2 \in M$ and $c_1, c_2 \in \mathbb{C}$. We shall use the words module and *representation* interchangeably.

A module M is *indecomposable* if there do not exist non zero A-modules M_1 and M_2 such that

$$M \cong M_1 \oplus M_2.$$

A module M is *irreducible* or *simple* if the only submodules of M are the zero module 0 and M itself. A module M is *semisimple* if it is the direct sum of simple submodules.

An algebra is *simple* if the only ideals of A are the zero ideal 0 and A itself. The *radical* $\mathrm{rad}(A)$ of an algebra A is the intersection of all the maximal left ideals of A. An algebra A is *semisimple* if all its modules are semisimple. An algebra A is *Artinian* if every decreasing sequence of left ideals of A stabilizes, that is for every chain

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

of left ideals of A there is an integer m such that $A_i = A_m$ for all $i \geq m$.

The following statements follow directly from the definitions.

Let A be an algebra.

(a) *Every irreducible A-module is indecomposable.*

(b) *The algebra A is semisimple if and only if every indecomposable A-module is irreducible.*

The proofs of the following statements are more involved and can be found in [Bou2] Chpt. VIII, §6, n^o4 and §5, n^o3.

Theorem A1.1.

(a) *If A is an Artinian algebra then the radical of A is the largest nilpotent ideal of A.*

(b) *An algebra A is semisimple if and only if A is Artinian and* $\mathrm{rad}(A) = 0$.

(c) *Every semisimple algebra is a direct sum of simple algebras.*

The case when A is not necessarily semisimple is often called *modular representation theory*. Let M be an A-module. A *composition series* of M is a chain

$$M = M_k \supseteq M_{k-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0,$$

such that, for each $1 \leq i \leq k$, the modules M_i/M_{i-1} are irreducible. The irreducible modules M_i/M_{i-1} are the *factors* of the composition series. The following theorem is proved in [CR1] (13.7).

Theorem A1.2. *(Jordan-Hölder) If there exists a composition series for M then any two composition series must have the same multiset of factors (up to module isomorphism).*

An important combinatorial point of view is as follows: The analogue of the subgroup lattice of a group can be studied for any A-module M. More precisely, the *submodule lattice* $L(M)$ of M is the lattice defined by the submodules of M with the order relations given by inclusions of submodules. The composition series are maximal chains in this lattice.

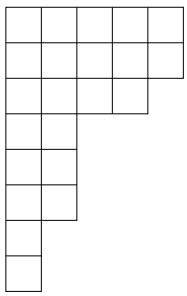
References

All of the above results can be found in [Bou2] Chapt. VIII and [CR1].

A2. Partitions and tableaux

Partitions

A *partition* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of integers such that $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. It is conventional to identify a partition with its Ferrers diagram which has λ_i boxes in the ith row. For example the partition $\lambda = (55422211)$ has Ferrers diagram



$\lambda = (55422211)$

We number the rows and columns of the Ferrers diagram as is conventionally done for matrices. If x is a box in λ then the *content* and the *hook length* of x are respectively given by

$$\begin{array}{cc} c(x) = j - i, & \text{if } x \text{ is in position } (i,j) \in \lambda, \text{ and} \\ h_x = \lambda_i - i + \lambda'_j - j + 1, & \text{where } \lambda'_j \text{ is the length of the } j\text{th column of } \lambda. \end{array}$$

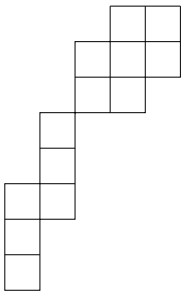
0	1	2	3	4
-1	0	1	2	3
-2	-1	0	1	
-3	-2			
-4	-3			
-5	-4			
-6				
-7				

Contents of the boxes

12	9	5	4	2
11	8	4	3	1
9	6	2	1	
6	3			
5	2			
4	1			
2				
1				

Hook lengths of the boxes

If μ and λ are partitions such that the Ferrers diagram of μ is contained in the Ferrers diagram of λ then we write $\mu \subseteq \lambda$ and we denote the difference of the Ferrers diagrams by λ/μ. We refer to λ/μ as a *shape* or, more specifically, a *skew shape*.



$\lambda/\mu = (55422211)/(32211)$

Tableaux

Suppose that λ has k boxes. A *standard tableau* of shape λ is a filling of the Ferrers diagram of λ with $1, 2, \ldots, k$ such that the rows and columns are increasing from left to right and from top to bottom respectively.

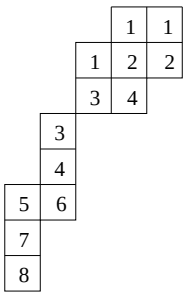
1	2	5	9	13
3	6	10	14	16
4	8	15	17	
7	12			
11	20			
18	21			
19				
22				

Let λ/μ be a shape. A *column strict tableau* of shape λ/μ filled with $1, 2, \ldots n$ is a filling of the Ferrers diagram of λ/μ with elements of the set $\{1, 2, \ldots, n\}$ such that the rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom. The *weight* of a column strict tableau T is the sequence of positive integers $\nu = (\nu_1, \ldots, \nu_n)$, where ν_i is the number of i's in T.

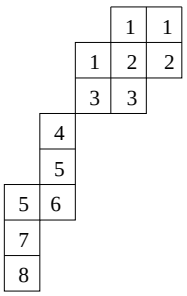
1	1	1	2	3
2	2	4	4	5
3	3	5	5	
6	7			
7	8			
8	9			
10				
11				

Shape $\lambda = (55422211)$
Weight $\nu = (33323122111)$

The *word* of a column strict tableau T is the sequence $w = w_1 w_2 \cdots w_p$ obtained by reading the entries of T from right to left in successive rows, starting with the top row. A word $w = w_1 \cdots w_p$ is a *lattice permutation* if for each $1 \leq r \leq p$ and each $1 \leq i \leq n-1$ the number of occurrences of the symbol i in $w_1 \cdots w_r$ is not less than the number of occurences of $i+1$ in $w_1 \cdots w_r$.



$w = 1122143346578$
Not a lattice permutation



$w = 1122133456578$
Lattice permutation

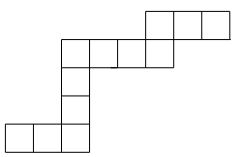
A *border strip* is a skew shape λ/μ which is

(a) connected (two boxes are connected if they share an edge), and
(b) does not contain a 2×2 block of boxes.

The weight of a border strip λ/μ is given by

$$\mathrm{wt}(\lambda/\mu) = (-1)^{r(\lambda/\mu)-1},$$

where $r(\lambda/\mu)$ is the number of rows in λ/μ.



$\lambda/\mu = (86333)/(5222)$
$\mathrm{wt}(\lambda/\mu) = (-1)^{5-1}$

Let λ and $\mu = (\mu_1, \ldots, \mu_\ell)$ be partitions of n. A *μ-border strip tableau* of shape λ is a sequence of partitions

$$T = (\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(\ell-1)} \subseteq \lambda^{(\ell)} = \lambda)$$

such that, for each $1 \leq i \leq \ell$,

(a) $\lambda^{(i)}/\lambda^{(i-1)}$ is a border strip, and
(b) $|\lambda^{(i)}/\lambda^{(i-1)}| = \mu_i$.

The weight of a μ-border strip tableau T of shape λ is

$$\mathrm{wt}(T) = \prod_{i=1}^{\ell-1} \mathrm{wt}(\lambda^{(i)}/\lambda^{(i-1)}). \tag{A2.1}$$

Theorem A2.2. *(Murnaghan-Nakayama rule) Let λ and μ be partitions of n and let $\chi^\lambda(\mu)$ denote the irreducible character of the symmetric group S_n indexed by λ evaluated at a permutation of cycle type μ. Then*

$$\chi^\lambda(\mu) = \sum_T \mathrm{wt}(T),$$

where the sum is over all μ-border strip tableaux T of shape λ and $\mathrm{wt}(T)$ is as given in (A2.1).

References

All of the above facts can be found in [Mac] Chapt. I. The proof of theorem (A2.2) is given in [Mac] Ch. I §7, Ex. 5.

A3. The flag variety, unipotent varieties, and Springer theory for $GL(n, \mathbb{C})$

Borel subgroups, Cartan subgroups, and unipotent elements

The groups

$$B_n = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix} \right\},$$

$$T_n = \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & * \end{pmatrix} \right\},$$

$$U_n = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\},$$

are the subgroups of $GL(n, \mathbb{C})$ consisting of upper triangular, diagonal, and upper unitriangular matrices, respectively.

A *Borel subgroup* of $GL(n,\mathbb{C})$ is a subgroup which is conjugate to B_n.

A *Cartan subgroup* of $GL(n,\mathbb{C})$ is a subgroup which is conjugate to T_n.

A matrix $u \in GL(n,\mathbb{C})$ is *unipotent* if it is conjugate to an upper unitriangular matrix.

The flag variety

There is a one-to-one correspondence between each of the following sets:

(1) $\mathcal{B} = \{\text{Borel subgroups of } GL(n,\mathbb{C})\}$,
(2) G/B, where $G = GL(n,\mathbb{C})$ and $B = B_n$,
(3) $\{\text{flags } 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \text{ such that } \dim(V_i) = i\}$.

Each of these sets naturally has the structure of a complex algebraic variety, which is called the *flag variety*.

The unipotent varieties

Given a unipotent element $u \in GL(n,\mathbb{C})$ with Jordan blocks given by the partition $\mu = (\mu_1, \ldots, \mu_\ell)$ of n, define an algebraic variety

$$\mathcal{B}_\mu = \mathcal{B}_u = \{\text{Borel subgroups of } GL(n,\mathbb{C}) \text{ which contain } u\}.$$

By conjugation, the structure of the subvariety $\mathcal{B}_u$ of the flag variety depends only on the partition μ. Thus $\mathcal{B}_\mu$ is well defined, as an algebraic variety.

Springer theory

It is a deep theorem of Springer [Spr] (which holds in the generality of semisimple algebraic groups and their corresponding Weyl groups) that there is an action of the symmetric group S_n on the cohomology $H^*(\mathcal{B}_u)$ of the variety $\mathcal{B}_u$. This action can be interpreted nicely as follows. The imbedding

$$\mathcal{B}_u \subseteq \mathcal{B} \quad \text{induces a surjective map} \quad H^*(\mathcal{B}) \longrightarrow H^*(\mathcal{B}_u).$$

It is a famous theorem of Borel that there is a ring isomorphism

$$H^*(\mathcal{B}) \;\cong\; \mathbb{C}[x_1, \ldots, x_n]/I^+, \tag{A3.1}$$

where I^+ is the ideal generated by symmetric functions without constant term. It follows that $H^*(\mathcal{B}_u)$ is also a quotient of $\mathbb{C}[x_1, \ldots, x_n]$. From the work of Kraft [Kr], DeConcini and Procesi [DP] and Tanisaki [Ta], one has that the ideal $\mathcal{T}_u$ which it is necessary to quotient by in order to obtain an isomorphism

$$H^*(\mathcal{B}_u) \;\cong\; \mathbb{C}[x_1, \ldots, x_n]/\mathcal{T}_u,$$

can be described explicitly.

The symmetric group S_n acts on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ by permuting the variables. It turns out that the ideal $\mathcal{T}_u$ remains invariant under this action, thus yielding a well defined action of S_n on $\mathbb{C}[x_1, \ldots, x_n]/\mathcal{T}_u$. This action coincides with

the Springer action on $H^*(\mathcal{B}_u)$. Hotta and Springer [HS] have established that, if u is a unipotent element of shape μ then, for every permutation $w \in S_n$,

$$\sum_i q^i \varepsilon(w) \text{trace}(w^{-1}, H^{2i}(\mathcal{B}_u)) = \sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(q) \chi^\lambda(w),$$

where

$\varepsilon(w)$ is the sign of the permutation w,
$\text{trace}(w^{-1}, H^{2i}(\mathcal{B}_u))$ is the trace of the action of w^{-1} on $H^{2i}(\mathcal{B}_u)$,
$\chi^\lambda(w)$ is the irreducible character of the symmetric group evaluated at w, and
$\tilde{K}_{\lambda\mu}(q)$ is a variant of the Kostka-Foulkes polynomial, see [Mac] III §7 Ex. 8, and §6.

It follows from this discussion and some basic facts about the polynomials $\tilde{K}_{\lambda\mu}(q)$ that the top degree cohomology group in $H^*(\mathcal{B}_\mu)$ is a realization of the irreducible representation of S_n indexed by μ,

$$S^\mu \;\cong\; H^{\text{top}}(\mathcal{B}_\mu).$$

This construction of the irreducible modules of S_n is the *Springer construction*.

References

See [Mac] II §3 Ex. 1 for a description of the variety $\mathcal{B}_u$ and its structure. The theorem of Borel stated in (A3.1) is given in [Bo] and [BGG]. The references quoted in the text above will provide a good introduction to the Springer theory. The beautiful combinatorics of Springer theory has been studied by Barcelo [Ba], Garsia-Procesi [GP], Lascoux [L], Lusztig [Lu12], Shoji [Shj], Spaltenstein [Sp], Weyman [Wm], and others.

A4. Polynomial and rational representations of $GL(n, \mathbb{C})$

If V is a $GL(n, \mathbb{C})$-module of dimension d then, by choosing a basis of V, we can define a map

$$\begin{array}{rccc} \rho_V : GL(n,\mathbb{C}) & \longrightarrow & GL(d,\mathbb{C}) \\ g & \longmapsto & \rho(g), \end{array}$$

where $\rho(g)$ is the transformation of V that is induced by the action of g on V. Let

g_{ij} denote the (i, j) entry of the matrix g, and
$\rho(g)_{kl}$ denote the (k, l) entry of the matrix $\rho(g)$.

The map ρ depends on the choice of the basis of V, but the following definitions do not.

The module V is a *polynomial representation* if there are polynomials $p_{kl}(x_{ij})$, $1 \le k, l \le d$, such that

$$\rho(g)_{kl} = p_{kl}(g_{ij}), \qquad \text{for all } 1 \le k, l \le d.$$

In other words $\rho(g)_{jk}$ is the same as the polynomial p_{kl} evaluated at the entries g_{ij} of the matrix g.

The module V is a *rational representation* if there are rational functions (quotients of two polynomials) $p_{kl}(x_{ij})/q_{kl}(x_{ij})$, $1 \leq k, l \leq d$, such that

$$\rho(g)_{kl} = p_{kl}(g_{ij})/q_{kl}(g_{ij}), \qquad \text{for all } 1 \leq k, l \leq n.$$

Clearly, every polynomial representation is a rational one.

The theory of rational representations of $GL(n, \mathbb{C})$ can be reduced to the theory of polynomial representations of $GL(n, \mathbb{C})$. This is accomplished as follows. The determinant $\det\colon GL(n, \mathbb{C}) \to \mathbb{C}$ defines a 1-dimensional (polynomial) representation of $GL(n, \mathbb{C})$. Any integral power

$$\begin{array}{rccc} \det^k\colon & GL(n,\mathbb{C}) & \longrightarrow & \mathbb{C} \\ & g & \longmapsto & \det(g)^k \end{array}$$

of the determinant also determines a 1-dimensional representation of $GL(n, \mathbb{C})$. **All irreducible rational representations $GL(n, \mathbb{C})$ can be constructed in the form**

$$\det{}^k \otimes V^\lambda,$$

for some $k \in \mathbb{Z}$ and some irreducible polynomial representation V^λ of $GL(n, \mathbb{C})$.

There exist representations of $GL(n, \mathbb{C})$ which are not rational representations, for example

$$g \mapsto \begin{pmatrix} 1 & \ln|\det(g)| \\ 0 & 1 \end{pmatrix}.$$

There is no known classification of representations of $GL(n, \mathbb{C})$ which are not rational.

References

See [Ste1] for a study of the combinatorics of the rational representations of $GL(n, \mathbb{C})$.

A5. Schur-Weyl duality and Young symmetrizers

Let V be the usual n-dimensional representation of $GL(n, \mathbb{C})$ on column vectors of length n, that is

$$V = \operatorname{span}\{b_1, \ldots, b_n\} \quad \text{where} \quad b_i = (0, \ldots, 0, 1, 0, \ldots, 0)^t,$$

and the 1 in b_i appears in the ith entry. Then

$$V^{\otimes k} = \operatorname{span}\{b_{i_1} \otimes \cdots \otimes b_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$$

is the span of the words of length k in the letters b_i (except that the letters are separated by tensor symbols). The general linear group $GL(n, \mathbb{C})$ and the symmetric group S_k act on $V^{\otimes k}$ by

$$g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k, \quad \text{and} \quad (v_1 \otimes \cdots \otimes v_k)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

where $g \in GL(n, \mathbb{C})$, $\sigma \in S_k$, and $v_1, \ldots, v_k \in V$. (We have chosen to make the S_k-action a right action here, one could equally well choose the action of S_k to be a left action but then the formula would be $\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$.)

The following theorem is the amazing relationship between the group S_k and the group $GL(n,\mathbb{C})$ which was discovered by Schur [Sc1] and exploited with such success by Weyl [Wy1].

Theorem A5.1. *(Schur-Weyl duality)*

(a) *The action of* S_k *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{Gl(n,\mathbb{C})}(V^{\otimes k})$.
(b) *The action of* $GL(n,\mathbb{C})$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{S_k}(V^{\otimes k})$.

This theorem has the following important corollary, which provides a intimate correspondence between the representation theory of S_k and *some* of the representations of $GL(n,\mathbb{C})$ (the ones indexed by partitions of k).

Corollary A5.2. *As* $GL(n,\mathbb{C}) \times S_k$ *bimodules*

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} V^\lambda \otimes S^\lambda,$$

where V^λ *is the irreducible* $GL(n,\mathbb{C})$*-module and* S^λ *is the irreducible* S_k*-module indexed by* λ.

If λ is a partition of k, then the irreducible $GL(n,\mathbb{C})$-representation V^λ is given by

$$V^\lambda \cong V^{\otimes k} P(T) N(T),$$

where T is a tableau of shape λ and $P(T)$ and $N(T)$ are as defined in Section 2, Question **C**.

A6. The Borel-Weil-Bott construction

Let $G = GL(n,\mathbb{C})$ and let $B = B_n$ be the subgroup of upper triangular matrices in $GL(n,\mathbb{C})$. A *line bundle on* G/B is a pair $(\mathcal{L}, p)$ where $\mathcal{L}$ is an algebraic variety and p is a map (morphism of algebraic varieties)

$$p : \mathcal{L} \longrightarrow G/B,$$

such that the fibers of p are lines and such that $\mathcal{L}$ is a locally trivial family of lines. In this definition, *fibers* means the sets $p^{-1}(x)$ for $x \in G/B$ and *lines* means one-dimensional vector spaces. For the definition of *locally trivial family of lines* see [Sh] Chapt. VI §1.2. By abuse of language, a line bundle $(\mathcal{L}, p)$ is simply denoted by $\mathcal{L}$. Conceptually, a line bundle on G/B means that we are putting a one-dimensional vector space over each point in G/B.

A *global section* of the line bundle $\mathcal{L}$ is a map (morphism of algebraic varieties)

$$s : G/B \to \mathcal{L}$$

such that $p \circ s$ is the identity map on G/B. In other words a global section is any possible "right inverse map" to the line bundle.

Each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ determines a character (i.e. 1-dimensional representation) of the group T_n of diagonal matrices in $GL(n, \mathbb{C})$ via

$$\lambda\left(\begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & t_n \end{pmatrix}\right) = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}.$$

Extend this character to be a character of $B = B_n$ by letting λ ignore the strictly upper triangular part of the matrix, that is $\lambda(u) = 1$, for all $u \in U_n$. Let $\mathcal{L}_\lambda$ be the fiber product $G \times_B \lambda$, i.e. the set of equivalence classes of pairs (g, c), $g \in G$, $c \in \mathbb{C}^*$, under the equivalence relation

$$(gb, c) \sim (g, \lambda(b^{-1})c), \qquad \text{for all } b \in B.$$

Then $\mathcal{L}_\lambda = G \times_B \lambda$ with the map

$$\begin{array}{rccc} p\colon & G \times_B \lambda & \longrightarrow & G/B \\ & (g, c) & \longmapsto & gB \end{array}$$

is a line bundle on G/B.

The Borel-Weil-Bott theorem says that the irreducible representation V^λ of $GL_n(\mathbb{C})$ is

$$V^\lambda \cong H^0(G/B, \mathcal{L}_\lambda),$$

where $H^0(G/B, \mathcal{L}_\lambda)$ is the space of global sections of the line bundle $\mathcal{L}_\lambda$.

References

See [FH] and G. Segal's article in [CMS] for further information and references on this very important construction.

A7. Complex semisimple Lie algebras

A *finite dimensional complex semisimple Lie algebra* is a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ such that $\mathrm{rad}(\mathfrak{g}) = 0$. The following theorem classifies all finite dimensional complex semisimple Lie algebras.

Theorem A7.1.

(a) *Every finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ is a direct sum of complex simple Lie algebras.*

(b) *There is one complex simple Lie algebra corresponding to each of the following types*

$$A_{n-1},\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

The complex simple Lie algebras of types A_n, B_n, C_n and D_n are the ones of *classical type* and they are

$$\text{Type } A_{n-1}: \qquad \mathfrak{sl}(n,\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \mathrm{Tr}(A) = 0\},$$

$$\text{Type } B_n: \qquad \mathfrak{so}(2n+1,\mathbb{C}) = \{A \in M_{2n+1}(\mathbb{C}) \mid A + A^t = 0\},$$

$$\text{Type } C_n: \qquad \mathfrak{sp}(2n,\mathbb{C}) = \{A \in M_{2n}(\mathbb{C}) \mid AJ + JA^t = 0\},$$

$$\text{Type } D_n: \qquad \mathfrak{so}(2n) = \{A \in M_{2n}(\mathbb{C}) \mid A + A^t = 0\},$$

where J is the matrix of a skew-symmetric form on a $2n$-dimensional space.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. If V is a finite dimensional $\mathfrak{g}$-module and $\mu\colon \mathfrak{h} \to \mathbb{C}$ is any linear function, define

$$V_\mu = \{v \in V \mid hv = \mu(h)v, \text{ for all } h \in \mathfrak{h}\}.$$

The space V_μ is the *μ-weight space* of V. It is a nontrivial theorem (see [Se2]) that

$$V = \bigoplus_{\mu \in P} V_\mu,$$

where P is a $\mathbb{Z}$-lattice in $\mathfrak{h}^*$ which can be identified with the $\mathbb{Z}$-lattice P which is defined below in Appendix A8. The vector space $\mathfrak{h}^*$ is the space of linear functions from $\mathfrak{h}$ to $\mathbb{C}$.

Let $\mathbb{C}[P]$ be the group algebra of P. It can be given explicitly as

$$\mathbb{C}[P] \;=\; \mathbb{C}\text{-span}\{e^\mu \mid \mu \in P\}, \quad \text{with multiplication } e^\mu e^\nu = e^{\mu+\nu}, \text{ for } \mu, \nu \in P,$$

where the e^μ are formal variables indexed by the elements of P. The *character* of a $\mathfrak{g}$-module is

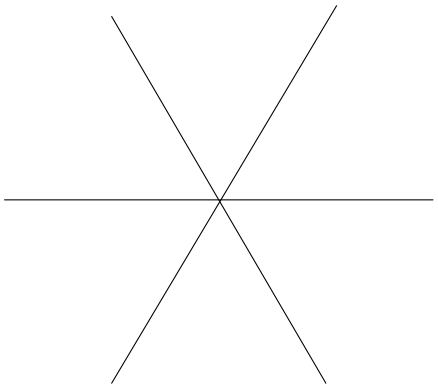
$$\mathrm{char}(V) = \sum_{\mu \in P} \dim(V_\mu) e^\mu.$$

References

Theorem (A7.1) is due to the founders of the theory, Cartan and Killing, from the late 1800's. The beautiful text of Serre [Se2] gives a review of the definitions and theory of complex semisimple Lie algebras. See [Hu1] for further details.

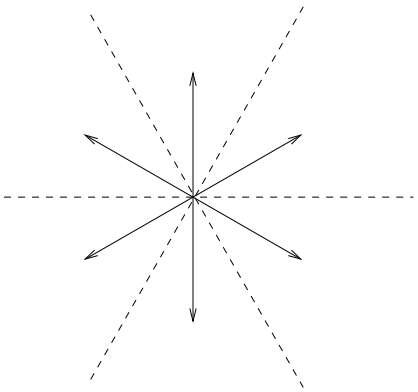
A8. Roots, weights and paths

To each of the "types", A_n, B_n, etc., there is an associated hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$.



Hyperplane arrangement for A_2

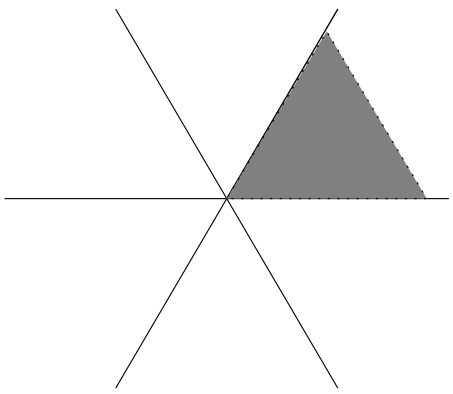
The space $\mathbb{R}^n$ has the usual Euclidean inner product $\langle\ ,\ \rangle$. For each hyperplane in the arrangement $\mathcal{A}$ we choose two vectors orthogonal to the hyperplane and pointing in opposite directions. This set of chosen vectors is called the *root system* R associated to $\mathcal{A}$.



Root system for A_2

There is a convention for choosing the lengths of these vectors but we shall not worry about that here.

Choose a chamber (connected component) C of $\mathbb{R}^n \setminus \bigcup_{H\in\mathcal{A}} H$.



A chamber for A_2

For each root $\alpha \in R$ we say that α is *positive* if it points toward the same side of the hyperplane as C is and *negative* if points toward the opposite side. It is standard notation to write

$$\alpha > 0, \qquad \text{if } \alpha \text{ is a positive root, and} \qquad \alpha < 0, \qquad \text{if } \alpha \text{ is a negative root.}$$

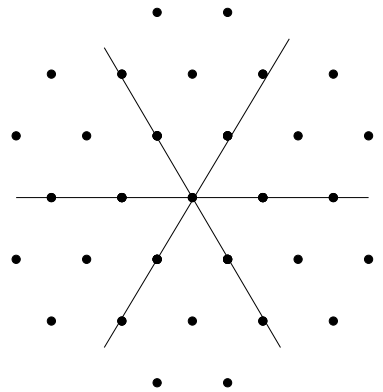
The positive roots which are associated to hyperplanes which form the walls of C are the *simple roots* $\{\alpha_1, \ldots, \alpha_n\}$. The *fundamental weights* are the vectors $\{\omega_1, \ldots, \omega_n\}$ in $\mathbb{R}^n$ such that

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad \text{where} \quad \alpha_j^\vee = \frac{2\alpha_j}{\langle \alpha_j, \alpha_j \rangle}.$$

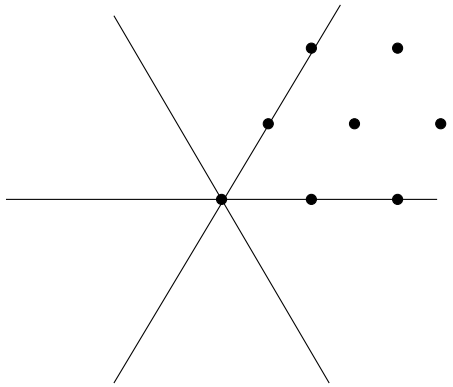
Then

$$P = \sum_{i=1}^{r} \mathbb{Z}\omega_i, \qquad \text{and} \qquad P^+ = \sum_{i=1}^{n} \mathbb{N}\,\omega_i, \qquad \text{where} \quad \mathbb{N} = \mathbb{Z}_{\geq 0},$$

are the lattice of *integral weights* and the cone of *dominant integral weights*, respectively.



Lattice of integral weights

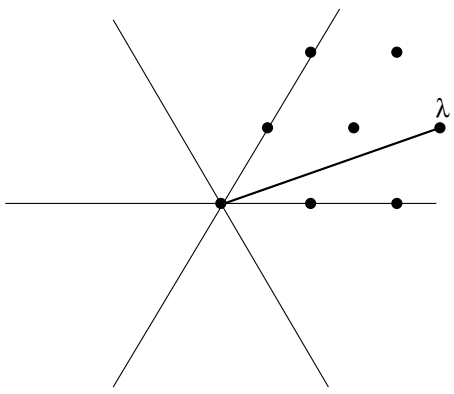


Cone of dominant integral weights

There is a one-to-one correspondence between the irreducible representations of $\mathfrak{g}$ and the elements of the cone P^+ in the lattice P.

Let λ be a point in P^+. Then the straight line path from 0 to λ is the map

$$\begin{array}{rccc} \pi_\lambda\colon & [0,1] & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & t\lambda. \end{array}$$

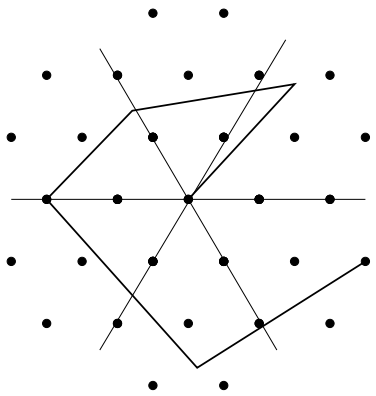


Path from 0 to λ

The set $\mathcal{P}\pi_\lambda$ is given by

$$\mathcal{P}\pi_\lambda = \{f_{i_1} \cdots f_{i_k} \pi_\lambda \mid 1 \leq i_1, \ldots, i_k \leq n\}$$

where $f_1, \ldots, f_n$ are the path operators introduced in [Li2]. These paths might look like



Path in $\mathcal{P}\pi_\lambda$

They are always piecewise linear and end in a point in P.

References

The basics of root systems can be found in [Hu1]. The reference for the path model of Littelmann is [Li2].

Appendix B

B1. Coxeter groups, groups generated by reflections, and Weyl groups

A *Coxeter group* is a group W presented by generators $S = \{s_1, \ldots, s_n\}$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } 1 \le i \le n, \\ (s_i s_j)^{m_{ij}} &= 1, && \text{for } 1 \le i \ne j \le n, \end{aligned}$$

where each m_{ij} is either ∞ or a positive integer greater than 1.

A *reflection* is a linear transformation of $\mathbb{R}^n$ which is a reflection in some hyperplane.

A *finite group generated by reflections* is a finite subgroup of $GL(n, \mathbb{R})$ which is generated by reflections.

Theorem B1.1. *The finite Coxeter groups are exactly the finite groups generated by reflections.*

A finite Coxeter group is *irreducible* if it cannot be written as a direct product of finite Coxeter groups.

Theorem B1.2. *(Classification of finite Coxeter groups)*

(a) *Every finite Coxeter group can be written as a direct product of irreducible finite Coxeter groups.*
(b) *There is one irreducible finite Coxeter group corresponding to each of the following "types"*

$$A_{n-1},\ B_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ H_3,\ H_4,\ I_2(m).$$

The irreducible finite Coxeter groups of *classical type* are the ones of types A_{n-1}, B_n, and D_n and the others are the irreducible finite Coxeter groups of *exceptional type*.

(a) The group of type A_{n-1} is the symmetric group S_n.
(b) The group of type B_n is the hyperoctahedral group $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$, the wreath product of the group of order 2 and the symmetric group S_n. It has order $2^n n!$.
(c) The group of type D_n is a subgroup of index 2 in the Coxeter group of type B_n.
(d) The group of type $I_2(m)$ is a dihedral group of order $2m$.

A finite group W generated by reflections in $\mathbb{R}^n$ is *crystallographic* if there is a lattice in $\mathbb{R}^n$ which is stable under the action of W. The crystallographic finite Coxeter groups are also called *Weyl groups*. The irreducible Weyl groups are the irreducible finite Coxeter groups of types

$$A_{n-1},\ B_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2 = I_2(6).$$

References

The most comprehensive reference for finite groups generated by reflections is [Bou1]. See also the book of Humphreys [Hu2].

B2. Complex reflection groups

A *complex reflection* is an invertible linear transformation of $\mathbb{C}^n$ of finite order which has exactly one eigenvalue that is not 1. A *complex reflection group* is a group generated by complex reflections in $\mathbb{C}^n$. The finite complex reflection groups have been classified by Shepard and Todd [ST]. Each finite complex reflection group is either

(a) $G(r,p,n)$ for some positive integers r,p,n such that p divides r, or
(b) one of 34 other "exceptional" finite complex reflection groups.

Let r,p,d and n be positive integers such that $pd = r$. The complex reflection group $G(r,p,n)$ is the set of $n \times n$ matrices such that

(a) The entries are either 0 or rth roots of unity,
(b) There is exactly on nonzero entry in each row and each column,
(c) The dth power of the product for the nonzero entries is 1.

The group $G(r,p,n)$ is a normal subgroup of $G(r,1,n)$ of index p and

$$|G(r,p,n)| = dr^{n-1}n!.$$

In addition

(a) $G(1,1,n) \cong S_n$ the symmetric group or *Weyl group of type* A_{n-1},
(b) $G(2,1,n)$ is the *hyperoctahedral group* or *Weyl group of type* B_n,
(c) $G(r,1,n) \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_n$, the wreath product of the cyclic group of order r with S_n,
(d) $G(2,2,n)$ is the *Weyl group of type* D_n.

Partial results for $G(r,1,n)$

The following are answers to the main questions **(Ia-c)** for the groups $G(r,1,n) \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_n$. For the general $G(r,p,n)$ case see [HR2].

I. What are the irreducible $G(r,1,n)$-modules?

(a) How do we index/count them?

There is a bijection

r-tuples $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of partitions such that $\sum_{i=1}^{r} |\lambda^{(i)}| = n$ $\overset{1-1}{\longleftrightarrow}$ **Irreducible representations C^λ.**

(b) What are their dimensions?

The dimension of the irreducible representation C^λ is given by

$$\begin{aligned}\mathbf{dim}(C^\lambda) &= \mathbf{\#\ of\ standard\ tableaux\ of\ shape}\ \lambda \\ &= n! \prod_{i=1}^{r} \prod_{x \in \lambda^{(i)}} \frac{1}{h_x},\end{aligned}$$

where h_x is the hook length at the box x. A standard tableau of shape $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ is any filling of the boxes of the $\lambda^{(i)}$ with the numbers $1, 2, \ldots, n$ such that the rows and the columns of each $\lambda^{(i)}$ are increasing.

(c) What are their characters?

A Murnaghan-Nakayama type rule for the characters of the groups $G(r, 1, n)$ was originally given by Specht [Spc]. See also [Osi] and [HR2].

References

The original paper of Shepard and Todd [ST] remains a basic reference. Further information about these groups can be found in [HR2]. The articles [OS], [Leh], [Ste3], [Mal] contain other recent work on the combinatorics of these groups.

B3. Hecke algebras and "Hecke algebras" of Coxeter groups

Let G be a finite group and let B be a subgroup of G. The *Hecke algebra* of the pair (G, B) is the subalgebra

$$\mathcal{H}(G,B) = \left\{ \sum_{g \in G} a_g g \;\middle|\; a_g \in \mathbb{C}, \text{ and } a_g = a_h \text{ if } BgB = BhB. \right\}$$

of the group algebra of G. The elements

$$T_w = \frac{1}{|B|} \sum_{g \in BwB} g,$$

as w runs over a set of representatives of the double cosets $B\backslash G/B$, form a basis of $\mathcal{H}(G, B)$.

Let G be a finite Chevalley group over the field $\mathbb{F}_q$ with q elements and fix a Borel subgroup B of G. The pair (G, B) determines a pair (W, S) where W is the Weyl group of G and S is a set of simple reflections in W (with respect to B). The *Iwahori-Hecke algebra* corresponding to G is the Hecke algebra $\mathcal{H}(G, B)$. In this case the basis elements T_w are indexed by the elements w of the Weyl group W corresponding to the pair (G, B) and the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w), \\ (q-1)T_w + qT_{sw}, & \text{if } \ell(sw) < \ell(w), \end{cases}$$

if s is a simple reflection in W. In this formula $\ell(w)$ is the *length* of w, i.e. the minimum number of factors needed to write w as a product of simple reflections.

A particular example of the Iwahori-Hecke algebra occurs when $G = GL(n, \mathbb{F}_q)$ and B is the subgroup of upper triangular matrices. Then the Weyl group W, is the symmetric group S_n, and the simple reflections in the set S are the transpositions $s_i = (i, i+1)$, $1 \le i \le n-1$. In this case the algebra $\mathcal{H}(G, B)$ is the *Iwahori-Hecke algebra of type* A_{n-1} and (as we will see later) can be presented by generators $T_1, \ldots, T_{n-1}$ and relations

$$\begin{aligned} &T_iT_j = T_jT_i, && \text{for } |i-j|>1,\\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && \text{for } 1\le i\le n-2,\\ &T_i^2 = (q-1)T_i + q, && \text{for } 2\le i\le n. \end{aligned}$$

See Section B5 for more facts about the Iwahori-Hecke algebras of type A. In particular, these Iwahori-Hecke algebras also appear as tensor power centralizer algebras, see Theorem B5.3. This is some kind of miracle: the Iwahori-Hecke algebras of type A are the only Iwahori-Hecke algebras which arise naturally as tensor power centralizers.

In view of the multiplication rules for the Iwahori-Hecke algebras of Weyl groups it is easy to define a "Hecke algebra" for all Coxeter groups (W,S), just by defining it to be the algebra with basis T_w, $w\in W$, and multiplication

$$T_sT_w = \begin{cases} T_{sw}, & \text{if } \ell(sw)>\ell(w),\\ (q-1)T_w + qT_{sw}, & \text{if } \ell(sw)<\ell(w), \end{cases}$$

if $s\in S$. These algebras are not true Hecke algebras except when W is a Weyl group.

References

For references on Hecke algebras see [CR2] (Vol I, Section 11). For references on Iwahori-Hecke algebras see [Bou1] Chpt. IV §2 Ex. 23-25, [CR2] Vol. II §67, and [Hu2] Chpt. 7. The article [Cu] is also very informative.

B4. "Hecke algebras" of the groups $G(r,p,n)$

Let q and $u_0,u_1,\dots,u_{r-1}$ be indeterminates. Let $H_{r,1,n}$ be the algebra over the field $\mathbb{C}(u_0,u_1,\dots,u_{r-1},q)$ given by generators $T_1,T_2,\dots,T_n$ and relations

$$\begin{aligned} &(1)\quad T_iT_j = T_jT_i, && \text{for } |i-j|>1,\\ &(2)\quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && \text{for } 2\le i\le n-1,\\ &(3)\quad T_1T_2T_1T_2 = T_2T_1T_2T_1,\\ &(4)\quad (T_1-u_0)(T_1-u_1)\cdots(T_1-u_{r-1}) = 0,\\ &(5)\quad (T_i-q)(T_i+q^{-1}) = 0, && \text{for } 2\le i\le n. \end{aligned}$$

Upon setting $q=1$ and $u_{i-1}=\xi^{i-1}$, where ξ is a primitive rth root of unity, one obtains the group algebra $\mathbb{C}G(r,1,n)$. In the special case where $r=1$ and $u_0=1$, we have $T_1=1$, and $H_{1,1,n}$ is isomorphic to an Iwahori-Hecke algebra of type A_{n-1}. The case $H_{2,1,n}$ when $r=2$, $u_0=p$, and $u_1=p^{-1}$, is isomorphic to an Iwahori-Hecke algebra of type B_n.

Now suppose that p and d are positive integers such that $pd=r$. Let $x_0^{1/p},\dots,x_{d-1}^{1/p}$ be indeterminates, let $\varepsilon=e^{2\pi i/p}$ be a primitive pth root of unity and specialize the variables $u_0,\dots,u_{r-1}$ according to the relation

$$u_{\ell d+kp+1} = \varepsilon^\ell x_k^{1/p},$$

where the subscripts on the u_i are taken mod r. The *"Hecke algebra" $H_{r,p,n}$ corresponding to the group $G(r,p,n)$* is the subalgebra of $H_{r,1,n}$ generated by the elements

$$a_0 = T_1^p, \quad a_1 = T_1^{-1}T_2T_1, \quad \text{and} \quad a_i = T_i, \quad 2 \le i \le n.$$

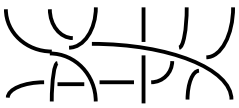
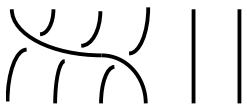
Upon specializing $x_k^{1/p} = \xi^{kp}$, where ξ is a primitive rth root of unity, $H_{r,p,n}$ becomes the group algebra $\mathbb{C}G(r,p,n)$. Thus $H_{r,p,n}$ is a "q-analogue" of the group algebra of the group $G(r,p,n)$.

References

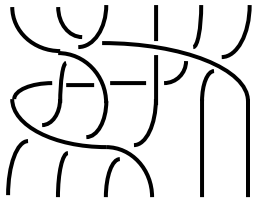
The algebras $H_{r,1,n}$ were first constructed by Ariki and Koike [AK], and they were classified as cyclotomic Hecke algebras of type B_n by Broué and Malle [BM] and the representation theory of $H_{r,p,n}$ was studied by Ariki [Ari]. See [HR2] for information about the characters of these algebras.

B5. The Iwahori-Hecke algebras $H_k(q)$ of type A

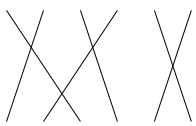
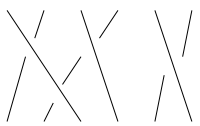
A *k-braid* is viewed as two rows of k vertices, one above the other, and k strands that connect top vertices to bottom vertices in such a way that each vertex is incident to precisely one strand. Strands cross over and under each other in three-space as they pass from one vertex to the next.

$t_1 =$  $t_2 =$ 

We multiply k-braids t_1 and t_2 using the concatenation product given by identifying the vertices in the top row of t_2 with the corresponding vertices in the bottom row of t_1 to obtain the product t_1t_2.

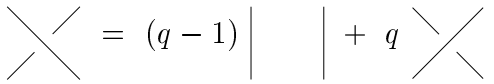
$t_1t_2 =$ 

Given a permutation $w \in S_k$ we will make a k-braid T_w by tracing the edges in order from left to right across the top row. Any time an edge that we are tracing crosses an edge that has been already traced we raise the pen briefly so that the edge being traced goes under the edge which is already there. Applying this process to all of the permutations in S_k produces a set of $k!$ braids.

$w =$  $T_w =$ 

Fix $q \in \mathbb{C}$. The *Iwahori-Hecke algebra $H_k(q)$ of type A_{k-1}* is the span of the $k!$ braids produced by tracing permutations in S_k with multiplication determined by

the braid multiplication and the following identity.



This identity can be applied in any local portion of the braid.

Theorem B5.1. *The algebra $H_k(q)$ is the associative algebra over $\mathbb{C}$ presented by generators $T_1, \ldots, T_{k-1}$ and relations*

$$\begin{aligned} &T_iT_j = T_jT_i, && \text{for } |i-j|>1,\\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && \text{for } 1 \le i \le n-2,\\ &T_i^2 = (q-1)T_i + q, && \text{for } 2 \le i \le n. \end{aligned}$$

The Iwahori-Hecke algebra of type A is a q-analogue of the group algebra of the symmetric group. If we allow ourselves to be imprecise (about the limit) we can write

$$\lim_{q\to 1} H_k(q) = \mathbb{C}S_k\,.$$

Let q be a power of a prime $G = GL(n, \mathbb{F}_q)$ where $\mathbb{F}_q$ is the finite field with q elements. Let B be the subgroup of upper triangular matrices in G and let $\mathbf{1}_B^G$ be the trivial representation of B induced to G, i.e. the G-module given by

$$\mathbf{1}_B^G = \mathbb{C}\text{-span}\{gB \mid g \in G\},$$

where G acts on the cosets by left multiplication. Using the description, see §B3, of $H_n(q)$ as a double coset algebra one gets an action of $H_n(q)$ on $\mathbf{1}_B^G$, by right multiplication. This action commutes with the G action.

Theorem B5.2. (a) *The action of $H_n(q)$ on $\mathbf{1}_B^G$ generates* $\mathrm{End}_G(\mathbf{1}_B^G)$.
(b) *The action of G on $\mathbf{1}_B^G$ generates* $\mathrm{End}_{H_n(q)}(\mathbf{1}_B^G)$.

This theorem gives a "duality" between $GL(n, \mathbb{F}_q)$ and $H_n(q)$ which is similar to a Schur-Weyl duality, but it differs in a crucial way: the representation $\mathbf{1}_B^G$ is not a tensor power representation, and thus this is not yet realizing $H_n(q)$ as a tensor power centralizer.

The following result gives a true analogue of the Schur-Weyl duality for the Iwahori-Hecke algebra of type A, it realizes $H_k(q)$ as a tensor power centralizer. Assume that $q \in \mathbb{C}$ is not 0 and is not a root of unity. Let $U_q\mathfrak{sl}_n$ be the Drinfel'd-Jimbo quantum group of type A_{n-1} and let V be the n-dimensional irreducible representation of $U_q\mathfrak{sl}_n$ with highest weight ω_1. There is an action, see [CP], of $H_k(q^2)$ on $V^{\otimes k}$ which commutes with the $U_q\mathfrak{sl}_n$ action.

Theorem B5.3. (a) *The action of $H_k(q^2)$ on $V^{\otimes k}$ generates* $\mathrm{End}_{U_q\mathfrak{sl}_n}(V^{\otimes k})$.
(b) *The action of $U_q\mathfrak{sl}_n$ on $V^{\otimes k}$ generates* $\mathrm{End}_{H_k(q^2)}(V^{\otimes k})$.

Theorem B5.4. *The Iwahori-Hecke algebra of type A_{k-1}, $H_k(q)$, is semisimple if and only if $q \neq 0$ and q is not a jth root of unity for any $2 \le j \le n$.*

Partial results for $H_k(q)$

The following results giving answers to the main questions **(Ia-c)** for the Iwahori-Hecke algebras of type A hold when q is such that $H_k(q)$ is semisimple.

I. What are the irreducible $H_k(q)$-modules?

(a) How do we index/count them?

There is a bijection

$$\textbf{Partitions } \lambda \textbf{ of } n \quad \overset{1-1}{\longleftrightarrow} \quad \textbf{Irreducible representations } H^\lambda$$

(b) What are their dimensions?

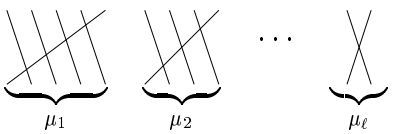
The dimension of the irreducible representation H^λ is given by

$$\begin{aligned}\textbf{dim}(H^\lambda) &= \#\textbf{ of standard tableaux of shape } \lambda \\ &= \frac{n!}{\prod_{x\in\lambda} h_x},\end{aligned}$$

where h_x is the hook length at the box x in λ.

(c) What are their characters?

For each partition $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ of k let $\chi^\lambda(\mu)$ be the character of the irreducible representation H^λ evaluated at the element T_{γ_μ} where γ_μ is the permutation

$\gamma_\mu =$  .

Then **the character $\chi^\lambda(\mu)$ is given by**

$$\chi^\lambda(\mu) = \sum_T \mathrm{wt}^\mu(T),$$

where the sum is over all standard tableaux T of shape λ and

$$\mathrm{wt}^\mu(T) = \prod_{i=1}^{n} f(i,T),$$

where

$$f(i,T) = \begin{cases} -1, & \textbf{if } i \notin B(\mu) \textbf{ and } i+1 \textbf{ is sw of } i, \\ 0, & \textbf{if } i, i+1 \notin B(\mu), \ i+1 \textbf{ is ne of } i, \textbf{ and } i+2 \textbf{ is sw of } i+1, \\ q, & \textbf{otherwise,} \end{cases}$$

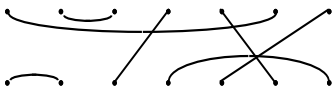
and $B(\mu) = \{\mu_1 + \mu_2 + \cdots + \mu_k | 1 \le k \le \ell\}$**.** In the formula for $f(i,T)$, **sw** means strictly south and weakly west and **ne** means strictly north and weakly east.

References

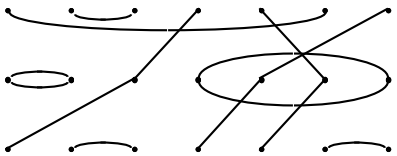
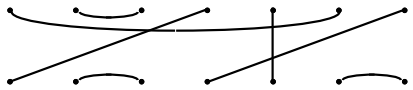
The book [CP] contains a treatment of the Schur-Weyl duality type theorem given above. See also the references there. Several basic results on the Iwahori-Hecke algebra are given in the book [GHJ]. The theorem giving the explicit values of q such that $H_k(q)$ is semisimple is due to Gyoja and Uno [GU]. The character formula given above is due to Roichman [Ro]. See [Ra3] for an elementary proof.

B6. The Brauer algebras $B_k(x)$

Fix $x \in \mathbb{C}$. A *Brauer diagram on k dots* is a graph on two rows of k-vertices, one above the other, and k edges such that each vertex is incident to precisely one edge. The product of two k-diagrams d_1 and d_2 is obtained by placing d_1 above d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. The resulting graph contains k paths and some number c of closed loops. If d is the k-diagram with the edges that are the paths in this graph but with the closed loops removed, then the product $d_1 d_2$ is given by $d_1 d_2 = \eta^c d$. For example, if

$d_1 =$  and $d_2 =$,

then

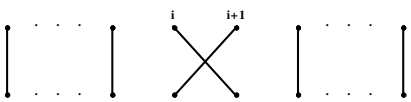
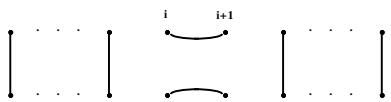
$d_1 d_2 =$  $= x^2$  .

The *Brauer algebra* $B_k(x)$ is the span of the k-diagrams with multiplication given by the linear extension of the diagram multiplication. The dimension of the Brauer algebra is

$$\dim(B_k(x)) = (2k)!! = (2k-1)(2k-3)\cdots 3 \cdot 1,$$

since the number of k-diagrams is $(2k)!!$.

The diagrams in $B_k(x)$ which have all their edges connecting top vertices to bottom vertices form a symmetric group S_k. The elements

$s_i =$  and $e_i =$  ,

$1 \le i \le k-1$, generate the Brauer algebra $B_k(x)$.

Theorem B6.1. *The Brauer algebra $B_k(x)$ has a presentation as an algebra by generators $s_1, s_2, \ldots, s_{k-1}$, $e_1, e_2, \ldots, e_{k-1}$ and relations*

$$s_i^2 = 1, \quad e_i^2 = x e_i, \quad e_i s_i = s_i e_i = e_i, \qquad 1 \le i \le k-1,$$

$$s_i s_j = s_j s_i, \quad s_i e_j = e_j s_i, \quad e_i e_j = e_j e_i, \qquad |i-j| > 1,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \qquad 1 \le i \le k-2,$$

$$s_i e_{i+1} e_i = s_{i+1} e_i, \quad e_{i+1} e_i s_{i+1} = e_{i+1} s_i, \qquad 1 \le i \le k-2.$$

There are two different Brauer algebra analogues of the Schur Weyl duality theorem, Theorem A5.1. In the first one the orthogonal group $O(n,\mathbb{C})$ plays the same role that $GL(n,\mathbb{C})$ played in the S_k-case, and in the second, the symplectic group $Sp(2n,\mathbb{C})$ takes the $GL(n,\mathbb{C})$ role.

Let $O(n,\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid AA^t = I\}$ be the orthogonal group and let V be the usual n-dimensional representation of the group $O(n,\mathbb{C})$. There is an action of the Brauer algebra $B_k(n)$ on $V^{\otimes k}$ which commutes with the action of $O(n,\mathbb{C})$ on $V^{\otimes k}$.

Theorem B6.2. (a) *The action of* $B_k(n)$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{O(n)}(V^{\otimes k})$.
(b) *The action of* $O(n,\mathbb{C})$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{B_k(n)}(V^{\otimes k})$.

Let $Sp(2n,\mathbb{C})$ be the symplectic group and let V be the usual $2n$-dimensional representation of the group $Sp(2n,\mathbb{C})$. There is an action of the Brauer algebra $B_k(-2n)$ on $V^{\otimes k}$ which commutes with the action of $Sp(2n,\mathbb{C})$ on $V^{\otimes k}$.

Theorem B6.3. (a) *The action of* $B_k(-2n)$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{Sp(2n,\mathbb{C})}(V^{\otimes k})$.
(b) *The action of* $Sp(2n,\mathbb{C})$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{B_k(-2n)}(V^{\otimes k})$.

Theorem B6.4. *The Brauer algebra* $B_k(x)$ *is semisimple if* $x \notin \{-2k+3, -2k+2, \ldots, k-2\}$.

Partial results for $B_k(x)$

The following results giving answers to the main questions **(Ia-c)** for the Brauer algebras hold when x is such that $B_k(x)$ is semisimple.

I. What are the irreducible $B_k(x)$-modules?

(a) How do we index/count them?

There is a bijection

Partitions of $k-2h$**,** $h = 0, 1, \ldots, \lfloor k/2 \rfloor$ $\quad\overset{1-1}{\longleftrightarrow}\quad$ **Irreducible representations** B^λ**.**

(b) What are their dimensions?

The dimension of the irreducible representation B^λ is given by

$$\begin{aligned}\mathbf{dim}(B^\lambda) &= \mathbf{\#\ of\ up\text{-}down\ tableaux\ of\ shape}\ \lambda\ \mathbf{and\ length}\ k\\ &= \binom{k}{2h}(2h-1)!!\frac{(k-2h)!}{\prod_{x\in\lambda} h_x},\end{aligned}$$

where h_x is the hook length at the box x in λ. An up-down tableau of shape λ and length k is a sequence $(\emptyset = \lambda^{(0)}, \lambda^{(1)}, \cdots \lambda^{(k)} = \lambda)$ of partitions, such that each partition in the sequence differs from the previous one by either addingor removing a box.

(c) What are their characters?

A Murnaghan-Nakayama type rule for the characters of the Brauer algebras was given in [Ra1].

References

(1) The Brauer algebra was defined originally by R. Brauer [Br] in 1937. H. Weyl treats it in his book [Wy].

(2) The Schur-Weyl duality type theorems are due to Brauer [Br], from his original paper. See also [Ra1] for a detailed description of these Brauer algebra actions.

(3) The theorem giving values of x for which the Brauer algebra is semisimple is due to Wenzl, see [Wz2].

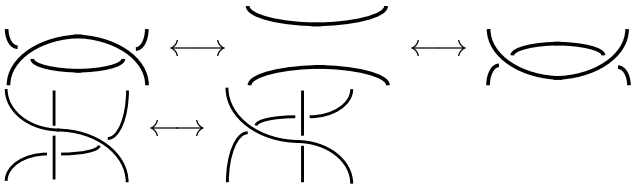
B7. The Birman-Murakami-Wenzl algebras $BMW_k(r,q)$

A *k-tangle* is viewed as two rows of k vertices, one above the other, and k strands that connect vertices in such a way that each vertex is incident to precisely one strand. Strands cross over and under each other in three-space as they pass from one vertex to the next. For example, the following are 7-tangles:

$t_1 =$ $t_2 =$.

We multiply k-tangles t_1 and t_2 using the concatenation product given by identifying the vertices in the top row of t_2 with the corresponding vertices in the bottom row of t_1 to obtain the product tangle t_1t_2. Then we allow the following "moves."

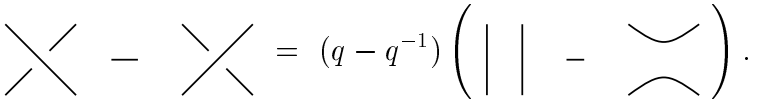
Reidemeister moves II and III:

(R2) (R3) 

Given a Brauer diagram d we will make a tangle T_d tracing the edges in order from left to right across the top row and then from left to right across the bottom row. Any time an edge that we are tracing crosses and edge that has been already traced we raise the pen briefly so that the edge being traced goes under the edge which is already there. Applying this process to all of the Brauer diagrams on k dots produces a set of $(2k)!!$ tangles.

$d =$ $T_d =$

Fix $r, q \in \mathbb{C}$. The *Birman-Murakami-Wenzl algebra* $BMW_k(r,q)$ is the span of the $(2k)!!$ tangles produced by tracing the Brauer diagrams with multiplication determined by the tangle multiplication and the Reidemeister moves and the following tangle identities.



The Reidemeister moves and the tangle identities can be applied in any appropriate local portion of the tangle.

Theorem B7.1. *Fix $r, q \in \mathbb{C}$. The Birman-Murakami-Wenzl algebra $BMW_k(r,q)$ is the algebra generated over $\mathbb{C}$ by $1, g_1, g_2, \ldots, g_{k-1}$, which are assumed to be invertible, subject to the relations*

$$\begin{aligned}
&g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},\\
&g_i g_j = g_j g_i \qquad \text{if } |i-j| \geq 2,\\
&(g_i - r^{-1})(g_i + q^{-1})(g_i - q) = 0,\\
&E_i g_{i-1}^{\pm 1} E_i \;=\; r^{\pm 1} E_i \quad \text{and} \quad E_i g_{i+1}^{\pm 1} E_i \;=\; r^{\pm 1} E_i,
\end{aligned}$$

where E_i is defined by the equation

$$(q - q^{-1})(1 - E_i) = g_i - g_i^{-1}.$$

The BMW-algebra is a q-analogue of the Brauer algebra in the same sense that the Iwahori-Hecke algebra of type A is a q-analogue of the group algebra of the symmetric group. If we allow ourselves to be imprecise (about the limit) we can write

$$\lim_{q \to 1} BMW_k(q^{n+1}, q) = B_k(n).$$

It would be interesting to sharpen the following theorem to make it an if and only if statement.

Theorem B7.2 (Wz3)**.** *The Birman-Murakami-Wenzl algebra is semisimple if q is not a root of unity and $r \neq q^{n+1}$ for any $n \in \mathbb{Z}$.*

Partial results for $BMW_k(r,q)$

The following results hold when r and q are such that $BMW_k(r,q)$ is semisimple.

I. What are the irreducible $BMW_k(r,q)$-modules?

(a) How do we index/count them?

There is a bijection

Partitions of $k-2h$, $h = 0, 1, \ldots, \lfloor k/2 \rfloor$ $\quad \overset{1-1}{\longleftrightarrow} \quad$ **Irreducible representations W^λ.**

(b) What are their dimensions?

The dimension of the irreducible representation W^λ is given by

$$\begin{aligned}\mathbf{dim}(W^\lambda) &= \mathbf{\#\ of\ up\text{-}down\ tableaux\ of\ shape}\ \lambda\ \mathbf{and\ length}\ k \\ &= \binom{k}{2h}(2h-1)!!\frac{(k-2h)!}{\prod_{x\in\lambda} h_x},\end{aligned}$$

where h_x is the hook length at the box x in λ, and up-down tableaux is as in the case of the Brauer algebra, see Section B6 **(Ib)**.

(c) What are their characters?

A Murnaghan-Nakayama rule for the irreducible characters of the BMW-algebras was given in [HR1].

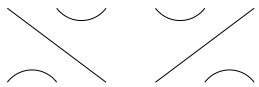
References

(1) The Birman-Murakami-Wenzl algebra was defined independently by Birman and Wenzl in [BW] and by Murakami in [Mu1]. See [CP] for references to the analogue of Schur-Weyl duality for the BMW-algebras. The articles [HR1], [LR], [Mu2], [Re], and [Wz3] contain further important information about the BMW-algebras.

(2) Although the tangle description of the BMW algebra was always in everybody's minds it was Kaufmann that really made it precise see [Ka2].

B8. The Temperley-Lieb algebras $TL_k(x)$

A TL_k*-diagram* is a Brauer diagram on k dots which can be drawn with no crossings of edges.



The *Temperley-Lieb algebra* $TL_k(x)$ is the subalgebra of the Brauer algebra $B_k(x)$ which is the span of the TL_k-diagrams.

Theorem B8.1. *The Temperley-Lieb algebra $TL_k(x)$ is the algebra over $\mathbb{C}$ given by generators $E_1, E_2, \ldots, E_{k-1}$ and relations*

$$\begin{aligned}&E_iE_j = E_jE_i, \quad \text{if } |i-j|>1,\\ &E_iE_{i\pm1}E_i = E_i, \quad \text{and}\\ &E_i^2 = xE_i\ .\end{aligned}$$

Theorem B8.2. *Let $q \in \mathbb{C}^*$ be such that $q+q^{-1}+2 = 1/x^2$ and let $H_k(q)$ be the Iwahori-Hecke algebra of type A_{k-1}. Then the map*

$$\begin{aligned}H_k(q) &\longrightarrow TL_k(x)\\ T_i &\longmapsto \frac{q+1}{x}E_i - 1\end{aligned}$$

is a surjective homomorphism and the kernel of this homomorphism is the ideal generated by the elements

$$T_iT_{i+1}T_i + T_iT_{i+1} + T_{i+1}T_i + T_i + T_{i+1} + 1, \quad \text{for } 1 \le i \le n-2.$$

The Schur Weyl duality theorem for S_n has the following analogue for the Temperley-Lieb algebras. Let $U_q\mathfrak{sl}_2$ be the Drinfeld-Jimbo quantum group corresponding to the Lie algebra $\mathfrak{sl}_2$ and let V be the 2-dimensional representation of $U_q\mathfrak{sl}_2$. There is an action, see [CP], of the Temperley-Lieb algebra $TL_k(q+q^{-1})$ on $V^{\otimes k}$ which commutes with the action of $U_q\mathfrak{sl}_2$ on $V^{\otimes k}$.

Theorem B8.3. (a) *The action of* $TL_k(q+q^{-1})$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{U_q\mathfrak{sl}_2}(V^{\otimes k})$.
(b) *The action of* $U_q\mathfrak{sl}_2$ *on* $V^{\otimes k}$ *generates* $\mathrm{End}_{TL_k(q+q^{-1})}(V^{\otimes k})$.

Theorem B8.4. *The Temperley-Lieb algebra is semisimple if and only if* $1/x^2 \neq 4\cos^2(\pi/\ell)$, *for any* $2 \leq \ell \leq k$.

Partial results for $TL_k(x)$

The following results giving answers to the main questions **(Ia-c)** for the Temperley-Lieb algebras hold when x is such that $TL_k(x)$ is semisimple.

I. What are the irreducible $TL_k(x)$-modules?

(a) How do we index/count them?

There is a bijection

Partitions of k with at most two rows $\overset{1-1}{\longleftrightarrow}$ **Irreducible representations T^λ.**

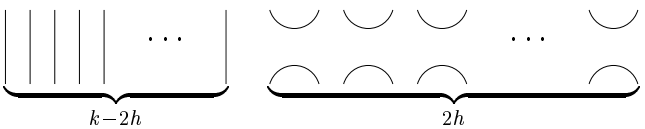
(b) What are their dimensions?

The dimension of the irreducible representation $T^{(k-\ell,\ell)}$ is given by

$$\begin{aligned}\dim(T^{(k-\ell,\ell)}) &= \#\text{ of standard tableaux of shape } (k-\ell,\ell)\\ &= \binom{k}{\ell} - \binom{k}{\ell-1}.\end{aligned}$$

(c) What are their characters?

The **character of the irreducible representation $T^{(k-\ell,\ell)}$ evaluated at the element**

d_{2h} = 

is

$$\chi^{(k-\ell,\ell)}(d_{2h}) = \begin{cases} \binom{k-2h}{\ell-h} - \binom{k-2h}{\ell-h-1}, & \text{if } \ell \geq h,\\ 0, & \text{if } \ell < h.\end{cases}$$

There is an algorithm for writing the character $\chi^{(k-\ell,\ell)}(a)$ of a general element $a \in TL_k(x)$ as a linear combination of the characters $\chi^{(k-\ell,\ell)}(d_{2h})$.

References

The book [GHJ] contains a comprehensive treatment of the basic results on the Temperley-Lieb algebra. The Schur-Weyl duality theorem is treated in the book [CP], see also the references there. The character formula given above is derived in [HR1].

B9. Complex semisimple Lie groups

We shall not define Lie groups and Lie algebras let us only recall that a complex Lie group is a differential $\mathbb{C}$-manifold and a real Lie group is a differential $\mathbb{R}$-manifold and that every Lie group has an associated Lie algebra, see [CMS].

If G is a complex Lie group then the word *representation* is usually used to refer to a *holomorphic representation*, i.e. the homomorphism

$$\rho\colon G \to GL(V)$$

determined by the module V should be a morphism of (complex) analytic manifolds. Strictly speaking there are representations which are not holomorphic but there is a good theory only for holomorphic representations, so one usually abuses language and assumes that representation means holomorphic representation. The terms holomorphic representation and *complex analytic representation* are used interchangeably. Similarly, if G is a real Lie group then representation usually means *real analytic representation*. See [Va] p. 102 for further details. Every holomorphic representation of $GL(n,\mathbb{C})$ is also rational representation, see [FH].

A *complex semisimple Lie group* is a connected complex Lie group G such that its Lie algebra $\mathfrak{g}$ is a complex semisimple Lie algebra.

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