NONHYPERBOLIC DEHN FILLINGS ON HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. In this paper we will give three infinite families of examples of nonhyperbolic Dehn fillings on hyperbolic manifolds. A manifold in the first family admits two Dehn fillings of distance two apart, one of which is toroidal and annular, and the other is reducible and ∂-reducible. A manifold in the second family has boundary consisting of two tori, and admits two reducible Dehn fillings. A manifold in the third family admits a toroidal filling and a reducible filling with distance 3 apart. These examples establish the virtual bounds for distances between certain types of nonhyperbolic Dehn fillings.

§1. INTRODUCTION

Given a slope r on a torus boundary component T_0 of a 3-manifold M, the *Dehn filling* of M along the slope r, denoted by $M(r)$, is the manifold obtained by gluing a solid torus V to M along ∂V and T_0 so that r bounds a meridian disk on V. A manifold is *simple* if it is irreducible, ∂-irreducible, atoroidal, and anannular. Thus a simple manifold is either hyperbolic, or a small Seifert fiber space, or it would be a counter example to the Geometrization Conjecture. In particular, if $M(r)$ has nonempty toroidal boundary, then it is simple if and only if it is hyperbolic [Th]. A Dehn filling $M(r)$ is of type S (resp. D, T, A) if $M(r)$ contains an essential S^2 (resp. D^2, T^2, A^2), so it is reducible (resp. ∂-reducible, toroidal, annular). The bound $\Delta(X,Y)$ is the least nonnegative number n such that if M is a hyperbolic manifold which admits two Dehn fillings $M(r_1), M(r_2)$ of type X, Y, respectively, then $\Delta(r_1, r_2) \leq n$. The bounds $\Delta(X,Y)$ have been established, via the work of many people, for all the 10 possible choices of (X,Y); see [GW2] for more details.

In some cases, the upper bound of $\Delta(X,Y)$ is reached only by a few manifolds. For example, it was shown in [GW1] that if $M(r_1)$ is annular and $M(r_2)$ is toroidal, then $\Delta(r_1, r_2) \leq 3$ unless M is one of three special manifolds, for which $\Delta(r_1, r_2)$ is

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4 or 5; moreover, there are infinitely many manifolds which admit two such Dehn fillings with $\Delta(r_1, r_2) = 3$. Thus $\Delta(A, T) = 5$, but the "virtual bound" to be defined below is 3. Similarly for $\Delta(T, T)$, see [Go]. The main results of this paper are the following.

Theorem 0.1. *There are infinitely many hyperbolic manifolds M which admit two nonhyperbolic Dehn fillings $M(r_1)$ and $M(r_2)$, such that $M(r_1)$ is toroidal and annular, $M(r_2)$ is reducible and ∂-reducible, and $\Delta(r_1, r_2) = 2$.*

Theorem 0.2. *There are infinitely many hyperbolic manifolds M with two torus boundary components, each of which admits two reducible Dehn fillings $M(r_1), M(r_2)$, with $\Delta(r_1, r_2) = 1$.*

Theorem 0.3. *There are infinitely many hyperbolic manifolds M which admit two nonhyperbolic Dehn fillings $M(r_1)$ and $M(r_2)$, such that $M(r_1)$ is reducible, $M(r_2)$ is toroidal, and $\Delta(r_1, r_2) = 3$.*

These theorems follow immediately from Theorems 2.6, 3.6 and 4.2 below. The first example satisfying the conditions in Theorem 0.1 was found by Hayashi and Motegi [HM], and the first example as in Theorem 0.2 was given by Gordon and Litherland [GLi].

Similar to $\Delta(X, Y)$, we define the *virtual bound* $\Delta_v(X, Y)$ of distances between type X and type Y Dehn fillings to be the maximal integer n such that there are infinitely many hyperbolic manifolds M which admit two Dehn fillings $M(r_1), M(r_2)$ of type X, Y respectively, with $\Delta(r_1, r_2) = n$. If no such infinite family exist, define $\Delta_v(X, Y) = 0$. Thus $\Delta_v(X, Y) \leq \Delta(X, Y)$. The above theorems and some known results determine the virtual bounds of distances between certain types of nonhyperbolic Dehn fillings. The following is a table of $\Delta_v(X, Y)$.

X \ Y	T	A	S	D
D	2	2	0	1
S	3	2	1	
A	3	3--5		
T	5			

Table 1.1. Virtual bound $\Delta_v(X, Y)$

As we can see, except for $\Delta_v(A, A)$, all the other $\Delta(X, Y)$ have been completely determined. In the table, $\Delta_v(T,T)$ is determined by Gordon [Go], $\Delta_v(T,A)$ by Gordon and Wu [GW1]. The upper bounds of the other entries in Table 1.1 are the same as that in [GW2], and the lower bounds of them are determined by Theorem 0.1 for $\Delta_v(D,T)$, $\Delta_v(D,A)$, and $\Delta_v(S,A)$; by Theorem 0.3 for $\Delta_v(S,T)$; by Gabai [Ga] and Berge [Be] for $\Delta_v(D,D)$; by Gordon and Wu [GW1] for $\Delta_v(A,A)$; and by Gordon and Litherland [GLi] for $\Delta_v(S,S)$. Theorem 0.2 gives a stronger result about type S-S fillings, manely the manifolds can be chosen to have an extra torus boundary components. Also, it provides infinitely many examples of two essential planar surfaces in 3-manifolds with distinct boundary slopes, and one of the surfaces has unbounded number of boundary components.

We would like to thank Cameron Gordon and John Luecke for some interesting discussion on this topic.

§2. Toroidal/annular fillings and reducible/∂-reducible fillings

In this section we prove Theorem 2.6, which shows that there are infinitely many hyperbolic manifolds which admit two Dehn fillings of distance two apart, one of which is toroidal and annular, and the other is reducible and ∂-reducible. Let $Y = S^2 \times I$. Consider the tangles T_p in Y as shown in Figure 2.1, where a rectangle labeled by an integer n denotes a rational tangle of slope $1/n$; in other words, it contains two vertical strings with n left hand half twists.

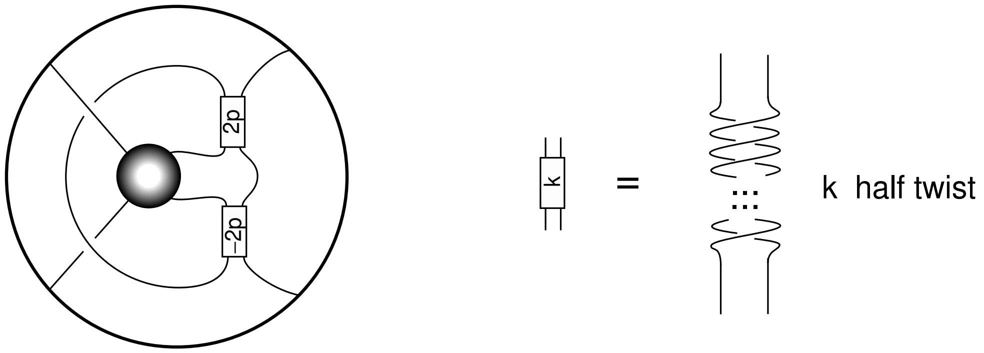

Figure 2.1

Let $T_p(r)$ be the tangle obtained by filling the inside sphere S_0 of Y with a rational tangle of slope r. The tangles $T_p(r)$ are drawn in Figure 2.2(a)–(d) for $r = \infty, 0, -1, -1/2$, respectively. From the pictures we have the following lemma. We use $T(r,s)$ to denote a Montesinos tangle consisting of two rational tangles associated to the rational numbers r and s respectively. See [Wu2] or [Mo1, Co] for more details about Montesinos tangles and algebraic tangles.

Lemma 2.1. *(1) $T_p(\infty)$ is the connected sum of a trivial tangle and a Hopf link.*

(2) $T_p(0)$ is the Montesinos tangle $T[\frac{1}{2p-1}, \frac{-1}{2p+1}]$.

(3) $T_p(-1)$ is the Montesinos tangle $T[\frac{1}{2p+1}, \frac{-1}{2p-1}]$.

(4) $T_p(\frac{-1}{2})$ is an algebraic tangle obtained by summing a Montesinos tangle $T[\frac{1}{2p}, \frac{-1}{2p}]$ with a rational tangle $T[\frac{1}{2}]$. It is not a Montesinos tangle. □

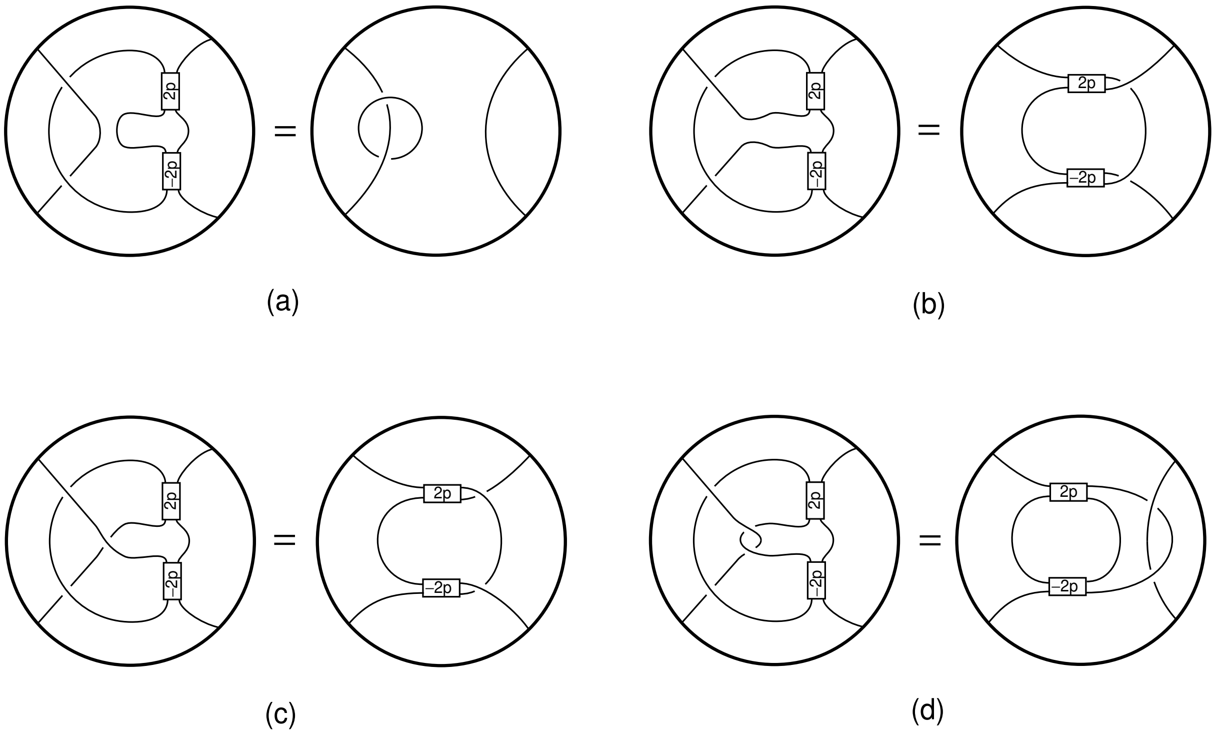

Figure 2.2

Let M_p be the double branched covering of Y with branch set the tangle T_p. Then M_p is a compact orientable 3-manifold with boundary consisting of two tori T_0 and T_1, where T_0 is the lift of the inside sphere S_0. The ∞ and 0 slopes on S_0 lifts to a meridian-longitude pair on T_0, with respect to which the Dehn filling manifold $M_p(r)$ is the double covering of the 3-ball branched along the tangle $T_p(r)$. See [Mo2] for more details. Denote by $Q(r,s)$ the double branched cover of a Montesinos tangle $T[\frac{1}{r}, \frac{1}{s}]$. Note that when $|r|, |s| > 1$, $Q(r,s)$ is a Seifert fiber space with orbifold $D(r,s)$, which by definition is a disk with two cone points of angle $2\pi/|r|$ and $2\pi/|s|$. Denote by $C(r,s)$ the cable space of type (r,s), that is, the exterior of a knot K in a solid torus V which is parallel to a curve on ∂V

representing $rl + sm$ in $H_1(\partial V)$, where (m, l) is a meridian-longitude pair of ∂V. The above facts and Lemma 2.1 lead to the following lemma.

Lemma 2.2. *Suppose $p \geq 2$. The manifolds M_p have the following properties.*

(1) $M_p(\infty)$ is the connected sum of a solid torus and the projective space RP^3;

(2) $M_p(0) = Q(2p-1, -2p-1)$;

(3) $M_p(-1) = Q(2p+1, -2p+1)$;

(4) $M_p(-1/2)$ is a non Seifert fibered graph manifold containing a unique essential torus T, cutting it into a cable space $C(2,1)$ and a Seifert fiber space $Q(2p, -2p)$.

Proof. (1) follows from the fact that the double branched cover of the Hopf link is RP^3, and connected sum of links and tangles downstairs corresponds to connected sum of manifolds upstairs. (2) and (3) follow from the definition of $Q(r, s)$.

To prove (4), notice that the Conway sphere in $T_p(-1/2)$ cutting off the tangle $T(2p, -2p)$ lifts to an essential torus T upstairs. Since $T_p(-1/2)$ is not a Montesinos tangle, $M_p(-1/2)$ is not a Seifert fiber space, so T is the Jaco-Shalen-Johannson decomposing torus because each side of it is a Seifert fiber space. Since each of $C(2,1)$ and $Q(2p, -2p)$ are atoroidal, T is the unique essential torus in $M_p(-1/2)$. $\square$

It is easy to see that if $|p| \leq 1$ then M_p is non hyperbolic. In the following, we will assume $M = M_p$ and $p \geq 2$, and show that M is hyperbolic. Since M has toroidal boundary, by [Th] we need only show that M is irreducible, ∂-irreducible, non Seifert fibered, and atoroidal.

Lemma 2.3. *If $p \geq 2$, then M is irreducible, ∂-irreducible, and non Seifert fibered.*

Proof. If M is reducible, let S be a reducing sphere. S is separating, otherwise it would be a reducing sphere in all $M(r)$, contradicting Lemma 2.2(2). Let W, W' be the two components of M cut along S, with W the one containing T_0. Let $\widehat{W}'$ be W' with S capped off by a 3-ball. Since $M(0)$ is the Seifert fiber space $Q(2p-1, -2p-1)$, which is irreducible, $W(0)$ must be a 3-ball, so $\widehat{W}' = M(0) = Q(2p-1, -2p-1)$. But then we have

$$M(\infty) = \widehat{W}' \# \widehat{W}(\infty) = Q(2p-1, -2p-1) \# \widehat{W}(\infty) \neq (S^1 \times D^2) \# RP^3,$$

which is a contradiction. Therefore M is irreducible.

If M is ∂-reducible, then after ∂-compression one of the T_i becomes a sphere separating the two component of ∂M, hence is a reducing sphere, contradicting the above conclusion.

If M is Seifert fibered, then $M(r)$ is Seifert fibered for all but at most one r, for which $M(r)$ is reducible. Since $M(-1/2)$ is irreducible and is not a Seifert fiber space, this is not possible. $\square$

Lemma 2.4. *Suppose T is an essential separating torus in an irreducible 3-manifold M, and suppose it is compressible in $M(r_1), M(r_2)$ with $\Delta(r_1, r_2) \geq 2$, where r_i are slopes on $T_0 \subset \partial M$. Then T and T_0 cobound a cable space in M, with cabling slope r_0 satisfying $\Delta(r_0, r_i) = 1$, $i = 1, 2$.*

Proof. Cut M along T and let X be the component containing T_0. Then T is compressible in $X(r_i)$ and $\Delta(r_1, r_2) \geq 2$, so by [Wu, Theorem 1] there is an essential annulus A in X with one boundary on T and the other on T_0, with slope r_0, say. Since T is essential in M, it is not parallel to T_0, so by [CGLS, Theorem 2.4.3] T is compressible in $X(r)$ only if $\Delta(r_0, r) \leq 1$. We must have $\Delta(r_0, r_i) = 1$, because if $r_0 = r_1$ then we would have $\Delta(r_0, r_2) = \Delta(r_1, r_2) = 2$, a contradiction. Now the manifold $X(r_i)$ is homeomorphic to the manifold Y obtained by cutting X along A, so the torus component of ∂Y corresponding to T under the homeomorphism is compressible in Y. Since M is irreducible, this implies that Y is a solid torus. It follows that X is a cable space with cabling slope r_0. $\square$

Lemma 2.5. *M is atoroidal.*

Proof. Assuming the contrary, let T be an essential torus in M. Then T must be separating, otherwise $M(r)$ would contain a nonseparating torus or, if T becomes compressible in $M(r)$, a nonseparating sphere, for all r, which contradicts Lemma 2.2(1).

Let W, W' be the two components of M cut along T, with W the one containing T_0. Since M contains no nonseparating essential torus, by the Haken finiteness theorem (cf. [Ja, Page 49]), we may choose T to be outermost in the sense that W' contains no essential torus.

CLAIM. *T is compressible in $M(-1/2)$.*

Recall from Lemma 2.2(4) that $M(-1/2)$ has a unique essential torus T'. So if T is incompressible in $M(-1/2)$ then either it is boundary parallel or it is isotopic to T'. The first case is impossible, otherwise T' would be an essential torus in W', contradicting the choice of T. Therefore T must be isotopic to T' in $M(-1/2)$. It follows that either $W' = C(2,1)$, or $W' = Q(2p, -2p)$.

Since $M(0)$ is atoroidal, either T is boundary parallel in $M(0)$ or it is compressible in $M(0)$. In the first case we would have $Q(2p-1, -2p-1) = M(0) = W' =$

$C(2,1)$ or $Q(2p,-2p)$, which is absurd. In the second case let D be a compressing disk of T in $W(0)$, and let $\widehat{W}'$ be the manifold obtained by capping off the sphere boundary component of $W' \cup N(D)$ with a 3-ball. Then $\widehat{W}'$ is a summand of $M(0) = Q(2p-1,-2p-1)$, so either $\widehat{W}' = Q(2p-1,-2p-1)$ or $\widehat{W}' = S^3$. However, this is impossible whether $W' = C(2,1)$ or $W' = Q(2p,-2p)$ because $\widehat{W}'$ is obtained from W' by Dehn filling on T along certain slope, and it is easily seen that when $p \geq 2$ none of the Dehn fillings on such W' could produce $Q(2p-1,-2p-1)$ or S^3. This completes the proof of the claim.

Since $M(\infty)$ contains no incompressible torus, T is compressible in $M(\infty)$. By the claim above, T is also compressible in $M(-1/2)$. Since $\Delta(\infty,-1/2) = 2$, it follows from Lemma 2.4 that W is a cable space $C(p,q)$ with cabling slope r_0 satisfying $\Delta(r_0,\infty) = \Delta(r_0,-1/2) = 1$. Solving these equalities, we have $r_0 = 0$ or -1. Now we have $W(r_0) = L(p,q)\#(S^1 \times D^2)$, so $M(r_0)$ should have a lens space summand. On the other hand, we have shown that $r_0 = 0$ or -1, and in either case by Lemma 2.2 $M(r_0)$ is a prime manifold with torus boundary. This contradiction completes the proof that M is atoroidal. □

Theorem 2.6. *The manifolds M_p, $p \geq 2$, are mutually distinct hyperbolic manifolds, each admitting two nonhyperbolic Dehn fillings $M(r_1)$ and $M(r_2)$, such that $M(r_1)$ is toroidal and annular, $M(r_2)$ is reducible and ∂-reducible, and $\Delta(r_1,r_2) = 2$.*

Proof. Consider the manifolds M_p which is the double cover of $Y = S^2 \times I$ branched along the tangle T_p in Figure 2.1. By Lemmas 2.3 and 2.5, M_p are hyperbolic for all $p \geq 2$. By Lemma 2.2, $M_p(\infty)$ is reducible and ∂-reducible, and $M_p(-1/2)$ is the union of $C(2,1)$ and $Q(2p,-2p)$ along a torus, hence is toroidal and annular because there is an essential annulus in $C(2,1)$ with both boundary on the outside torus T_1. Since $\Delta(\infty,-1/2) = 2$, M_p satisfy all the conditions of the theorem. It remains to show that M_p and M_q are non homeomorphic when $p,q \geq 2$ and $p \neq q$.

Let T_0 (resp. T_0') be the torus of ∂M_p (resp. ∂M_q) on which the Dehn fillings are performed. Let (m,l) (resp. (m',l')) be the meridian-longitude pair on T (resp. T') chosen as in Lemma 2.2. Let $f : M_p \to M_q$ be a homeomorphism.

There is a homeomorphism of Y interchanging the two sphere boundary components, and leaving T_p invariant, which induces a self homeomorphism of M_p interchanging the two boundary components. This can be seen by redrawing the tangle in Figure 2.1 as in Figure 2.3(a), where the sphere S_0 represents the inside sphere in Figure 2.1, and S_1 the outside sphere. After an isotopy the picture becomes that in Figure 2.3(b). (Note that the isotopy have changed the position of the endpoints of the tangle on the spheres, but that does not matter.) Now blow up the sphere S_0, we get the same picture as that in Figure 2.1, with S_0 and S_1 interchanged. Thus without loss of generality we may assume that f maps T_0 to T_0'.

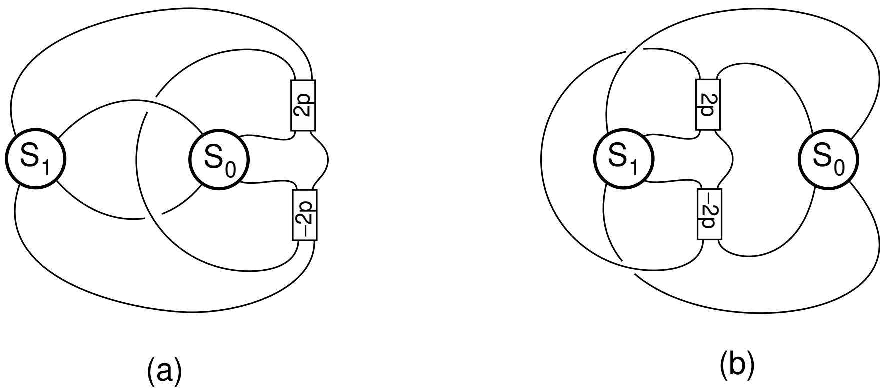

Figure 2.3

Since $M_p(\infty)$ is ∂-reducible, by [Sch] $M_p(r)$ is irreducible for all $r \neq \infty$. Hence the reducing slope ∞ is unique, so f must send m to m'. Assume $f(l) = l' + km'$. Because of uniqueness of Seifert fibration, neither of $M_p(0)$ or $M_p(-1)$ is homeomorphic to $M_q(0)$ or $M_q(-1)$ when $p, q \geq 2$ and $p \neq q$. Hence $k \neq 0, \pm 1$. Now f sends the slope $-1/2$ to $(2k-1)/2$, so both $M_q(-1/2)$ and $M_q((2k-1)/2)$ are toroidal. We have $\Delta(-1/2, (2k-1)/2) = |4k| \geq 8$. On the other hand, by [Go], this happens only if M_q is the figure 8 knot complement or the Whitehead sister link complement. Since M_q have two boundary components, this is impossible. $\square$

§3. Manifolds admitting two reducible Dehn fillings

In this section we will show that there are infinitely many hyperbolic manifolds with two torus boundary components, each admitting two reducible Dehn fillings. Consider the tangles T_p in $Y = S^2 \times I$ as shown in Figure 3.1, where, as in Figure 2.1, a rectangle labeled by an integer n denotes a rational tangle of slope $1/n$.

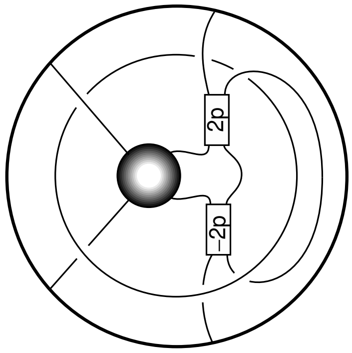

Figure 3.1

As in Section 2, we denote by M_p the double branched cover of Y branched along T_p, and by $T_p(r)$ the tangle obtained by filling the inside sphere S_0 with a rational tangle of slope r. Then the Dehn filling manifold $M_p(r)$ is the double cover of Y branched along $T_p(r)$. The tangles $T_p(\infty)$ and $T_p(0)$ are drawn in Figure 3.2(a)–(b). We can see that $T_p(\infty)$ is the connected sum of $T(1/2, -1/2)$ and a Hopf link, while $T_p(0)$ is the connected sum of a Montesinos tangle $T(1/2p, -1/2p)$ and a Hopf link. Recall that $Q(r, s)$ denotes the Seifert fiber space which double branch covers the tangle $T(1/r, 1/s)$, and the double branched cover of a Hopf link is the projective space RP^3. Therefore we have the following lemma.

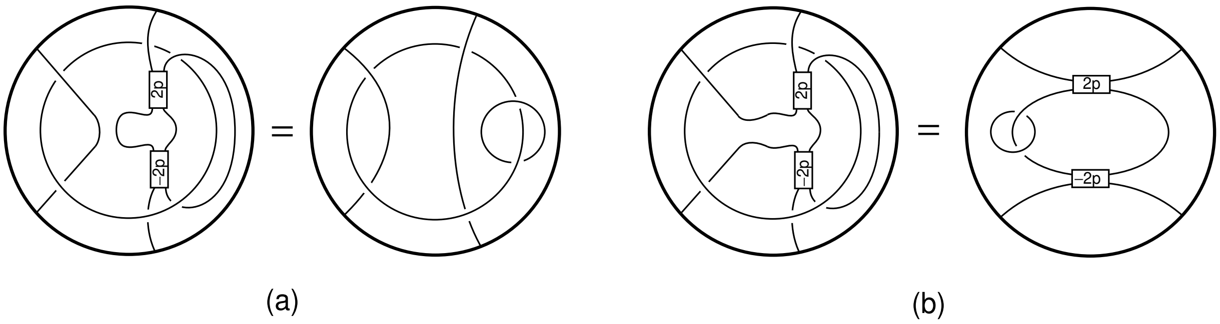

Figure 3.2

Lemma 3.1. *The manifolds M_p, $p \neq 0$, have the following properties.*

(1) $M_p(\infty) = Q(2, -2)\#RP^3$;

(2) $M_p(0) = Q(2p, -2p)\#RP^3$. □

Thus each M_p admits two reducible Dehn fillings. In below, we will assume $M = M_p$, and $p \geq 2$. We need to show that M is hyperbolic. Let T_0 be the component of ∂M on which the Dehn fillings are performed. Thus T_0 covers the inside sphere S_0 in Figure 3.1. Let T_1 be the component of ∂M covering the outside sphere S_1.

Lemma 3.2. *M is irreducible.*

Proof. Assuming the contrary, let S be a reducing sphere of M. Clearly S is separating, otherwise $M(0)$ would contain a nonseparating reducing sphere, contradicting Lemma 3.1. Let W, W' be the components of M cut along S, with W the one containing T_0. Denote by $\widehat{W}$ the manifold W with the sphere boundary capped off by a 3-ball. Similarly for $\widehat{W}'$. Then $\widehat{W}'$ is a summand of both $M(0)$ and $M(\infty)$, so by Lemma 3.1 we must have $\widehat{W}' = RP^3$. This also shows that the reducing sphere in M is unique up to isotopy, because if S and S' bound different punctured RP^3, then tubing them together would give a sphere which does not bound a punctured RP^3.

Let ρ be the involution of M which induces the branched covering. Since the reducing sphere S is unique up to isotopy, by the equivariant sphere theorem [MSY], it can be chosen to be invariant under the involution ρ, hence it double branch covers a sphere S' in the manifold Y downstairs, which must cut off a 3-ball B because one side of S is W', which does not contain the preimage of S_0 or S_1. Extending the involution $\rho|_S$ trivially over a 3-ball D, we get a double branched cover $\widehat{W}' \to S^3 = B \cup D'$, with branch set L the union of $T' = T_p \cap B$ and a trivial arc in the attached 3-ball D', which is the image of D under the branched covering map. Since $\widehat{W}' = RP^3 = L(2,1)$, the link L is the 2-bridge link associated to the number $1/2$, which is the Hopf link. Therefore, $T' = T_p \cap B$ is a tangle in B consisting of an unknotted arc and a trivial circle C around it.

We want to shown that no such pair (B, T') exists in (Y, T_p). Assuming the contrary, then (B, T') would remain the same after filling the sphere boundaries S_0, S_1 of Y with any rational tangles. The tangle T_p has two circle components C_1, C_2, where C_1 denotes the one on the left in Figure 3.1. The circle component C of T' must be one of the C_i. However, after filling both S_i with 0-tangle, C_2 has linking number $p \geq 2$ with one of the components of the resulting link, while after filling S_0 with 1-tangle and S_1 with ∞-tangle the circle C_1 has linking number 2 with one of the components of the resulting link, either case contradicting the fact that C bounds a disk in B intersecting the resulting link only once. $\square$

Lemma 3.3. *M is ∂-irreducible, and is not a Seifert fiber space.*

Proof. Since ∂M consists of two tori, M being ∂-reducible would imply that it is reducible, which would contradict Lemma 3.2. If M is Seifert fibered (with two torus boundary components), then $M(r)$ would be reducible for at most one r, which would contradict Lemma 3.1. □

Lemma 3.4. *Let W be an irreducible and ∂-irreducible 3-manifold. If both $W(r_1)$ and $W(r_2)$ are reducible and ∂-reducible, then $r_1 = r_2$.*

Proof. Let T_0 be the Dehn filling component of ∂W. Assume $r_1 \neq r_2$. Since $W(r_1)$ is ∂-reducible and $W(r_2)$ is reducible, by Scharlemann's theorem [Sch, Theorem 6.1], r_2 is a cabling slope, so there is an essential annulus A_2 in W with boundary two copies of r_2 of opposite orientations. Similarly, we have an essential annulus A_1 in W with boundary consisting of two copies of r_1 of opposite orientations. Isotope A_1 to intersect A_2 essentially. Then $A_1 \cap A_2$ consists of essential arcs on A_i, running from one boundary component to the other. By the parity rule on [CGLS, Page 279], if an arc component of $A_1 \cap A_2$ connects two components of ∂A_1 which have opposite orientations on T_0, then it must connect two components of ∂A_2 with the same orientation on T_0. This is a contradiction because the two boundary components of each A_i have opposite orientations on T_0. □

Lemma 3.5. *M is atoroidal.*

Proof. Consider an essential torus T in M. Clearly T is separating, otherwise $M(0)$ would contain a nonseparating torus or sphere, which would contradict Lemma 3.1. Let W, W' be the two components of M cut along T, where W contains T_0. Note that T cannot be boundary parallel in $M(0)$ or $M(\infty)$, otherwise W', and hence M, would be reducible, which would contradict Lemma 3.2. Hence T is compressible in both $W(0)$ and $W(\infty)$ because by Lemma 3.1 they are atoroidal. After compression, T becomes a sphere in $W(0)$ and $W(\infty)$, so if W contained T_1, then both $W(0)$ and $W(\infty)$ would also be reducible, which is impossible by Lemma 3.3. Hence we conclude that any essential torus in M must separate the two boundary components of M.

Let $\rho : M \to M$ be the involution which induces the branch covering, and let X be the fixed point set of ρ. Then X covers the tangle T_p in the manifold Y downstairs. Since T_p contains four arcs running from S_0 to S_1, X has four arcs running from T_0 to T_1, hence each essential torus T intersect X at least four times.

By the equivariant torus theorem [MS, Theorem 8.6], there is a set of essential torus $\mathcal{T}$ in M such that $\rho(\mathcal{T}) = \mathcal{T}$. Let T be a component of $\mathcal{T}$. Since X intersects

T in at least four points, we must have $\rho(T) = T$. Calculating the Euler number of T/ρ, we see that X cannot intersect T in more than four points. Hence T intersects X exactly four times, and $S = T/\rho$ is a sphere in Y which intersects each of the four arc components of T_p exactly once, and is disjoint from the circle components of T_p. Since the two circle components of T_p have linking number 1, they must lie on the same side of S.

Let Y_1, Y_2 be the two components of Y cut along S, with Y_1 the one disjoint from the circle components of T_p. Let W_1, W_2 be the components of M cut along T, with W_i covering Y_i. Consider the tangle T'_p consisting of the arc components of T_p. Let M' be the double cover of Y branched along T'_p, let T' be the torus in M' that covers S, and let W'_i be the part of M' that covers Y_i. It can be seen from Figure 3.1 that T'_p is isotopic to four straight arcs running from S_0 to S_1; hence $M' = T^2 \times I$. Since T' is a torus separating the two components of $\partial M'$, it is isotopic to a horizontal torus $T^2 \times x$, so each W'_i is also homeomorphic to $T^2 \times I$. Now we have $T_p \cap Y_1 = T'_p \cap Y_1$, therefore W_1, as the double cover of Y_1 branched along $T_p \cap Y_1$, is the same as W'_1, hence is a product $T^2 \times I$. But then T is boundary parallel, contradicting the assumption that T is an essential torus in M. $\square$

Theorem 3.6. *The manifolds M_p, $p \geq 2$, are distinct hyperbolic manifolds, each admiting two reducible Dehn fillings $M(r_1), M(r_2)$ with $\Delta(r_1, r_2) = 1$.*

Proof. We have shown in Lemmas 3.1–3.5 that M_p are hyperbolic manifolds admitting two reducible Dehn fillings $M_p(0)$ and $M_p(\infty)$, so it remains to show that the manifolds are all different.

Suppose $f : M_p \to M_q$ is a homeomorphism, $p > q \geq 2$. As in the proof of Theorem 2.6, it is easy to see that there is a self homeomorphism of M_p interchanging the two boundary components, hence we may assume that f maps T_0 to T'_0, where T'_0 and T'_1 are the boundary tori of M_q, with T'_0 the one covering the inside sphere.

By [GLu1], M_i admits at most three reducible Dehn fillings, with mutual distance 1. Since $M_p(0) = Q(2p, -2p) \# RP^3$ is homeomorphic to neither $M_q(0)$ nor $M_q(\infty)$, f maps the slope 0 to another reducing slope of M_q, which must be ± 1 because it has distance 1 from 0 and ∞. Thus the only reducible Dehn filling of M_q homeomorphic to $M_p(\infty)$ is $M_q(\infty)$, so f sends the ∞ slope on T_0 to ∞ on T'_0. Similarly, it sends the ∞ slope on T_1 to ∞ on T'_1. Denote by $M_p(r, s)$ the manifold obtained by r filling on T_0 and s filling on T_1. Then we have $M_p(0, \infty) = M_q(\pm 1, \infty)$.

The manifold $M_k(r, s)$ is a double cover of $T_k(r, s)$, which is obtained from T_k by filling the inside sphere with a rational tangle of slope r and the outside sphere with one of slope s. One can check that $T_p(0,\infty)$ is the split union of a Hopf link and a trivial knot, while $T_q(\pm 1,\infty)$ is the connected sum of a Hopf link and a 2-bridge link associated to the rational number $\pm\frac{1}{4}$. Thus $M_p(0,\infty) = S^1 \times S^2 \# RP^3$, and $M_q(\pm 1,\infty) = L(4,\pm 1)\# RP^3$. Since these two manifolds are not homeomorphic, this is a contradiction. □

§4. Reducible and toroidal fillings

In this section we show that there are infinitely many hyperbolic manifolds which admit a reducible filling and a toroidal filling of distance 3 apart. Consider the tangles T_p $(p \geq 2)$ in Y, as shown in Figure 4.1(a), where Y is the 3-ball obtained by deleting the interior of the 3-ball B in the figure from S^3. As before, let $T(r)$ be the union of (Y, T_p) with a rational tangle of slope r, and let $M_p(r)$ be the double branched cover of S^3 branched along $T_p(r)$.

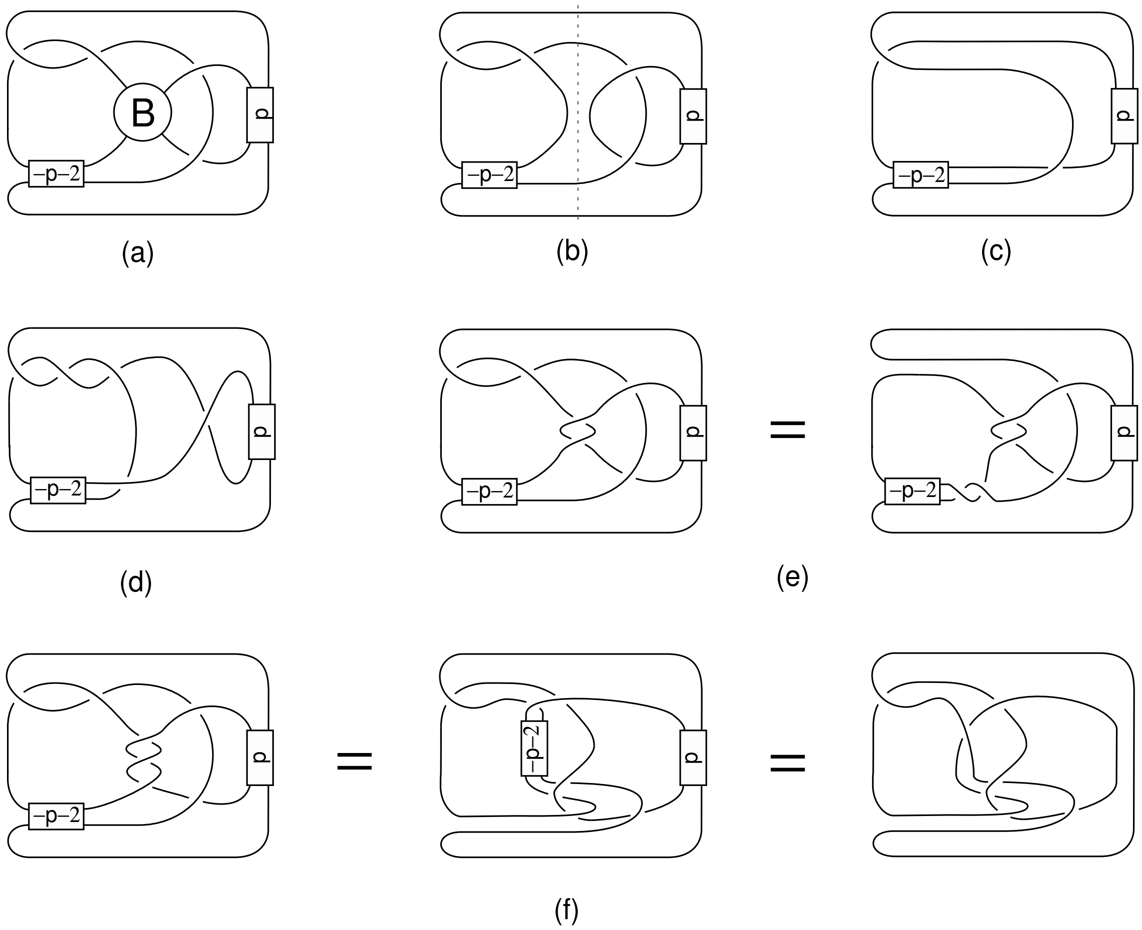

Figure 4.1

Lemma 4.1. *The manifold M_p admits the following Dehn fillings.*

(1) $M_p(\infty)$ is a non Seifert fibered, irreducible, toroidal manifold;

(2) $M_p(0)$ is a lens space $L((p-1)(p+3)+1, p+3)$;

(3) $M_p(1)$ and $M_p(1/2)$ are small Seifert fibered manifolds, but not lens spaces;

(4) $M_p(1/3) = L(3,1)\#L(2,1)$.

Proof. The tangles $T(\infty), T(0), T(1), T(1/2), T(1/3)$ are shown in Figure 4.1(b)–(f), respectively. We can see that $T(\infty)$ is the union of $T[\frac{1}{2}, \frac{1}{-(p+2)}]$ and $T[\frac{1}{2}, \frac{1}{p}]$, and is not a Montesinos link; $T(0)$ is a 2-bridge link associated to the rational number $1/((p-1)+1/(p+3)) = (p+3)/((p+3)(p-1)+1)$; $T(1)$ and $T(1/2)$ are Montesinos links consisting of three rational tangles; and $T(1/3)$ is the connected sum of a trefoil knot and a Hopf link. The result now follows by taking the double cover of S^3 branched along the corresponding links. □

Theorem 4.2. *The manifolds $M = M_p$, $p \geq 2$, are mutually distinct hyperbolic manifolds, each admitting two Dehn fillings $M(r_1)$ and $M(r_2)$, such that $M(r_1)$ is reducible, $M(r_2)$ is toroidal, and $\Delta(r_1, r_2) = 3$.*

Proof. Let $r_1 = 1/3$, and $r_2 = \infty$. Then $\Delta(r_1, r_2) = 3$, and by Lemma 4.1, $M(r_1)$ is reducible, $M(r_2)$ is toroidal. We need to show that M_p are hyperbolic and mutually distinct.

M is irreducible, otherwise a closed summand would survive after all Dehn fillings; but since $M(0)$ and $M(1)$ are non homeomorphic prime manifolds, this is impossible. M is not a Seifert fiber space because two Dehn fillings $M(\infty)$ and $M(1/3)$ are non Seifert fibered. These imply that M is ∂-irreducible. To prove M is hyperbolic, it remains to show that M is atoroidal.

If T is an essential annulus in M, then it is compressible in $M(0)$, $M(1)$, $M(1/2)$ and $M(1/3)$. Since $M(0)$ is irreducible, T must be separating. Let W, W' be the components of M cut along T, with W the one containing T_0. Since $\Delta(1, 1/3) = 2$, by Lemma 2.4, W is a cable space $C(r,s)$, with cabling slope r_0 satisfying $\Delta(r_0, 1) = \Delta(r_0, 1/3) = 1$. Solving these equalities, we have $r_0 = 0$ or $1/2$; but since $M(r_0)$ contains a lens space $L(r,s)$, we must have $r_0 = 0$. Let δ_0 and δ_1 be the slopes on T which bound disks in $W(0)$ and $W(1/3)$, respectively. Since 0 is the cabling slope, we have $\Delta(\delta_0, \delta_1) = |r| > 1$. Now $W(0)$ is the connected sum of a solid torus and $L(r,s)$, while $W(1/3)$ is a solid torus, so we have

$$M(0) = L(r,s)\#W'(\delta_0),$$
$$M(1/3) = W'(\delta_1).$$

Comparing the first equation with Lemma 4.1(2), we see that W' is the exterior of a knot in S^3 with δ_0 the meridional slope. But then since $\Delta(\delta_0, \delta_1) > 1$, by [GLu2] the manifold $M(1/3)$ would be irreducible, which would contradict Lemma 4.1(4). This completes the proof that M is atoroidal, and hence hyperbolic.

It remains to show that the manifolds M_p are mutually distinct. Assume there is a homeomorphism $f : M_p \cong M_q$, $p > q \geq 2$. Let (m, l) and (m', l') be the meridian-longitude pair of M_p and M_q, respectively. By [CGLS], [GLu1] and [BZ, Theorem 0.1], a hyperbolic manifold admits a total of at most three reducible or cyclic Dehn fillings, with mutual distance 1. Thus two of the four slopes $0, 1/3, f(0), f(1/3)$ on ∂M_q must be the same. But since $M_p(0)$ is not homeomorphic to $M_q(0)$ or $M_q(1/3)$, we must have $f(1/3) = 1/3$, and $f(0)$ is of distance 1 from 0 and 1/3, so $f(0) = 1/2$ or $1/4$. The first is impossible because $M_q(1/2)$ is not a lens space. Hence $f(0) = 1/4$. Now $f(m) = f((m+3l) - 3l) = (m'+3l') \pm 3(m'+4l')$, and we have $\Delta(m', f(m)) \geq 9$. Since both m' and $f(m)$ are toroidal Dehn fillings slopes on ∂M_q, this contradicts [Go]. $\square$

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