BROWNIAN MOTION IN A WEYL CHAMBER, NON-COLLIDING PARTICLES, AND RANDOM MATRICES

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Abstract. Let n particles move in standard Brownian motion in one dimension, with the process terminating if two particles collide. This is a specific case of Brownian motion constrained to stay inside a Weyl chamber; the Weyl group for this chamber is A_{n-1}, the symmetric group. For any starting positions, we compute a determinant formula for the density function for the particles to be at specified positions at time t without having collided by time t. We show that the probability that there will be no collision up to time t is asymptotic to a constant multiple of $t^{-n(n-1)/4}$ as t goes to infinity, and compute the constant as a polynomial of the starting positions. We have analogous results for the other classical Weyl groups; for example, the hyperoctahedral group B_n gives a model of n independent particles with a wall at $x = 0$.

We can define Brownian motion on a Lie algebra, viewing it as a vector space; the eigenvalues of a point in the Lie algebra correspond to a point in the Weyl chamber, giving a Brownian motion conditioned never to exit the chamber. If there are m roots in n dimensions, this shows that the radial part of the conditioned process is the same as the $n + 2m$-dimensional Bessel process. The conditioned process also gives physical models, generalizing Dyson's model for A_{n-1} corresponding to $\mathfrak{su}_n$ of n particles moving in a diffusion with a repelling force between two particles proportional to the inverse of the distance between them.

1. Introduction

Let n particles move in standard Brownian motion in one dimension, with the process terminating if two particles collide. Given the starting positions, we can use a reflection argument to calculate the density function for the particles to be at specified positions at time t without having collided by time t. Using this density function and the theory of Lie algebras, we can prove the following results; Theorems 3 and 4 were first proved by Dyson [10].

Theorem 1. *For any starting positions, the probability that there will be no collision up to time t is asymptotic to a constant multiple of $t^{-n(n-1)/4}$ as t goes to infinity; the constant is a known polynomial in the starting positions.*

Supported by an NSF Graduate Fellowship and NSF Postdoctoral Fellowship, and a postdoctoral fellowship from the Center for Discrete Mathematics and Theoretical Computer Science at the Hebrew University of Jerusalem. Some of this work was done while visiting the Mathematical Sciences Research Institute; MSRI is supported by NSF Grant DMS-9022140.

Theorem 2. *Given that there is no collision up to time t, the distribution of the radius of the vector whose coordinates are the positions of the particles, divided by the square root of t, converges in measure to the distribution of the Bessel process with parameter $n(n+1)/2$ at time 1, which is the radial part of an $n(n+1)/2$-dimensional standard normal.*

Theorem 3. *We can construct an n-dimensional Brownian motion which is conditioned for no two particles ever to collide. If we take Brownian motion on the space of Hermitian matrices, the induced process on the eigenvalues is the same process. If the starting point is appropriately chosen at the starting radius, the process given by the radial part of the conditioned Brownian motion is the same process as the n^2-dimensional Bessel process.*

Theorem 4. *This conditioned process is identical to the process obtained by n particles moving in a one-dimensional diffusion with constant infinitesimal variance, with a repelling force between two particles proportional to the inverse of the distance between them; that is, its infinitesimal generator has $\sigma_{ij}^2(\vec{x},t) = \delta_{ij}$, $\mu_i(\vec{x},t) = \sum_{i\neq j} 1/(x_i - x_j)$.*

This is a specific case of Brownian motion in a Weyl chamber; the vector whose coordinates are the locations of the n particles is constrained to stay inside the chamber. The Weyl groups is A_{n-1}, the symmetric group.

We have similar results for other Weyl groups; Theorems 1 and 2 have analogues for the classical Weyl groups, and Theorems 3 and 4 have analogues for all Weyl groups. For example, the Weyl group B_n models n particles in independent Brownian motion with an absorbing wall at $x = 0$. In this case, the asymptotic probability is $ct^{-n^2/2}$ (and the constant is again known), with the radial part corresponding to the $n(n+1)$-dimensional Bessel process; the conditioned process has a radial part which is the $n(2n+1)$-dimensional Bessel process. This generalizes the results of Pitman and Williams [24, 28] that one-dimensional Brownian motion conditioned to stay positive is the same process as the three-dimensional Bessel process.

Our reflection argument is a generalization of the *reflection principle*, a standard argument in the analysis of both discrete random walks and Brownian motion. In the discrete case, it is used in the classical formula for the Catalan numbers, which enumerate the arrangements of n $+1$'s and n -1's so that none of the partial sums are negative. Similarly, it can be used to study Brownian motion in one dimension with an absorbing barrier at $x = 0$ and a known starting point [14]. It has also been extended to multiple reflections in one dimension to study Brownian motion with two absorbing barriers [11].

The reflection principle has been generalized to multiple dimensions. For example, the *ballot problem*, a classical problem in random walks, asks how many ways there are to walk from the origin to a point $(\lambda_1, \dots, \lambda_n)$, taking k unit-length steps in the positive coordinate directions while staying in the region $x_1 \geq x_2 \geq \cdots \geq x_n$.

The solution is known in terms of the hook-length formula for Young tableaux; a combinatorial proof, using a reflection argument, is given in [27, 29].

The same reflection argument has also been applied to the case of n independent diffusions, or discrete processes which cannot pass each other without first colliding. Using this method, Karlin and McGregor [17] give a determinant formula for the probability or measure for the n particles, starting at known positions, not to have collided up to time t and to be in given positions. Hobson and Werner [16] generalize this argument to n particles in an interval or circle, and use this to prove a result analogous to Theorem 4 for n particles on the circle.

Gessel and Zeilberger [12], and independently Biane [3] give a further generalization. For certain "reflectable" random walk-types, we can count the number of k-step walks between two points of a lattice, staying within a chamber of a Weyl group, in terms of numbers of unconstrained walks. The steps must have certain allowable lengths and directions. In [13], all cases in which this method applies are enumerated, and determinant formulas are given for many important cases, including walks in the classical Weyl chambers.

The argument of [12] can be generalized to Brownian motion in any Weyl chamber or chamber of a Coxeter group, with either absorbing or reflecting boundary conditions. We prove this generalization, and then use the result to compute determinant formulas for Brownian motion in the Weyl chambers of A_{n-1}, $B_n = C_n$, and D_n. The A_{n-1} and B_n cases are applicable to the independent motion of n particles in one dimension. The A_{n-1} formula appears in [17], using the model of n independent particles rather than motion in a Weyl chamber. The cases of the affine Weyl groups $\tilde{A}_{n-1}$ and $\tilde{B}_n$ are studied in [16], also as models of n independent particles, in a circle or interval.

These determinants factor into forms which can be easily analyzed; this allows us to find the asymptotic probability that there will be no collision up to time t for these three cases, for any starting point. In the case of A_{n-1}, we get a simple formula for the actual measure as well.

Weyl chambers arise naturally as the spaces of eigenvalues of elements of Lie algebras. Also, Brownian motion on a Lie algebra is symmetric under an orthonormal change of basis. There is thus a natural correspondence between standard Brownian motion on the Lie algebra and some diffusion on the Weyl chamber. Dyson [10] showed that this diffusion has the same generator as the conditioned Brownian motion for A_{n-1}; this generalizes naturally to other Lie algebras. We can thus use the known properties of random matrices to study the distribution.

DeBlassie [8] uses a different approach to give a more general formula for asymptotics and density functions for a general class of cones, which include the cases discussed here. A cone is defined as the union of all rays from the origin which intersect the unit sphere in a connected open set C. The Laplace-Beltrami operator $L_{S^{n-1}}$ on the unit sphere is the non-radial portion of the ordinary Laplacian. If λ_1 is the eigenvalue of $L_{S^{n-1}}$ with smallest absolute value on the space of all L^2 functions on C which vanish continuously on the boundary of C, then the asymptotic probability is a constant multiple of

$$t^{-\left[-(n-2)+\sqrt{(n-2)^2+4\lambda_1^2}\right]/4}.$$

This result gives asymptotics for a large class of cones. The coefficients depend on the eigenvalues of the Laplace-Beltrami operator rather than on explicit coefficients; these eigenvalues are known for Weyl chambers [2]. The asymptotics can thus be computed from these formulas; in theory, the explicit density functions can also be computed, but as infinite series, This formula also shows that the asymptotics for all such cones exist and are powers of t, with no other terms such as logarithms.

The asymptotic probability is known to be a constant multiple of $t^{-m/2}$ for a wedge of angle π/m in two dimensions [7]. The result holds in general, although the region is only a chamber of a Coxeter group (the dihedral group) if m is an integer.

O'Connell and Unwin [23] study the opposite asymptotic problem to ours, computing an asymptotic for the probability that n particles in independent Brownian motion will have a collision up to time t when t is small compared to the initial separation of the particles.

Our results are organized as follows. Section 2 contains the basic definitions. In section 3, we prove the basic reflection result, and in section 4, we apply this result to get the determinant formulas. In section 5, we prove Theorems 1 and 2, and their analogues for B_n and D_n. In section 6, we construct the conditioned motion, both by Lie theory and by h-transformation, proving Theorems 3 and 4 and their analogues, and use the h-transformation to find a physical model for all finite Coxeter groups.

2. Definitions

We will study a process with continuous sample paths in $\mathbb{R}^n$, either unconstrained or constrained by a chamber. In the constrained case, we may have either an absorbing boundary condition, causing the process to terminate when it hits a wall, or a reflecting boundary condition. All references to the analogous discrete problem are discussed in [13].

We require that our chamber C be a chamber of a finite or affine *Coxeter group.* In the finite case, C is defined by a system of simple roots $\Delta \subset \mathbb{R}^n$ as

$$C = \{\vec{x} \in \mathbb{R}^n \mid (\alpha, \vec{x}) \geq 0 \text{ for all } \alpha \in \Delta\}, \tag{1}$$

and the orthogonal reflections $r_\alpha\colon\ \vec{x} \mapsto \vec{x} - \frac{2(\alpha,\vec{x})}{(\alpha,\alpha)}\alpha$, generate a finite group W of linear transformations, the *Coxeter group.* In the affine case, the hyperplanes of reflection which define C do not all pass through the origin, and the group W is infinite, but if T is the subgroup of all translations, W/T must be finite. In the analogous discrete problem, the steps of the the random walk must generate a lattice L which is stable under the action of W; in this case, C is a *Weyl chamber* and W a *Weyl group*.

Let $X(t)$ be a Markov process with continuous sample paths with values in $\mathbb{R}^n$; that is, the distribution of $X(t_2)$ given $X(t_1)$ is independent of $X(t)$ at any t outside the interval $[t_1, t_2]$. We say that the constrained motion is *reflectable* if the increments of the unconstrained motion are symmetric under the Coxeter group; that is, the distribution of $X(t_2)$ given $X(t_1)$ is the same as the distribution of $w(X(t_2))$ given the w-image $w(X(t_1))$ as the starting point for any $w \in W$.

Standard Brownian motion is reflectable for any Coxeter group. For a finite Coxeter group, in which all planes of reflection pass through the origin, any diffusion with variance dependent only on time and the radius, and drift dependent on time and symmetric with respect to rotations and reflections about the origin, is reflectable; for example, there could be an absorbing or reflecting barrier at $|\vec{x}| = R$.

As another example, consider the case in which each coordinate $x_i(t)$ is an independent identical diffusion; this could model n independent particles instead of one particle in n dimensions. If our Coxeter group is the symmetric group $A_{n-1} = S_n$ (giving the chamber $x_1 > x_2 > \cdots > x_n$), it permutes the particles, so the process is reflectable under this action; this case is discussed in [17]. If the individual diffusions are symmetric about $x_i = 0$, then the product process will also be reflectable under the hyperoctahedral group B_n, which includes all permutations with any number of sign changes.

In the discrete case, reflectability requires the additional condition that the walk cannot go from inside the chamber to outside it without stopping on a wall [12]; this is our condition of continuous sample paths, which is satisfied by any diffusion.

For fixed t, this process defines a probability measure $P_t(A) = P\{X(t) \in A\}$ which represents the chance that this process, if started at 0, will be in a set A at time t. We let $c_t(\vec{x})$ be the density function of this probability measure with respect to Lebesgue measure on $\mathbb{R}^n$; that is,

$$\int_{\vec{x} \in A} c_t(\vec{x}) = P_t(A).$$

Now, we study the case in which the motion is constrained by a chamber, with either absorbing or reflecting boundary conditions. We must now fix the starting point η, since the process now depends on it in a non-trivial way. This generates new stochastic processes $Y(\eta, t)$ for absorbing boundary conditions and $Y'(\eta, t)$ for reflecting boundary conditions. These give probability measures $Q_t(\eta, A)$ and $Q'_t(\eta, A)$ which give the probability that the process, if started at η, will be in the set A at time t, and these have density functions $b_t(\eta, \lambda)$ and $b'_t(\eta, \lambda)$. Note that the total measure $Q_t(\eta, \mathbb{R}^n)$ will be less than 1 because the process terminates when it reaches a wall of the chamber. The total measure $Q'_t(\eta, \mathbb{R}^n)$ for reflecting boundary conditions will still be 1.

If our process is a Brownian motion, it has drift μ_i and variance σ_i^2 in each coordinate direction. If the Coxeter group contains any reflection in the x_i, x_j-plane other than a sign change of x_i or x_j, the reflectability condition requires that $\sigma_i = \sigma_j$. Thus, for an irreducible Coxeter group, all of the σ_i must be equal; for a reducible Coxeter group acting on $\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \oplus \cdots \oplus \mathbb{R}^{n_k}$, we can multiply the coordinates in each $\mathbb{R}^{n_j}$ by a constant factor so that the σ_i are all equal. We can then re-scale time so that all $\sigma_i = 1$.

Reflectability also requires the Coxeter group to fix the vector μ whose coordinates are the drifts μ_i. In all non-trivial cases except for A_{n-1} on $\mathbb{R}^n$, this requires that all $\mu_i = 0$, giving standard Brownian motion. For A_{n-1}, it requires that the μ_i all be equal; we can then change coordinates to $x_i' = x_i - \mu_i t$ to get an equivalent process in which all of the μ_i' are zero. Thus, if the unconstrained process has stationary increments, and is thus a Brownian motion [14], we may assume that it is standard Brownian motion; however, we can state the theorems just as easily in terms of the more general reflectable process.

3. The formulas

Theorem 5. *If c_t is the density function for a reflectable continuous stochastic process, then for absorbing boundary conditions, we have*

$$b_t(\eta, \lambda) = \sum_{w \in W} \operatorname{sgn}(w) c_t(w(\lambda) - \eta), \tag{2}$$

and for reflecting boundary conditions,

$$b_t'(\eta, \lambda) = \frac{\sum_{w \in W} c_t(w(\lambda) - \eta)}{\#\{w \in W : w(\lambda) = \lambda\}}. \tag{3}$$

If W is an affine group rather than a finite group, these may be infinite sums. The integrals over the images of any region must converge absolutely, since the measure of unconstrained motion over the whole space, the set of all W-images of all points in the chamber, is 1.

Proof. The discrete result analogous to (2) is proved in [12] and [3]; the proof which follows is essentially identical to that in [12] except that the discrete terms "walk" and "step" are replaced by their continuous analogues "path" and "time."

Every path from η to any $w(\lambda)$ which does touch at least one wall of the chamber has some first time t_0 at which it touches a wall; let the wall be the hyperplane perpendicular to α_i, choosing the largest i if there are several choices [25]. Reflect the path after time t_0 across that hyperplane; the resulting path is a path from η to $r_{\alpha_i} w(\lambda)$ which also first touches wall i at time t_0. This clearly gives a measure-preserving bijection of paths, and since r_{α_i} has sign -1, all such paths cancel out in (2). The only paths which do not cancel in these pairs are the paths which stay within the Weyl chamber, and since $w(\lambda)$ is inside the Weyl chamber only if w is the identity, this is the desired measure.

For (3), we note that the map on all paths starting at η which takes every point to its unique image in the chamber C is measure-preserving, since we have reflecting boundary conditions and increments which are stable under the group W. This map takes all paths which end at any $w(\lambda)$ to paths which end at λ itself. If λ is on a wall of the chamber, paths to λ may be counted multiple times, so we must divide by the size of its stabilizer in W.

In practice, we can ignore this constant factor; it is 1 except for λ on a wall, and this is a set of measure zero unless $t = 0$. Eliminating the denominator thus changes the density function only on a set of measure zero, and thus does not change the measure of any measurable set.

4. Determinant formulas for the density functions

We can now apply this theorem to standard Brownian motion, in the Weyl chambers of A_{n-1}, $B_n = C_n$, and D_n, with either absorbing or reflecting boundary conditions. The measure for unconstrained standard Brownian motion is $c_t(\vec{x}) = \prod_{i=1}^{n} N_t(x_i)$, where N_t is the normal distribution function with mean 0 and variance t. Since this factors into separate terms for the individual coordinates, we can use the same techniques to compute determinant formulas as in the discrete case [13].

The most interesting case is $A_{n-1} = S_n$, the symmetric group. The Weyl chamber is $x_1 > x_2 > \cdots > x_n$. This Brownian motion thus models n independent particles in one dimension. With absorbing boundary conditions, collisions are forbidden (the process terminates if one occurs); with reflecting boundary conditions, particles collide elastically with one another.

For absorbing boundary conditions, we write the sum (2) as

$$b_t(\eta, \lambda) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) c_t(\sigma(\lambda) - \eta) \tag{4}$$

and use the value of c_t to write this as

$$b_t(\eta, \lambda) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(N_t(\lambda_{\sigma(i)} - \eta_i) \right). \tag{5}$$

This sum can be written as a determinant, which gives

$$b_t(\eta, \lambda) = \det_{n \times n} \left| N_t(\lambda_i - \eta_j) \right|. \tag{6}$$

This determinant gives the measure for n particles which start at positions η_i and are in independent Brownian motion to be at positions λ_i at time t without having collided.

For a more general product of n independent diffusions, with individual density $p_t(x \to y)$ for the diffusion which started at point x to be at point y at time t, we can apply the same argument. This gives the following generalization of (6), which first appears in [17].

$$b_t(\eta, \lambda) = \det_{n\times n} |p_t(\eta_j \to \lambda_i)| . \tag{7}$$

For reflecting boundary conditions, the calculations are the same except that there is no term for $\mathrm{sgn}(\sigma)$, and thus we get permanents rather than determinants. This result can also be seen by observing that an elastic collision between two identical particles is equivalent to the two particles passing through each other with no collision, and thus the reflected particles will be at the positions indicated by λ if the unreflected particles are at any permutation of the coordinates of λ.

For B_n, the hyperoctahedral group, which includes permutations with any number of sign changes, the Weyl chamber is $x_1 > x_2 > \cdots > x_n > 0$. This also models n independent particles in one dimension, with an additional wall at $x = 0$.

We write $w \in W$ as a product of an ϵ which negates some coordinates and a σ in the symmetric group. We get

$$b_t(\eta, \lambda)(x) = \sum_{\sigma\in S_n} \sum_{\epsilon_i=\pm 1} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} \epsilon_i \prod_{i=1}^{n} \left(N_t(\epsilon_i \lambda_{\sigma(i)} - \eta_i)\right) . \tag{8}$$

Using the multilinearity of the products in the determinant, we can again write this sum as a determinant, with separate terms for $\epsilon_i = 1$ and $\epsilon_i = -1$ in each entry. We use $N_t(x) = N_t(-x)$ to keep the signs of λ positive and get a more elegant formula. This gives

$$b_t(\eta, \lambda) = \det_{n\times n} |N_t(\lambda_i - \eta_j) - N_t(\lambda_i + \eta_j)| . \tag{9}$$

This determinant gives the measure for n particles which start at η_i to be at λ_i at time t, neither having collided nor having touched $x = 0$.

Again, the same argument applies if n particles are in general independent diffusions, provided that the diffusions are symmetric about $x = 0$. The more general formula is

$$b_t(\eta, \lambda) = \det_{n\times n} |p_t(\eta_j \to \lambda_i) - p_t(\eta_j \to -\lambda_i)| . \tag{10}$$

For reflecting boundary conditions, we lose the sign of the σ, which makes the determinant into a permanent, and the sign of the ϵ_i, which turns the minus sign between the two N_t in (9) or p_t in (10) into a plus sign. Again, the resulting formula is the same as would be obtained by treating elastic collisions as though the particles passed through each other, and allowing particles to pass through the wall at $x = 0$ instead of bouncing. (In the transformed model, particles at positions x and $-x$ no longer collide, but since they collided elastically in the original model and could be considered to pass through each other instead, the effect is the same.)

For D_n, the even hyperoctahedral group, which includes permutations with an even number of sign changes, the Weyl chamber is $x_1 > x_2 > \cdots > x_n$, $x_{n-1} > -x_n$. This does not give a natural model for n particles in one dimension.

Again, we can write $w = \epsilon\sigma$. We take our sum over all possible ϵ, and then add an additional factor of $(1 + \prod_{i=1}^n \epsilon_i)/2$ to annihilate those ϵ which are not allowed in D_n.

$$b_t(\eta, \lambda)(x) = \sum_{\sigma \in S_n} \sum_{\epsilon_i = \pm 1} \operatorname{sgn}(\sigma) \frac{1 + \prod_{i=1}^n \epsilon_i}{2} \prod_{i=1}^n \left(\epsilon_i N_t(\epsilon_i \lambda_{\sigma(i)} - \eta_i)\right). \tag{11}$$

We now take the $\frac{1}{2}$ and the $(\prod_{i=1}^n \epsilon_i)/2$ terms separately. The $(\prod_{i=1}^n \epsilon_i)/2$ term is half the sum we had in (9); the $\frac{1}{2}$ term gives half of (9), but with a plus sign between the terms. Thus we get

$$\begin{aligned} b_t(\eta, \lambda)(x) &= \frac{1}{2}\Big[\det_{n\times n} |N_t(\lambda_i - \eta_j) - N_t(\lambda_i + \eta_j)| \\ &\qquad + \det_{n\times n} |N_t(\lambda_i - \eta_j) + N_t(\lambda_i + \eta_j)|\Big]. \end{aligned} \tag{12}$$

If we let λ' be obtained from λ by changing the sign of λ_n, this will change the sign of the first term but preserve the second term. Thus the first term alone, with no factor of $1/2$, is $b_t(\eta, \lambda)(x) - b_t(\eta, \lambda')(x)$, and the second term alone is $b_t(\eta, \lambda)(x) + b_t(\eta, \lambda')(x)$. If $\lambda_n = 0$ or $\eta_n = 0$, the first term is zero, so $b_t(\eta, \lambda)(x)$ is the second term alone, with the factor of $1/2$.

For reflecting boundary conditions, we ignore the sign of σ. Thus the determinants become permanents, but we keep the minus signs because they came from the factor $(1 + \prod_{i=1}^n \epsilon_i)/2$, which was not from $\operatorname{sgn}(w)$.

5. Asymptotics

We can use these formulas to find asymptotics for the probability that the motion will not hit a wall of the chamber by time t, and for its distribution at time t given that it has not hit a wall.

5.1. **Calculating the individual values for A_{n-1}.** We can eliminate the determinant to get a more explicit formula for (6) at a single point λ if all the coordinates of our starting point η are rational. Re-scaling by $x_i \to cx_i$, $t \to c^2 t$ will make all the coordinates integers, and we can then translate all coordinates by $-\eta_n$ so that we have $\eta_n = 0$. (Both of these transformations leave the Weyl chamber $x_1 > x_2 > \cdots > x_n$ unchanged.)

We now write out the normal distributions in (6) explicitly as exponentials, and expand $(\lambda_i - \eta_j)^2$ as $\lambda_i^2 - 2\lambda_1\eta_j + \eta_j^2$:

$$b_t(\eta, \lambda) = \frac{1}{(2\pi t)^{n/2}} \det_{n\times n} \left| \exp\left(\frac{-\lambda_i^2 + 2\lambda_i\eta_j - \eta_j^2}{2t} \right) \right| . \tag{13}$$

Row i of this matrix contains a constant factor $\exp(-\lambda_i^2)$, and column j a constant factor $\exp(-\eta_j^2)$, so we can take these out, and put them in a constant term, which simplifies further because $|\lambda|^2 = \sum \lambda_i^2$. This gives us

$$b_t(\eta, \lambda) = \frac{1}{(2\pi t)^{n/2}} \exp\left(\frac{-|\lambda|^2 - |\eta|^2}{2t} \right) \det_{n\times n} \left| \exp\left(\frac{\lambda_i\eta_j}{t} \right) \right| . \tag{14}$$

Since the η_j are all integers, we can write the determinant as the generalized Vandermonde determinant

$$\det_{n\times n} \left| [\exp(\lambda_i/t)]^{\eta_j} \right| . \tag{15}$$

If $\eta_j = n - j$, this is the standard Vandermonde determinant, equal to

$$\prod_{i>j} [\exp(\lambda_i/t) - \exp(\lambda_j/t)]. \tag{16}$$

And for any non-negative integers η_j, it is the product of this Vandermonde determinant and the Schur function [19]

$$s_{\eta_1-n+1,\eta_2-n+2,\dots,\eta_n}(\exp(\lambda_1/t), \dots, \exp(\lambda_n/t))$$

The Schur function can also be defined combinatorially [19], with the coefficient of $\prod x_i^{n_i}$ in s_μ the number of ways to fill in the partition diagram of μ, using the number i exactly n_i times, such that the entries are non-decreasing in each row and strictly increasing in each column.

In particular, we can let $C_\eta = S_\mu(1, 1, \dots, 1)$ be the total number of such tableaux; this is important because it is the approximate value of the Schur function when the λ_i are much less than t. (This holds because the Schur function is a homogeneous polynomial of degree $\eta_1 + \cdots + \eta_n - (n(n-1))/2$, with positive coefficients.) This will show the dependence of the asymptotics on the starting point.

This constant is known [19]; it is

$$C_\eta = \prod_{i<j} (\eta_i - \eta_j) \Big/ \prod_{i<j} (j - i). \tag{17}$$

This allows us to compute the constant term in the asymptotics.

For any η with the same sum of the coordinates, all multiples of $1/c$, our rescaling gives a Schur function whose index is the transformed vector $c\eta$. Rescaling to restore the old time values gives a Schur function which is a homogeneous polynomial in the $\exp(\lambda_i/t)^{1/c}$ whose degree in the $\exp(\lambda_i/t)$ is $\eta_1 + \cdots + \eta_n - (n(n-1))/2c$. Thus, for any point η, even one with fractional coordinates, the degree will be bounded by the sum of its coordinates. Thus the determinants, and therefore the $b_t(\eta,\lambda)$, for different starting points η and η' both of radius less than a known δ, will be in the approximate ratio of C_η to $C_{\eta'}$, with an error of $O(\delta/t)$. If δ is fixed, then when we calculate the asymptotics as $t\to\infty$, the Schur function will converge to C_η at a rate of $O(1/t)$.

5.2. **Asymptotic probability of no collisions.** The integral of the value in (14) over the whole Weyl chamber is the probability that n particles starting at the positions η_j will have no collisions up to time t. We can use this formula to show that the asymptotic probability as $t\to\infty$ is a constant multiple of $t^{-n(n-1)/4}$, with the constant depending on η. We will also show that, given that the n-dimensional Brownian motion has not hit a wall, its radial distribution, rescaled by multiplying the radius by $1/\sqrt{t}$, converges to the distribution of the Bessel process with parameter $n(n+1)/2$ at time 1.

Fix t very large compared with $|\eta|^2$. If $|\lambda|^2$ is much larger than t, then the exponential of $-|\lambda|^2/2t$ in (14) will decay exponentially fast. In particular, λ_i/t can be assumed to be $O(t^{\epsilon-1/2})$, so we only need to look at the leading nonzero terms in (15); the error will be a factor of this order when compared with the value of the determinant. In (16), the first nonzero term of the factor $[\exp(\lambda_i/t)-\exp(\lambda_j/t)]$ is $(\lambda_i-\lambda_j)/t$; we can then take out the factor of $t^{-n(n-1)/2}$ and leave only a term involving the λ. Likewise, in the Schur function from (15), we need only keep the constant term C_η.

Thus our probability is asymptotic to the integral over the Weyl chamber of

$$\frac{C_\eta}{(2\pi t)^{n/2}}\exp\left(\frac{-|\lambda|^2-|\eta|^2}{2t}\right)t^{-n(n-1)/2}\prod_{i>j}(\lambda_i-\lambda_j). \tag{18}$$

And this integral can be computed by using Selberg's integral [20, 22]; we have

$$\int_{\mathbb{R}^n}\exp(-|x|^2/2)\left|\prod_{i>j}(x_i-x_j)\right|dx_i=2^{3n/2}\prod_{k=1}^{n}\Gamma((k/2)+1). \tag{19}$$

This corresponds to our desired integral when we set $\vec{x}=\lambda/\sqrt{t}$, and divide by $n!$ because we are taking our integral over only one of the $n!$ different Weyl chambers. Dropping the $\exp(-|\eta|^2/2t)$ (which goes to 1 as $t\to\infty$ and thus doesn't affect the leading term), and writing out C_η explicitly again gives our full asymptotic:

$$\frac{\prod_{i<j}(\eta_i-\eta_j)}{\prod_{i<j}(j-i)}(2\pi)^{n/2}2^{3n/2}\prod_{k=1}^{n}[\Gamma((k/2)+1)]t^{-n(n-1)/4}. \tag{20}$$

We can also note that the exponential in (18) is spherically symmetric, while $\prod_{i<j}(\lambda_i - \lambda_j)$ is homogeneous of degree $n(n-1)/2$. Thus the density, integrated over the sphere of radius r at a fixed t, is a constant multiple of

$$\exp(-r^2/2t)r^{((n(n-1)/2)-1},$$

and thus a constant multiple of the radial distribution of $(n(n+1)/2)$-dimensional Brownian motion at time t. Thus, given that no two particles have collided, the distribution would be exactly the same if it were given by (18). This proves Theorem 2; if we restrict to $|\lambda| < ct^{1/2+\epsilon}$, the λ_i/t terms are all $O(t^{\epsilon-1/2})$. Thus, in this region, the ratio of the radial distribution for the constrained motion to the radial distribution for unconstrained motion in $n(n+1)/2$ dimensions converges uniformly to 1 at a rate of $O(t^{\epsilon-1/2})$.

Equivalently, we could fix time and λ, and for a scalar δ, take $\hat{\eta} = \delta\eta$ as our starting point. By the scaling properties of Brownian motion, this is equivalent to keeping η fixed, taking $\hat{\lambda} = \lambda/\delta$, and $\hat{t} = t/\delta^2$. Thus, as $\delta \to 0$, the ratio of the radial distributions converges uniformly to 1 within the region $|\lambda| < ct^{1/2+\epsilon}$ at a rate of $O(\delta^{1-\epsilon})$, and the probability that either distribution is outside that region goes to 0 exponentially fast.

5.3. **Asymptotics for B_n: no collisions and a wall.** We can use the same technique to get asymptotics for B_n as for A_{n-1}; here, we get a constant multiple of $t^{-n^2/2}$ as the probability of no collision, and the radial distribution given no collision is the n^2+n-dimensional Bessel process.

Here, we require that the coordinates all be odd integers. Again, we write out the determinant (9) explicitly:

$$b_t(\eta, \lambda) = \frac{1}{(2\pi t)^{n/2}} \det_{n\times n} \left| \exp\left(\frac{-\lambda_i^2 + 2\lambda_i\eta_j - \eta_j^2}{2t}\right) - \exp\left(\frac{-\lambda_i^2 - 2\lambda_i\eta_j - \eta_j^2}{2t}\right) \right|. \tag{21}$$

As before, we remove the constant factors to get

$$b_t(\eta, \lambda) = \frac{1}{(2\pi t)^{n/2}} \exp\left(\frac{-|\lambda|^2 - |\eta|^2}{2t}\right) \det_{n\times n} \left| \exp\left(\frac{\lambda_i\eta_j}{t}\right) - \exp\left(\frac{-\lambda_i\eta_j}{t}\right) \right|. \tag{22}$$

The determinant here is not an actual Vandermonde determinant. However, in the specific case $\eta_j = 2n+1-2j$, we can make it a Vandermonde determinant by elementary operations. Adding $(-1)^k\binom{2n+1-2j}{k}$ times column $j+k$ to column j does not change the determinant, but it changes the entries in column j to

$$\sum_{k=0}^{2n+1-2j} (-1)^k \binom{2n+1-2j}{k} \exp((2n+1-2j-2k)\lambda_i/t)$$
$$= \left[\exp(\lambda_i/t) - \exp(-\lambda_i/t)\right]^{2n+1-2j}.$$

This is a generalized Vandermonde determinant; we can make it an actual Vandermonde determinant by dividing row i by $\exp(\lambda_i/t) - \exp(-\lambda_i/t)$. The resulting determinant is

$$\det_{n\times n} \left| [\exp(\lambda_i/t) - \exp(-\lambda_i/t)]^{2(n+1-j)} \right|, \tag{23}$$

and its value is

$$\prod_{i>j} \left[(\exp(\lambda_i/t) - \exp(-\lambda_i/t))^2 - (\exp(\lambda_j/t) - \exp(-\lambda_j/t))^2 \right]. \tag{24}$$

Putting this together with the constants we have taken out, we get

$$\begin{aligned} b_t(\eta, \lambda) \;\; = \;\; & \frac{1}{(2\pi t)^{n/2}} \exp\left(\frac{-|\lambda|^2 - |\eta|^2}{2t} \right) \\ & \times \left(\prod_{i=1}^{n} [\exp(\lambda_i/t) - \exp(-\lambda_i/t)] \right) \\ & \times \prod_{i>j} \left[(\exp(\lambda_i/t) - \exp(-\lambda_i/t))^2 - (\exp(\lambda_j/t) - \exp(-\lambda_j/t))^2 \right]. \end{aligned}$$

For a more general starting point, with all coordinates odd integers, we note that $(\exp(\lambda_i\eta_j/t) - \exp(-\lambda_i\eta_j/t))$ is a polynomial in $\exp(\lambda_i/t) - \exp(-\lambda_i/t)$ with no constant term. We can break these polynomials into their individual terms, giving a large number of determinants, each one a generalized Vandermonde determinant in the $\exp(\lambda_i/t) - \exp(-\lambda_i/t)$. Each individual determinant is thus the product of (24) and a Schur function; it also contains the product of the $\exp(\lambda_i/t) - \exp(-\lambda_i/t)$ as a factor, since these are constant factors in row i. Thus each determinant is a product of these factors with some symmetric function in the $\exp(\lambda_i/t) - \exp(-\lambda_i/t)$. As with A_{n-1}, the error we get in approximating the Schur functions by their constant term C_η is a factor of $O(1/t)$.

And as with A_{n-1}, we can get asymptotics by integrating this over the Weyl chamber. For large t, the radial exponential will be exponentially small if $|\lambda| > t^{1/2+\epsilon}$, so we can assume that all of the λ_i/t are very small. Thus $\exp(\lambda_i/t) - \exp(-\lambda_i/t)$ can be approximated by its leading nonzero term, $2\lambda_i/t$. Thus, as with (18), we get

$$\frac{1}{(2\pi t)^{n/2}} \exp\left(\frac{-|\lambda|^2 - |\eta|^2}{2t} \right) (2/t)^{n^2} \prod_{i=1}^{n} (\lambda_i) \prod_{i>j} (\lambda_i^2 - \lambda_j^2). \tag{25}$$

This integral can also be computed by using Selberg's integral [20, 22]; we have

$$\int_{\mathbb{R}^n} \exp(-|x|^2/2) \prod_{i=1}^{n} |x_i| \left| \prod_{i>j} (x_i^2 - x_j^2) \right| dx_i$$
$$= \frac{2^{(n^2+3n)/2}}{\pi^{n/2}} \prod_{i=1}^{n} \left[\Gamma(1+i/2)\Gamma((1+i)/2)\right].$$

This corresponds to our desired integral when we set $\vec{x} = \lambda/\sqrt{t}$, and divide by $2^n n!$ because that is the number of Weyl chambers. Dropping the $\exp(-|\eta|^2/2t)$ because it goes to 1 as $1/t$, we get our asymptotic probability that Brownian motion started at the specific point η will not hit a wall up to time t:

$$\frac{2^{(3n^2/2)}}{\pi^n n!} \prod_{k=1}^{n} \left[\Gamma(1+k/2)\Gamma((1+k)/2)\right] t^{-n^2/2}. \tag{26}$$

For a more general starting point, each individual determinant in the sum gives an integral of a homogeneous polynomial in the $\exp(\lambda_i/t) - \exp(-\lambda_i/t)$ which is of degree at least n^2, since it contains the previous determinant as a factor. The same technique as above gives an asymptotic which is thus at most $t^{-n^2/2}$. Thus the asymptotic is at most a constant multiple of $t^{-n^2/2}$, and is less if and only if the coefficient of $t^{-n^2/2}$ is zero. But the coefficient cannot be zero. Let $m = \max((\eta_i - \eta_{i+1}/2), \eta_n)$, and $\eta'_i = (2n+1-2i)m$. Then we know that Brownian motion starting at η' decays asymptotically as $t^{-n^2/2}$, but Brownian motion starting at η will always hit a wall if it is translated to start at η', because η' is at least as far as η from every wall. Thus the asymptotic probability will be some constant C_η times the formula (26); we will compute C_η in Section 5.5.

A result analogous to Theorem 2 also holds for B_n. The exponential in numbbnasymp is spherically symmetric, while the product is homogeneous of degree n^2. The same argument as for A_{n-1} shows that the ratio between the radial part of this distribution and the radial distribution for unconstrained Brownian motion in n^2+n dimensions converges uniformly to 1 at a rate of $O(t^{\epsilon-1/2})$ inside the region $|\lambda| < ct^{1/2+\epsilon}$, and the probability that either distribution is outside that region goes to 0 exponentially fast. Again, we can rescale by multiplying the radius by $1/\sqrt{t}$ to get convergence to a fixed distribution.

5.4. **Asymptotics for D_n.** The process for D_n is almost the same as for B_n, so we won't work it out in full detail; we get a similar result with the same error terms and convergence properties.

Here, it is most natural to let $\eta_i = n - i$. For this value of η, we have only the second determinant in (12), with a plus sign between the terms; the last row of the other determinant is zero. For general η, the first determinant is the B_n determinant, which we know is $O(t^{-n^2/2})$, and we will show that the second determinant is asymptotically larger.

The determinant that we get is

$$\det_{n\times n}\left|\exp\left(\frac{\lambda_i\eta_j}{t}\right)+\exp\left(\frac{-\lambda_i\eta_j}{t}\right)\right|. \tag{27}$$

Again, this isn't a Vandermonde determinant, but $\exp(\lambda_i/t)+\exp(-\lambda_i/t)^n$ is equal to $\exp(\lambda_i\eta_j/t)+\exp(-\lambda_i\eta_j/t)$ plus a sum of lower order terms, so elementary operations which do not change the determinant give us the Vandermonde determinant

$$\det_{n\times n}\left|[\exp(\lambda_i/t)+\exp(-\lambda_i/t)]^{n-j}\right|, \tag{28}$$

and its value is

$$\prod_{i>j}\left[\exp(\lambda_i/t)+\exp(-\lambda_i/t)-\exp(\lambda_j/t)-\exp(-\lambda_j/t)\right]. \tag{29}$$

The leading nonzero term is

$$\prod_{i>j}\frac{(\lambda_i/t)^2-(\lambda_j/t)^2}{2},$$

and the full integral is

$$\frac{1}{(2\pi t)^{n/2}}\exp\left(\frac{-|\lambda|^2-|\eta|^2}{2t}\right)(2/t)^{n^2-n}\prod_{i>j}(\lambda_i^2-\lambda_j^2). \tag{30}$$

Again, we get a result which can be obtained from Selberg's integral [20, 22]; we have

$$\int_{\mathbb{R}^n}\exp(-|x|^2/2)\left|\prod_{i>j}(x_i^2-x_j^2)\right|dx_i=\frac{2^{(n^2+2n)/2}}{\pi^{n/2}}\prod_{i=1}^{n}\left[\Gamma(1+i/2)\Gamma(i/2)\right]. \tag{31}$$

Here, there are $2^{n-1}n!$ Weyl chambers, which gives the asymptotic

$$\frac{2^{(3n^2-3n+2)/2}}{\pi^n n!}\prod_{k=1}^{n}\left[\Gamma(1+k/2)\Gamma((1+k)/2)\right]t^{(-n^2+n)/2}. \tag{32}$$

For a general starting point whose coordinates are all integers, we use the same technique as for B_n. The terms in the determinant are all polynomials in $\exp(\lambda_i/t)+\exp(-\lambda_i/t)$, so we can again split the sum into individual determinants, each of which is the product of (29) and a Schur function of the $\exp(\lambda_i/t)+\exp(-\lambda_i/t)$. Since λ_i/t is small, these are all close to 2, so the full asymptotic is a sum of terms of order $t^{(-n^2+n)/2}$. As with B_n, we can translate to η' which is further from any wall than η to show that the coefficient of $t^{(-n^2+n)/2}$ cannot be zero.

5.5. **The constant factor for a general starting point.** Since we know the constant factor in the asymptotic probability of no collision up to time t for one specific starting point η, and the asymptotic distribution for an arbitrary starting point, we can use the time-reversibility of Brownian motion to compute the asymptotic probability of no collisions for an arbitrary starting point. The argument is the same for all of the Weyl groups.

In each case, the density $b_t(\eta, \lambda)$ for Brownian motion starting at η to be at λ at time t, not having collided with a wall up to time t, is asymptotic for large t to a product of the form

$$C_\eta C'_\lambda f(t) \exp\left(\frac{-|\lambda|^2 - |\eta|^2}{2t}\right), \tag{33}$$

in which C'_λ and $f(t)$ are known. Since Brownian motion is symmetric in time, the density $b_t(\lambda, \eta)$ must be equal to $b_t(\eta, \lambda)$. As long as t is large enough compared to $|\eta|$ and $|\lambda|$ for the formula (33) to be valid (which it will be for large t because of the exponentials), we can reverse the roles of η and λ. Thus we have $C_\eta C'_\lambda = C_\lambda C'_\eta$, and since the formula for C' is known, we see that C_η must be a constant multiple of C'_η; it is thus a constant multiple of $\prod(\eta_i - \eta_j)$ for A_n, of $\prod(\eta_i^2 - \eta_j^2)\prod \eta_i$ for B_n, and $\prod(\eta_i^2 - \eta_j^2)$ for D_n.

We know the value of the constant from the formulas (20), (26), and (32). For A_n, we already have the value for a general starting point because we used the Schur functions to compute C_η in (20); we could have instead used this technique. For B_n, we get

$$\prod_{i<j} \frac{\eta_i^2 - \eta_j^2}{(2j-1)^2 - (2i-1)^2} \prod_{i=1}^{n} \frac{\eta_i}{2n+1-2i}$$
$$\times \frac{2^{(3n^2/2)}}{\pi^n n!} \prod_{k=1}^{n} [\Gamma(1+k/2)\Gamma((1+k)/2)]\, t^{-n^2/2}.$$

as the asymptotic probability of no collision; for D_n, we get

$$\prod_{i<j} \frac{\eta_i^2 - \eta_j^2}{(2j-1)^2 - (2i-1)^2}$$
$$\times \frac{2^{(3n^2-3n+2)/2}}{\pi^n t^{n^2/2} n!} \prod_{k=1}^{n} [\Gamma(1+k/2)\Gamma((1+k)/2)]\, t^{(-n^2+n)/2}.$$

6. Random matrices and conditioned Brownian motion

6.1. **Brownian motion on a Lie algebra.** Instead of viewing our Weyl chamber as the chamber of a Weyl group, we can view it as the space of eigenvalues of the Lie algebra corresponding to that Weyl group, and then use the theory of Lie algebras to study it. The following construction was first developed by Dyson [10] for $\mathfrak{su}_n$; he computed the properties of the Brownian motion from the specific data rather than using the general Lie theory. The results we use from Lie theory are given in [1, 15].

Given a Lie group G and its Lie algebra $\mathfrak{g}$, we can define a normal distribution or Brownian motion on $\mathfrak{g}$ by viewing it as a vector space and taking an orthonormal basis with respect to a G-invariant inner product. In particular, on $\mathfrak{so}_n(\mathbb{R})$, the Lie algebra of skew-symmetric matrices, we can take $1/\sqrt{2}$ times standard Brownian motion on each matrix entry M_{ij} with $i < j$, and take $M_{ji} = -M_{ij}$; this makes the Hilbert-Schmidt norm of M equivalent to the radius of a Brownian motion in $n^2 - n$ dimensions. Likewise, for $\mathfrak{su}_n(\mathbb{C})$, the Lie algebra of skew-Hermitian matrices, we can take standard Brownian motion on the imaginary part of each diagonal entry M_{ii}, and $1/\sqrt{2}$ times standard Brownian motion on each matrix entry M_{ij} with $i < j$, with $M_{ji} = -\overline{M_{ij}}$; this again gives us the Hilbert-Schmidt norm, this time in n^2 dimensions. For any Lie algebra, the Brownian motion at time t will have a Gaussian distribution on the space of matrices in that Lie algebra. This allows us to use all of the known results about random matrices [22] to study the motion, and its eigenvalues in particular.

Since this Brownian motion is invariant under conjugation by G, it induces a diffusion on the space of eigenvalues of the matrices. The Weyl chamber is in a correspondence with the space of eigenvalues, obtained by dividing the independent eigenvalues by i and arranging them in decreasing order. To study this diffusion, we consider the measure on the Weyl chamber which is induced by the standard measure on the Lie algebra. We can study this measure by proving a version of the Weyl Integration Formula for Lie algebras, which can be proved directly from the analogous formula for Lie groups [1].

Lemma 1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with m roots, and $f(X) = \hat{f}(\lambda_1, \dots, \lambda_n)$ a function on $\mathfrak{g}$ with compact support which depends only on the eigenvalues of X. Let δ' be $(2\pi)^m$ times the product of all the roots of $\mathfrak{g}$. Then the integral of f over $\mathfrak{g}$ is equal to the integral of $\delta'\bar{\delta}'\hat{f}$ over the Weyl chamber.*

Proof. Let G be the Lie group corresponding to $\mathfrak{g}$. We can apply the exponential map $h : X \to \exp(\pi i t X)$ to translate a neighborhood of zero in $\mathfrak{g}$ which contains the support of f to an arbitrarily small neighborhood of the origin in G by choosing t sufficiently small. The derivative of this map can be made arbitrarily close to $(\pi i t)^n$ (which becomes $(\pi t)^n$ when we consider volume elements) by taking t sufficiently small. We now apply the Weyl Integration Formula to $h \circ f$, which is a class function on the Lie group G; it says that the integral of $h \circ f$ over G is equal to the integral of $\delta\bar{\delta}$ over a maximal torus, where

$$\delta = \prod_{\alpha \in \Delta} \left(e^{\pi i \alpha(\vec{x})} - e^{-\pi i \alpha(\vec{x})} \right). \tag{34}$$

For $\vec{x}$ sufficiently close to zero, we can approximate δ by its leading nonzero term, which is

$$\delta \approx \delta' = (2\pi)^m \prod_{\alpha\in\Delta} \alpha(\vec{x}). \tag{35}$$

This allows us to proceed with the integration.

We can apply this formula to the characteristic functions of small intervals for the eigenvalues; this shows that the measure at a point $(x_1, \dots, x_n)$ in the Weyl chamber is proportional to $\delta\bar{\delta}$. In particular, we note that δ is positive real on the interior of the Weyl chamber, and zero on the walls; thus we have a diffusion which is conditioned never to leave the Weyl chamber.

Note that this proves Theorem 3 for this process, and an analogous theorem for any Weyl group. If we view $\mathfrak{g}$ as a vector space, the Hilbert-Schmidt norm of a matrix is the radius of a vector. If there are m positive roots, then $\mathfrak{g}$ has dimension $n+2m$ [15]. The Hilbert-Schmidt norm is thus given by the Bessel process in $n+2m$ dimensions. The Hilbert-Schmidt norm is unchanged under diagonalization, so the diffusion in the Weyl chamber has its radius given by the same process.

To properly state this result for Brownian motion conditioned never to leave the Weyl chamber, we cannot allow the process to start at the origin, which is not a point in the chamber. However, the square of the product of the roots is a homogeneous function of degree $2m$; it thus gives an identical distribution on any fixed radius. Thus, if we start the motion on the Lie algebra at 0, then at any later time, given the fixed radius, the distribution on the sphere will be given by the square of the product of the roots, and the process can be continued from that time on as a Bessel process. This allows us to state the general theorem, formalizing Theorem 3.

Theorem 6. *For any Weyl group W acting on $\mathbb{R}^n$ with m roots, consider the process which starts at a fixed radius r_0, with the starting point chosen on the sphere of radius r_0 by a distribution with density proportional to the square of the product of the roots. Then the radius of the position of transformed motion at time t gives a Bessel process with parameter $n+2m$.*

The simplest case of this theorem is for B_1, which is one-dimensional motion with an absorbing boundary at $x=0$. In this case, the only root is x, so $m=1$, and we have the result of [24] and [28] that Brownian motion conditioned never to hit 0 is the same as the three-dimensional Bessel process. In our more general cases, the number of dimensions for the Bessel process is $n+2m=n^2$ for A_{n-1}, $n(2n+1)$ for B_n, and $n(2n-1)$ for D_n.

6.2. **Construction by h-transformation.** We could have alternatively constructed this diffusion by h-transformation [9]. We will first construct the conditioned Brownian motion in this way, and then show that the two processes are actually identical; this allows us to develop the physical model of Theorem 4.

6.2.1. *General properties.* Given standard Brownian motion in any number of dimensions, we can use the process of h-transformation to construct a Brownian motion satisfying certain conditions. For any non-negative harmonic function h, the measure for the transformed Brownian motion to go to λ at time t after starting at η is $h(\lambda)$ times the measure for untransformed Brownian motion with the same starting point, renormalized by an appropriate constant so that the total measure on all paths is 1.

In particular, suppose that we have a connected region D and a harmonic function h which is zero on the boundary of the region and positive on the interior. Then the h-transformed Brownian motion will be conditioned to stay in the interior of the region.

The density function for this transformed motion will be h times the density function for the untransformed motion on the same region, normalized appropriately. That is, its value at a point λ and time t will be $h(\lambda)$ times the value $b_t(\eta, \lambda)$ which gives the measure for unconstrained motion to go from η to λ in time t while staying within the region, since the transformation puts a new measure on the same set of paths.

The same technique can also be applied to a discrete random walk with a set S of steps, provided that h is harmonic in the discrete lattice; that is, we need

$$\sum_{s \in S} h(x+s)/|S| = h(x). \tag{36}$$

The transformed discrete walk now has probability $h(x+s)/(h(x)|S|)$ instead of $1/|S|$ of going from x to $x+s$ in a given step. Thus, given a starting point η, the probability of going to λ in a given number of steps of the transformed random walk is $h(\lambda)/h(\eta)$ times the probability for the untransformed walk to go to λ while staying within the region in which h is positive.

In order to make it impossible for the walk to leave the region D, we need h to be zero on all points which can be reached from the interior of D in a single step. Thus, only if the discrete walk is reflectable (as defined above) is it sufficient for h to be zero on the continuous boundary of D.

6.2.2. *Finding the function.* The properties of conditioning, as well as our asymptotics, suggest that δ itself should be our h. It can be checked algebraically that h is harmonic for each group; however, it can also be proved naturally.

Theorem 7. *For any finite Coxeter group W, the product of all the positive roots is a harmonic function, both for the continuous Laplacian and for the discrete Laplacian*

$$L_S h(\vec{x}) = \frac{1}{|S|} \sum_{\vec{s} \in S} [h(\vec{x}+\vec{s}) - h(\vec{x})],$$

for any set S which is symmetric under the group W.

Proof. This result in the continuous case is due to [2]; the discrete argument is a simple generalization which is mentioned in [5].

By the properties of root systems, a reflection in any simple root changes the sign of only that root, while permuting the other positive roots. Thus the product h of all m roots is antisymmetric in all the simple roots, and by applying repeated reflections, we see that it is antisymmetric in every root. It is also of degree m.

The continuous Laplacian is spherically symmetric, and thus symmetric under W. The discrete Laplacian is symmetric under W because the set S is. Applying the continuous Laplacian to a polynomial decreases the degree by 2, while applying the discrete Laplacian decreases the degree by at least 1 since $h(\vec{x}+\vec{s})-h(\vec{x})$ is of lower degree than h. Thus the application of the Laplacian to h gives a polynomial which is of degree less than m which is still antisymmetric in W.

Now, any polynomial which is antisymmetric in W must be zero on every one of the hyperplanes of reflection. If it is not identically zero, it must have all m roots as factors, so it must be of degree at least m. Thus the Laplacian must annihilate our polynomial h, so h is harmonic for either the discrete or continuous walk.

It follows that the transformation by this h gives the same process as the process generated by Lie theory. We have already computed the asymptotic density that Brownian motion started at a fixed η will remain in the Weyl chamber for time t and be at λ at that time. Transforming by h has the effect of multiplying the measure of all paths from a fixed η to an arbitrary λ which stay within the chamber by a factor of $h(\lambda)$ (and a normalizing constant). Thus the density function for the transformed Brownian motion which starts at η to be at λ at time t is proportional to $h(\lambda)b_t(\eta,\lambda)$, and this converges to a constant multiple of $h(\lambda)^2$ as t becomes large. This is the same factor $\delta'\bar{\delta}'$ which we obtained from the Weyl Integration Formula.

6.3. **The infinitesimal generator and physical models.** We have constructed the conditioned process as a transformation of Brownian motion. We can also construct it as a diffusion with its known infinitesimal generator. In this form, both the multidimensional models and the models of n independent particles lead to natural physical models.

We will use the notation of [18] for infinitesimal generators of diffusions. The drift vector μ is defined by

$$\mu_i(\vec{x},t)=\lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\{E(X_i(t+\Delta t))-X_i(t)|\vec{X}(t)=\vec{x}\},$$

and the infinitesimal variance matrix is defined by

$$\sigma_{i,j}(\vec{x},t)=\lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\{[E(X_i(t+\Delta t))-X_i(t)][E(X_j(t+\Delta t))-X_j(t)]|\vec{X}(t)=\vec{x}\},$$

We will omit the variable t in the infinitesimal generators of our diffusions, because they are independent of time.

It can easily be checked that Brownian motion transformed by a harmonic (and thus necessarily C^2) function h has infinitesimal drift $\mu = \nabla h/h$ and infinitesimal variance $\sigma_{ij} = \delta_{ij}$.

Note that $\nabla h/h = \nabla(\log h)$. This makes the computation easy, because our h is the product $\prod_{\alpha\in\Delta}\alpha$ of all the roots, viewed as linear functions of the x_i. If we write $\alpha(\vec{x})$ as the dot-product $(\alpha, \vec{x})$. then we have

$$\frac{\nabla h}{h} = \sum_{\alpha\in\Delta} \nabla(\alpha, \vec{x}) = \sum_{\alpha\in\Delta} \alpha \frac{1}{(\alpha, \vec{x})}. \tag{37}$$

Since $(\alpha, \vec{x})/|\alpha|$ is the distance from $\vec{x}$ to the hyperplane orthogonal to α, while $\alpha/|\alpha|$ is the unit vector in the direction of α, this term in the drift is the inverse of the distance between $\vec{x}$ and the hyperplane, directed away from the hyperplane. We thus have the following physical model.

Theorem 8. *For any finite Coxeter group, Brownian motion conditioned to stay within a chamber is equivalent to the motion of a particle in a diffusion with constant infinitesimal velocity, and a repulsive force from every hyperplane of reflection (not merely the walls of the chamber) inversely proportional to the distance from that hyperplane.*

We can also look at the n coordinates as individual motions in one dimension. If the root α contains cx_i, we get a term $c/(\alpha, \vec{x})$ in the sum.

In particular, for A_n, the drift μ_i is $\sum_{i\neq j} 1/(x_i - x_j)$. Thus each particle is subject to a repelling force from every other particle (not merely its neighbors), inversely proportional to the distance between them. This proves Theorem 4; this result is originally due to Dyson [10].

The model for B_n is not as natural as a model of particles. The drift μ_i is

$$\frac{1}{x_i} + \sum_{i\neq j}\left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\right)$$

That is, each particle is repelled by every other particle, and by the wall at 0 (the $1/x_i$ term), but also by the mirror image of every other particle (the $1/(x_i+x_j)$ term), as if the wall at $x = 0$ was also a mirror reflecting all forces. For D_n, we have only the terms of $1/(x_i - x_j)$ and $1/(x_i + x_j)$; this means that the mirror reflects forces but is itself permeable to particles.

These models are more natural as models of the eigenvalues. For D_n, a matrix in the Lie algebra $\mathfrak{so}_{2n}$ has eigenvalues $\pm i\lambda_j$; this model thus says that the eigenvalues in different pairs $\pm i\lambda_j$ and $\pm i\lambda_k$ for $j \neq l$ repel one another other, although the pair $\pm i\lambda_j$ do not repel each other. For B_n, a matrix in the Lie algebra $\mathfrak{so}_{2n+1}$ has eigenvalues $\pm i\lambda_j$ and 0; this model thus says that the unconnected eigenvalues $i\lambda_j$ and $i\lambda_k$ for $j \neq k$ repel each other, and each eigenvalue is also repelled by the fixed eigenvalue at 0.

7. Open problems

The discussion of Brownian motion on a Lie algebra is valid for the exceptional Lie groups as well, but the techniques for computing the specific asymptotics do not appear to work. For a chamber of a general Coxeter group, the Lie algebra technique is not meaningful. In either case, is it possible to get the same type of asymptotics, with constant terms in particular?

We have shown that the same harmonic functions which we used to transform Brownian motion can be used to transform discrete random walks. Is it possible to use these results to compute asymptotics for the discrete walks, including the constants on leading terms?

Brownian motion can be defined on a general manifold [21]. This allows us to apply the argument of Theorem 5 whenever we have a suitable chamber. As before, the Brownian motion must be symmetric under the reflections in any wall of the chamber, and the reflections in the walls must generate a discrete group which partitions the manifold into chambers. For example, since 2-dimensional Brownian motion is conformally invariant, we can define a Brownian motion on the modular surface [21]. Our chamber can be the standard fundamental domain; if we use the standard map of the modular surface to the upper half-plane, our chamber is bounded by $x > -1/2$, $x < 1/2$, and $x^2 + y^2 > 1$. Can the resulting formulas be used to compute properties of this Brownian motion, such as asymptotic survival probabilities, hitting times, and physical models?

Acknowledgements. Sections 2-4 are part of my thesis, "Walks and Representation Theory," which was written at Harvard University. I would like to thank Peter Magyar for his collaboration in work on the discrete problem; Itai Benjamini and Gil Kalai, for suggestions related to the asymptotic results; Arun Ram, for a suggestion which simplified the computation of the determinant formulas; Dan Stroock, for suggesting that I study reflecting boundary conditions; the referee of an earlier version of this paper, for suggesting the Lie theory argument and connection to random matrices; Wendelin Werner, for suggesting some important references; my thesis advisors, Persi Diaconis and Richard Stanley, for their guidance in my thesis research; and Yuval Peres, for his guidance in the research beyond my thesis.

References

[1] Adams, J. F., *Lectures on Lie Groups.* University of Chicago Press, 1969.

[2] Bérard, P., and Besson, G., Spectres et groupes cristallographiques II: domaines sphériques. *Ann. Inst. Fourier* **30**(1980):3, 237–248.

[3] Biane, P., Minuscule weights and random walks on lattices. *Quant. Prob. Rel. Topics* **7**(1992) 51–65.

[4] Biane, P., Quelques proprietés du mouvement Brownien dans un cone. *Stoch. Proc. Appl.* **53**(1994) 233–240.

[5] Biane, P., Permutation model for semi-circular systems and quantum random walks. *Pacific J. Math.* **171**(1995) 373–387.
[6] Bourbaki, N., *Groupes et Algebres de Lie,* Chapters 4,5,6. Hermann, Paris, 1968.
[7] Burkholder, D. L., Exit times of Brownian motion, harmonic majorization, and Hardy spaces. *Adv. Math.* **26**(1977) 182–205.
[8] DeBlassie, R. D., Exit times from cones in $\mathbb{R}^n$ of Brownian motion. *Prob. Th. Rel. Fields* **74**(1987) 1–29.
[9] Durett, R. *Brownian Motion and Martingales in Analysis.* Wadsworth, Belmont, California, 1984.
[10] Dyson, F. J.. A Brownian-motion model for the eigenvalues of a random matrix, *J. Math. Phys.***3**(1962), 1191–1198.
[11] Freedman, D., *Brownian Motion and Diffusion.* Holden-Day, San Francisco; second edition published by Springer-Verlag, New York, 1971)
[12] Gessel, I. M., and Zeilberger, D. Random walk in a Weyl chamber. *Proc. Amer. Math. Soc.* **115**(1992), 27–31.
[13] Grabiner, D. J., and Magyar, P. Random walks in Weyl chambers and the decomposition of tensor powers. *J. Alg. Combin.* **2**(1993) 239–260.
[14] Harrison, J. M. *Brownian Motion and Stochastic Flow Systems.* John Wiley and Sons, New York, 1985.
[15] Humphreys, J. E. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag, New York, 1972.
[16] Hobson, D. and Werner, W. Non-colliding Brownian motion on the circle. *Bull. London. Math. Soc.***28**(1996), 643–650.
[17] Karlin, S. P., and MacGregor, G. Coincidence probabilities. *Pacific. J. Math.* **9**(1959),1141–1164.
[18] Karlin, S. P., and Taylor, H. M., *A Second Course in Stochastic Processes.* Academic Press, New York, 1981.
[19] Macdonald, I. G. *Symmetric Functions and Hall Polynomials.* Oxford University Press, 1979.
[20] Macdonald, I. G. Some conjectures for root systems. *SIAM J. Math. Anal.* **13**(1982), 988-1007.
[21] McKean, H. P. *Stochastic Integrals.* Academic Press, New York, 1969.
[22] Mehta, M. L. *Random Matrices: Second Edition.* Academic Press, New York, 1991.
[23] O'Connell, N. and Unwin, A. Collision times and exit times from cones: A duality. *Stoch. Proc. Appl.* **43**(1992), 291–301.
[24] Pitman, J. One-dimensional Brownian motion and the three-dimensional Bessel process. *Ann. Appl. Prob.* **7**(1975), 511–526.
[25] Proctor, R. A. Reflection and algorithm proofs of some more Lie group dual pair identities. *J. Combin. Th. A* **62**(1993) 107–127.
[26] Selberg, A. Bemerkinger om et multipelt integral. *Norsk Mathematisk Tidsskrift* **26**(1944) 71-78.
[27] Watanabe, T. and Mohanty, S. G., On an inclusion-exclusion formula based on the reflection principle. *Discrete Math* **64**(1987), 281–288.
[28] Williams, D. (1974). Path decomposition and continuity of local time for one-dimensional diffusions. *Proc. London Math. Soc.* **28**(1974), 738–768.
[29] Zeilberger, D. Andre's reflection proof generalized to the many-candidate ballot problem. *Discrete Math* **44**(1983) 325–326.

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