MANIFOLDS NOT CONTAINING GOMPF NUCLEI

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Abstract. In this note we show that there are 4-manifolds not containing Gompf nucleus N_2; in this way we answer Problem 4.98 of Kirby's problem list (see [K]) in the negative.

1. Introduction

This note is devoted to answer a question in Kirby's problem list [K] asking whether every simply connected smooth 4-manifold with $b^+ \geq 3$ contains a Gompf nucleus N_2 (Problem 4.98 in [K]). We prove that by doing logarithmic transformations on three linearly independent tori in the *K3*-surface we get a 4-manifold not contaning N_2. To make our statements more precise we need a few definitions.

The hypersurface $X = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid \sum_{i=0}^3 z_i^4 = 0\} \subset \mathbb{CP}^3$ is a simply connected, smooth 4-manifold with $c_1(X) = 0$, hence it is a *K3-surface*. It is known that all simply connected complex surfaces with vanishing first Chern class are diffeomorphic, consequently from the differential topological point of view X is *the K3*-surface. The complex surface X admits a holomorphic fibration $\pi \colon X \to \mathbb{CP}^1$ such that the generic fiber is a smooth elliptic curve — a 2-dimensional torus. Such fibrations are called *elliptic fibrations*. It can be assumed that π has a singular fiber homeomorphic to the 2-dimensional sphere S^2 — such fibers are called *cusp fibers*. The fibration also has a section $\sigma \colon \mathbb{CP}^1 \to X$, the image of which is a sphere $S \subset X$ with square $[S]^2 = -2$. The *Gompf nucleus* N_2 is by definition the tubular neighborhood of the union of a cusp fiber and a section in X. The manifold N_2 admits a handle decomposition with one 0-handle and two 2-handles, where these two 2-handles are attached to $S^3 = \partial(0 - \mathit{handle})$ according to the Kirby diagram shown in Figure 1.

It can be shown that the *K3*-surface X contains three disjoint copies of N_2. If a 4-manifold M^4 contains a 2-dimensional torus T with square 0, then we can perform a *logarithmic transformation* on T: deleting the tubular neighborhood of T (which is diffeomorphic to $D^2 \times T^2$) and regluing it via a diffeomorphism $\varphi \colon \partial(M \setminus D^2 \times T^2) \to \partial(D^2 \times T^2)$ we get a new manifold M_φ. It turnes out that if T is the fiber in a Gompf nucleus $N_2 \subset M$, then the diffeomorphism type of M_φ will depend only on one nonnegative number p associated to φ. This number is called the *multiplicity* of the logarithmic transformation. For more about elliptic surfaces, nuclei and logarithmic transformations see [G], [FS1] and [GS].

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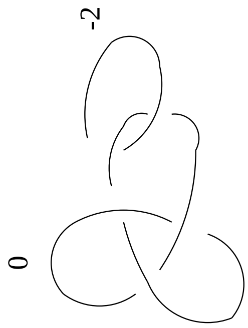

FIGURE 1. Kirby picture for N_2

Now perform logarithmic transformations of multiplicity 2 on the three tori contained by the three disjoint nuclei in the *K3*-surface X. The resulting manifold will be denoted by $X_{2,2,2}$.

Theorem 1.1. *The 4-manifold $X_{2,2,2}$ does not contain a Gompf nucleus N_2.*

Remark 1.2. One of the most interesting questions in 4-manifold theory is whether all 4-manifolds are of simple type or not. (For the definition of simple type see Section 2.) It is known that if M^4 contains a homologically essential torus with square 0, then M is of simple type. There is no example of a 4-manifold with $b^+ \geq 3$ and not containing a torus with square 0. One can ask whether every 4-manifold contains a cusp neeighborhood (a tubular neighborhood of a cusp fiber) — or even a Gompf nucleus N_2. The above theorem shows an example of a manifold which contains no N_2; $X_{2,2,2}$ is still of simple type, however.

One can modify the question by trying to find more general nuclei N_n in 4-manifolds. The manifold N_n is described by the Kirby diagram shown in Figure 2 — it can be defined alternatively as the tubular neighborhhod of the union of a cups fiber and a section in a simply connected elliptic surface (admitting a section) with Euler characteristic $12n$ (see [G]).

Theorem 1.3. *For every n there is a manifold Y_n such that Y_n does not contain N_n.*

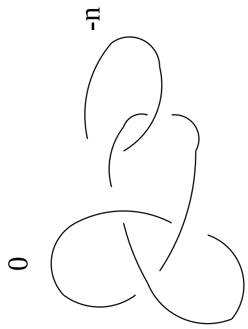

FIGURE 2. Kirby picture for N_n

A related question would be to find a 4-manifold M with the property that M does not contain $N_2(p,q)$; here $N_2(p,q)$ denotes the 4-manifold with boundary we get by performing two logarithmic transformations (of multiplicity p and q) along the fiber of the Gompf nucleus N_2. Using a recent construction of Fintushel and Stern [FS3] such M can be found, see [SSz].

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2. Basic classes

In studying differential topological properties of smooth 4-manifolds, *Seiberg-Witten basic classes* turn out to be very powerful tools. For the definition of these objects see [A], [M] or [GS], here we restrict ourselves to a very short outline.

Assume that M^4 is a smooth, oriented, simply connected, closed 4-manifold with $b_M^+ \geq 3$. A cohomology class $K \in H^2(M;\mathbb{Z})$ with $K \equiv w_2(M)$ $(mod\ 2)$ uniquely determines a spin^c structure on M, and for such a structure a certain pair of partial differential equations (the so-called Seiberg-Witten equations) can be described — involving a choice of metric on M, a coupled Dirac operator and a perturbation 2-form. By a delicate "counting argument" of the solutions of these equations a number $SW_M(K)$ can be associated to the cohomology class K. It turnes out that this number (up to sign) is a smooth invariant of the manifold M, more precisely

Theorem 2.1. *If $f\colon M' \to M$ is an orientation preserving diffeomorphism then $SW_M(K) = \pm SW_{M'}(f^*K)$. Moreover, for a fixed 4-manifold M there are only finitely many classes K with $SW_M(K) \neq 0$, and $SW_M(-K) = \pm SW_M(K)$.* □

Definition 2.2. The cohomology class $K \in H^2(M;\mathbb{Z})$ is called a *Seiberg-Witten basic class* if $SW_M(K) \neq 0$. The 4-manifold M is of simple type if $SW_M(K) \neq 0$ implies that $K^2 = 3\sigma(M) + \chi(M)$; here $\sigma(M)$ and $\chi(M)$ stand for the signature and Euler characteristic of M.

The most important relation between the smooth topology of a 4-manifold M and its basic classes $\{K_i \mid i = 1,...,n\}$ is shown by the *generalized adjunction formula*:

Theorem 2.3. (Kronheimer-Mrowka) *If $\Sigma^2 \subset M^4$ is a smooth, connected 2-dimensional surface of genus $g(\Sigma)$, $[\Sigma] \neq 0$ and $[\Sigma]^2 \geq 0$, then for every basic class K we have*

$$2g(\Sigma) - 2 \geq [\Sigma]^2 + |K([\Sigma])|.$$

□

3. Proofs of Theorems 1.1 and 1.3

Assume that T_1, T_2 and T_3 are three tori in the *K3*-surface lying in three disjoint Gompf nuclei. Perform logarithmic transformations of multiplicity 2 on each T_i. The basic classes of the resulting manifold $X_{2,2,2}$ are determined by Fintushel and Stern [FS2].

Proposition 3.1. *The basic classes of $X_{2,2,2}$ are the Poincaré duals of the homology classes $\pm[\frac{T_1}{2}] \pm [\frac{T_2}{2}] \pm [\frac{T_3}{2}]$.* □

Proof of Theorem 1.1: Assume that $N_2 \subset X_{2,2,2}$. The homology class of the fiber and the section in N_2 will be denoted by f and s respectively; note that f and $g = f + s$ have square 0 and can be represented by tori. Consequently (by the generalized adjunction formula) a basic class K of $X_{2,2,2}$ evaluates trivially on f and g. Now $f \cdot ([\frac{T_i}{2}] + [\frac{T_j}{2}] + [\frac{T_k}{2}]) = 0$ and $f \cdot (-[\frac{T_i}{2}] + [\frac{T_j}{2}] + [\frac{T_k}{2}])0$ ($\{i, j, k\} = \{1, 2, 3\}$) implies that $f \cdot [\frac{T_i}{2}] = 0$, similarly $g \cdot [\frac{T_i}{2}] = 0$ for $i = 1, 2, 3$. Since the complement of the three disjoint nuclei in X have negative definite intersection form, the above equalities imply that $f = \sum_{i=1}^{3} \alpha_i [T_i]$ and similarly $g = \sum_{i=1}^{3} \beta_j [T_j]$. These two latter equations, however, give a contradiction, since $f \cdot g = 1$ but $(\sum_{i=1}^{3} \alpha_i [T_i]) \cdot (\sum_{i=1}^{3} \beta_j [T_j]) = 0$. Consequently N_2 does not embed in $X_{2,2,2}$. $\square$

Proof of Theorem 1.3: Perform logarithmic transsformations of multiplicity $2n$ on T_1, T_2 and T_3 as above; the resulting 4-manifold $X_{2n,2n,2n}$ will be denoted by Y_n. The basic classes of Y_n are determined in [FS2]: these are the Poincaré duals of the homology classes $\alpha[\frac{T_1}{2n}] + \beta[\frac{T_2}{2n}] + \gamma[\frac{T_3}{2n}]$ where $\alpha, \beta, \gamma \in \{\pm(2j-1) \mid j = 1, ..., n\}$. Assume now that $N_n \subset Y_n$. As before, f and s will denote the homology classes of the fiber and the section in N_n respectively. Note that $f^2 = 0$ and $s^2 = -n$. The class f can be represented by a torus, while $g = s + nf$ can be represented by a surface of genus n; moreover $g^2 = n$. The same argument as in the proof of Theorem 1.1 gives that $f = \sum_{i=1}^{3} \alpha_i [T_i]$. To show that every basic class of Y_n evaluates trivially on g involves a little trick: take the basic classes $\pm[\frac{T_1}{2}] \pm [\frac{T_2}{2}] \pm [\frac{T_3}{2}]$ and evaluate them on g. If one of them, say K, evaluates nontrivially on g, then for $L = (2n-1)K$ (which is also a basic class) we have $|L(g)| \geq 2n - 1$, but then L and g would violate the generalized adjunction formula. Consequently $(\pm[\frac{T_1}{2}] \pm [\frac{T_2}{2}] \pm [\frac{T_3}{2}]) \cdot g = 0$ implying that $g \cdot [T_i] = 0$ ($i = 1, 2, 3$). These latter equations show that $g = \sum_{j=1}^{3} \beta_j [T_j]$, and we have the same contradiction as before. $\square$

Remark 3.2. Note that the proof given above also shows that no N_k with $k \leq n$ embeds in $Y_n = X_{2n,2n,2n}$.

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