GAUGE SYMMETRY AND INTEGRABLE MODELS

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Abstract. We establish the isomorphism between a nonlinear σ-model and the abelian gauge theory on an arbitrary curved background, which allows us to derive integrable models and the corresponding Lax representations from gauge theoretical point of view. In our approach the spectral parameter is related to the global degree of freedom associated with the conformal or Galileo transformations of the spacetime. The Bäcklund transformations are derived from Chern-Simons theory where the spectral parameter is defined in terms of the additional compactified space dimension coordinate.

1. Introduction

The aim of this paper is to present in a clear form the systematic method outlined in Ref. [17] of generating Lax-form and the spectral parameter of integrable models through the gauge theory. For the gauge group $SU(2)$, the method leads both the non-relativistic and relativistic integrable equations in 1-space and 1-time dimension, whose fundamental examples are respectively the celebrated nonlinear Schrödinger (NLS) equation and the Liouville equation. In the theory of classical integrable systems, the zero-curvature formulation has been a crucial step to elucidate the hidden symmetry of the equation [19] [6] [20]. A large class of solvable systems originated in the gauge theory. Of course such systems represent only a restricted, but nevertheless rich, class of integrable equations of physical interests. It has been well-known that the relation of gauge equivalence exists between various models like the continuum Heisenberg spin chain and NLS [8]. Historically, these like relations were established on the level of Lax or zero-curvature representations [22] [23] [18], but without admitting an extention to the nontrivial backgroud of higher dimensions. From another site the $\mathbb{CP}^n$ σ-model representation [1] allows one to associate the model with a gauge field theory using the auxiliary fields. In this paper we shall focus on some well-known models of which the Lax forms already exist in literature, and establish the equivalent relations among them through the $SU(2)$ or $SU(1,1)$-gauge theory. As it is known for integrable models, the Lax representation is the fundamental procedure for the solvablity of the equation. However the achievement of Lax forms with a spectral parameter usually relies on the experience of expertise. It is desirable to have some additional mathematical or physical grounds for the understanding of the qualitative nature of existing results. In the gauge theory approach of integrable systems, we are able to associate a spectral parameter with the global degree of freedom of the equations connected to some special automorphisms of spacetime, e.g. the Galileo transformations for NLS. Furthermore the Bäcklund transformations can be derived from the higher dimensional Chern-Simons theory with the interpretation of the extract space dimension coordinate as the spectral parameter.

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Our paper is organized as follows: In Section 2 we review the general properties of connections and homogenous spaces, which will be relevent to the discussion of this paper. In what follows, we present a systematic approach to integrable models related to the gauge theory of $SU(2)$ or $SU(1,1)$. In Section 3 we establish the general correspodence between S^2 σ-models and abelian gauge theories with background fields derived from the $SU(2)$-flat connections. Models with certain special constraints will be treated in the next three sections. In Section 4 we discuss the self-dual equation of a Riemann surface and its corresponding abelian gauge system. In particular for the complex plane, it gives rise the solutions of Liouville equation. In Section 5 the (conformal) sinh-Gordon equation and and its various equivalent forms will be derived. We shall relate the spectral parameter of the sinh-Gordon equation with conformal symmetries of the equation. In Section 6 we give a gauge theory description of NLS equation, and establish its equivalent relation with the Heisenberg model as the corresponding σ-model. The Zakharov-Shabat representation of NLS equation will be derived from the Galileo invariance of the equation. In Section 7 we present an interpretation of the Bäcklund transformation of NLS through the three dimensional Chern-Simons theory by the technique of dimensional reduction, a treatment in the framework of BF-type gauge theory in [16] [17]. We reinterpret the spectral parameter of NLS equation as the extract compactified space dimension coordinate. In Section 8 we present the correspodence bewteen hyperbolic non-compact σ-models and the abelian gauge models derived from $SU(1,1)$-flat connections, in particular the Liouville equation, and its relation with the cylindrical symmetry solutions of the 4-dimensional self-dual Yang-Mills equation obtained by E. Witten in [24]. In the conclusion we discuss some physical ideas to explain our results.

Notations.
We shall use the following notations unless we mention specifically. Let X be a (differentiable) manifold, and $\mathbf{L}$ be a complex Hermitian line bundle over X. We use x^μ as the local coordinates of X. By a local open subset of X, we shall always mean a contractible open set U endowed with a coordinate system. Denote:

$\Omega^k(X) =$ the vector space of real k-forms on X.

$\Omega^k(X)_{\mathbb{C}} = \Omega^k(X) \otimes_{\mathbb{R}} \mathbb{C}$.

$\Omega^k(X, \mathbf{L})$= the vector space of $\mathbf{L}$-valued k-forms on X.

For $s \in \Omega^k(X, \mathbf{L})$, we shall denote the complex conjugate of s by $s^\dagger \in \Omega^k(X, \mathbf{L}^*)$. For manifolds X, Y, with base elements $x_0 \in X, y_0 \in Y$, we denote

$\mathcal{C}^\infty((X, x_0), (Y, y_0))$ = the set of all $\mathcal{C}^\infty$-maps φ from X to Y with $\varphi(x_0) = y_0$. For simplicity, we shall call (X, x_0) a marked manifold X (with the base element x_0), and φ a marked differentiable map from X to Y. In what follows, we systematically write $\mathcal{C}^\infty(X, Y)$ instead of $\mathcal{C}^\infty((X, x_0), (Y, y_0))$ if no confusion could arise. For a Lie group the base element will always be the identity element.

$\mathrm{Diff}(X)$ ($= \mathrm{Diff}(X, x_0)$) = the group of automorphisms of X preserving the base element x_0.

2. Preliminaries : Connection and Curvature

A systematic review of connection and curvature cannot be presented in a regular paper. Here we simply recall some basic concepts and a few definitions, in order to fix the notations used in this work. We follow in this respect Ref. [4] [9]. Let G be a semi-simple Lie group with the Lie algebra $\mathbf{g}$, and $\mathrm{Aut}(\mathbf{g})$ be the identity component of the group of Lie-automorphisms of $\mathbf{g}$. The negative Killing form defines an inner product structure $< *, * >$ of $\mathbf{g}$, invariant under the adjoint action of G. Let $\mathrm{Der}(\mathbf{g})$ be the algebra of derivations of $\mathbf{g}$. It is known that

$$\mathbf{g} \simeq \mathrm{ad}(\mathbf{g}) = \mathrm{Der}(\mathbf{g}) \ ,$$

and one has the exact sequence:

$$1 \longrightarrow \mathrm{Z}(G) \longrightarrow G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathbf{g}) \longrightarrow 1 \tag{1}$$

where $\mathrm{Z}(G)$ is the center of G. Then we have

$$\begin{array}{l} \mathrm{Aut}(\mathbf{g}) \subset SO(\mathbf{g}) := \{\rho \in SL(\mathbf{g}) \mid \ < \rho(x), \rho(y) >=< x, y >, \ \forall x, y \in \mathbf{g}\} \ , \\ \mathrm{Der}(\mathbf{g}) \subset so(\mathbf{g}) = \text{Lie algebra of } SO(\mathbf{g}) \ , \end{array}$$

and we shall always consider $\mathrm{Aut}(\mathbf{g})$ as a closed subgroup of $SO(\mathbf{g})$ in what follows. Let $(e^1, \ldots, e^n)$ be an orthonormal basis of $\mathbf{g}$ with respect to $< *, * >$. The Lie algebra $\mathbf{g}$ is defined by

$$[e^i, e^j] = e^k c_k^{ij} \ , \quad 1 \le i, j, k \le n \ ,$$

where the symbol c_k^{ij} is a structure constant of $\mathbf{g}$ and is anti-symmetric in its indices. Let $(e_1, \ldots, e_n)$ be the basis of $\mathbf{g}^*$ dual to $(e^1, \ldots, e^n)$. As $\mathbf{g}$ is isomorphic to $\mathbf{g}^*$ via $< *, * >$, equivariant with respect to G-adjoint actions, we shall make the identification: $\mathbf{g} = \mathbf{g}^*, e^j = e_j$. Using the basis e^j, one regards $\mathrm{Aut}(\mathbf{g})$ as a subgroup of $SO(n_+, n_-)$ by the following relation, where (n_+, n_-) is the signature of $< *, * >$,

(2)

$$\mathrm{Ad}(g) \leftrightarrow \quad R(g) = (R(g)_j^k), \quad \text{with} \quad \mathrm{Ad}(g)(e^1, \ldots, e^n) = (e^1, \ldots, e^n)R(g) \ , \quad g \in G \quad .$$

Now it is easy to see the following lemma.

Lemma 1. For a linear map l of $\mathbf{g}$,

$$l : (e^1, \ldots, e^n) \mapsto (e^1, \ldots, e^n)M, \quad M = (M_i^j) \ ,$$

the following conditions are equivalent:

(I) $l \in \mathrm{Der}(\mathbf{g})$.

(II) $M_k^s c_s^{ij} = M_s^i c_k^{sj} - M_s^j c_k^{si}$ for all i, j, k.

(II) There exist real numbers, $\alpha_1, \ldots, \alpha_n$, such that $M_k^i = c_k^{ji} \alpha_j$ for all i, k. □

Let P be a principal G-bundle over a manifold X,

$$G \longrightarrow P \longrightarrow X \ .$$

The bundle P induces a principal $\mathrm{Aut}(\mathbf{g})$-bundle P' and a vector bundle $\mathrm{ad}(P)$ over X associated to the adjoint representation of G on $\mathbf{g}$:

$$P' = P \times_G \mathrm{Aut}(\mathbf{g}) \ , \quad \mathrm{ad}(P) = P \times_G \mathbf{g} \ .$$

Denote $\Omega^*(X, \mathrm{ad}(P))$ the set of differential forms with values in $\mathrm{ad}(P)$. The fibers of $\mathrm{ad}(P)$ are endowed with a natural Lie algebra structure from the Lie algebra $\mathbf{g}$. The $\mathrm{Aut}(\mathbf{g})$-bundle P' can be regarded as the bundle of frames $(v^1, \ldots, v^n)$ of $\mathrm{ad}(P)$ with the fixed Lie structure constants. Let J be a connection on P over a manifold X. It gives rise to a $\mathrm{Aut}(\mathbf{g})$-conncetion J_{ad} on the bundle P', and a covariant differental D_{ad} on $\Omega^*(X, \mathrm{ad}(P))$,

$$D_{\mathrm{ad}}(\phi) = d(\phi) + [J, \phi] \ , \quad \phi \in \Omega^*(X, \mathrm{ad}(P)) \ .$$

For a local open subset U of X, the local expressions of $J, J_{\rm ad}$ and $D_{\rm ad}$ are given by

(3)
$$\begin{array}{rl} J = & e^j\omega_j = J_\mu dx^\mu \in \mathbf{g}\otimes\Omega^1(U), \qquad \omega_j = J_{j,\mu}dx^\mu \in \Omega^1(U) \ , \ J_\mu = e^j J_{j,\mu} \in \Omega^0(U)\otimes\mathbf{g} \ , \\ J_{\rm ad} = & e^k e_i(\omega_{\rm ad})^i_k \in {\rm Der}(\mathbf{g})\otimes\Omega^1(U) \ , \quad (\omega_{\rm ad})^i_k = (J_{\rm ad})^i_{k,\mu}dx^\mu \ , \ (J_{\rm ad})^i_{k,\mu} := c^{ji}_k J_{j,\mu} \ , \end{array}$$

and

$$\begin{array}{l} D_{\rm ad}(e^i f_i) = e^i d(f_i) + (D_{\rm ad}e^i)f_i \ , \qquad f_j \in \Omega^*(X) \ , \\ D_{\rm ad}e^i = [e^j\omega_j, e^i] = e^k(J_{\rm ad})^i_{k,\mu}dx^\mu \in \mathbf{g}\otimes\Omega^1(X). \end{array}$$

The curvature $F(A)$ is a 2-form with values in ${\rm ad}(P)$:

$$F(J) = dJ + \frac{1}{2}[J,J] \in \Omega^2(X, {\rm ad}(P)) \ .$$

Recall that two connections

$$J = e^j\omega_j \ , \quad \tilde{J} = e^j\tilde{\omega}_j \ ,$$

are gauge equivalent if there exists an element $h \in \mathcal{C}^\infty(X,G)$ such that

$$\tilde{J} = h^{-1}dh + {\rm Ad}(h)^{-1}(e^j)\omega_j \ , \quad \tilde{\omega}_j = \tilde{J}_{j,\mu}dx^\mu \ , \quad \tilde{J}_{j,\mu} = J(h)_{j,\mu} + R(h^{-1})^k_j J_{k,\mu} \ .$$

This implies the corresponding ${\rm Aut}(\mathbf{g})$-connections, $(\tilde{J}_{\rm ad})^i_{k,\mu}dx^\mu$, $(J_{\rm ad})^i_{k,\mu}dx^\mu$, are also equivalent. It is known that the diffeomorphism group ${\rm Diff}(X)$ acts on the space of G-connectons in a canonical manner. In fact, for $\varphi \in {\rm Diff}(X)$ and a connection J on a principal G-bundle, φ^*J is a conncection on the pull-back bundle φ^*P. For a function $g \in \mathcal{C}^\infty(U,G)$, it defines a current by

$$J(g)_\mu dx^\mu := g^{-1}dg \in \mathbf{g}\otimes\Lambda^1 \ , \quad J(g)_\mu = e^j J(g)_{j,\mu} \ .$$

A connection J is flat if $F(J) = 0$. Its local expression is given by

$$\begin{array}{rl} & J = e^j\omega_j = g^{-1}dg \ , \qquad \text{for } \ g \in \mathcal{C}^\infty(U,G) \ , \ \text{ i.e. } J_\mu = J(g)_\mu \ , \\ \Longleftrightarrow & d\omega_i + \frac{1}{2}c^{jk}_i\omega_j\wedge\omega_k = 0 \ , \ \text{ i.e. } \ \partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0 \ . \end{array}$$

For two gauge equivalent connections, $J, \tilde{J}$, the flat condition of one implies the other. In fact, one has

$$J = g^{-1}dg \quad \Longleftrightarrow \quad \tilde{J} = (gh)^{-1}d(gh) \ .$$

As the G-bundle P is a finite cover over the ${\rm Aut}(\mathbf{g})$-bundle P' with the covering group $Z(G)$,

$$\begin{array}{ccc} P & \longrightarrow & P/{\rm Z}(G) = P' \\ \downarrow & & \downarrow \\ X & = & X \ , \end{array}$$

the flat conditions of connections, J , $J_{\rm ad}$, are equivalent. For a simply-connected manifold X with a base element x_0, there is an one-to-one correspondence of the following data, equivariant under the action of ${\rm Diff}(X)$:

(I) A flat G-connection on P.
(II) A flat ${\rm Aut}(\mathbf{g})$-connection on P'.
(III) $g \in \mathcal{C}^\infty((X,x_0),(G,1))$.
(IV) $m \in \mathcal{C}^\infty((X,x_0),({\rm Aut}(\mathbf{g}),1))$.

Here an element $\varphi \in {\rm Diff}(X)$ acts on g, m by

$$\begin{array}{ll} (\varphi,g) \mapsto \varphi^*g \ , & \varphi^*g(x) := (g(\varphi(x_0))^{-1}g(\varphi(x)) \ , \ \text{for } x \in X \ , \\ (\varphi,m) \mapsto \varphi^*m \ , & \varphi^*m(x) := (m(\varphi(x_0))^{-1}m(\varphi(x)) \ , \ \text{for } x \in X \ . \end{array}$$

The functions g, m, in (III), (IV), are the gauge functions of flat connections of (I), (II), respectively. Note that under this situation, the bundle P is necessarily a trivial one. With $R(g) = (m^i_k)$ in (2), we have the following result.

Proposition 1. Let X be a simply-connected marked manifold (with a base element x_0), and $J_\mu = e^j J_{j,\mu} \in \Omega^0(X) \otimes \mathbf{g}, 1 \leq j \leq n$. Then the following conditions are equivalent:

(I)

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0 \quad , \quad \text{i.e.} \quad [\partial_\mu + J_\mu, \partial_\nu + J_\nu] = 0 \ , \quad \text{for all } \mu, \nu. \tag{4}$$

(II) There exists an element $m = e^k e_i m^i_k \in \mathcal{C}^\infty(X, \mathrm{Aut}(\mathbf{g}))$ ($:= \mathcal{C}^\infty((X, x_0), (\mathrm{Aut}(\mathbf{g}), 1))$) such that

$$(\partial_\mu m^i_k) = (m^i_k)((J_{\mathrm{ad}})^i_{k,\mu}) \ , \quad (J_{\mathrm{ad}})^i_{k,\mu} = c^{ji}_k J_{j,\mu} \ ,$$

i.e.

$$\partial_\mu(m^1, \ldots, m^n) = (m^1, \ldots, m^n)((J_{\mathrm{ad}})^i_{k,\mu}) \ , \tag{5}$$

where m^j is the j-th column vector of the matrix (m^i_k). Furthermore, the above correspondence is equivariant under actions of Diff(X). □

For the group $G = U(1)$, we shall identified the Lie algebra of $U(1)$ with $i\mathbb{R}$ through the usual exponential map,

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow i\mathbb{R} \xrightarrow{\exp} U(1) \longrightarrow 1 \ .$$

For a connection A on a principal $U(1)$-bundle P over a manifold X, ad(P) is the trivial bundle with D_{ad}= the ordinary differential d. However, through the canonical emdedding $U(1) \subset \mathbb{C}^* = GL_1(\mathbb{C})$, there is a Hermitian (complex) line bundle $\mathbf{L}$ over X associated to P with a covariant differential d_A on sections of $\mathbf{L}$ and its extension on $\mathbf{L}$-valued forms,

$$d_A : \Omega^*(X, \mathbf{L}) \longrightarrow \Omega^*(X, \mathbf{L}) \ .$$

The Chern class $c_1(\mathbf{L})$ of $\mathbf{L}$ is the integral class represented by the curvature $\frac{i}{2\pi}F(A)$:

$$c_1(\mathbf{L}) = \frac{i}{2\pi}F(A) \ \in \ \mathrm{H}^2(X, \mathbb{Z}) \ .$$

On a local open set U, A is expressed by an 1-form iV for $V \in \Omega^1(U)$. In this situation, we have $F(A) = dA$, and the covariant differential d_A,

$$d_A := d + A : \Omega^*(U)_{\mathbb{C}} \longrightarrow \Omega^*(U)_{\mathbb{C}} \ .$$

In a local coordinate system $(x^1, \ldots, x^n)$, we shall write

$$d_A(f) = dx^\mu(\partial_A)_\mu f \ , \quad \text{for } f \in \Omega^0(U)_{\mathbb{C}} \ .$$

Let H be a closed subgroup of G, S be the H-orbit space G/H with the base element e equal to the identity coset, and $\pi : G \longrightarrow S$ the natural projection. We have the exact sequence of sets of marked maps, equivariant under the actions of Diff(X),

$$0 \longrightarrow \mathcal{C}^\infty(X, H) \longrightarrow \mathcal{C}^\infty(X, G) \xrightarrow{\pi_*} \mathcal{C}^\infty(X, S) \xrightarrow{\delta} \mathrm{H}^1(X, \Omega^0(H)) \longrightarrow \cdots \ .$$

For $s \in \mathcal{C}^\infty(X, S)$ with $\delta(s) = 0$, we have $s = \pi \cdot g$ for a function $g \in \mathcal{C}^\infty(X, G)$, unique up to the multiplication of an element h in $\mathcal{C}^\infty(X, H)$, i.e. $g \sim gh$. One can assign the element s a flat G-connection $e^j J_{j,\mu} dx^\mu$ corresponding to g, then modulo the qauge equivalent relation induced by elements of $\mathcal{C}^\infty(X, H)$. Hence we have the following result:

Proposition 2. For a simply-connected marked manifold X, there is an one-to-one correspondence of the following data, which are equivariant under the actions of Diff(X):

(I) $s \in \mathcal{C}^\infty(X, S)$ with $\delta(s) = 0$.

(II) $J_\mu = e^j J_{j,\mu} dx^\mu \in \mathbf{g} \otimes \Omega^0(X)$, a flat connection modulo the relation, $J_\mu \sim \tilde{J}_\mu$,

$$\tilde{J}_{j,\mu} = J(h)_{j,\mu} + R(h^{-1})_j^k J_{k,\mu} \ , \quad \text{for } h \in \mathcal{C}^\infty(X, H) \ .$$

The relation of s, J_μ, is given by

$$s = \pi_*(g) \ , \quad g^{-1}dg = e^j J_{j,\mu} dx^\mu \quad \text{for some } g \in \mathcal{C}^\infty(X, G) \ .$$

$\square$

Remark. Under the situation where H is abelian and $\mathrm{H}^2(X, \mathbb{Z}) = 0$, one has $\mathrm{H}^1(X, \Omega^0(H)) = 0$, then the condition (I) simply means $s \in \mathcal{C}^\infty(X, S)$.

3. Flat SU(2)-Connection and S^2 σ-model

In this section, we set the Lie algebra $\mathbf{g} = su(2)$, where the negative Killing form is defined by

$$< x, y >= \frac{-1}{2} \mathrm{Tr}(\mathrm{ad}x \ \mathrm{ad}y) \ , \quad x, y \in \mathbf{g},$$

An orothonormal basis of $\mathbf{g}$ consists of the elements:

$$e^j = \frac{i}{2}\sigma^j \ , \quad j = 1, 2, 3 \ .$$

Here σ^j are Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that $\sigma^\pm$, $\frac{\sigma^3}{2}$, form the standard basis of the Lie algebra $sl_2(\mathbb{C})$ with

$$\sigma^+ = \frac{\sigma^1 + i\sigma^2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \frac{\sigma^1 - i\sigma^2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Lie algebra $\mathbf{g}$ is given by

$$[e^j, e^k] = -\epsilon_{jkl} e^l \ ,$$

where ϵ_{jkl} are antisymmetric with $\epsilon_{123} = 1$. We have the equality, Aut($\mathbf{g}$)$= SO(\mathbf{g})$, which will be identified with $SO(3)$ via the basis e^j. The sequence (1) becomes

$$1 \longrightarrow \{\pm 1\} \longrightarrow SU(2) \longrightarrow SO(3) \longrightarrow 1 \ .$$

Associated to an element of $SO(3)$, $(m_k^i) = (m^1, m^2, m^3)$, we shall denote

$$\vec{S} = m^3 \ ,$$

which is an element of the unit sphere S^2 in $\mathbf{g}$. We regard (m^1, m^2) as a positive orthonormal frame of the tangent space of S^2 at $\vec{S}$. Note that $m^3 = m^1 \times m^2$. Hence we can identify $SO(3)$ with the unit tangent bundle of S^2, i.e.

$$SO(3) = \{(\vec{S}, \vec{t}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \vec{S}^2 = \vec{t}^2 = 1 \ , \ \vec{S} \cdot \vec{t} = 0 \ \}.$$

For an element $(\vec{S}, \vec{t}) \in SO(3)$, we shall denote

$$\vec{t}_+ = \vec{t} + i(\vec{S} \times \vec{t}) \ , \ \vec{t}_- = \vec{t} - i(\vec{S} \times \vec{t}) \quad .$$

The following relations hold:

$$\begin{array}{l} \vec{t}_+ \cdot \vec{t}_+ = \vec{t}_- \cdot \vec{t}_- = 0 \ , \qquad \vec{t}_+ \cdot \vec{t}_- = 2 \ , \\ \vec{t}_+ \times \vec{S} = i\vec{t}_+ \ , \quad \vec{t}_- \times \vec{S} = -i\vec{t}_- \ , \quad \vec{t}_- \times \vec{t}_+ = 2i\vec{S} \ . \end{array}$$

The standard metric of S^2 together with the involution,

$$\vec{t} \mapsto \vec{S} \times \vec{t} \ , \quad \vec{S} \times \vec{t} \mapsto -\vec{t} \ ,$$

defines a complex Hermitian structure on the tangent space of S^2 at $\vec{S}$. In what follows, we shall regard the tangent bundle of S^2 as a Hermitian (complex) line bundle, denoted by $\mathbf{T}_{S^2}$, with $SO(3)$ as the unit sphere bundle. The map, $\vec{t} \mapsto \vec{t}_-$, defines a $\mathbb{C}$-linear isomorphism between $\mathbf{T}_{S^2}$ and the complex line subbundle of $\mathbf{T}_{S^2} \otimes_{\mathbb{R}} \mathbb{C}$ generated by $\vec{t}_-$.

Let X be a simply-connected marked manifold. An $SU(2)$-connection J on X is expressed by

$$J = 2Im(q)e^1 - 2Re(q)e^2 + Ve^3 = q\sigma^- - \overline{q}\sigma^+ + Ve^3$$

with $q = q_\mu dx^\mu \in \Omega^1(X)_{\mathbb{C}}$, $V = V_\mu dx^\mu \in \Omega^1(X)$. The associated $SO(3)$-connection J' is given by

$$J' = (e^1, e^2, e^3) \begin{pmatrix} 0 & V & 2Re(q) \\ -V & 0 & 2Im(q) \\ -2Re(q) & -2Im(q) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} .$$

It is known that the zero curvature of the connection J, $F(J) = 0$, is the consistency condition of the linear system :

$$dg = g(q\sigma^- - \overline{q}\sigma^+ + Ve^3) \ , \quad g \in \mathcal{C}^\infty(X, SU(2)) \ , \tag{6}$$

which is equivalent to the system (5), now in terms of $\vec{S}, \vec{t}$, expressed by

$$\left\{ \begin{array}{ll} d\vec{S} & = q\vec{t}_- + \overline{q}\vec{t}_+ \\ d_A\vec{t}_+ & = -2q\vec{S}, \qquad \text{for} \ \ A = -iV \ . \end{array} \right. \tag{7}$$

By Proposition 1, the 1-forms q, A, satisfy the relation (4), which is equivalent to the expression:

$$\left\{ \begin{array}{l} F(A) = 2q\overline{q}, \\ d_A q = 0 \qquad \text{for} \ , \ \ A = -iV \ . \end{array} \right. \tag{8}$$

The curvature $F(A)$ is related to $\vec{S}$ by the formula:

$$\tfrac{1}{4\pi}\vec{S} \cdot (d\vec{S} \times d\vec{S}) = \tfrac{i}{\pi} q\overline{q} = \tfrac{i}{2\pi} F(A) \ . \tag{9}$$

In the case of a 2-dimensional Riemannian manifold X, one has the inequality:

$$dS \cdot *dS \geq \pm\vec{S} \cdot (d\vec{S} \times d\vec{S}) \ \ , \ \ (\Leftrightarrow \quad 4q(*\bar{q}) \geq \pm 4iq\bar{q} \ ) \ .$$

The following lemma is implied by Proposition 1.

Lemma 2. Through the relation (7), there is an one-to-one correspondence between the following data for a simply-connected marked manifold X:

(I)$(\vec{S}, \vec{t}) \in \mathcal{C}^\infty(X, SO(3))$.

(II) (A, q), a solution of (8) with $A \in i\Omega^1(X)$, $q \in \Omega^1(X)_{\mathbb{C}}$. □

For $X=\mathbb{R}^2$, the local expressions of (7), (8), are:

$$\left\{\begin{array}{l}\partial_\mu\vec{S}=q_\mu\vec{t}_-+\overline{q_\mu}\vec{t}_+ \ ,\\ (\partial_\mu-iV_\mu)\vec{t}_+=-2q_\mu\vec{S}\ , \quad \mu=0,1\ ,\end{array}\right. \qquad \left\{\begin{array}{l}\partial_0V_1-\partial_1V_0=2i(q_0\overline{q_1}-\overline{q_0}q_1)\ ,\\ (\partial_0-iV_0)q_1=(\partial_1-iV_1)q_0\ .\end{array}\right. \tag{10}$$

Let H be the Cartan subgroup consisting of diagonal elements of $SU(2)$. As H is the isotropic subgroup of the element e^3 for the adjoint action of $SU(2)$ on $\mathbf{g}$,

$$H=\{g\in SU(2)\mid \mathrm{Ad}(g)(e^3)=e^3\}\ ,$$

the homogeneous space $SU(2)/H$ can be identified with the unit sphere S^2 in $\mathbf{g}$ with the base element $e=e^3$. Note that the exponential of an element $i\xi\in i\mathbb{R}$ (= the Lie algebra of H) acts on $\mathbf{g}$ by

$$(e^1,e^2,e^3)\mapsto(e^1,e^2,e^3)\begin{pmatrix}\cos 2\xi & \sin 2\xi & 0\\ -\sin 2\xi & \cos 2\xi & 0\\ 0 & 0 & 1\end{pmatrix}\ .$$

For $h\in\mathcal{C}^\infty(X,H)$, we have

$$h^{-1}dh=(d\alpha)e^3 \quad \text{for} \quad \alpha\in\Omega^0(X)\ .$$

Under the action of h, the transformations of $\vec{S},\vec{t},V,q$, in (7) are given by

$$\begin{array}{ll}\vec{S}\mapsto\vec{S}\ , & \vec{t}_-\mapsto \mathrm{e}^{-i\alpha}\vec{t}_-.\\ q\mapsto \mathrm{e}^{i\alpha}q, & V\mapsto V+(d\alpha)\ .\end{array} \tag{11}$$

Consider A $(:=-iV)$ as an $U(1)$-connection on the trivial complex line bundle $\mathbf{L}$ over X with the canonical Hermitian structure, q as a section of 1-forms with values in $\mathbf{L}$. There is a covariant differential d_A induced by A on $\mathbf{L}$-valued differential forms. The transformations of q,V, in (11) are described by a different choice of trivialization of the line bundle $\mathbf{L}$. By Proposition 2, we have the following result:

Proposition 3. Let X be a simply-connected marked manifold with $\mathrm{H}^2(X,\mathbb{Z})=0$, $\mathbf{L}$ be the trivial complex line bundle over X. There is an one-to-one correspondence of the following data:

(I) $\vec{S}\in\mathcal{C}^\infty(X,S^2)$.

(II) (V,q), $V\in\Omega^1(X), q\in\Omega^1(X)_{\mathbb{C}}$, a solution of (8) modulo the relation $(V,q)\sim(\tilde{V},\tilde{q})$:

$$(\tilde{V},\tilde{q})=(V+(d\alpha),\mathrm{e}^{i\alpha}q), \quad \text{for } \alpha\in\Omega^0(X)\ .$$

(III) (A,s), where A is an $U(1)$-conncetion on $\mathbf{L}$, $s\in\Omega^1(X,\mathbf{L})$, satisfying the following relation:

$$\left\{\begin{array}{l}F(A)=2ss^\dagger,\\ d_A(s)=0.\end{array}\right. \tag{12}$$

The above $\vec{S}$, (V,q), are related by (7) for some $\vec{t}\in\mathcal{C}^\infty(X,S^2)$, and (A,s), (V,q), by the relation: $(A,s)=(-iV,q)$, via a trivialization of $\mathbf{L}=X\times\mathbb{C}$ (as $U(1)$-bundles). □

Note that the vector-valued 1-form $q\vec{t}_-$ in the above proposition is independent of the choice of gauge transformation α in (II), hence it defines the section s of (III). The Hermitian line bundle $\mathbf{L}$ is isomorphic to the pull-back of line bundle $\mathbf{T}_{S^2}$ via the map $\vec{S}$. Therefore one can formulate the conclusion of Proposition 3 in the following instrinsic form.

Theorem 1. Let $\mathbf{L}$ be a Hermitian line bundle over a simply-connected marked manifold X. There is an one-to-one correspondence of the following data:

(I) $\vec{S} \in \mathcal{C}^\infty(X, S^2)$ with $\vec{S}^*(\mathbf{T}_{S^2}) = \mathbf{L}$.

(II) (A, s), a solution of (12), where A is an $U(1)$-connection on $\mathbf{L}$, $s \in \Omega^1(X, \mathbf{L})$.

In fact, (A, s) in (II) is expressed by $(-iV, q)$ via a trivialization of $\mathbf{L}$ over a local open set U, which is related to the $\vec{S}$ in (I) by (7) for some $\vec{t} \in \mathcal{C}^\infty(X, S^2)$. Furthermore, the above correspondence satisfies the following functorial property. Namely, let X_j, $(j = 1, 2)$, be two simply-connected marked manifolds with the corresponding elements, (A_j, s_j) in (II), $\vec{S}_j \in \mathcal{C}^\infty(X_j, S^2)$ in (I). Assume $\vec{S}_1 = \vec{S}_2\varphi$ for some marked map $\varphi : X_1 \longrightarrow X_2$. Then $(A_1, s_1) = (\varphi^*(A_2), \varphi^*(s_2))$.

Proof. Let $\{U_\alpha\}$ be a simple open cover of X, u_α be a base element of U_α. By Proposition 3, an element (A, s) of (II) gives rise a collection of maps, $\vec{S}_\alpha : (U_\alpha, u_\alpha) \longrightarrow (S^2, e)$, with $\vec{S}^*_\alpha((\mathbf{T}_{S^2}) = \mathbf{L}|_{U_\alpha}$, and vice versa. For each pair (α, β), there is an unique element $g_{\alpha,\beta}$ in $SO(3)$ such that

$$\vec{S}_\alpha(x) = g_{\alpha,\beta}(\vec{S}_\beta(x)) \ , \quad \text{for } x \in U_\alpha \cap U_\beta \ .$$

Then $\{g_{\alpha,\beta}\}$ forms a cocycle for the cover $\{U_\alpha\}$ with values in $SO(3)$. By the simply-connectness of X, there exists an 1-cochain $\{g_\alpha\}$ such that $g_{\alpha,\beta} = g_\alpha^{-1} g_\beta$. Then the function,

$$\vec{S}(x) := g_\alpha(\vec{S}_\alpha(x)) \ ,$$

is the marked map from X to S^2 in (I). The functorial property follows easily from Proposition 3. □

To illustrate the correspondence of Theorem 1, we are going to give the explicit construction of a solution of (II) for the complex projective plane $\mathbb{P}^1$, corresponding to the canonical identification of S^2 with the extended complex plane via the stereographic projection.

Example. The Fubini-Study metric on $\mathbb{P}^1$, which is identified with the one-point compactification of the complex plane,

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \ , \ [1, 0] \leftrightarrow \infty, \ [\zeta, 1] \leftrightarrow \zeta \in \mathbb{C} \ . \tag{13}$$

The stereographic projection defines the diffeomorphism,

$$p : \mathbb{P}^1 \longrightarrow S^2 \ , \quad \zeta \mapsto \vec{S} \ , \text{ with } \quad S_1 + iS_2 = \tfrac{2\zeta}{1+|\zeta|^2} \ , \qquad S_3 = \tfrac{1-|\zeta|^2}{1+|\zeta|^2} \ , \tag{14}$$

which sends the origin, $\zeta = 0$, to the base element of S^2. The Fubini-Study metric on $\mathbb{P}^1$ is a metric invariant under fractional tranformations of $SU(2)$ on $\mathbb{P}^1$,

$$\zeta \mapsto \zeta' = \tfrac{\alpha\zeta+\beta}{\gamma\zeta+\delta} \ , \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2) \ . \tag{15}$$

In the coordinate ζ, the metric is expressed by

$$\frac{d\zeta \otimes d\overline{\zeta}}{(1 + |\zeta|^2)^2} \ ,$$

with the 2-form,

$$(1 + |\zeta|^2)^{-2} d\zeta d\overline{\zeta} = -\overline{\partial}\partial \log(1 + |\zeta|^2) \ .$$

Furthermore, the expressions in the above hold also for any coordinate system ζ' related to ζ by (15). With the canonical metric on S^2 and the Riemannian metric $4(1+|\zeta|^2)^{-2}Re(d\zeta \otimes$

$d\overline{\zeta}$) on $\mathbb{P}^1$, the map (14) is an $SU(2)$-equivariant isometry. One can set $\vec{t}_\pm$ on the ζ-plane by

$$\vec{t}_- = (1+|\zeta|^2)\partial_\zeta \vec{S} \ , \quad \vec{t}_+ = (1+|\zeta|^2)\partial_{\bar{\zeta}} \vec{S} \ ,$$

hence

$$\vec{S} = \frac{1}{1+|\zeta|^2}\begin{pmatrix} \zeta+\bar{\zeta} \\ -i(\zeta-\bar{\zeta}) \\ 1-|\zeta|^2 \end{pmatrix} , \ \vec{t}_+ = \frac{1}{1+|\zeta|^2}\begin{pmatrix} 1-\zeta^2 \\ i(1+\zeta^2) \\ -2\zeta \end{pmatrix} , \ \vec{t}_- = \frac{1}{1+|\zeta|^2}\begin{pmatrix} 1-\bar{\zeta}^2 \\ -i(1+\bar{\zeta}^2) \\ -2\bar{\zeta} \end{pmatrix} .$$

By computation, one obtains the expressions of q, A, in (7), denoted by $\breve{q}, \breve{A}$, as follows:

$$\breve{q} = \frac{d\zeta}{1+|\zeta|^2} \ , \quad \breve{A} = -\partial \log(1+|\zeta|^2) + \overline{\partial}\log(1+|\zeta|^2) \ .$$

By the $SU(2)$-invariant properties of metrics on $\mathbb{P}^1$ and S^2, the expressions of $\breve{q}$, $\breve{A}$, in the above formula hold also for any coordinate system ζ' in (15). The connection $\breve{A}$ is characterized as an $SU(2)$-invariant $U(1)$-connection on the tangent bundle of $\mathbb{P}^1$. Note that the $U(1)$-bundle $p^*\mathbf{T}_{S^2}$ is the anti-canonical bundle $\mathbf{K}^{-1}$ over $\mathbb{P}^1$ endowed with the Fubini-Study metric. The local frame for Eq.(7) is $(1+|\zeta'|^2)^{-1}\frac{\partial}{\partial\zeta'}$. Hence the element (A,s) in Theorem 1 is given by the $SU(2)$-invariant connection $\breve{A}$ and the "tautological" section $\breve{\Phi}$,

$$\begin{array}{lcl} d_{\breve{A}}((1+|\zeta'|^2)^{-1}\frac{\partial}{\partial\zeta'}) & = & (-\partial\log(1+|\zeta'|^2)+\overline{\partial}\log(1+|\zeta'|^2))(1+|\zeta'|^2)^{-1}\frac{\partial}{\partial\zeta'} \ , \\ \breve{\Phi} & = & d\zeta'\frac{\partial}{\partial\zeta'} \ \in \ \Omega^{1,0}(\mathbb{P}^1,\mathbf{K}^{-1}) \ . \end{array} \tag{16}$$

By the universal property of Theorem 1, the above $(\breve{A},\breve{\Phi})$ can be served as the universal object of solutions of (12). Hence we obtain the following statement:

Corollary. The expression of the corresponding element (A,s) in Theorem 1 (II) for an element $\vec{S} \in \mathcal{C}^\infty(X,S^2)$ is given by

$$(A,s) = (\zeta^*(\breve{A}),\zeta^*(\breve{\Phi})) \ , \quad \text{where } \zeta := p^*(\vec{S}) : X \longrightarrow \mathbb{P}^1 \ .$$

□

4. Self-Dual Equation and Liouville Equation

In this section, X will always denote a Riemann surface, i.e. an one-dimensional complex manifold, with a local coordinate system, $z = x+iy$. With the holomorphic and anti-holomorphic differentials,

$$dz = dx + idy \in \Omega^{1,0}(X), \quad d\overline{z} = dx - idy \in \Omega^{0,1}(X) \ ,$$

one decomposes differential forms of X into (p, q)-forms,

$$\Omega^*(X)_{\mathbb{C}} = \bigoplus_{\mathrm{p,q}} \Omega^{\mathrm{p,q}}(X) \ ,$$

hence the differential $d = d' + d''$,

$$d' : \Omega^{\mathrm{p,q}}(X) \longrightarrow \Omega^{\mathrm{p+1,q}}(X) \ , \quad d'' : \Omega^{\mathrm{p,q}}(X) \longrightarrow \Omega^{\mathrm{p,q+1}}(X) \ .$$

For $q \in \Omega^1(X)_{\mathbb{C}}$, we have

$$q = q' + q'' \ , \quad q' \in \Omega^{1,0}(X) \ , \quad q'' \in \Omega^{0,1}(X) \ ,$$

with the local expressions,

$$q = q_x dx + q_y dy \ , \quad q' = q_z dz \ , \quad q'' = q_{\overline{z}} d\overline{z} \quad \text{where} \ \ q_z := \frac{q_x - iq_y}{2} \ , \ q_{\overline{z}} := \frac{q_x + iq_y}{2} \ .$$

For an $U(1)$-connection $A \in i\Omega^1(X)$, one has the decomposition of d_A:

$$d_A = d'_A + d''_A, \quad d'_A := d' + A' \ , \quad d''_A := d'' + A'' \ .$$

Note that $A'' = -\overline{A'}$. In a local coordinate z, we shall write

$$d'_A(f) = dz(\partial_A)_z f \ \ , \ d''_A(f) = d\bar{z}(\partial_A)_{\bar{z}} f \quad \text{for} \ \ f \in \Omega^0(U)_{\mathbb{C}} \ .$$

Furthermore one has the decomposition of the $\mathbf{L}$-valued forms, where $\mathbf{L}$ is a Hermitian line bundle over X with an $U(1)$-conncetion A,

$$\begin{array}{ll} \Omega^*(X, \mathbf{L}) = \bigoplus_{\mathrm{p,q}} \Omega^{\mathrm{p,q}}(X, \mathbf{L}) \ , & d_A = d'_A + d''_A \ , \\ d'_A : \Omega^{\mathrm{p,q}}(X, \mathbf{L}) \longrightarrow \Omega^{\mathrm{p+1,q}}(X, \mathbf{L}) \ , & d''_A : \Omega^{\mathrm{p,q}}(X, \mathbf{L}) \longrightarrow \Omega^{\mathrm{p,q+1}}(X, \mathbf{L}) \ . \end{array}$$

For $s \in \Omega^1(X, \mathbf{L})$, we denote

$$s = s' + s'' \ , \quad s' \in \Omega^{1,0}(X, \mathbf{L}) \ , \ \ s'' \in \Omega^{0,1}(X, \mathbf{L}) \ .$$

Hence

$$F(A) = d''A' + d'A'' \ , \quad d_A s = d''_A s' + d'_A s'' \ , \ \ ss^\dagger = s's'^\dagger - s''^\dagger s'' \ .$$

Locally we have the expressions,

$$A = A_z dz + A_{\bar{z}} d\bar{z} \ , \quad s = q_z dz + q_{\bar{z}} d\bar{z} \ ,$$

then (12) becomes

$$\left\{ \begin{array}{l} \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = 2(|q_z|^2 - |q_{\bar{z}}|^2), \\ (\partial_A)_z q_{\bar{z}} = (\partial_A)_{\bar{z}} q_z \ , \end{array} \right. \tag{17}$$

which is the compactibility condition of the linear system (6), now taking the form:

$$\left\{ \begin{array}{l} d'g = g(q'\sigma^- - \overline{q''}\sigma^+ + V'e^3) \ , \\ d''g = g(q''\sigma^- - \overline{q'}\sigma^+ + \overline{V'}e^3) \ , \end{array} \right.$$

with $g \in \mathcal{C}^\infty(X, SU(2))$. An equivalent expression is the system:

$$\partial_z g = g \begin{pmatrix} \frac{i}{2} V_z & -\overline{q_{\bar{z}}} \\ q_z & \frac{-i}{2} V_z \end{pmatrix} \ , \ \partial_{\bar{z}} g = g \begin{pmatrix} \frac{i}{2} \overline{V_z} & -\overline{q_z} \\ q_{\bar{z}} & \frac{-i}{2} \overline{V_z} \end{pmatrix} \ . \tag{18}$$

Note that the r.h.s. in the above is a $SL_2(\mathbb{C})$-connection.

Consider the following constraint, called the self-dual equation, of functions $\vec{S}$ from a marked Riemann surface X to S^2:

$$d'\vec{S} = i\vec{S} \times d'\vec{S} \ , \quad (\Leftrightarrow \quad d'\zeta = 0 \ ), \tag{19}$$

or

$$d''\vec{S} = i\vec{S} \times d''\vec{S} \quad (\Leftrightarrow \quad d''\zeta = 0 \ ), \tag{20}$$

where ζ is the composition of $\vec{S}$ with the stereographic projection (14), $\zeta : X \longrightarrow \mathbb{P}^1$. By the formula of vector product in $\mathbb{R}^3$,

$$v_1 \times (v_2 \times v_3) = (v_1 \cdot v_3) v_2 - (v_1 \cdot v_2) v_3 \ ,$$

Eqs. (19), (20), have the following local expressions:

$$\begin{array}{l} (19) \ \ \Leftrightarrow \ \ \partial_x \vec{S} = \vec{S} \times \partial_y \vec{S} \ , \\ (20) \ \ \Leftrightarrow \ \ \partial_x \vec{S} = -\vec{S} \times \partial_y \vec{S} \ . \end{array}$$

For a solution $\vec{S}$ of (19), let $\mathbf{L}$ be the Hermitian line bundle $\vec{S}^*\mathbf{T}_{S^2}$, (A, s) be the solution of (12) corresponding to $\vec{S}$ in Theorem 1. With a local trivilization of $\mathbf{L}$ over an open set U, s is represented by an 1-form $q \in \Omega^1(U)_{\mathbb{C}}$. By (7), Eq.(19) is equivalent to the relation, $q' = 0$, i.e. $q_x = iq_y$, and the same for Eq. (20) and $q'' = 0$. Hence

$$\begin{array}{lcl} (19) & \Leftrightarrow & s' = 0 \ , \ \text{i.e} \ \ s = s'' \ , \\ (20) & \Leftrightarrow & s'' = 0 \ , \ \text{i.e} \ \ s = s' \ . \end{array}$$

Replace $\mathbf{L}, A$, by the complex conjuate $\mathbf{L}^\dagger, A^\dagger$, in the case of (19), and set

$$\Phi = \begin{cases} s^\dagger & \text{for} \ \ (19) \ , \\ s & \text{for} \ \ (20) \ . \end{cases}$$

We have

$$d_A\Phi = d''_A\Phi \ ,$$

and Eq.(12) becomes

$$\begin{cases} F(A) = 2\Phi\Phi^* \ , \\ d''_A\Phi = 0 \ , \end{cases} \tag{21}$$

where A is an $U(1)$-connection of a Hermitian line bundle $\mathbf{L}$ over X, and $\Phi \in \Omega^{1,0}(X, \mathbf{L})$. By Theorem 1, we obtain the following result:

Theorem 2. Let $\mathbf{L}$ be a Hermitian complex line bundle over a simply-connected marked Riemann surface X. There is an one-to-one correspondence between the following data:
(I) $\vec{S} \in \mathcal{C}^\infty(X, S^2)$, a solution of (19) with $\vec{S}^*(\mathbf{T}^\dagger_{S^2}) = \mathbf{L}$.
(II) $\vec{S} \in \mathcal{C}^\infty(X, S^2)$, a solution of (20) with $\vec{S}^*(\mathbf{T}_{S^2}) = \mathbf{L}$.
(III) (A, Φ), a solution of Eq. (21), where A is an $U(1)$-connection on $\mathbf{L}$, $\Phi \in \Omega^{1,0}(X, \mathbf{L})$.
The $\vec{S}$'s in (I), (II), are related by the transformation,

$$\vec{S} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{S} \ .$$

For (A, Φ) of (III), the corresponding $\vec{S}$ in (I) (or (II)) is the element related to $(A^\dagger, \Phi^\dagger)$ (or (A, Φ) respectively) in Proposition 3 (I). □

Remark. (i) Replacing X of the above theorem by an arbitrary Riemann surface, by the relation,

$$d_A\Phi = 0 \Longrightarrow d''_A\Phi = 0 \ , \ \ \text{for} \ \Phi \in \Omega^{1,0}(X, \mathbf{L}) \ ,$$

one obtains the following relations among (I) (II) and (III),

$$(\mathrm{I}) \Leftrightarrow (\mathrm{II}) \Longrightarrow (\mathrm{III}) \ .$$

(ii) The general solution of (II) in Theorem 2 for a simply-connected marked Riemann surface X (with a base element x_0) is given by

$$(A, \Phi) = (\varphi^*(\breve{A}), \varphi^*(\breve{\Phi})) \ , \ \ \varphi : (X, x_0) \longrightarrow (\mathbb{P}^1, 0) \ \ \text{holomorphic} \ ,$$

with $\mathbf{L} = \varphi^*(\mathbf{K}^{-1})$. Here $(\breve{A}, \breve{\Phi})$ is the solution over $\mathbb{P}^1$ described by (16). For $X = \mathbb{P}^1$, the map φ is a rational function of X. By (9), the Chern class of $\varphi^*(\mathbf{K}^{-1})$ is given by

$$c_1(\varphi^*(\mathbf{K}^{-1})) = \frac{1}{4\pi} \int \vec{S} \cdot (d\vec{S} \times d\vec{S}) = 2 \ (\text{degree of } \varphi) \ .$$

□

Corollary. For $X = \mathbb{C}$ with the coordinate $z = x + iy$, let $\mathbf{L}$ be the trivial line bundle over X with the canonical Hermitian structure. Then there is an one-to-one correspondence between the following data:

(I) $\vec{S} : X \longrightarrow S^2$, a solution of (19) (or (20) respectively) with $\vec{S}(0,0) = e$.

(II) (A, Φ), a solution of (21) where A is an $U(1)$-connection of $\mathbf{L}$, $\Phi \in \Omega^{1,0}(X, \mathbf{L})$.

(III) $\phi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$, a solution of Liouville equation:

$$\text{(LE)} \ : \ -(\partial_x^2 + \partial_y^2)\phi = 8\mathrm{e}^{\phi} \ . \tag{22}$$

The relation of (I), (II), is given by Theorem 2. The relation of (II), (III), is given by

$$\begin{aligned} &A = A_z dz + A_{\bar{z}} d\bar{z} \ , \quad A_{\bar{z}} = \tfrac{-1}{2}\partial_{\bar{z}}\phi \ , \\ &\Phi = \mathrm{e}^{\frac{\phi}{2}} dz \ , \end{aligned} \tag{23}$$

and (I), (III), by $2\mathrm{e}^{\phi} = |\partial_z \vec{S}|^2$ (or $|\partial_{\bar{z}}\vec{S}|^2$ respectively).

Proof. As X is a contractable space, we may assume the line bundle $\mathbf{L}$ in Theorem 2 is the trivial one. Hence it follows the relation between (I), (II). For an element (A, Φ) of (II), by a suitable frame of $\mathbf{L}$ outsider the zeros of Φ, one can write $\Phi = \mathrm{e}^{\frac{\phi}{2}} dz$ for a function ϕ with values in $\mathbb{R} \cup \{-\infty\}$. Then Eq. (21) is equivalent to the relation, $A_{\bar{z}} = \frac{-1}{2}\partial_{\bar{z}}\phi$, where ϕ satisfies Eq. (22) of (III). The relation between (I), (III), follows easily from (10). □

Remark. By Corollary of Theorem 1 and the above relation between (II), (III), one obtains the well-known expression of a general solution of Liouville equation:

$$\phi(z) = \log \frac{|\partial_z \zeta(z)|^2}{(1 + |\zeta(z)|^2)^2} \ , \quad \zeta : \mathbb{C} \longrightarrow \mathbb{P}^1 \ \text{holomorphic} \ .$$

5. Conformal Sinh-Gordon Equation

In this section, X is a simply-connected marked Riemann surface. We are going to discuss the following equation of $\vec{S} \in \mathcal{C}^\infty(X, S^2)$,

$$\vec{S} \times d''d'\vec{S} = 0 \ , \ (\Longleftrightarrow \ d''d'\vec{S} + (d''\vec{S}, d'\vec{S})\vec{S} = 0 \ \Longleftrightarrow \ d'd''\zeta = \tfrac{2\bar{\zeta}}{1+|\zeta|^2} d'\zeta d''\zeta \ ) \ , \tag{24}$$

where $\zeta = p^*(\vec{S})$. The local expression of (24) is given by

$$\vec{S} \times (\partial_x^2 + \partial_y^2)\vec{S} = 0 \ , \ (\Longleftrightarrow \ \partial_{\bar{z}}\partial_z\vec{S} + (\partial_{\bar{z}}\vec{S} + \partial_z\vec{S})\vec{S} = 0 \ \Longleftrightarrow \ \partial_z\partial_{\bar{z}}\zeta = 2\bar{\zeta}\tfrac{\partial_z\zeta\partial_{\bar{z}}\zeta}{1+|\zeta|^2} \ ) \ . \tag{25}$$

Let (A, s) be the solution of (12) corresponding to $\vec{S}$ in Theorem 1. Eq.(24) gives rise a constraint of (A, s), which is the expressions,

$$d''_A s' + d'_A s'' = d''_A s' - d'_A s'' = d''_A s' = d'_A s'' = 0 \ \ .$$

Note that the zeros of any two of the above four sections imply the others. Locally they are described by the relations:

$$(\partial_A)_x q_y - (\partial_A)_y q_x = (\partial_A)_x q_x + (\partial_A)_y q_y = 0 \ .$$

Obviously these conditions are preserved under holomorphic transformations of the Riemann surface X.

Theorem 3. Let $\mathbf{L}$ be a Hermitian line bundle over a simply-connected marked Riemann surface X. Under the correspondence of Theorem 1, the following data are equivalent:

(I) $\vec{S} \in \mathcal{C}^\infty(X, S^2)$, a solution of (24) with $\vec{S}^*(\mathbf{T}_{S^2}) = \mathbf{L}$.

(II) (A, s), where A is a $U(1)$-connection on $\mathbf{L}$, $s \in \Omega^1(X, \mathbf{L})$, satisfying the equation

$$\left\{ \begin{array}{l} F(A) = 2ss^\dagger \ , \\ d''_A s' = d'_A s'' = 0 \ . \end{array} \right. \tag{26}$$

Furthermore, the $\vec{S}$'s of (I) corresponding to those elements of (II) with $s' = 0$ (or $s'' = 0$) are solutions of (19) (or (20) respectively) described by Theorem 2 before. □

Now we are going to establish the relation between Eq.(26) and the (conformal) sinh-Gordon equation:

$$(\mathrm{shG})_u : \quad -(\partial_x^2 + \partial_y^2)\phi = 8(u\mathrm{e}^\phi - \mathrm{e}^{-\phi}) \ , \quad \phi : \mathbb{C} \longrightarrow \mathbb{R} \cup \{-\infty\} \ , \tag{27}$$

where $u = |U|^2$ for a (fixed) holomorphic function U of the complex plane $\mathbb{C}$. For $u = 0$, Eq. (27) is the Liouville equation, which posseses the conformal invariant property. Our main discussion here will be on the case of a non-zero u. Note that by Corollary of Theorem 1, the function,

$$\phi(z) = \log \frac{|\partial_z \zeta(z)|^2}{(1 + |\zeta(z)|^2)^2} \qquad \text{with} \ \ \partial_z \partial_{\bar{z}} \zeta = 2\bar{\zeta} \frac{\partial_z \zeta \partial_{\bar{z}} \zeta}{1 + |\zeta|^2} \quad ,$$

is a solution of (27) for

$$u = |U|^2 \ , \quad U = \frac{\partial_z \zeta \partial_z \bar{\zeta}}{(1 + |\zeta|^2)^2} \ .$$

Proposition 4. Let $X = \mathbb{C}$ with the coordinate $z = x + iy$, and $\mathbf{L}$ be the trivial line bundle over $\mathbb{C}$ endowed with the natural Hermitian structure. Let $\mathbf{K}$ be the canonical bundle over X with the vector space of holomorphic sections, $\Gamma(X, \mathbf{K})$. Then there is an one-to-one correspondence between the following data:

(I) $\vec{S} : \mathbb{C} \longrightarrow S^2$, a solution of (25), but not a solution of (20), with $\vec{S}(0) = e$.

(II) (A, s), a solution of (26), where A is a $U(1)$-connection of $\mathbf{L}$, $s \in \Omega^1(\mathbb{C}, \mathbf{L})$ with $s'' \neq 0$.

(III) (h, σ), where $\sigma \in \Gamma(X, \mathbf{K})$, $h =< dz, dz >$ for a Hermetian metric $<,>$ of $\mathbf{K}$, satisfying the relation:

$$d'' d' \log h = h\sigma\sigma^\dagger - h^{-1} dz d\bar{z} \ . \tag{28}$$

(IV) (ϕ, U), where U is an entire function, ϕ is a solution of (27) with $u = |U|^2$.

The correspondence between (I), (II), is given in Theorem 1. The relations between (II), (III), (IV), are described by

$$\begin{array}{ll} A = \frac{1}{2}(h^{-1}d'h - h^{-1}d''h) \ , & s = h^{\frac{1}{2}}\sigma + h^{-\frac{1}{2}} d\bar{z} \ , \\ A = \frac{\partial_z \phi}{2} dz - \frac{\partial_{\bar{z}} \phi}{2} d\bar{z} \ , & s = U\mathrm{e}^{\frac{\phi}{2}} dz + \mathrm{e}^{-\frac{\phi}{2}} d\bar{z} \ , \\ h = \mathrm{e}^\phi \ , & \sigma = U dz \ . \end{array}$$

Furthermore, the above correspondences are equivariant under the following actions of analytic automorphisms f of $\mathbb{C}$ with $f(0) = 0$:

$$f^*\vec{S} \longleftrightarrow (f^*A, f^*s) \longleftrightarrow (|\partial_z f|^2 f^*h, f^*\sigma) \longleftrightarrow (\tilde{\phi}, \tilde{U}) \ ,$$

where $\tilde{\phi}, \tilde{U}$, are defined by

$$\tilde{\phi}(z) = \phi(f(z)) + 2\log|\partial_z f(z)|, \quad \tilde{U}(z) = U(f(z))(\partial_z f(z))^2 \ . \tag{29}$$

Proof. One may assume the line bundle $\mathbf{L}$ in Theorem 3 to be the trivial one, hence the relation between (I) and (II) follows immediately . We are going to show the equivalence of (II) and (IV). In terms of a suitable frame of $\mathbf{L}$, the element (A, s) of (II) is expressed by

$$s = q_z dz + q_{\bar{z}} d\bar{z} \ , \quad A = A_z dz + A_{\bar{z}} d\bar{z} \ .$$

Then (26) becomes

$$\left\{ \begin{array}{l} \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = 2(|q_z|^2 - |q_{\bar{z}}|^2) \ , \\ (\partial_{\bar{z}} + A_{\bar{z}}) q_z = (\partial_z + A_z) q_{\bar{z}} = 0 \ . \end{array} \right.$$

Define

$$U = q_z \overline{q_{\bar{z}}} \ .$$

By the property $A_{\bar{z}} = -\overline{A_z}$, we have

$$\partial_{\bar{z}} U = (\partial_{\bar{z}} + A_{\bar{z}}) q_z = (\partial_z + A_z) q_{\bar{z}} = 0 \ .$$

Since the vanishing of any two in the above implies the third one, one may replace $(A, q_z, q_{\bar{z}})$ by $(A, U, q_{\bar{z}})$. Then the equation becomes

$$\left\{ \begin{array}{l} \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = 2(|q_z|^2 - |q_{\bar{z}}|^2) \ , \\ \partial_{\bar{z}} U = (\partial_z + A_z) q_{\bar{z}} = 0 \ . \end{array} \right.$$

By a suitable unitary frame of $\mathbf{L}$, we may assume $q_{\bar{z}} = \mathrm{e}^{\frac{-\phi}{2}}$, where $\phi : \mathbb{C} \longrightarrow \mathbb{R}$. Then we have

$$A_z = \frac{1}{2} \partial_z \phi \ ,$$

where ϕ is a solution of Eq. (27). Hence we obtain the correspondence between (II), (IV). The rest relations follow immediately. $\square$

Remark. With the same argument as in the above proposition, the equivalent form of Eq. (26) with $s' \neq 0$ gives rise the equation of (ϕ, U) in [2]:

$$\left\{ \begin{array}{l} -(\partial_x^2 + \partial_y^2)\phi = 8(\mathrm{e}^{\phi} - |U|^2 \mathrm{e}^{-\phi}) \ , \\ \partial_{\bar{z}} U = 0 \ . \end{array} \right.$$

$\square$

By Proposition 4, one can derive the zero-curvature representation of the sinh-Gordon equation (27) as follows. Let U be an entire function with $|U|^2 = u$. Then the data in Proposition 4 (IV) for a fixed ϕ are given by the collection of $(\phi, \lambda U)$ with $\lambda \in \mathbb{C}, |\lambda| = 1$, each of which corresponds to an element (A, s) in (II), then by (18), gives rise a $SU(2)$-connection with the linear system,

$$\partial_z g = g \begin{pmatrix} \frac{-1}{4}\partial_z \phi & -\mathrm{e}^{\frac{-\phi}{2}} \\ \lambda U \mathrm{e}^{\frac{\phi}{2}} & \frac{1}{4}\partial_z \phi \end{pmatrix} \ , \quad \partial_{\bar{z}} g = g \begin{pmatrix} \frac{1}{4}\partial_{\bar{z}} \phi & -\frac{1}{\lambda}\bar{U} \mathrm{e}^{\frac{\phi}{2}} \\ \mathrm{e}^{\frac{-\phi}{2}} & \frac{-1}{4}\partial_{\bar{z}} \phi \end{pmatrix} \ , \tag{30}$$

for $|\lambda| = 1$, hence $\lambda \in \mathbb{C}^*$ by extension. The variable λ is called the spectral parameter. There is another intepretation of the spectral parameter by using the $*$-involuion of $\Omega^1(X)_{\mathbb{C}}$:

$$* : \Omega^1(X)_{\mathbb{C}} \longrightarrow \Omega^1(X)_{\mathbb{C}} \ , \quad *dx = dy \ , \quad *dy = -dx \ .$$

In fact, for a trivialization of $\mathbf{L}$ in Proposition 4, one has the identification, $\Omega^1(X,\mathbf{L}) = \Omega^1(X)_{\mathbb{C}}$. Then Eq. (26) becomes

$$\begin{cases} F(A) = 2q\overline{q} \ , \\ d_A q = d_A(*q) = 0 \ , \end{cases}$$

which is equivalent to the system for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$,

$$\begin{cases} F(A) = 2q_\theta\overline{q_\theta} \ , \\ d_A q_\theta = d_A(*q_\theta) = 0 \ , \end{cases}$$

where $q_\theta := q\cos\theta + (*q)\sin\theta$. The corresponding element of Proposition 4 (III) is given by

$$\begin{array}{llll} (h,\sigma) = & (\mathrm{e}^\phi, Udz), & \longleftrightarrow & (A,q) \ , \\ (h,\sigma_\theta) = & (\mathrm{e}^\phi, \mathrm{e}^{-2i\theta}Udz), & \longleftrightarrow & (A,q_\theta) \ . \end{array}$$

The family $\{q_\theta\}$ provides a role of the hidden symmetry of Eq.(30). On the other hand, one can also associate the spectral parameter λ with the symmetries of Eq.(29). For $u=0$, it is the Liouville equation, which possesses the holomorphic symmetries of $\mathbb{C}$. For $u \neq 0$ in $(\mathrm{shG})_u$, the relation (29) determines the following holomorphic functions f which preserve the function u of the form u_k:

$$\begin{cases} u(z) = u_k(z) := |z|^{2k}, & k \in \mathbb{Z}_{\geq 0}, \\ f(z) = f_\lambda(z) := \lambda z \ , & \lambda \in \mathbb{C}^* \ , \ |\lambda| = 1 \ . \end{cases}$$

Hence one obtains the following statement as a corollary of Proposition 4:

Proposition 5. Let $\mathbf{L}, \mathbf{K}$ be the same as in Proposition 4. For a non-negative integer k, there is an one-to-one correspondence between the following data:

(I) (h,σ), a solution of (28) with $\sigma\sigma^\dagger = |z|^{2k}dzd\bar{z}$, modulo the relation $(h,\sigma) \sim (f_\lambda^* h, f_\lambda^*\sigma)$ with $|\lambda| = 1$.

(II) ϕ, a solution of (27) for $u = u_k$.

For a function ϕ in (II), the corresponding element in (I) is represented by

$$h = \mathrm{e}^\phi \ , \ \sigma = z^k dz \ .$$

□

For Eq. (27) with $u = u_k$, a solution ϕ generates an one-parameter family of solutions as follows:

$$\phi(z,\bar{z}) \mapsto \phi_\theta(z,\bar{z}) := \phi(\mathrm{e}^{i\theta}z, \mathrm{e}^{-i\theta}\bar{z}) \ , \quad \theta \in \mathbb{R}/2\pi\mathbb{Z} \ .$$

The linear system corresponding to ϕ_θ is given by

$$\partial_z g = g \begin{pmatrix} \frac{-1}{4}\partial_z\phi_\theta & -\mathrm{e}^{\frac{-\phi_\theta}{2}} \\ z^k\mathrm{e}^{\frac{\phi_\theta}{2}} & \frac{1}{4}\partial_z\phi_\theta \end{pmatrix} \ , \quad \partial_{\bar{z}} g = g \begin{pmatrix} \frac{1}{4}\partial_{\bar{z}}\phi_\theta & -\bar{z}^k\mathrm{e}^{\frac{\phi_\theta}{2}} \\ \mathrm{e}^{\frac{-\phi_\theta}{2}} & \frac{-1}{4}\partial_{\bar{z}}\phi_\theta \end{pmatrix} \ .$$

By Proposition 4 and the relation (29), using the transformation $f_{\mathrm{e}^{-i\theta}}$, one can change the above system into (30) with $U = z^k$, $\lambda = \mathrm{e}^{-(k+2)i\theta}$.

Replacing the coordinate (x, iy) by $(t,x) \in \mathbb{R}^2$ in previous arguments of this section, one is able to derive the sine-Gordon equation. In fact, we define the light-cone coordinate,

$$t^+ = t + x \ , \quad t^- = t - x \ ,$$

with the differential operators,

$$\partial_+ = \frac{1}{2}(\partial_t + \partial_x) \ , \quad \partial_- = \frac{1}{2}(\partial_t - \partial_x) \ .$$

The equation is defined by

$$\vec{S} \times (\partial_t^2 - \partial_x^2)\vec{S} = 0 \ , \ (\iff \ \partial_+\partial_-\zeta = 2\bar{\zeta}\tfrac{\partial_+\zeta\partial_-\zeta}{1+|\zeta|^2} \ ) \ \ , \ \zeta = p^*(\vec{S}) \ . \tag{31}$$

The constraint of the corresponding (A, s) of Proposition 3 is given by

$$d_A^- s^+ + d_A^+ s^- = d_A^- s^+ - d_A^+ s^- = d_A^- s^+ = d_A^+ s^- = 0 \ .$$

Locally, s is an 1-form q which satisfies the relation,

$$(\partial_A)_t q_t = (\partial_A)_x q_x \ .$$

Then (31) is related to the sine-Gordon equation:

$$(\mathrm{sG})_W : \ \left\{ \begin{array}{l} \partial_+\partial_-\Phi = -4\mathrm{e}^W \sin\Phi \ , \\ \partial_+\partial_- W = 0 \ . \end{array} \right. \tag{32}$$

Indeed, by the same argument as in Proposition 4 one can derive the following fact.

Proposition 6. Let $\mathbf{L}$ be the trivial line bundle over $\mathbb{R}^2$ with the natural Hermitian structure. Then there is an one-to-one correspondence between the following data:

(I) $\vec{S} \in \mathcal{C}^\infty(\mathbb{R}^2, S^2)$, a solution of (31) with $\vec{S}^*(\mathbf{T}_{S^2}) = \mathbf{L}$.

(II) (A, s), where A is a $U(1)$-connection on $\mathbf{L}$, $s \in \Omega^1(\mathbb{R}^2, \mathbf{L})$, satisfying the equation

$$\left\{ \begin{array}{l} F(A) = 2ss^\dagger \ , \\ (\partial_+ + A_+)s_- = (\partial_- + A_-)s_+ = 0 \ . \end{array} \right. \tag{33}$$

(III) (Φ, r_+, r_-), where $r_\pm$ are non-negative functions on $\mathbb{R}^2$ with $\partial_- r_+ = \partial_+ r_- = 0$, Φ a solution of (32) for $W = \log r_+ + \log r_-$

The correspondence between (I), (II), is given in Theorem 1, and (II) , (III), by

$$A = -i\partial_-\Phi dt^- \ , \quad s = r_+\mathrm{e}^{i\Phi}dt^+ + r_- dt^- \ .$$

Furthermore, the correspondences are equivariant under action of the following automorphisms of $\mathbb{R}^2$:

$$f : (t^+, t^-) \mapsto (\alpha(t^+), \beta(t^-)) \ \text{ with } \ f(0,0) = (0,0) \ , \quad \frac{d\alpha}{dt^+} \geq 0 \ , \ \frac{d\beta}{dt^-} \geq 0 \ ,$$

via the relations:

$$f^*\vec{S} \longleftrightarrow (f^*A, f^*s) \longleftrightarrow (\tilde{\Phi}, \tilde{r}_+, \tilde{r}_-) := (f^*\Phi \ , \ \frac{d\alpha}{dt^+}\alpha^* r_+ \ , \ \frac{d\beta}{dt^-}\beta^* r_-) \ .$$

□

When W is the constant, $\log m^2$, the linear system associated to (32) gives rise to the family with a spectral parameter $\lambda \in \mathbb{C}^*$:

$$\partial_+ g = g \begin{pmatrix} 0 & \frac{-m^2}{\lambda}\mathrm{e}^{-i\Phi} \\ \frac{m^2}{\lambda}\mathrm{e}^{i\Phi} & 0 \end{pmatrix} \ , \quad \partial_- g = g \begin{pmatrix} \frac{i}{2}\partial_-\Phi & -\lambda \\ \lambda & \frac{-i}{2}\partial_-\Phi \end{pmatrix} \ .$$

6. Heisenberg Model and Nonlinear Schrödinger Equation

In this section, we set $X = \mathbb{R}^2$ with the origin as the base element, (t, x) as the coordinates. We are going to establish the equivalent relation between Heisenberg model and nonlinear Schrödinger equation:

$$\begin{array}{lll} \text{(HM)} & : \ \partial_0 \vec{S} = \vec{S} \times \partial_1^2 \vec{S} \ , & \vec{S} \in \mathcal{C}^\infty(\mathbb{R}^2, S^2) \ , \\ \text{(NLS)} & : \ i\partial_0 Q + \partial_1^2 Q + 2|Q|^2 Q = 0 \ , & Q \in \Omega^0(\mathbb{R}^2)_{\mathbb{C}} \ . \end{array}$$

For an element $(\vec{S}, \vec{t})$ of $\mathcal{C}^\infty(X, SO(3))$, let (V, q) be the corresponding pair of 1-forms in Lemma 2. By (7), we have

$$\partial_1^2 \vec{S} = (\partial_A)_1 q_1 \vec{t}_- + (\overline{(\partial_A)_1 q_1}) \vec{t}_+ - 4|q_1|^2 \vec{S} \ , \qquad A = -iV \ .$$

Hence one obtains the following constraint of the corresponding (V, q) for a solution $\vec{S}$ of the equation (HM):

$$q_0 = i(\partial_A)_1 q_1 \ .$$

Besides the local H-gauge symmetries, the above condition is invariant under the following transformations:

$$q \mapsto \mathrm{e}^{2ivx} q \ , \qquad (t, x) \mapsto \varphi_v(t, x) := (t, x - 2vt) \ ,$$

here φ_v is the Galileo tranformation of $\mathbb{R}^2$ with $v \in \mathbb{R}$. Note that we have the relation:

$$(\varphi_v)_* \begin{pmatrix} \partial_0 \\ \partial_1 \end{pmatrix} = \begin{pmatrix} \partial_0 - 2v\partial_1 \\ \partial_1 \end{pmatrix} \ .$$

One can eliminate q_0 in (10), which is reduced to a system of functions q_1, V_0, V_1:

$$\begin{cases} \partial_0 A_1 - \partial_1 A_0 = 2i\partial_1 |q_1|^2 \ , \\ i(\partial_A)_0 q_1 + (\partial_A)_1^2 q_1 = 0 \ , \quad \text{for } A = -iV \ . \end{cases}$$

By the change of variables,

$$W_0 = V_0 - 2|q_1|^2 \ , \quad W_1 = V_1 \ ,$$

the above system is equivalent to

$$\begin{cases} \partial_0 B_1 - \partial_1 B_0 = 0 \ , \\ i(\partial_B)_0 q_1 + (\partial_B)_1^2 q_1 + 2|q_1|^2 q_1 = 0 \ , \quad \text{for } B = -iW \ . \end{cases} \tag{34}$$

Under a gauge transformation $h \in \mathcal{C}^\infty(X, H)$ with $h^{-1}dh = (d\alpha)e^3$, the change of V_μ in (11) gives rise the transformations of W_μ, q_1,

$$q_1 \mapsto \mathrm{e}^{i\alpha} q_1, \qquad W_\mu \mapsto W_\mu + (\partial_\mu \alpha) \ , \ \mu = 0, 1.$$

As $W_\mu = \partial_\mu \phi$ for some $\phi \in \Omega^0(X)$, a solution of (34) can always be transfomed into a special form of the following type through a local H-gauge:

$$(q_1, W_0, W_1) = (Q, 0, 0) \quad (\Longleftrightarrow (q_1, V_0, V_1) = (Q, 2|Q|^2, 0) \ ) \ . \tag{35}$$

The above function Q satisfies the equation (NLS), and it is unique up to a multiplicative constant of modulo 1. Hence we have obtained the following result:

Theorem 4. There is an one-to-one correspondence of the following data:
(I) $\vec{S}(t, x)$, a solution (HM) with $\vec{S}(0, 0) = e$.
(II) $Q(t, x)$, a solution (NLS), modulo the relation $Q(t, x) \sim cQ(t, x)$ with $c \in \mathbb{C}, |c| = 1$.

The above functions $\vec{S}, Q$, are related by

$$\begin{cases} \partial_1 \vec{S} & = Q\vec{t}_- + \overline{Q}\vec{t}_+ \ , \\ \partial_1 \vec{t}_+ & = -2Q\vec{S} \ , \\ \partial_0 \vec{S} & = i(\partial_1 Q)\vec{t}_- + \overline{i(\partial_1 Q)}\vec{t}_+ \ , \\ \partial_0 \vec{t}_+ & = -2i(\partial_1 Q)\vec{S} + 2i|Q|^2\vec{t}_+ \ , \end{cases}$$

for some lifting pair $(\vec{S}, \vec{t}) \in \mathcal{C}^\infty(X, SO(3))$ of $\vec{S}$. (Note that such a lifting always exists, which is unique up to a rotation on tangent planes of S^2 by a constant angle). □

If the functions W_0, W_1 in (34) are constant functions, say $W_0 = \rho$, $W_1 = v$, the equation of q_1 has the form

$$i(\partial_0 - 2v\partial_1)q_1 + \partial_1^2 q_1 + 2|q_1|^2 q_1 + (\rho - v^2)q_1 = 0 \ . \tag{36}$$

Using the H-gauge tranformation by setting $W + d(\rho t + vx) = 0$, one obtains a solution of (NLS),

$$Q(t,x) = \mathrm{e}^{-i(\rho t + vx)} q_1(t,x) \ .$$

In the case of $\rho = v^2$, one can reduce Eq.(36) to (NLS) by the Galileo tranformation φ_v. Hence by starting from one solution Q of (NLS), one produces another solution by the relation:

$$Q_v(t,x) := (\varphi_v)^*(\mathrm{e}^{i(v^2 t + vx)} Q) = \mathrm{e}^{i(-v^2 t + vx)} Q(t, x - 2vt) \ .$$

Therefore we have shown the following result:

Proposition 7. The map

$$Q(t,x) \mapsto Q_v(t,x) := \mathrm{e}^{i(-v^2 t + vx)} Q(t, x - 2vt) \ , \quad v \in \mathbb{R} \ ,$$

defines an 1-parameter family of solutions of the equation (NLS). □

One can describe the above family through the following symmetries of the (q, A)-system:

$$q \mapsto \mathrm{e}^{i(v^2 t + vx)} q \ , \quad A \mapsto A + i(v^2 dt + v dx) \ , \quad \varphi_v : (t,x) \mapsto (t, x - 2vt) \ .$$

The expression of Q by Q_v is given by

$$Q(t,x) = (\varphi_{-v})^*(\mathrm{e}^{i(v^2 t - vx)} Q_v) \ . \tag{37}$$

By (35), one can obtain the zero-curvature representation of a solution Q of (NLS) through the process of Section 3. In fact, the linear system (6) associated to Q has the form:

$$\begin{aligned} \partial_1 g &= g U_0(t,x) \ , \quad U_0(t,x) = \begin{pmatrix} 0 & -\bar{Q} \\ Q & 0 \end{pmatrix} \ , \\ \partial_0 g &= g V_0(t,x), \quad V_0(t,x) = i \begin{pmatrix} |Q|^2 & \partial_1 \bar{Q} \\ \partial_1 Q & -|Q|^2 \end{pmatrix} \ , \end{aligned} \tag{38}$$

here $g \in \mathcal{C}^\infty(\mathbb{R}^2, SU(2))$. The curvature of the above connection J is given by

$$F(J) = -i \begin{pmatrix} 0 & -i\partial_0 \bar{Q} + \partial_1^2 \bar{Q} + 2|Q|^2 \bar{Q} \\ i\partial_0 Q + \partial_1^2 Q + 2|Q|^2 Q & 0 \end{pmatrix}$$

The Zakharov-Shabat spectral representation of Q is the expression,

$$\begin{aligned} \partial_1\psi_\lambda = \psi_\lambda U(t,x;\lambda)\ , \quad U(t,x;\lambda) &= \begin{pmatrix} \frac{i}{2}\lambda & -\bar{Q} \\ Q & \frac{-i}{2}\lambda \end{pmatrix} = U_0(t,x) + \lambda U_1(t,x)\ , \\ \partial_0\psi_\lambda = \psi_\lambda V(t,x;\lambda)\ , \quad V(t,x;\lambda) &= i \begin{pmatrix} |Q|^2 - \frac{1}{2}\lambda^2 & \partial_1\bar{Q} - i\lambda\bar{Q} \\ \partial_1 Q + iQ\lambda & -|Q|^2 + \frac{1}{2}\lambda^2 \end{pmatrix} \\ &= V_0(t,x) + \lambda V_1(t,x) + \lambda^2 V_2(t,x)\ , \end{aligned} \tag{39}$$

where $U_1 = e^3, V_1 = -U_0, V_2 = -U_1$ and $\lambda \in \mathbb{C}$, (Note that the Q here is equal to $-i\Psi$ in [8]). One can relate the above representation to the Galileo symmetries of (NLS) and derive it form the Q_v-family of Proposition 7. In fact, there associates a linear system (38) for each Q_v. With the local H-gauge transformation,

$$h = \mathrm{e}^{\alpha e^3}\ , \quad \alpha(t,x) = v^2 t - vx\ ,$$

the linear system is transformed into the following one:

$$\partial_1 g_v = g_v \begin{pmatrix} -\frac{i}{2}v & -\mathrm{e}^{-i(v^2t-vx)}\bar{Q}_v \\ \mathrm{e}^{i(v^2t-vx)}Q_v & \frac{i}{2}v \end{pmatrix}\ , \ \partial_0 g_v = g_v \begin{pmatrix} i|Q_v|^2 + \frac{i}{2}v^2 & i\mathrm{e}^{-i(v^2t-vx)}\partial_1\bar{Q}_v \\ i\mathrm{e}^{i(v^2t-vx)}\partial_1 Q_v & -i|Q_v|^2 - \frac{i}{2}v^2 \end{pmatrix}\ .$$

Then apply the Galileo transformation φ_{-v} to the above system. By the relation (37), we obtain the expression (39) for $\lambda = -v$ with $\psi_\lambda = \varphi_\lambda^*(g_{-\lambda})$.

7. Chern-Simons Theory and Bäcklund Transformation

We are going to discusss the nonlinear Schrödinger equation of the previous section from the 3-dimensional Chern-Simons theory point of view. The spectral parameter will be interpreted as a 3 dimensional zero-curvature condition through the dimensional reduction. We set $X = \mathbb{R}^3$ in the discussion of this section with the coordinates (x^0, x^1, x^2). Consider the equation

$$\partial_0\vec{S} = \vec{S} \times (\partial_1^2 + \partial_2^2)\vec{S}\ , \quad \vec{S} \in \mathcal{C}^\infty(X, S^2)\ . \tag{40}$$

Let (A, q) be the solution of (8) corresponding to $\vec{S}$ in Proposition 3. Introduce the complex coordinate of (x^1, x^2)-plane,

$$z = x^1 + ix^2\ .$$

An 1-form q of X is decomposed into $dx^0, dz, d\bar{z}$-components:

$$q = q^0 + q' + q''\ , \quad q^0 = q_0 dx^0\ ,\ q' = q_z dz\ , \quad q'' = q_{\bar{z}} d\bar{z}\ .$$

Hence one has the corresponding decomposition of the differential d on forms of X,

$$d = d^0 + d' + d''\ .$$

In particular, we have the decomposition of a $U(1)$-connection A and the covariant differential d_A:

$$\begin{aligned} &A = A^0 + A' + A'' \qquad \text{with } \overline{A^0} = -A^0\ ,\ \ \overline{A'} = -A''\ , \\ &d_A = d_A^0 + d_A' + d_A''\ ,\ \ d_A^0 = d^0 + A^0\ ,\ \ d_A' = d' + A'\ ,\ \ d_A'' = d'' + A''\ . \end{aligned}$$

For a solution (A, q) of Eq.(8), its $(z\bar{z}), (x^0 z), (x^0\bar{z})$-components are the expressions:

$$(41) \qquad \begin{cases} d'A'' + d''A' = 2(q'\overline{q'} + q''\overline{q''}) \ , \\ d'_A q'' + d''_A q' = 0 \ , \end{cases} \text{i.e.} \begin{cases} \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = 2(|q_z|^2 - |q_{\bar{z}}|^2) \ , \\ (\partial_A)_z q_{\bar{z}} = (\partial_A)_{\bar{z}} q_z \ , \end{cases}$$
$$\begin{cases} d^0 A' + d'A^0 = 2(q^0\overline{q''} + q'\overline{q^0}) \ , \\ d^0_A q' + d'_A q^0 = 0 \ , \end{cases} \text{i.e.} \begin{cases} \partial_0 A_z - \partial_z A_0 = 2(q_0\overline{q_{\bar{z}}} - \overline{q_0} q_z) \ , \\ (\partial_A)_0 q_z = (\partial_A)_z q_0 \ , \end{cases}$$
$$\begin{cases} d^0 A'' + d''A^0 = 2(q^0\overline{q'} + q''\overline{q^0}) \ , \\ d^0_A q'' + d''_A q^0 = 0 \ , \end{cases} \text{i.e.} \begin{cases} \partial_0 A_{\bar{z}} - \partial_{\bar{z}} A_0 = 2(q_0\overline{q_z} - \overline{q_0} q_{\bar{z}}) \ , \\ (\partial_A)_0 q_{\bar{z}} = (\partial_A)_{\bar{z}} q_0 \ . \end{cases}$$

The corresponding constraint of (A, q) to a solution of (40) is given by

$$q_0 = i(\partial_A)_1 q_1 + i(\partial_A)_2 q_2 = 4i(\partial_A)_z q_{\bar{z}} = 4i(\partial_A)_{\bar{z}} q_z \ .$$

By the relation

$$4(\partial_A)_z(\partial_A)_{\bar{z}} = (\partial_A)_1^2 + (\partial_A)_2^2 + 2(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z) \ ,$$

and the change of variables,

$$B_0 = A_0 + 4i(|q_z|^2 + |q_{\bar{z}}|^2) \ , \quad B_1 = A_1, \ B_2 = A_2 \ ,$$

Eq. (41) is transformed into the system:

$$(42) \qquad \begin{cases} q_0 = 4i(\partial_B)_z q_{\bar{z}} = 4i(\partial_B)_{\bar{z}} q_z \ , \\ \partial_z B_{\bar{z}} - \partial_{\bar{z}} B_z = 2(|q_z|^2 - |q_{\bar{z}}|^2) \ , \\ \partial_0 B_z - \partial_z B_0 = 8i(\overline{q_{\bar{z}}}(\partial_B)_z q_{\bar{z}} + q_z\overline{(\partial_B)_{\bar{z}} q_z}) - 4i\partial_z(|q_z|^2 + |q_{\bar{z}}|^2) \ , \quad z \leftrightarrow \bar{z} \ , \\ i(\partial_B)_0 q_z + ((\partial_B)_1^2 + (\partial_B)_2^2) q_z + 8|q_z|^2 q_z = 0 \ , \quad z \leftrightarrow \bar{z} \ . \end{cases}$$

Following arguments in [16], we perform the dimensional reduction, and look for solutions independent of x_2. Denote

$$\Psi_+ = \frac{1}{2} q_{\bar{z}} \ , \quad \Psi_- = \frac{1}{2} q_z \ .$$

By the change of variables,

$$C_0 = B_0 - iB_2^2 \ , \quad C_1 = B_1 \ , \quad C_2 = B_2 \ ,$$

the system (42) becomes

$$(43) \qquad \begin{cases} \partial_0 C_1 - \partial_1 C_0 = 0, \\ i(\partial_C)_0 \Psi_\pm + (\partial_C)_1^2 \Psi_\pm + 2|\Psi_\pm|^2 \Psi_\pm = 0 \ , \\ q_0 = 4i(\partial_1 + 2C_z)\Psi_+ = 4i(\partial_1 + 2C_{\bar{z}})\Psi_- \ , \\ \partial_1 C_2 = 16i(|\Psi_+|^2 - |\Psi_-|^2) \ , \\ \partial_0 C_2 = -16[(\overline{\Psi_+}(\partial_C)_1\Psi_+ - \Psi_+\overline{(\partial_C)_1\Psi_+}) - (\overline{\Psi_-}(\partial_C)_1\Psi_- - \Psi_-\overline{(\partial_C)_1\Psi_-})] \ . \end{cases}$$

By a suitable $U(1)$-gauge trasnformation, i.e. a real-value function $\alpha = \alpha(t, x)$ with

$$C_0 - i\partial_0\alpha = 0 \ , \quad C_1 - i\partial_1\alpha = 0 \ , \quad \mathrm{e}^{i\alpha}\Psi_\pm = Q_\pm \ ,$$

one reduces (43) into the special form with $C_0 = C_1 = 0$,

$$(44) \qquad \begin{cases} i\partial_0 Q_\pm + \partial_1^2 Q_\pm + 2|Q_\pm|^2 Q_\pm = 0 \ , \\ \partial_1(Q_+ - Q_-) = iC_2(Q_+ + Q_-) \ , \\ \partial_1 C_2 = 16i(|Q_+|^2 - |Q_-|^2) \ , \\ \partial_0 C_2 = -16[(\overline{Q_+}\partial_1 Q_+ - Q_+\overline{\partial_1 Q_+}) - (\overline{Q_-}\partial_1 Q_- - Q_-\overline{\partial_1 Q_-})] \ , \\ q_0 = 4i(\partial_1 - iC_2)Q_+ \quad . \end{cases}$$

Solve the above C_2 by the expression:

$$iC_2 = \pm 4\sqrt{\eta^2 - |Q_+ - Q_-|^2} \ ,$$

where η is a constant depending on Q_+ and Q_-. Then the system (44) is equivalent to the Bäcklund transformation of (NLS) for $Q_\pm$:

$$\left\{ \begin{array}{l} i\partial_0 Q_\pm + \partial_1^2 Q_\pm + 2|Q_\pm|^2 Q_\pm = 0 \ , \\ \partial_1 Q_+ - \partial_1 Q_- = \pm 4\eta (Q_+ + Q_-)\sqrt{1 - |(Q_+ - Q_-)/\eta|^2} \ . \end{array} \right. \tag{45}$$

Following the above arguments, one can derive a linear system on X with the consistency condition (45). In fact, the three components of the connection J for the corresponding linear system (6) are the expressions:

$$\begin{aligned} J_0 &= 4i \begin{pmatrix} 2\eta^2 + 2(Q_+\overline{Q_-} + Q_-\overline{Q_+}) & \partial_1 \overline{Q_+} + 4\eta a \overline{Q_+} \\ \partial_1 Q_+ + 4\eta a Q_+ & -2\eta^2 - 2(Q_+\overline{Q_-} + Q_-\overline{Q_+}) \end{pmatrix} , \\ J_1 &= \begin{pmatrix} 0 & -2\overline{(Q_+ + Q_-)} \\ 2(Q_+ + Q_-) & 0 \end{pmatrix} , \\ J_2 &= -2\eta i \begin{pmatrix} a & \bar{b} \\ b & -a \end{pmatrix} , \end{aligned}$$

where $a = \mp\sqrt{1 - |(Q_+ - Q_-)/\eta|^2}$, $b = (Q_+ - Q_-)/\eta$. A key point of this approach is on the relation of J_2 and the global gauge tranformations for Zakharov-Shabat spectral representations (39) of $Q_\pm$:

$$\left\{ \begin{array}{l} \partial_1 \psi_+ = \psi_+ U_+(x^0, x^1; \lambda), \\ \partial_0 \psi_+ = \psi_+ V_+(x^0, x^1; \lambda) \ , \end{array} \right. \qquad \left\{ \begin{array}{l} \partial_1 \psi_- = \psi_- U_-(x^0, x^1; \lambda), \\ \partial_0 \psi_- = \psi_- V_-(x^0, x^1; \lambda) \ . \end{array} \right.$$

The Bäcklund transformation is described by the existence of a $SU(2)$-valued function $g(x^0, x^1; \lambda)$ which transforms ψ_- to ψ_+:

$$\psi_+(x^0, x^1; \lambda) = \psi_-(x^0, x^1; \lambda) g(x^0, x^1; \lambda) \ .$$

An equivalent form is

$$g^{-1}\partial_1 g = U_+ - g^{-1}U_- g \ , \quad g^{-1}\partial_0 g = V_+ - g^{-1}V_- g \ .$$

By the following identification of the spectral parameter λ with the coordinate x^2,

$$\frac{4\eta}{\lambda} = \tan x^2 \ ,$$

one then obtains the expression of $g(x^0, x^1; \lambda)$,

$$g(x^0, x^1; \lambda) = \mathrm{e}^{-x^2 \frac{J_2}{2\eta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos x^2 + i \begin{pmatrix} a & \bar{b} \\ b & -a \end{pmatrix} \sin x^2.$$

8. Flat SU(1,1)-Connection and Non-compact σ-Models

In this section we discuss models related to the $SU(1,1)$-connections. A notable one is the example of E. Witten on the cylindrical symmetry solutions of the 4-dimensional self-dual Yang-Mills equation in [24]. We are going to follow the same arguments of previous sections by replacing the group $SU(2)$ by $SU(1,1)$. An orthogonal basis of the Lie algebra $su(1,1)$ (w.r.t. the negative Killing bilinear form with signature $(-,-,+)$) consists of the elements,

$$e^1 = \frac{-1}{2}\sigma^1, \quad e^2 = \frac{-1}{2}\sigma^2, \quad e^3 = \frac{i}{2}\sigma^3.$$

The sequence (1) becomes

$$1 \longrightarrow \{\pm 1\} \longrightarrow SU(1,1) \longrightarrow SO(1,2) \longrightarrow 1 \ .$$

Consider the diagonal subgroup H of $SU(1,1)$, and identify the homogenous space $SU(1,1)/H$ with the hyperboloid in $su(1,1)$ having the base element $e = e^3$,

$$S = \{\sum_{j=1}^{3} x_j e^j \mid -x_1^2 - x_2^2 + x_3^2 = 1\} \ .$$

The tangent bundle $\mathbf{T}_S$ of S is a Hermitian line bundle with the metric induced by the Killing form of $su(1,1)$. Via the stereographic projection,

$$p : \mathbb{B}^1 \longrightarrow S \ , \quad \zeta \mapsto \vec{S} \ , \quad \text{with} \quad S_1 + iS_2 = \frac{2\zeta}{1-|\zeta|^2} \ , \quad S_3 = \frac{1+|\zeta|^2}{1-|\zeta|^2} \ ,$$

the hyperboloid S is isometric to the unit disc $\mathbb{B}^1$ with the Poincaré metric

$$\frac{d\zeta \otimes d\overline{\zeta}}{(1-|\zeta|^2)^2} \quad .$$

An equivalent model is the upper-half plane,

$$\mathbb{H}^1 = \{\tau = x + iy \in \mathbb{C} \mid y > 0\} \ , \quad ds^2 = \frac{d\tau \otimes d\overline{\tau}}{4y^2} \ .$$

Let J be a $SU(1,1)$-connection on a simply-connected manifold X,

$$J = -2Re(q)e^1 - 2Im(q)e^2 + Ve^3 = q\sigma^- + \overline{q}\sigma^+ + Ve^3 \ , \quad q \in \Omega^1(X)_{\mathbb{C}} \ , \quad V \in \Omega^1(X) \ .$$

The zero curvature condition of J can be formulated by the expression:

$$\left\{ \begin{array}{ll} F(A) = -2q\overline{q} \ , & \\ d_A q = 0 \ , & \text{for} \quad A = -iV \ , \end{array} \right. \tag{46}$$

which, by (5), is the consistency condition of the linear system of $\vec{S}, \vec{t}$:

$$\left\{ \begin{array}{ll} d\vec{S} & = q\vec{t}_- + \overline{q}\vec{t}_+ \ , \\ d_A\vec{t}_+ & = 2q\vec{S} \ . \end{array} \right. \tag{47}$$

With the same arguments as in Theorems 1 and 2, one has the following results:

Theorem 5. Let $\mathbf{L}$ be a Hermitian complex bundle over a simply-connected marked manifold X. There is an one-to-one correspondence of the following data:

(I) $\vec{S} \in \mathcal{C}^\infty(X, S)$ with $\vec{S}^*(\mathbf{T}_S) = \mathbf{L}$.

(II) (A, s), where A is a $U(1)$-connection on $\mathbf{L}$, $s \in \Omega^1(X, \mathbf{L})$, satisfying the relation

$$\left\{ \begin{array}{l} F(A) = -2ss^\dagger, \\ d_A(s) = 0. \end{array} \right. \tag{48}$$

□

Corollary. For $X = \mathbb{H}$ with the coordinate $z = t + ir$, there is an one-to-one correspondence between the following data:

(I) $\vec{S} : X \longrightarrow S$ with $\vec{S}(0,0) = e$, a solution of the equation

$$\partial_0 \vec{S} = -\vec{S} \times \partial_1 \vec{S} \ .$$

(II) (A, Φ), where A is a $U(1)$-connection, $\Phi \in \Omega^{1,0}(X)$, satisfying the relation:

$$\left\{ \begin{array}{l} F(A) = -2\Phi\Phi^* \ , \\ d''_A \Phi = 0 \ . \end{array} \right. \tag{49}$$

(III) $\phi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$, a solution of Liouville equation:

$$(\partial_0^2 + \partial_1^2)\phi = 8e^{\phi} \ . \tag{50}$$

□

One can derive the expression of a general solution of (50) :

$$\phi(z) = \log \frac{|\partial_z \zeta(z)|^2}{(1 - |\zeta(z)|^2)^2} \quad , \quad \zeta : \mathbb{H} \longrightarrow \mathbb{B} \ \text{ holomorphic} \ ,$$

or equivalently,

$$\phi(z) = \log \frac{|\partial_z \tau(z)|^2}{4Im(\tau(z))^2} \quad , \quad \tau : \mathbb{H} \longrightarrow \mathbb{H} \ \text{ holomorphic} \ .$$

We are going to relate the equations of the above Corollary with the self-dual Yang-Mills equation. By the change of variables,

$$A_0 = B_0 + \frac{i\kappa}{r} \ , \ A_1 = B_1 \ , \ \ \Phi = \frac{\phi}{r} dz \ ,$$

Eq.(49) is transformed into

$$\left\{ \begin{array}{l} ir^2(\partial_0 B_1 - \partial_1 B_0) = \kappa - 4\phi\overline{\phi} \\ (\partial_{\bar{z}} + B_{\bar{z}})\phi + \frac{i(\kappa-1)}{2r}\phi = 0 \ , \end{array} \right. \tag{51}$$

where κ is a real constant. One may apply a gauge transformation $h \in \mathcal{C}^\infty(X, H)$ with $h^{-1}dh = (d\alpha)e^3$ to the system (49). It gives rise the transformations of functions B_0, B_1, ϕ, in (51):

$$B_\mu \mapsto B_\mu - i\partial_\mu \alpha \qquad (\mu = 0, 1) \ , \quad \phi \mapsto e^{i\alpha}\phi \ .$$

By setting

$$\phi = e^{\frac{\psi}{2}} \quad \text{ with } \ \psi : \mathbb{H} \longrightarrow \mathbb{R} \ ,$$

one obtains the expression of B,

$$B = (\frac{\partial_z \psi}{2} - \frac{i(\kappa - 1)}{2r})dz - (\frac{\partial_{\bar{z}} \psi}{2} + \frac{i(\kappa - 1)}{2r})d\bar{z} \ .$$

Then ψ satisfies the following Liouville-like equation on a curved space-time for all κ :

$$r^2(\partial_0^2 + \partial_1^2)\psi = -2 + 8e^{\psi} \ .$$

With the identification,

$$iB_x = W_x \ , \ iB_y = W_y, \quad 2\phi = \varphi_1 + i\varphi_2 \ ,$$

Eq.(51) for $\kappa = 1$ is now equivalent to the system,

$$\left\{ \begin{array}{l} r^2(\partial_0 W_1 - \partial_1 W_0) = 1 - \varphi_1^2 - \varphi_2^2 \ , \\ \partial_0\varphi_1 + W_0\varphi_2 = \partial_1\varphi_2 - W_1\varphi_1 \ , \\ \partial_1\varphi_1 + W_1\varphi_2 = -(\partial_0\varphi_2 - W_0\varphi_1) \ , \end{array} \right.$$

which is the cylindrical symmetric self-dual Yang-Mills equation in [24].

9. Conclusions

We have treated in the present paper only models related to the gauge theory of $SU(2)$ or $SU(1,1)$, though the formulizm is valid for a general gauge group. The equivalent relation, established in this work, from one site is the well known gauge equivalence between integrable models on an arbitary background manifold in the framework of Lax pair, while from another site describes the σ-model representations. An important point is that there exists a systematic way of producing these Lax forms from symmetries of the equation. It suggests a possibile unified BF gauge theoretical approach to an integrable system and its underlying spectral problem [3] [5] [10] [12] [13]. Namely, any σ-model (on a curved background) is equivalent to a BF gauge field theory in a special gauge. If the model is integrable, then, the theory provides a complete description in terms of the Lax pair. Furthermore the global gauge structure appears as the spectral parameter, while Bäcklund transformations have the origin in the higher dimensional Chern-Simons theory. In this paper we have shown that the hidden symmetry of some integrable models with physical interests has its structure revealed much clearly in a mathematical formulation using the language of fiber bundles and connections.

Another source of main interests in our investigation stems from the similarity of Eq.(21) with the Seiberg-Witten equation in the electric-magnetic duality of $N = 2$ SUSM Yang-Mills theory [21]. Recently, the dimensional reduced Seiberg-Witten equations were studied and the set of equations shares a similar structure with the Hitchin equation except the distinction of Higgs field and Weyl spinor [14] [15]. In this paper, we have treated the cylindrical symmetric self-dual Yang-Mills equation of [24] through the framework of integrable systems. It is plausible that a similar argument could also apply to the equivalent rellation in [7] on the axially symmetric Bogomoly monopole equation and Ernst equation of general relativity. The hope is that our approach based on the exactly soluble models could possibly help to illuminate the analytical structure of Seiberg-Witten theory. Such a program is now under our investigation.

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