LOW REGULARITY SEMI-LINEAR WAVE EQUATIONS

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ABSTRACT. We prove local well-posedness results for the semi-linear wave equation for data in H^γ, $0 < \gamma < \frac{n-3}{2(n-1)}$, extending the previously known results for this problem. The improvement comes from an introduction of a two-scale Lebesgue space $X_k^{r,p}$.

1. INTRODUCTION

We consider the initial value problem for the semi-linear wave equation

$$\begin{aligned} \Box u &= F(u) \\ u(0,\cdot) &= f \in H^\gamma(\mathbf{R}^n) \\ \partial_t u(0,\cdot) &= g \in H^{\gamma-1}(\mathbf{R}^n) \end{aligned} \tag{1}$$

where $n \geq 2$, u is scalar or vector valued on $\mathbf{R}^+ \times \mathbf{R}^n$, $\Box = -\frac{\partial^2}{\partial t^2} + \Delta$ is the D'Alembertian, $p > 1$, $\gamma \geq 0$ and the nonlinearity $F = F_p \in C^0$ satisfies[1]

$$F_p(0) = 0, \quad |F_p(u) - F_p(v)| \lesssim |u - v| \, (|u|^{p-1} + |v|^{p-1}). \tag{2}$$

We say that the problem (1) is locally well-posed in H^γ if, for every $(f, g) \in H^\gamma \times H^{\gamma-1}$, one can find a time[2] $T > 0$ and a unique weak solution $u \in C([0,T]; H^\gamma) \cap X$ to (1) which depend continuously on the data, where X is some additional Banach space.

The question of determining the triples (γ, p, n) for which (1) is locally well-posed in H^γ was studied for higher dimensions and nonlinearities by several authors, including [2], [9], [14], [13], [12]. We summarize the known results below.

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[1]When γ is large (e.g. $\gamma > 1/2$, or $\gamma > 3/2$) more regularity may be needed on F_p; see [14]. However, we will only be concerned with the low-regularity problem, and such issues will not arise.

[2]We will not concern ourselves with the exact dependence of T on the data. In practice, one can control T by the $H^\gamma \times H^{\gamma-1}$ norm of the data unless (3) is satisfied with equality, in which case T depends on the data itself rather than its norm.

Proposition 1.1. [9, 14, 12, 13] *In order for* (1) *to be locally well-posed in* H^γ *for general non-linearities* F *satisfying* (2) *the following two conditions are necessary:*

$$p(\frac{n}{2} - \gamma) \leq \frac{n+4}{2} - \gamma \qquad \textit{(Scaling)} \tag{3}$$

$$p(\frac{n+1}{4} - \gamma) \leq \frac{n+5}{4} - \gamma \qquad \textit{(Concentration)} \tag{4}$$

Conversely, if the above two conditions are satisfied and

$$p(\frac{n+1}{4} - \gamma) \leq \frac{n+1}{2n}(\frac{n+3}{2} - \gamma) \tag{5}$$

then (assuming sufficient regularity on F *if* γ *is large)* (1) *is locally well-posed in* H^γ*, with the exception of the case*

$$n = 3, \quad p = 2, \quad \gamma = 0, \tag{6}$$

which can be locally ill-posed.

For $n \geq 3$ one has the following simultaneous endpoint of (4) and (5):

$$\gamma = \gamma_0 = \frac{n-3}{2(n-1)} \qquad p = p_0 = \frac{(n+1)^2}{(n-1)^2 + 4}. \tag{7}$$

For $n = 3$ this is (6), which was shown in [13] to be locally ill-posed for $F(u) = -|u|^2$. For $n > 3$ (7) was shown to be locally well-posed in [12]. The other results in the above proposition may be found in [14], and also to a large extent in [9].

When $n \leq 3$ or when $\gamma \geq \gamma_0$ the above results form a complete answer to the question posed earlier, at least for general power-type non-linearities. In this paper we consider the high dimension, low-regularity case $n > 3$, $0 < \gamma < \gamma_0$. Our main result is the following.

Theorem 1.2. *Suppose* $0 < \gamma < \gamma_0$*. Then if* (4) *holds and*

$$p(\frac{n}{4} - \gamma) \leq \frac{1}{2}(\frac{n+3}{2} - \gamma), \tag{8}$$

then (1) *is locally well-posed in* H^γ *for all non-linearities satisfying* (2)*, with the possible exception of the simultaneous endpoint of* (4) *and* (8)

$$\gamma = \frac{n+1}{4} - \frac{1}{p-1} = \frac{n+3-\sqrt{n^2-2n+33}}{8}. \tag{9}$$

We note in passing that identical results can be obtained for the semi-linear Klein-Gordon equation by treating the mass term as an additional "non-linearity", which can be treated by (e.g.) energy estimates.

These results are compared with the existing results in Figure 1 in the case $n = 4$, which is already typical. The scaling example (which gives (3)) shows that ill-posedness is possible in the region E, while for non-radial data the concentration example (which gives (4)) shows ill-posedness is possible in F. (For the radial problem one has well-posedness everywhere above E; see [14]). In [9] well-posedness was shown for a certain region A, and extended to include B in [14], including all of the boundary except for the endpoint c corresponding to (7), which was shown to be well-posed in [12]. Our results extend the positive results to the region C including the boundary, with the exception of the endpoint d corresponding to (9). The points a and b represent the well-studied H^1-critical problem and conformally invariant problem $(\gamma, p) = (1, \frac{n+2}{n-2}), (\frac{1}{2}, \frac{n+3}{n-1})$ respectively.

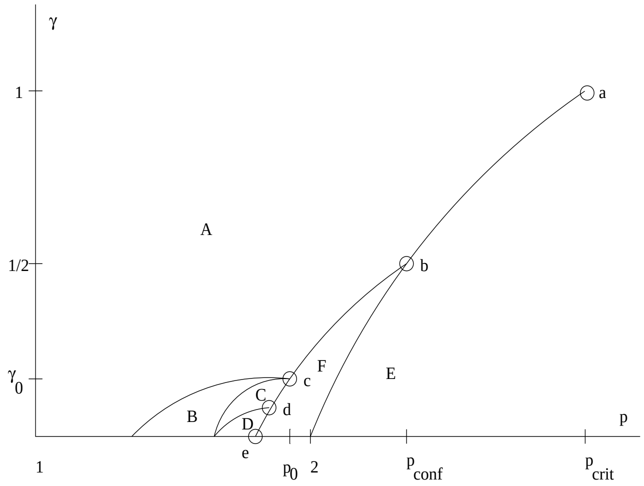

FIGURE 1. Local well-posedness results for $n = 4$.

We now motivate our attack strategy. We start with the observation that one can use standard Strichartz estimates to obtain well-posedness for the frequency-localized equation

$$\Box u = S_j F(u), \quad u(0) = S_j f, \quad u_t(0) = S_j g \tag{10}$$

all the way down to (3) and (4); here S_j is a Littlewood-Paley projection onto a fixed frequency range $|\xi| \sim 2^j$. We illustrate this with the problem

$$n = 4, \quad \gamma = 0, \quad p = \frac{9}{5},$$

which is the endpoint e in Figure 1. We will use a judiciously chosen Strichartz estimate[3] for the linear wave equation (see [12]) applied to (10), namely

$$\|\sqrt{-\Delta}^{-\frac{5}{54}} u\|_{L_t^{18} L_x^{54/25}} + \|u(T)\|_2 \lesssim \|\sqrt{-\Delta}^{-\frac{1}{6}} S_j F(u)\|_{L_t^2 L_x^{6/5}} + \|S_j f\|_2 + \|S_j g\|_{H^{-1}},$$

where time is restricted to $t \in [0, T]$ for some $T > 0$. Because we are localizing to frequencies $|\xi| \sim 2^j$, this estimate becomes

$$2^{-\frac{5}{54} j} \|u\|_{L_t^{18} L_x^{54/25}} + \|u(T)\|_2 \lesssim 2^{-\frac{1}{6} j} \||u|^{9/5}\|_{L_t^2 L_x^{6/5}} + \|f\|_2 + \|g\|_{H^{-1}}.$$

[3]The choice of exponents here is not unique; we are using the endpoint exponents $(2, 6)$ for the sake of concreteness only.

Also, Hölder's inequality gives

$$\| |u|^{\frac{9}{5}} \|_{L^2_t L^{6/5}_x} \lesssim T^{\frac{2}{5}} \|u\|^{\frac{9}{5}}_{L^{18}_t L^{54/25}_x}.$$

Combining these two inequalities we obtain

$$M + \|u(T)\|_2 \lesssim T^{\frac{2}{5}} M^{\frac{9}{5}} + \|f\|_2 + \|g\|_{H^{-1}}$$

where $M = 2^{-\frac{5}{54}j} \|u\|_{L^{18}_t L^{54/25}_x}$. Thus a continuity argument shows that the L^2 norm of $u(T)$ is controlled by the data for sufficiently small T. By adapting this inequality to differences of solutions and setting up an iteration scheme one can also obtain local well-posedness for this frequency-localized problem; we omit the details.

We have just seen that there are no obstructions to local well-posedness other than concentration and scaling if the frequencies are prevented from interacting. To deal with the original problem (1), we must therefore control the extent to which the 2^k frequency piece (say) of $F(u)$ is affected by the 2^j frequency piece of u, where j is much larger or much smaller than k. Because this is a low regularity problem, the high frequencies are less well behaved than the low frequencies, so one expects the worst type of interaction to be when $j \gg k$. This interaction cannot be adequately controlled by the norms used above for the problem (10), because of the presence of negative derivatives. This explains the presence of conditions such as (5) in previous work on the low regularity problem.

Fortunately, one can partially control this interaction with the smoothing effect of low-frequencies. A portion of $F(u)$ at frequency 2^k must necessarily be spread out at the spatial scale of 2^{-k}, according to the uncertainty principle. Thus, if one takes a portion of u with frequency $2^j \gg 2^k$ which is concentrated on a set which is much "thinner" than 2^{-k}, then its contribution to the 2^k-frequency portion of $F(u)$ will be moderated by this averaging effect at scale 2^{-k}. From examining the shape of standard examples such as the Knapp example, we see that it is indeed reasonable to expect the high-frequency portions of u to be "thin", at least for the linear problem.

To take advantage of this effect we need a measure of how thin the support of u is compared to the spatial scale 2^{-k}. To this end we introduce a two-scale Lebesgue space $X^{r,p}_k(\mathbf{R}^n)$ defined for $1 \leq r, p \leq \infty$ and non-negative integers k by

$$\|u\|_{X^{r,p}_k} = \left(\sum_Q \|u\|^r_{L^p(Q)} \right)^{1/r}, \tag{11}$$

where Q ranges over all dyadic cubes in $\mathbf{R}^n$ of sidelength 2^{-k}. (A similar norm, albeit in frequency space rather than physical space, has appeared in [4], [16]). The above heuristic about the high-frequency portion of solutions being "thin" can then be captured by some Strichartz estimates for the $X^{r,p}_k$ spaces that improve upon what can be obtained by the usual L^r_x estimates and elementary inequalities. The smoothing effect alluded to above is captured by an easy reverse Hölder inequality for the low-frequency pieces of functions in $X^{r,p}_k$. These improvements allow us to relax (5) to (8).

In the region D in Figure 1, (8) fails, and the $X^{r,p}_k$ estimates are not powerful enough to effectively control the frequency-interference behaviour of the non-linearity. Indeed, it seems that one cannot go below (8) using norms that rely only on the size and shape of (various frequency pieces of) u and $F(u)$. Nevertheless, one may still conjecture that one has well-posedness in the region D (except perhaps for the endpoint e). One possibility is that the solution exhibits some additional regularity along null directions, so that one may control it by (say) the $X^{s,b}$ spaces as employed in [1], [5], [10], [11] and elsewhere; however the non-algebraic nature of the non-linearity F seems to place this approach beyond the level of current technology, as one cannot work exclusively in frequency space.

This paper is organized as follows. In the next section we set out our notation and collect many basic properties of the $X^{r,p}_k$ spaces and the Littlewood-Paley decomposition that we will need. For technical reasons concerning endpoint results we will also need a somewhat refined bilinear interpolation theorem. In the third section we prove the Strichartz estimate we will need for this problem, which involves both $X^{r,p}_k$ and L^r_x spaces. In the last section we use this estimate together with estimates on the non-linearity to prove the local well-posedness results.

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2. Notation and preliminaries

Throughout the paper, we will be working in a fixed dimension $n > 3$, and r_0, r_0', γ_0 will denote the exponents

$$r_0 = \frac{2(n-1)}{n-3}, \quad r_0' = \frac{2(n-1)}{n+1}, \quad \gamma_0 = \frac{n-3}{2(n-1)}.$$

Note that $2 < r_0 < \infty$.

Definition 2.1. If $n > 3$, then an pair of exponents (q, r) is called *sharp wave-admissible* if

$$\frac{1}{q} + \frac{(n-1)/2}{r} = \frac{(n-1)/2}{2} \tag{12}$$

and $2 \leq q, r \leq \infty$, or (equivalently) if $(\frac{1}{q}, \frac{1}{r})$ lies on the closed line segment between $(\frac{1}{2}, \frac{1}{r_0})$ and $(0, \frac{1}{2})$.

Most of our estimates will involve sharp wave-admissible pairs of exponents; estimates using other pairs are certainly possible, but they can usually be obtained from the sharp estimates via Sobolev embedding or Hölder's inequality.

For any radial function m, define the multiplier $m(\sqrt{-\Delta})$ by

$$(m(\sqrt{-\Delta})f)\check{}(\xi) = m(\xi)\hat{f}(\xi).$$

Define a Littlewood-Paley cutoff to be any non-negative radial bump function supported on an annulus of the form $\{|\xi| \sim 1\}$ which is positive on $\{\frac{1}{4} \leq |\xi| \leq 4\}$. If f is a function and j is an integer, we use $S_j f$ to denote the 2^j Littlewood-Paley frequency piece of f:

$$S_j f = \beta(2^j\sqrt{-\Delta})f;$$

for technical reasons the exact choice of β used to define S_j may vary from line to line, but this is not a serious problem since a Littlewood Paley projection for one β can always be controlled (in virtually any space) by a finite number of such projections for any other β. Henceforth we will ignore this technicality.

We also define the projection $P_0 = \phi(\sqrt{-\Delta})$, where ϕ is a non-negative radial bump function which equals 1 on the ball $\{|\xi| \leq 4\}$.

The projections P_0 and S_j are bounded on every L^r_x space and every $X^{r,p}_k$ space, $1 \leq r, p \leq \infty$. In particular, we have the estimate

$$\|f\|_{X^{r,p}_k} \lesssim \|P_0 f\|_{X^{r,p}_k} + \sum_{j \geq 0} \|S_j f\|_{X^{r,p}_k} \tag{13}$$

from the triangle inequality, some multiplier calculus, and the above observation.

We now collect some useful facts about the spaces defined in (11). Firstly, when $p = r$ these spaces are just the Lebesgue spaces $X^{r,r}_k = L^r$. Since $l^a \subset l^b$ for $a < b$ (by e.g. Young's inequality) one has the inclusion

$$\|f\|_{X^{b,p}_k} \lesssim \|f\|_{X^{a,p}_k} \quad \text{for } a < b. \tag{14}$$

By Hölder's inequality we have a similar inclusion for the p index:

$$\|f\|_{X^{r,p}_k} \lesssim 2^{(\frac{1}{q}-\frac{1}{p})nk}\|f\|_{X^{r,q}_k} \quad \text{when } p < q. \tag{15}$$

In particular, we have

$$\|f\|_{X^{r,p}_k} \lesssim 2^{(\frac{1}{r}-\frac{1}{p})nk}\|f\|_r \quad \text{when } p < r. \tag{16}$$

If we localize in frequency we can reverse the above Hölder inequality and improve[4] on (14).

[4]These two lemmas can also be viewed as special cases of Sobolev embedding.

Lemma 2.2. *(Reverse Hölder inequality) If $1 \leq a \leq \infty$, then for any Schwarz function f and any $j \leq k$ we have*

$$\|S_j f\|_a \lesssim 2^{\frac{nk}{a'}} \|f\|_{X_k^{a,1}}.$$

Proof This is trivial for $a = 1$, so it suffices to verify the case $a = \infty$. By dilation invariance we may take $k = 0$. Since $j \leq 0$ we have the reproducing formula

$$S_j f = S_j(f * \phi)$$

where ϕ is a Schwarz function whose Fourier transform equals 1 on $\{|\xi| \leq 4\}$. Since S_j is bounded on L^∞, we have reduced ourselves to showing that

$$\sup_x |f * \phi(x)| \lesssim \|f\|_{X_0^{\infty,1}}.$$

Fix x. From trivial estimates we have

$$|f * \phi(x)| \leq \sum_Q \int_Q |f(y)||\phi(x-y)| \, dy \lesssim \|f\|_{X_0^{\infty,1}} \sum_Q \sup_{y \in Q} |\phi(x-y)|,$$

where Q ranges over unit cubes. But from the rapid decrease of ϕ we have

$$\sum_Q \sup_{y \in Q} |\psi(x-y)| \lesssim 1$$

uniformly in x_0, and we are done. ∎

Lemma 2.3. *(Young's inequality) If $1 \leq a \leq b \leq \infty$, $1 \leq p \leq \infty$, and $k \geq 0$, then*

$$\|P_0 f\|_{X_k^{b,p}} \lesssim 2^{-nk(\frac{1}{a} - \frac{1}{b})} \|f\|_{X_k^{a,p}}.$$

Proof By interpolation it suffices to prove this for $p = 1$ or $p = \infty$; by duality we need only consider $p = 1$. Since the estimate is trivial for $a = b$, we only need consider the case $a = 1$, $b = \infty$. The estimate now becomes

$$\|P_0 f\|_{X_k^{\infty,1}} \lesssim 2^{-nk} \|f\|_1.$$

But this is an immediate consequence of (15) and the trivial estimate

$$\|P_0 f\|_{X_k^{\infty,\infty}} \lesssim \|f\|_1.$$

∎

Finally we observe that while the spaces $X_k^{r,p}$ are not perfectly translation invariant, they are almost invariant in the sense that the translation operators are uniformly bicontinuous in $X_k^{r,p}$.

We define the space-time function spaces $L_t^q L_x^r$ and $L_t^q X_k^{r,p}$ by

$$\|F\|_{L_t^q L_x^r} = \left(\int \|F(t)\|_r^q \, dt \right)^{1/q}$$

and

$$\|F\|_{L^q_t X^{r,2}_k} = \left(\int \|F(t)\|^q_{X^{r,2}_k} \, dt \right)^{1/q},$$

with the obvious modification for $q = \infty$. The time integration will usually be on a compact interval such as $0 \le t \le 1$. Also we use H^γ to denote the inhomogeneous Sobolev spaces $(1 + \sqrt{-\Delta})^{-\gamma} L^2$, and $C(H^\gamma)$ to denote those spacetime functions which are in H^γ continuously with respect to the time variable; we give $C(H^\gamma)$ the same norm as $L^\infty_t H^\gamma$. We will not use the homogeneous spaces $\dot H^\gamma = \sqrt{-\Delta}^{-\gamma} L^2$ much, although most of our results can be transferred to these spaces.

We now address the problem of interpolation between the $X^{r,p}_k$ spaces, for fixed k; such interpolation was already used in the above lemmas. Since these spaces are equivalent to mixed Lebesgue spaces $l^r(L^p(Q))$ for a fixed 2^{-k}-cube Q, the standard interpolation theorems (e.g. the Riesz convexity theorem) apply. In particular the spaces $X^{r,2}_k$ behave like Hilbert-space valued L^r spaces, and so obey virtually all the interpolation identities that the scalar L^r spaces do.

Finally, we will also need a certain bilinear real interpolation theorem[5] which we state as follows. One can also prove this theorem by more explicit methods; see [12].

Proposition 2.4. *Fix $k \in \mathbf{Z}$ and $2 < a_0, b_0 < \infty$, and suppose that $\{T_i(F,G) : i \in \mathbf{Z}\}$ are a family of bilinear forms such that one has the estimate*

$$|2^{\beta(a,b)i} T_i(F,G)| \lesssim \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2 X^{b',2}_k}$$

uniformly in i for all $(\frac{1}{a}, \frac{1}{b})$ in a neighbourhood of $(\frac{1}{a_0}, \frac{1}{b_0})$, where $\beta(a,b)$ is an affine function of $\frac{1}{a}$ and $\frac{1}{b}$ which is not constant with respect to either of the two variables. Then one has

$$\sum_i |2^{\beta(a_0,b_0)i} T_i(F,G)| \lesssim \|F\|_{L^2_t L^{a_0'}_x} \|G\|_{L^2 X^{b_0',2}_k}.$$

Proof We introduce some notation, following [3] and [25]. If A_0, A_1 are Banach spaces contained in some larger space A, we define the real interpolation spaces $(A_0, A_1)_{\theta,q}$ for $0 < \theta < 1$, $1 \le q \le \infty$ via the norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \Big(\int_0^\infty (t^{-\theta} K(t,a))^q \frac{dt}{t} \Big)^{1/q},$$

where

$$K(t,a) = \inf_{a = a_0 + a_1} \|a_0\|_{A_0} + t \|a_1\|_{A_1}.$$

We have the inclusions

$$(L^2_t L^{p_0}_x, L^2_t L^{p_1}_x)_{\theta,2} = L^2_t L^{p,2}_x \subset L^2_t L^p_x$$

[5] It is possible to recover the non-endpoint results in this paper without recourse to this Proposition, or to the endpoint Strichartz estimates in [12]. More precisely, one can prove Theorem 1.2 using more standard interpolation methods provided that (4) and (8) are satisfied with strict inequality. We omit the details.

whenever $p_0 \neq p_1$, $p_0, p_1 \leq 2$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$; see [25] Sections 1.18.2 and 1.18.6 for the interpolation identity, and [17] for the Lorentz space inclusion. One also has the vector-valued analogue of the above inclusion:

$$(L^2_t X^{p_0,2}_k, L^2_t X^{p_1,2}_k)_{\theta,2} \subset L^2_t X^{p,2}_k.$$

Similarly, we have

$$(l^{s_0}_\infty, l^{s_1}_\infty)_{\theta,1} = l^s_1$$

whenever $s_0 \neq s_1$ and $s = (1-\theta)s_0 + \theta s_1$, where $l^s_q = L^q(\mathbf{Z}, 2^{js}\ dj)$ are weighted sequence spaces and dj is counting measure. See [3] Section 5.6.

We will use the following bilinear interpolation theorem:

Lemma 2.5. *([3], Section 3.13.5(b)) If A_0,A_1,B_0,B_1,C_0,C_1 are Banach spaces, and the bilinear operator T is bounded from*

$$T : A_0 \times B_0 \to C_0$$
$$T : A_0 \times B_1 \to C_1$$
$$T : A_1 \times B_0 \to C_1,$$

then one has

$$T : (A_0, A_1)_{\theta_0,2} \times (B_0, B_1)_{\theta_1,2} \to (C_0, C_1)_{\theta,1}$$

whenever $0 < \theta_0, \theta_1 < \theta < 1$ are such that $\theta = \theta_0 + \theta_1$.

Let $T(F, G)$ denote the sequence-valued bi-linear operator

$$T(F, G) = \{T_i(F, G)\}_{i \in \mathbf{Z}}.$$

Then we have

$$T : L^2_t L^{a'}_x \times L^2 X^{b',2}_k \to l^{\beta(a,b)}_\infty$$

for all $(\frac{1}{a}, \frac{1}{b})$ in a neighbourhood of $(\frac{1}{a_0}, \frac{1}{b_0})$. Applying the above lemma for suitable values of (a, b) and using the above inclusions, one obtains

$$T : L^2_t L^{a'}_x \times L^2 X^{b',2}_k \to l^{\beta(a,b)}_1$$

for all $(\frac{1}{a}, \frac{1}{b})$ in a neighbourhood of $(\frac{1}{a_0}, \frac{1}{b_0})$. Applying this to $(a, b) = (a_0, b_0)$ one obtains the desired result. ■

3. Two-scale Strichartz estimates

In this section time will always be localized to the interval $0 \leq t \leq 1$, f, g, and F will denote Schwarz functions on $\mathbf{R}^n$, $\mathbf{R}^n$, and $[0, 1] \times \mathbf{R}^n$ respectively, and j and k will denote *non-negative* integers.

If u is the solution to the linear Cauchy problem

$$\Box u = F, \quad u(0) = f, \quad u_t(0) = g \tag{17}$$

then we can write u explicity as

$$u = u_0 + \Box^{-1} F, \tag{18}$$

where

$$u_0(t) = \cos(t\sqrt{-\Delta}) f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g$$

$$\Box^{-1} F(t) = \int_{s<t} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s)\ ds.$$

One can localize these explicit formulae in frequency to obtain

$$\begin{aligned} S_j u_0(t) &= \sum_{\pm} U_j^{\pm}(t) S_j f \pm i 2^{-j} U_j^{\pm}(t) S_j g \\ S_j \Box^{-1} F(t) &= \sum_{\pm} \pm i 2^{-j} \int_{s<t} U_j^{\pm}(t) U_j^{\pm}(s)^* S_j F(s)\ ds \end{aligned} \tag{19}$$

for each integer j, where

$$U_j^{\pm}(t) = \beta(2^{-j}\sqrt{-\Delta}) e^{\pm it\sqrt{-\Delta}}$$

is a frequency localized evolution operator, and β is a Littlewood-Paley cutoff that varies from line to line. Henceforth we will suppress the $\pm$ symbols on $U_j^{\pm}$.

In [12] the following estimates[6] were proven:

Proposition 3.1. [12] *If* (q, r), $(\tilde{q}, \tilde{r})$ *are sharp wave-admissible pairs, and* $u(t)$ *is the solution to* (17), *then we have the one-sided estimates*

$$2^{-\frac{(n+1)j}{(n-1)q}} \|U_j(t) f\|_{L_t^q L_x^r} \lesssim \|f\|_2$$

$$\|\int_{s<t} U_j(t) U_j(s)^* F(s)\ ds\|_{C(L_x^2)} \lesssim 2^{\frac{(n+1)j}{(n-1)\tilde{q}}} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

together with the two-sided estimate

$$2^{-\frac{(n+1)j}{(n-1)q}} \|S_j u\|_{L_t^q L_x^r} + \|S_j u\|_{C(L_x^2)} \lesssim \|S_j f\|_2 + 2^{-j} \|S_j g\|_2 + 2^{-j} 2^{\frac{(n+1)j}{(n-1)\tilde{q}}} \|S_j F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

For examples and applications of these estimates, see [23], [14], [8], [15], [7], [21], [6].

The aim of this section is to prove the analogue of this propositon for the $X_k^{r,p}$ spaces. We begin with the basic energy and decay estimates we will need.

Lemma 3.2. *If* $j \geq k$, *we have the energy estimate*

$$\|U_j(t) f\|_{X_k^{2,2}} \lesssim \|f\|_2, \tag{20}$$

the decay estimate

$$\|U_j(t) U_j(s)^* F\|_{X_k^{\infty,2}} \lesssim (2^{2k-j}|t-s|)^{-\frac{n-1}{2}} \|F\|_{X_k^{1,2}}, \tag{21}$$

[6]Strictly speaking, these estimates was only proven (without time being localized) in [12] for $j = 0$, but the general result can be recovered by scaling.

and the asymmetric decay estimate

$$\|U_j(t)U_j(s)^*F\|_{X_k^{\infty,2}} \lesssim (2^{k-\frac{n}{n-1}j}|t-s|)^{-\frac{n-1}{2}}\|F\|_1, \tag{22}$$

for all $s \neq t$.

Proof The energy estimate follows immediately from Plancherel's theorem sincee $X_k^{2,2} = L^2$. To prove (21), it suffices by self-adjointness and interpolation to show that

$$\|U_j(t)U_j(s)^*F\|_{X_k^{\infty,1}} \lesssim (2^{2k-j}|t-s|)^{-\frac{n-1}{2}}\|F\|_{X_k^{1,1}}.$$

Since $X_k^{1,1} = L^1$ it suffices to verify this when F is a delta function, which we may place at the origin since the space $X_k^{\infty,1}$ is almost translation invariant. It now suffices to show that

$$\|U_j(t)U_j(s)^*\delta_0\|_{L^1(Q)} \lesssim (2^{2k-j}|t-s|)^{-\frac{n-1}{2}}.$$

for all 2^{-k}-cubes Q. Similarly, (22) will follow from

$$\|U_j(t)U_j(s)^*\delta_0\|_{L^2(Q)} \lesssim (2^{k-\frac{n}{n-1}j}|t-s|)^{-\frac{n-1}{2}}.$$

But these estimates are consequence of the standard stationary phase estimate

$$U_j(t)U_j(s)^*\delta_0(x) \leq C_N 2^{nj}(1+2^j|t-s|)^{-\frac{n-1}{2}}\left(1+2^j(|t-s|-|x|)\right)^{-N},$$

valid for any $N > 0$. Indeed, from these estimates we see that $U_j(t)U_j(s)^*\delta$ when restricted to Q has a sup norm of $O(2^{nj}(2^j|t-s|)^{-\frac{n-1}{2}})$ and is rapidly decreasing outside a set of measure $O(2^{-j}2^{-(n-1)k})$, and the claimed estimates follow from some algebraic manipulation. ∎

The estimates (20) and (21) imply the following family of one-sided Strichartz estimates:

Proposition 3.3. *If (q,r) is a sharp wave-admissible pair and $j \geq k$, then*

$$\|U_j(t)f\|_{L_t^q X_k^{r,2}} \lesssim 2^{-\frac{2k-j}{q}}\|f\|_2 \tag{23}$$

for all test functions f on $\mathbf{R}^n$.

Proof This is a special case of the abstract interpolation theorem [12], Theorem 10.1, although strictly speaking one must first rescale the time variable by 2^{2k-j} to satisfy the conditions of that theorem. Here we will present the proof for $q > 2$, so that we are excluding the endpoint $(2, r_0)$. At any rate, the endpoint of (23) is not essential for our regularity results.

By duality and the TT^* method the estimate is equivalent to the bilinear form estimate

$$\int_{|s|\lesssim 1}\int_{|t|\lesssim 1} |\langle U_j(s)^*F(s), U_j(t)G(t)\rangle|\, dtds \lesssim 2^{-2\frac{2k-j}{q}}\|F\|_{L_t^{q'}X_k^{r',2}}\|G\|_{L_t^{q'}X_k^{r',2}}.$$

From the Hardy-Littlewood inequality

$$\int\int |t-s|^{-\frac{2}{q}} f(t)g(s)\ dtds \lesssim \|f\|_{q'}\|g\|_{q'},$$

valid for $q > 2$, we see that it suffices to show that

$$|\langle U_j(s)^* F(s), U_j(t)G(t)\rangle| \lesssim 2^{-2\frac{2k-j}{q}} |t-s|^{-\frac{2}{q}} \|F(s)\|_{X_k^{r',2}} \|G(t)\|_{X_k^{r',2}}.$$

But this estimate is true for $q = \infty$, $r = 2$ by the energy estimate (20) and Cauchy-Schwarz, while for $q = 4/(n-1)$, $r = \infty$ the result follows from the decay estimate (21) and duality (the fact that q may be less than 1 is irrelevant). The general case then follows from interpolation and the assumption (12). ■

We are almost ready to state the frequency-localized two-sided Strichartz estimates from $L^2_t L^{r_0'}_x$ to $L^q_t X^{r,2}_k$. Unfortunately the optimal exponents for these estimates depend in a complicated way on the frequency scales j and k. Define the convex piecewise linear function $\alpha(j,k)$ for $j, k \geq 0$ as

$$\alpha(j,k) = \begin{cases} \frac{2n}{n-1}k - \frac{n+1}{n-1}j & \text{for } 0 \leq j \leq k \\ 2k - j & \text{for } k \leq j \leq 2k \\ 0 & \text{for } 2k \leq j \end{cases}$$

Equivalently, we may define $\alpha(j,k)$ to be the largest convex function such that

$$\alpha(0,k) = \frac{2n}{(n-1)}k, \quad \alpha(k,k) = k, \quad \alpha(2k,k) = 0, \quad \alpha(k,0) = 0 \tag{24}$$

for all $k \geq 0$.

Proposition 3.4. *If (q,r) is a sharp wave-admissible pair, j,k are non-negative integers, and u is the solution to (17), then*

$$2^{\frac{\alpha(j,k)}{q}} \|S_j u\|_{L^q_t X^{r,2}_k} + \|S_j u\|_{C(L^2_x)} \lesssim \|S_j f\|_2 + 2^{-j}\|S_j g\|_2 + 2^{-\gamma_0 j}\|S_j F\|_{L^2_t L^{r_0'}_x}.$$

Most of these estimates are proved by the existing Strichartz estimates and the embeddings mentioned in the previous section. The gain occurs when $k \leq j \leq 2k$, so that $\alpha(j,k) = 2k - j$. One can show using bump function examples and a combination of parallel Knapp examples that the estimates above are sharp, but we will not do so here.

Proof The claim involving $\|S_j u\|_{C(L^2_x)}$ follows directly from Proposition 3.1, since $(2, r_0)$ is sharp wave-admissible and

$$2^{-j} 2^{\frac{n+1}{2(n-1)j}} = 2^{-\gamma_0 j}.$$

Thus it remains to treat the contribution of $2^{\frac{\alpha(j,k)}{q}} \|S_j u\|_{L^q_t X^{r,2}_k}$. When $\alpha(j,k) = \frac{2n}{n-1}k - \frac{n+1}{n-1}j$ this follows from Proposition 3.1 and the estimate

$$\|S_j u\|_{L^q_t X^{r,2}_k} \lesssim 2^{\frac{2nk}{(n-1)q}} \|S_j u\|_{L^q_t L^r_x}$$

which follows from (15) and (12).

Similarly when $\alpha(j,k) = 0$ this follows from Proposition 3.1 and the estimate

$$\|S_j u\|_{L^q_t X^{r,2}_k} \lesssim \|S_j u\|_{L^\infty_t X^{2,2}_k} \lesssim \|S_j u\|_{C(L^2_x)},$$

which follows from Hölder's inequality, the time localization, and the inclusion (14).

Thus it remains to consider the case when $\alpha(j,k) = 2k - j$, so that $k \leq j \leq 2k$. The contribution of u_0 is dealt with in Proposition 3.3, so to finish the argument it suffices by (19) to show that

$$2^{\frac{2k-j}{q}} 2^{-j} \| \int_{s<t} U_j(t) U_j(s)^* F(s)\, ds \|_{L^q_t X^{r,2}} \lesssim 2^{-\gamma_0 j} \|F\|_{L^2_t L^{r_0'}_x} \tag{25}$$

for all Schwarz functions F. As is unfortunately the case in these types of estimates, the retarded integral (25) requires far more technical manipulation than the one-sided estimates proved earlier.

When $q = \infty, r = 2$ (25) follows from Proposition 3.1, so it suffices to verify (25) for the endpoint $q = 2, r = r_0$. We will adapt the argument in [12]. By duality (25) now becomes

$$\int \int_{s<t} |\langle U_j(s)^* F(s), U_j(t)^* G(t)\rangle|\, dtds \lesssim 2^{\frac{1}{2}(j-2k)} 2^{\frac{(n+1)}{2(n-1)} j} \|F\|_{L^2_t L^{r_0'}_x} \|G\|_{L^2_t X^{r_0',2}_k}.$$

It will suffice to show that

$$\sum_i |T_i(F,G)| \lesssim 2^{\frac{1}{2}(j-2k)} 2^{\frac{(n+1)}{2(n-1)} j} \|F\|_{L^2_t L^{r_0'}_x} \|G\|_{L^2_t X^{r_0',2}_k},$$

where for $i \leq 0$, $T_i(F,G)$ denotes the bilinear form

$$T_i(F,G) = \int \int_{t-s\sim 2^i} |\langle U_j(s)^* F(s), U_j(t)^* G(t)\rangle|\, dtds.$$

By Proposition 2.4 it will suffice to show that

$$2^{\beta(a,b) i} T_i(F,G) \lesssim 2^{\gamma(a,b,k,j)} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t X^{b',2}_k}$$

for all $i \in \mathbf{Z}$ and $(\frac{1}{a}, \frac{1}{b})$ in a neighbourhood of $(\frac{1}{r_0}, \frac{1}{r_0})$, where

$$\beta(a,b) = \frac{n-1}{2}\left(\frac{2}{r_0} - \frac{1}{a} - \frac{1}{b}\right).$$

$$\gamma(a,b,k,j) = \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{b}\right)(j-2k) + \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{a}\right) j.$$

By localization and time translation invariance it suffices to show that

$$|T_i(F,G)| \lesssim 2^{-\beta(a,b) i} 2^{\gamma(a,b,k,l)} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t X^{b',2}_k} \tag{26}$$

whenever $F(t), G(s)$ are supported on the time interval $|t|, |s| \lesssim 2^i$. We will prove this for the exponent pairs $(a,b) = (\infty,\infty)$, $(2,2)$, $(r_0, 2)$, and $(2, r_0)$, since the claim then follows by interpolation and the fact that $2 < r_0 < \infty$ (cf. [12]).

To prove the estimate when $(a,b) = (\infty,\infty)$ we use (22) and duality to obtain

$$|\langle U_j(s)^* F(s), U_j(t)^* G(t)\rangle| \lesssim (2^{k-\frac{n}{n-1}j}|t-s|)^{-\frac{n-1}{2}} \|F(s)\|_1 \|G(t)\|_{X_k^{1,2}}.$$

Integrating this over $|t-s| \sim 2^i$ we obtain

$$|T_i(F,G)| \lesssim 2^{-\frac{n-1}{2}i} 2^{-\frac{n-1}{2}k} 2^{\frac{n}{2}j} \|F\|_{L^1_t L^1_x} \|G\|_{L^1_t X_k^{1,2}},$$

and (26) follows from Hölder's inequality and some algebra.

Similarly, when $(a,b) = (2,2)$, we use Cauchy-Schwarz and energy estimates to obtain

$$|\langle U_j(s)^* F(s), U_j(t)^* G(t)\rangle| \lesssim \|F(s)\|_2 \|G(t)\|_{X_k^{2,2}},$$

which after integration becomes

$$|T_i(F,G)| \lesssim \|F\|_{L^1_t L^2_x} \|G\|_{L^1_t X_k^{2,2}},$$

and (26) again follows from Hölder's inequality.

When $(a,b) = (r_0, 2)$ we write

$$|T_i(F,G)| = |\int_t \left\langle \int_{t-s\sim 2^i} U_j(s)^* F(s)\ ds, U_j(t)^* G(t) \right\rangle\ dt|$$

and use Cauchy-Schwarz and (20) to obtain

$$|T_i(F,G)| \lesssim \sup_t \| \int_{t-s\sim 2^i} U_j(s)^* F(s)\ ds\|_2 \|G\|_{L^1_t X_k^{2,2}}.$$

However, from Proposition 3.1 we obtain

$$\| \int_{t-s\sim 2^i} U_j(s)^* F(s)\ ds\|_2 \lesssim 2^{\frac{(n+1)j}{2(n-1)}} \|F\|_{L^2_t L^{r_0'}_x},$$

and by inserting this into the previous estimate we obtain (26) after using Hölder's inequality.

The case $(a,b) = (2, r_0)$ is similar. Proceeding in analogy with the previous case we have

$$|T_i(F,G)| \lesssim \sup_s \| \int_{t-s\sim 2^i} U_j(t)^* G(t)\ dt\|_2 \|F\|_{L^1_t L^2}.$$

But from the adjoint of (3.3) we obtain

$$\| \int_{t-s\sim 2^i} U_j(t)^* G(t)\ dt\|_2 \lesssim 2^{-\frac{2k-j}{2}} \|G\|_{L^2_t X_k^{r_0,2}},$$

and inserting this into the previous estimate we obtain (26) after using Hölder's inequality. ∎

4. Proof of main theorem

Suppose that $n > 3$, $0 < \gamma < \gamma_0$, and (4) and (8) hold. Since $\gamma < \gamma_0$ we have from (4) and some algebra that

$$p < p_0 = \frac{(n+1)^2}{(n-1)^2 + 4}.$$

We also make the technical assumption that $p > \frac{n+1}{n-1}$; the low power case $p \leq \frac{n+1}{n-1}$ can be handled by Proposition 1.1, and appears in [9]. Since $n \geq 4$, our assumptions on p thus yield

$$\frac{n+2}{n}, \frac{n+1}{n-1} < p < \frac{n+3}{n-1}, 2. \tag{27}$$

Let f, g be data such that

$$\|f\|_{H^\gamma} + \|g\|_{H^{\gamma-1}} \lesssim M, \tag{28}$$

for some $M > 0$. We will show that there exists a time $0 < T \ll 1$ that depends only on M, n, γ, p, and the constant in (2), such that a solution u to (1) exists in $C(H^\gamma)$.

We write the equation (1) as an integral equation

$$u = u_0 + \Box^{-1} F(u), \tag{29}$$

where the notation is as in the previous section.

By the method of Picard iteration, to show the existence of a solution u to (1) it suffices to show that the map $u \to u_0 + \Box^{-1} F(u)$ is a contraction in some metric space that contains u_0. This space will be constructed using the numerology used to solve (10). Let $r = pr_0'$, and let q be defined by (12). From (27) we see that (q, r) is sharp wave-admissible. We also have the inequalities

$$2 \leq \frac{q}{p} \tag{30}$$

$$\frac{\gamma_0 - \gamma}{p} - \frac{2n}{(n-1)q} + \gamma + \frac{1}{q} \geq 0 \tag{31}$$

$$\frac{\gamma_0 - \gamma}{p} - \frac{2n}{(n-1)q} + 2\gamma \geq 0; \tag{32}$$

indeed, (30) simplifies to $p \leq \frac{n+3}{n-1}$, while (31), (32) are equivalent to (4) and (8) respectively. Since we are explicitly excluding the endpoint (9), we see that at least one of (31), (32) is satisfied with strict inequality.

We now iterate in the ball $\{u : \|u\|_* \lesssim M\}$, where the Besov-like norm $\|\|_*$ is given by

$$\|u\|_* = \|u\|_{C(H^\gamma)} + (\sum_{j \geq 0} \|S_j u\|_{*,j}^2)^{1/2},$$

and the partial norms $\|\|_{*,j}$ are given by

$$\|u\|_{*,j} = 2^{\gamma j} \sup_k 2^{\frac{\alpha(j,k)}{q}} \|u\|_{L^q_t X^{r,2}_k}.$$

From Proposition 3.4 we see that

$$\|S_j u_0\|_{*,j} \lesssim 2^{\gamma j}(\|S_j f\|_2 + 2^{-j}\|S_j g\|_2)$$

uniformly in j. Thus from (28) and Plancherel's theorem we thus have that $\|u_0\|_* \lesssim M$, as desired.

It remains to show that the above map is a contraction; note that this will give existence and uniqueness in $\|\|_*$, with the solution depending continuously on the data in $\|\|_*$, and hence in $C(H^\gamma)$.

It suffices to show that

$$\|\Box^{-1}(F(u) - F(v))\|_* \ll \|u - v\|_* \tag{33}$$

whenever

$$\|u\|_*, \|v\|_* \lesssim M. \tag{34}$$

By using Proposition 3.4 as before, we obtain

$$\|S_j \Box^{-1} F\|_{*,j} \lesssim 2^{(\gamma-\gamma_0)j} \|S_j F\|_{L^2_t L^{r_0'}_x}$$

and from Proposition 3.1 we obtain

$$\|S_j \Box^{-1} F\|_{C(H^\gamma)} \lesssim \left(\sum_j (2^{(\gamma-\gamma_0)j} \|S_j F\|_{L^2_t L^{r_0'}_x})^2 \right)^{1/2},$$

for all functions F and $j \geq 0$. Also, from the energy estimate and Sobolev embedding we have

$$\|P_0 \Box^{-1} F\|_{C(H^\gamma)} \sim \|P_0 \Box^{-1} F\|_{C(L^2)} \lesssim \|P_0 F\|_{L^1_t \dot H^{-1}} \lesssim \|P_0 F\|_{L^1_t L_x^{\frac{2n}{n+2}}}.$$

Applying all these estimates to $F(u) - F(v)$ and using Plancherel's theorem one obtains

$$\|\Box^{-1}(F(u)-F(v))\|_* \lesssim \|P_0(F(u) - F(v))\|_{L^1_t L_x^{\frac{2n}{n+2}}} + \left(\sum_{j \geq 0} (2^{(\gamma-\gamma_0)j} \|S_j(F(u) - F(v))\|_{L^2_t L^{r_0'}_x})^2 \right)^{1/2}.$$

Thus (33) will follow from the non-linear estimates

$$\|P_0(F(u) - F(v))\|_{L^1_t L_x^{\frac{2n}{n+2}}} \ll \|u - v\|_{C(H^\gamma)} \tag{35}$$

and

$$\sum_{j\geq 0}(2^{(\gamma-\gamma_0)j}\|S_j(F(u)-F(v))\|_{L^2_t L^{r_0'}_x})^2 \ll \|u-v\|^2_{C(H^\gamma)} + \sum_{j\geq 0}\|S_j(u-v)\|^2_{*,j}. \tag{36}$$

We first deal with the low-frequency estimate (35), which is very easy. From (2), Hölder's inequality we have

$$\|F(u)-F(v)\|_{C(L^{2/p}_x)} \lesssim \|u-v\|_{C(L^2)}(\|u\|^{p-1}_{C(L^2)} + \|v\|^{p-1}_{C(L^2)}).$$

u By another application of Hölder's inequality, (34), and the inclusion $L^2 \subset H^\gamma$ we thus have

$$\|F(u)-F(v)\|_{L^1_t L^{2/p}_x} \lesssim TM^{p-1}\|u-v\|_{C(H^\gamma)}.$$

But since P_0 is given by convolution with a bump function, (35) follows from Young's inequality (if T is sufficiently small), since one has $1 < \frac{2}{p} < \frac{2n}{n+2}$ from (27).

We now turn to the high-frequency estimate (36). We require the following estimates.

Lemma 4.1. *There exists an $\varepsilon > 0$ such that*

$$\|S_k(F(u)-F(v))\|_{L^2_t L^{r_0'}_x} \lesssim T^\varepsilon 2^{\frac{2nkp}{(n-1)q}}\|u-v\|_{L^q_t X^{r,2}_k}(\|u\|^{p-1}_{L^q_t X^{r,2}_k} + \|v\|^{p-1}_{L^q_t X^{r,2}_k}). \tag{37}$$

for any u, v.

Proof From (30), Hölder's inequality, and the definition of r we have

$$\|S_k(F(u)-F(v))\|_{L^2_t L^{r_0'}_x} \lesssim T^\varepsilon \|S_k(F(u)-F(v))\|_{L^{q/p}_t L^{r/p}_x}$$

for some $\varepsilon > 0$. From Lemma 2.2 we have

$$\|S_k(F(u)-F(v))\|_{L^{q/p}_t L^{r/p}_x} \lesssim 2^{nk(1-\frac{p}{r})}\|F(u)-F(v)\|_{L^{q/p}_t X^{r/p,1}_k}.$$

But from (2) and Hölder's inequality we have

$$\|F(u)-F(v)\|_{L^{q/p}_t X^{r/p,1}_k} \lesssim \|u-v\|_{L^q_t X^{r,p}_k}(\|u\|^{p-1}_{L^q_t X^{r,p}_k} + \|v\|^{p-1}_{L^q_t X^{r,p}_k}).$$

By (15) and (27) the right-hand side is dominated by

$$2^{-nk(1-\frac{p}{2})}\|u-v\|_{L^q_t X^{r,2}_k}(\|u\|^{p-1}_{L^q_t X^{r,2}_k} + \|v\|^{p-1}_{L^q_t X^{r,2}_k}).$$

Combining all these estimates and using (12) the lemma follows. ∎

Lemma 4.2. *There exist $i \in \{1,2\}$ and $\varepsilon > 0$ such that*

$$\|f\|_{L^q_t X^{r,2}_k} \lesssim 2^{\frac{\gamma_0-\gamma}{p}k}2^{-\frac{2nk}{(n-1)q}}(2^{-\varepsilon k}\|f\|_{C(H^\gamma)} + \sum_j 2^{-\varepsilon|j-ik|}\|S_j f\|_{*,j}). \tag{38}$$

for all f.

Proof From (13) it suffices to show that

$$\|P_0 f\|_{L^q_t X^{r,2}_k} \lesssim 2^{\frac{\gamma_0-\gamma}{p}k} 2^{-\frac{2nk}{(n-1)q}} 2^{-\varepsilon k} \|f\|_{C(L^2)} \tag{39}$$

and

$$\|S_j f\|_{L^q_t X^{r,2}_k} \lesssim 2^{-\frac{\gamma-\gamma_0}{p}k} 2^{-\frac{2nk}{(n-1)q}} 2^{-\varepsilon|j-ik|} \|S_j f\|_{*,j}. \tag{40}$$

We first consider the low-frequency estimate (39). From Proposition 2.3 and Hölder's inequality we have

$$\|P_0 f\|_{L^q_t X^{r,2}_k} \lesssim 2^{-nk(\frac{1}{2}-\frac{1}{r})} \|f\|_{C(L^2)},$$

and so (39) reduces to

$$-n(\frac{1}{2}-\frac{1}{r}) \leq \frac{\gamma_0-\gamma}{p} - \frac{2n}{(n-1)q} - \varepsilon,$$

which reduces using (12) to the hypothesis $\gamma < \gamma_0$.

We now turn to (40). From the definition of $\|\|_{*,j}$ it suffices to show that

$$1 \leq 2^{-\frac{\gamma-\gamma_0}{p}k} 2^{-\frac{2nk}{(n-1)q}} 2^{-\varepsilon|j-ik|} 2^{\gamma j} 2^{\frac{\alpha(j,k)}{q}}.$$

uniformly in j and k, for some $\varepsilon > 0$. This reduces to showing that

$$(\frac{\gamma_0-\gamma}{p} - \frac{2n}{(n-1)q})k + \gamma j + \frac{\alpha(j,k)}{q} \geq \varepsilon|j-ik|.$$

By the convexity of α it suffices to verify this inequality for the four ranges in (24). Dividing by k, it thus suffices to verify that

$$\begin{aligned}
(\frac{\gamma_0-\gamma}{p} - \frac{2n}{(n-1)q}) + \frac{2n}{(n-1)q} &\geq \varepsilon|0-i| \\
(\frac{\gamma_0-\gamma}{p} - \frac{2n}{(n-1)q}) + \gamma + \frac{1}{q} &\geq \varepsilon|1-i| \\
(\frac{\gamma_0-\gamma}{p} - \frac{2n}{(n-1)q}) + 2\gamma &\geq \varepsilon|2-i| \\
\gamma &\geq \varepsilon|1-0i|.
\end{aligned}$$

The first and fourth inequality follow from the hypothesis $0 < \gamma < \gamma_0$. From (31) and (32) we see that the second and third inequalities are satisfied with $\varepsilon = 0$, and since at least one of these inequalities is assumed to hold with strict inequality one can make $\varepsilon > 0$ by choosing i appropriately. ∎

Applying (38) to u,v we obtain

$$\|u\|_{L^q_t X^{r,2}_k}, \|v\|_{L^q_t X^{r,2}_k} \leq T^\varepsilon M 2^{-\frac{\gamma-\gamma_0}{p}k} 2^{-\frac{2nk}{(n-1)q}}, \tag{41}$$

since we have from (34) that

$$2^{-\varepsilon k}\|u\|_{C(H^\gamma)} + \sum_j 2^{-\varepsilon|j-ik|} \|S_j u\|_{*,j} \lesssim \|u\|_* \lesssim M,$$

and similarly for v. If we also apply (38) to $u-v$, and insert the resulting inequality and (41) into (37), one obtains

$$\|S_k(F(u)-F(v))\|_{L_t^2 L_x^{r_0'}} \leq T^{\varepsilon p} M^p 2^{-(\gamma-\gamma_0)k}$$
$$(2^{-\varepsilon k}\|u-v\|_{C(H^\gamma)} + \sum_j 2^{-\varepsilon|j-ik|}\|S_j(u-v)\|_{*,j}).$$

Thus, the left-hand side of (36) is majorized by

$$T^{2\varepsilon p} M^{2p}\left(\|u-v\|^2_{C(H^\gamma)} + \sum_{k\geq 0}(\sum_{j\geq 0} 2^{-\varepsilon|j-ik|}\|S_j u - S_j v\|_{*,j})^2\right).$$

By the Cauchy-Schwarz inequality this is majorized by

$$T^{2\varepsilon p} M^{2p}\left(\|u-v\|^2_{C(H^\gamma)} + \sum_{j\geq 0}\sum_{k\geq 0} 2^{-\frac{1}{2}\varepsilon|j-ik|}\|S_j u - S_j v\|^2_{*,j}\right),$$

and (36) follows by evaluating the k-summation if T is sufficiently small. This concludes the proof.

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